THE SYZYGY STABILITY OF CURVES OF GENUS 7

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We show the following about the syzygy points of genus 7 curves. (See [DFS14] for the definition of syzygy points.)

(1) A general curve of genus 7 has a stable first syzygy point.
(2) A general tetragonal curve of genus 7 has (at least) a semistable first syzygy point.
(3) A Casnati–Ekedahl special tetragonal curve of genus 7 has a strictly semistable first syzygy point. The syzygy points of all such curves coincide and this syzygy point is the syzygy point of the anti-canonically embedded del Pezzo surface of degree 6.

1. Generic stability

Proposition 1. *A general curve of genus 7 has a stable first syzygy point.*

Proof. It suffices to exhibit one canonical curve with stable first syzygy point. Let C be the *Fricke–Macbeath curve* of genus 7 [Mac65]. It has automorphism group $\mathrm{PSL}_8(\mathbf{F}_2)$ (whose order realizes the Hurwitz bound $84(g-1)$ for the maximal number of automorphisms). The $\mathrm{Aut}(C)$ representation $H^0(C, \omega_C)$ must be irreducible (simply because $\mathrm{PSL}(2,8)$ does not have any non-trivial irreducible representation of smaller dimension). By Kempf's criterion [Kem78, § 5], the syzygy point of C is stable. □

2. Syzygies of tetragonal curves

Let E be the vector bundle $O(3) \oplus O(3) \oplus O(4)$ on $\mathbf{P}^1$. Let ζ on $\mathbf{P}E$ be the class of $O_{\mathbf{P}E}(1)$. A general tetragonal curve of genus 7 is embedded naturally in $\mathbf{P}E$ via the relative canonical bundle $\omega_{C/\mathbf{P}^1}$. Its image in $\mathbf{P}E$ is a complete intersection of two surfaces. For a general tetragonal curve, the two surfaces have class $2\zeta - 5f$. There is codimension one sub-locus of the space of tetragonal curves where the two surfaces have classes $2\zeta - 4f$ and $2\zeta - 6f$, respectively. We call the tetragonal curves of the later kind *Casnati–Ekedahl special* and call their locus the *Casnati–Ekedahl* locus.

2.1. Generators and relations for the ideal of $\mathbf{P}E$.

We first describe the generators and relations for the ideal of $C \subset \mathbf{P}^6$ coming from the scroll $\mathbf{P}E$. Observe that

$$\omega_C = (\zeta - 2f)|_C.$$

Set $\omega = O(\zeta - 2f)$. We have

$$H^0(\mathbf{P}E, \omega) = H^0(\mathbf{P}^1, O(1) \oplus O(1) \oplus O(2)) = \langle U_1, U_2, V_1, V_2, W_1, W_2, W_3 \rangle.$$

The line bundle ω embeds $\mathbf{P}E$ as a quartic scroll in $\mathbf{P}^6$. The ideal of $\mathbf{P}E \subset \mathbf{P}^6$ is generated by the six 2×2 minors of the matrix

$$\begin{pmatrix} U_1 & V_1 & W_1 & W_2 \\ U_2 & V_2 & W_2 & W_3 \end{pmatrix}.$$

Denote by $Q_{i,j}$ the determinant of the minor formed by columns i and j. There are eight syzygies among these quadrics. These are obtained by choosing three of the four columns, one of the two rows, and using that the 3×3 matrix formed by the three chosen columns with the chosen row repeated has zero determinant. For example, choosing the first three columns and repeating the first row gives the relation

$$U_1\cdot Q_{2,3}-V_1\cdot Q_{1,3}+W_1\cdot Q_{1,2}=0.$$

There are no further relations among the $Q_{i,j}$. Thus, the scroll $\mathbf{P}E$ accounts for six quadric generators out of ten, and eight syzygies out of sixteen of the canonically embedded curve $C\subset\mathbf{P}^6$.

2.2. **Generators and relations for the ideal of C.** We now describe the remaining quadrics and relations for the ideal of C. We must make two cases: one for a general tetragonal curve, and another for a Casnati–Ekedahl special curve. First assume that C is general. Then $I_{C/\mathbf{P}E}$ is generated by two sections, say α and β, of $O(2\zeta-5f)$. Observe that

$$O(2\zeta-5f)=\omega^{\otimes 2}\otimes O(-f).$$

Let $\langle s,t\rangle=H^0(\mathbf{P}E,O(f))=H^0(\mathbf{P}^1,O(1))$. Then $s\alpha$, $s\beta$, $t\alpha$, and $t\beta$ are the remaining four quadric generators of the canonical ideal of C. In other words,

$$I_{C/\mathbf{P}^6}(2)=\langle Q_{i,j},s\alpha,s\beta,t\alpha,t\beta\rangle.$$

In addition to the eight syzygies described above, there must be eight additional syzygies among these quadrics. To describe them, let ξ be an element of $H^0(\mathbf{P}E,\omega\otimes O(-f))=H^0(\mathbf{P}^1,O\oplus O\oplus O(1))$. Each such ξ gives two relations

$$s\xi\otimes t\alpha-t\xi\otimes s\alpha,\quad s\xi\otimes t\beta-t\xi\otimes s\beta.$$

Since ξ is chosen from a four dimensional space, we get eight additional syzygies, as required. In summary, the quadrics and syzygies of a general tetragonal curve are given by

$$\begin{aligned}\text{Quadrics}&=\langle\text{Quadrics } Q_{i,j}\text{ of }\mathbf{P}E\subset\mathbf{P}^6\rangle\oplus\langle s\alpha,s\beta,t\alpha,t\beta\rangle.\\ \text{Syzygies}&=\langle\text{Syzygies among } Q_{i,j}\rangle\oplus\langle s\xi\otimes t\alpha-t\xi\otimes s\alpha,s\xi\otimes t\beta-t\xi\otimes s\beta\rangle,\\ &\qquad\text{where }\xi\in H^0(\mathbf{P}E,O(\zeta-3f)).\end{aligned}\tag{1}$$

Next, assume that C is Casnati–Ekedahl special. In this case $I_{C\mathbf{P}E}$ is generated by a section, say α, of $O(2\zeta-4f)$, and a section β of $O(2\zeta-6f)$. Since $O(2\zeta-4f)=\omega^{\otimes 2}$ and $O(2\zeta-6f)=\omega^{\otimes 2}\otimes O(-2f)$, the four quadrics in $\mathbf{P}E$ that we get from α and β are

$$\alpha,s^2\beta,st\beta,\text{ and }t^2\beta.$$

As before, let ξ be an element of $H^0(\mathbf{P}E,\omega(-f))$. Each such ξ gives two relations

$$s\xi\otimes s^2\beta-t\xi\otimes st\beta,s\xi\otimes t^2\beta-t\xi\otimes st\beta.$$

Again, with ξ coming from a four dimensional space, we recover the remaining eight syzygies. In summary, the quadrics and syzygies of a Casnati–Ekedahl special tetragonal curve are

$$\begin{aligned}\text{Quadrics}&=\langle\text{Quadrics } Q_{i,j}\text{ of }\mathbf{P}E\subset\mathbf{P}^6\rangle\oplus\langle \alpha,s^2\beta,st\beta,t^2\beta\rangle.\\ \text{Syzygies}&=\langle\text{Syzygies among } Q_{i,j}\rangle\oplus\langle s\xi\otimes s^2\beta-t\xi\otimes st\beta,s\xi\otimes t^2\beta-t\xi\otimes st\beta\rangle,\\ &\qquad\text{where }\xi\in H^0(\mathbf{P}E,O(\zeta-3f)).\end{aligned}\tag{2}$$

Observe that the syzygies only depend on $\mathbf{P}E$ and β.

Keep the above setup of a Casnati–Ekedahl special C. Let $X\subset\mathbf{P}E$ be the vanishing locus of β. Recall that $\omega=O_{\mathbf{P}E}(\zeta-2f)$.

Proposition 2. *The surface X is a del Pezzo surface of degree 6. The line bundle ω restricts to the anti-canonical line bundle of X. The ideal of $X \subset \mathbf{P}^6$ is generated by 9 quadrics and the module of syzygies is generated by 16 linear syzygies. In particular, the linear syzygies among the generators of $I_{X/\mathbf{P}^6}$ coincide with the linear syzygies among the generators of $I_{C/\mathbf{P}^6}$.*

Proof. Let $\pi\colon \mathbf{P}E \to \mathbf{P}^1$ be the projection. The relative Euler sequence

$$0 \to \Omega_{\mathbf{P}E/\mathbf{P}^1} \to \pi^* E \otimes O_{\mathbf{P}E}(-1) \to O_{\mathbf{P}E} \to 0$$

gives

$$K_{\mathbf{P}E} = O_{\mathbf{P}E}(-3) \otimes \pi^* \det E \otimes \pi^* K_{\mathbf{P}^1} = O_{\mathbf{P}E}(-3\zeta + 8f).$$

Since $O_{\mathbf{P}E}(X) = O(2\zeta - 6f)$, by adujnction we get

$$K_X = K_{\mathbf{P}E} \otimes O_{\mathbf{P}E}(X) = O(-\zeta + 2f)|_X = \omega^{-1}|_X.$$

Note that $\omega = \zeta - 2f$ is ample on $\mathbf{P}E$. Therefore, X is a surface with ample anti-canonical divisor $\omega|_X$. Using $\zeta^3 = 10\zeta^2$, we compute the degree

$$\omega|_X^2 = (\zeta - 2f)^2(2\zeta - 6f) = 6.$$

It is easy to verify that the anti-canonical image of $X \subset \mathbf{P}^6$ is cut out by 9 quadrics with 16 linear syzygies. It follows that these must be the 9 quadrics and 16 syzygies in (2) (excluding the quadric α). In particular, the linear syzygies among the generators of the ideal of X are the same as those of C. □

3. Semistability

Proposition 3. *The del Pezzo surface of degree six has a semi-stable first syzygy point in the anti-canonical embedding.*

Proof. Let X be the del Pezzo surface. Note that the torus $\mathbf{G}_m^2$ acts on X. The action makes $H^0(X, -K_X)$ a $\mathbf{G}_m^2$ representation. It is easy to check that this representation is multiplicity free. Concretely, we think of X as the blow up of $\mathbf{P}^2 = \operatorname{Proj} k[X, Y, Z]$ at the three torus fixed points. Then $H^0(X, -K_X)$ is identified with the cubics passing through these three points, that is, with the vector space

$$\langle X^2Y, XY^2, X^2Z, XZ^2, Y^2Z, YZ^2, XYZ \rangle.$$

Let the torus act by

$$(s, t) \cdot [X : Y : Z] \mapsto [sX : tY : Z].$$

Then the characters of the sections are given in order by

$$(1, 0), (0, 1), (1, -1), (0, -1), (-1, 1), (-1, 0), (0, 0).$$

By the Kempf–Morisson theorem, the syzygy point of X is SL_6 semi-stable if and only if it is T semi-stable, where $T \subset \mathrm{SL}_6$ is the maximal torus that acts diagonally on the basis vectors of $H^0(X, -K_X)$ listed above. This torus stability can be checked explicitly by a computer. See `delpezzo.m2` and `mcsyzygy.m2` for the Macaulay2 code that does this computation. □

Corollary 1. *A Casnati–Ekedahl special tetragonal curve of genus 7 has a semi-stable first syzygy point in the canonical embedding.*

Proof. The first syzygy point of such curve coincides with the first syzygy point of the del Pezzo surface of degree six. □

Corollary 2. *A general tetragonal curve of genus 7 has a semi-stable first syzygy point in the canonical embedding.*

Proof. Since semi-stability is an open condition, the assertion follows from the semi-stability of the first syzygy point of a Casnati–Ekedahl special curve. □

References

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