COUNTING 3-UPLE VERONESE SURFACES

ANAND DEOPURKAR & ANAND PATEL

1. Introduction

The classical fact that $d+3$ general points in d-dimensional projective space $\mathbb{P}^d$ determine a unique rational normal curve can be seen in many ways:

- By explicit algebraic construction.
- By Steiner's geometric construction.
- By an elementary degeneration argument.
- By an application of Goppa's lemma from the theory of association or Gale duality.

By a *d-uple Veronese n-fold* we mean any variety in $\mathbb{P}^{\binom{n+d}{n}-1}$ projectively equivalent to the standard d-uple Veronese image of $\mathbb{P}^n$. A parameter count uncovers an infinite array of enumerative problems, with the rational normal curves occupying only the first column:

Problem 1.1. *Determine the number of d-uple Veronese n-folds passing through $\binom{n+d}{d}+n+1$ general points.*

These numbers will be denoted $v_{d,n}$. Theorem 1.1 has seen virtually no advancement beyond the case of rational normal curves. Arthur Coble, about a century ago, used his new theory of association to find that 9 general points in $\mathbb{P}^5$ determine precisely 4 2-uple Veronese surfaces [Cob22, Theorem 19]. The configuration of these four surfaces is as special as Coble's argument showing $v_{2,2}=4$. He discovered, using what is now called Goppa's lemma, that a unique elliptic normal sextic curve $E \subset \mathbb{P}^5$ interpolates through the 9 points. This implies that the 2-uple Veronese surfaces containing all 9 points must entirely contain E, an exceptional circumstance. This means that each surface corresponds to choosing a square root of the degree 6 line bundle $\mathcal{O}_E(1)$. And so, Coble established a correspondence

$$\left\{ \begin{array}{c} \textit{2-uple Veronese surfaces} \\ \textit{containing the 9 points} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Line bundles } L \textit{ on } E \\ \textit{satisfying } L^2 \simeq \mathcal{O}_E(1) \end{array} \right\}, \tag{1}$$

from which he deduced $v_{2,2}=4$. There is currently no explanation for the number 4 without the curve E. Specifically, no approach parallel to those available for the $n=1$ case is known.

Our work in this paper solves the next case of Theorem 1.1.

Theorem 1.2. *Thirteen general points in $\mathbb{P}^9$ determine 4246 3-uple Veronese surfaces, i.e. $v_{3,2}=4246$.*

A caricature of the proof best serves to explain the contents of the paper. The first step is to use a correspondence like (1), though more intricate. It is the content of Theorem 2.3 in §2. This correspondence trades the counting of 3-uple Veronese surfaces for the counting of certain triples of points in the plane called *singular triads* $T \in \mathrm{Hilb}_3 \mathbb{P}^2$. It first appeared in the work of the second author and A. Landesman [LP19] on interpolation [1]. We give a thorough account to keep things self-contained.

In order to count singular triads we need access to a vector bundle which to a point $T \in \mathrm{Hilb}_3 \mathbb{P}^2$ assigns the vector space $H^0(\mathbb{P}^2, \mathcal{I}_T^2(5))$ of quintic forms singular at T. Unfortunately this vector bundle doesn't exist because of a familiar failure of flatness: the scheme obtained by squaring the ideal of a length 3 scheme is not always a length 9 scheme. And so the second step is to deal with this non-vector-bundle. We swap out the Hilbert scheme for a birational modification we call the space of *complete triangles* CT. The name is chosen because of many similarities it shares with the space of complete conics. We study the geometry of CT in §3. While it can be shown that CT is the quotient of the space of *ordered* triangles studied by S. Keel in [Kee93] by the natural symmetric group action, our construction of CT is of independent interest as it does not require first ordering triangles. The vector bundle we sought in the previous paragraph exists over CT, and we gain an enumerative understanding of it using Atiyah-Bott localization. At least at first glance, the set of singular triads is the degeneracy scheme in CT of a vector bundle map involving our newfound bundle. Porteous' formula, implemented using `sage`, then suggests that the number we seek is 57728.

All is not well, however, because there is still a gnarly excess in the Porteous setup. Our third step is to circumvent this new complication by further linearizing the problem, switching to a 26-dimensional Grassmannian bundle over CT which we call the space of *singular quintic pencils* SQP. Only in SQP do we finally find an excess-free vantage point. *Proving* the lack of excess is painful, requiring a combination of dimension counting and limit linear series arguments. This verification is the subject of §4, and is the content behind Theorem 4.2 which expresses $\nu_{3,2}$ as an integral:

$$\nu_{3,2} = \int_{\mathsf{SQP}} [\mathsf{Dom}(p)]^{13}. \tag{2}$$

The details are not so important right now, but $\mathsf{Dom}(p)$ is a relevant codimension 2 subvariety of SQP. In §5 we compute the integral (2) using Atiyah-Bott localization, performed with the help of `sage`. Finally, in §6 we discuss some of the many questions emerging from our investigation. We've included the sage code we used for the calculations in §7.

1.1. **Relation to other work.** Apart from the obvious connection to Coble's work, the present paper is related to some other work which deserves mentioning. Despite much of the progress on curve counting, there aren't many examples of counts of higher dimensional varieties. The closest work in this sense is due to Coskun in [Cos06b, Cos06a]. While our construction of the space of (unordered) complete triangles is novel, it has an *ordered* antecedent in the work of Collino and Fulton in [CF+89] and in the work of Keel in [Kee93].

[1] It is quite plausible that Coble, the foremost expert on relationships between association and Cremona transformations, knew about the correspondence in Theorem 2.3. One hypothesis as to why he didn't write about this particular correspondence might be that the resulting problem of counting singular triads posed too many complications given the technology available at the time.

1.2. **Notation and conventions.** Our ground field $\Bbbk$ is algebraically closed and of characteristic 0. All schemes considered in the paper are separated and of finite-type over $\Bbbk$. If A is a $\Bbbk$-vector space, and if X is a scheme then $\underline{A}$ will denote the constant vector bundle on X whose fibers are A. If Z is a closed subscheme of a scheme X then $\mathcal{I}_Z$ will denote its ideal sheaf.

If V is a vector bundle, then $\mathbb{P}V$ denotes its projectivization which represents lines in V. In particular, $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)) = V^*$ canonically. Similarly $\mathrm{Gr}(k, V)$ denotes the Grassmannians representing k-dimensional subspaces.

2. The correspondence

The fundamental trick for computing $\nu_{3,2}$ is to switch to the task of counting corresponding *triples* of non-collinear points $\{a, b, c\} \subset \mathbb{P}^2$ satisfying certain conditions relative to 13 prescribed, general points $\Gamma_{13} \subset \mathbb{P}^2$. It is the content of Theorem 2.3 in this section. Whether similar useful correspondences are available for determining other $\nu_{d,n}$'s remains an intriguing question.

The correspondence critically uses Coble's theory of association, and we begin by reviewing this theory following the incisive account in [EP00].

2.1. **Association.** We let R be a Gorenstein 0-dimensional $\Bbbk$-algebra of length d, and we let $\Gamma = \operatorname{Spec} R$, and write

$$\pi : \Gamma \longrightarrow \operatorname{Spec} \Bbbk$$

for the structural morphism. The Gorenstein condition says that the dualizing sheaf ω_π, associated to the R-module

$$\mathrm{Hom}_{\Bbbk}(R, \Bbbk),$$

is invertible and in fact generated by the trace functional. The evaluation map

$$ev : \pi_* \omega_\pi \longrightarrow \Bbbk$$

sends a functional $f \in \mathrm{Hom}_{\Bbbk}(R, \Bbbk)$ to $f(1)$.

Given a line bundle L on Γ (equivalently a rank 1 free R-module), one obtains a natural pairing

$$\langle , \rangle : \pi_* L \times \pi_* \left(L^{-1} \otimes_{\mathcal{O}_\Gamma} \omega_\pi \right) \longrightarrow \Bbbk$$

which is a perfect pairing between two d-dimensional $\Bbbk$-vector spaces, thanks to the Gorenstein property. So, if $V \subset H^0(L) = \pi_* L$ is any $r+1$-dimensional vector subspace, then we can define its *associated subspace*

$$V^\perp \subset H^0(L^{-1} \otimes \omega_\pi)$$

to be V's orthogonal complement with respect to $\langle , \rangle$.

Remark 2.1. The passage from V to $V^\perp$ can also be done in a relative setting: If $\pi : \mathcal{G} \to S$ is a finite, degree d Gorenstein morphism of schemes, and if $\mathcal{L}$ is a line bundle on $\mathcal{G}$, then a rank $r+1$ sub-bundle

$$\mathcal{V} \subset \pi_* \mathcal{L}$$

(with locally free quotient) has an associated rank $d-r-1$ sub-bundle

$$\mathcal{V}^{\perp} \subset \pi_*\left(\mathcal{L}^{-1} \otimes \omega_{\mathcal{G}/S}\right)$$

with locally free quotient.

How does one identify $V^{\perp}$ in practice? Goppa's theorem provides an answer in a common geometric situation:

Lemma 2.2 (Goppa)**.** *Let C be smooth projective curve, L a non-special line-bundle on C, and $\Gamma \subset C$ a finite subscheme such that the restriction map $H^0(C, L) \to H^0(\Gamma, L_\Gamma)$ is injective. Then the image of the vector space $H^0(C, \omega_C(\Gamma) \otimes L^{-1})$ in $H^0(\Gamma, \omega_\Gamma \otimes L^{-1})$ induced by the adjunction isomorphism $\omega_C(\Gamma)|_\Gamma \otimes L^{-1} \to \omega_\Gamma \otimes L^{-1}$ is the associated space to $H^0(C, L)$.*

Armed with Goppa's theorem, we're now ready to switch problems.

2.2. **The correspondence.** Let $\mathcal{H}$ denote the variety parametrizing 3-uple Veronese surfaces in $\mathbb{P}^9$, and let

$$\mathcal{X} \subset \mathcal{H} \times (\mathbb{P}^9)^{13}$$

denote the irreducible, closed subvariety parametrizing tuples $([V], q_1, \ldots, q_{13})$ satisfying $q_i \in V$ for all i. We let $\mathcal{X}^\circ \subset \mathcal{X}$ denote the open subset parametrizing tuples

$$([V], q_1, \ldots, q_{13})$$

which satisfy the following conditions:

(1) The points q_i should be distinct;
(2) When we think of V as a projective plane $\mathbb{P}^2$, the points q_i should define a pencil of plane quartic curves whose base scheme consists of $\{q_1, \ldots, q_{13}\}$ together with three distinct non-collinear points $R \subset \mathbb{P}^2$. Furthermore, the triangle spanned by R should not contain any of the points q_i.

We let

$$\pi : \mathcal{X} \to (\mathbb{P}^9)^{13}$$

denote the map sending $([V], q_1, \ldots, q_{13})$ to $(q_1, \ldots, q_{13})$, and we note that

$$\nu_{3,2} = \#\pi^{-1}(\{(q_1, \ldots, q_{13})\})$$

for general choices of points q_i.

Next, we let $\operatorname{Hilb}_3 \mathbb{P}^2$ denote the Hilbert scheme parametrizing length 3 subschemes of $\mathbb{P}^2$, and we define

$$\mathcal{Y} \subset \operatorname{Hilb}_3 \mathbb{P}^2 \times (\mathbb{P}^2)^{13}$$

to be the locally closed subvariety parametrizing tuples $([T], p_1, \ldots, p_{13})$ satisfying the following conditions:

(1) The length 3 subscheme $T \subset \mathbb{P}^3$ is reduced, and is not contained in any line.
(2) The triangle spanned by T does not contain any of the points p_i.
(3) The points p_i are distinct.

(4) There exist two reduced, irreducible degree 5 curves containing all points p_i and singular at the three points of T.

If $([T], p_1, \dots, p_{13})$ is an element of $\mathcal{Y}$, we say T *is a* **singular triad** *for* $p_1, \dots, p_{13}$. We write

$$\eta : \mathcal{Y} \to (\mathbb{P}^2)^{13}$$

for the map sending $([T], p_1, \dots, p_{13})$ to $(p_1, \dots, p_{13})$. Observe that the groups PGL(10) and PGL(3) act on $\mathbb{P}^9$ and $\mathbb{P}^2$, respectively, and furthermore induce natural actions on $\mathcal{X}, \mathcal{X}^\circ$ and $\mathcal{Y}$.

Theorem 2.3. *Let $(q_1, \dots, q_{13}) \in (\mathbb{P}^9)^{13}$ be a general tuple with associated tuple $(p_1, \dots, p_{13}) \in (\mathbb{P}^2)^{13}$. There exists a bijective correspondence*

$$\left\{ \begin{array}{c} \text{3-uple Veronese} \\ \text{surfaces } V \subset \mathbb{P}^9 \\ \text{containing } q_1, \dots, q_{13} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Singular triads} \\ T \subset \mathbb{P}^2 \\ \text{for } p_1, \dots, p_{13} \end{array} \right\}.$$

Proof. A and B will denote the sets $\pi^{-1}(\{(q_1, \dots, q_{13})\})$ and $\eta^{-1}(\{(p_1, \dots, p_{13})\})$, respectively – the objective is to show A and B are in bijection. A simple dimension count, which we omit, shows that A and B are finite sets. As $\mathcal{X}$ and $(\mathbb{P}^9)^{13}$ are both irreducible and 117-dimensional, and because $(q_1, \dots, q_{13}) \in (\mathbb{P}^9)^{13}$ is general, it follows that $A \subset \mathcal{X}^\circ$.

First we describe a function

$$\Phi : A \to B.$$

Choose any $([V], q_1, \dots, q_{13}) \in A$ to begin with. When we interpret V as a projective plane, the thirteen points $q_i \in V$ determine a general pencil of quartic plane curves $C_t \subset V$, $t \in \mathbb{P}^1$. Let $R \subset V$ denote the three points residual to $\{q_1, \dots, q_{13}\}$ in the base locus of the pencil C_t. Since $A \subset \mathcal{X}^\circ$, when we view V as a plane, the triangle in V spanned by R does not contain any of the points q_i, and the general member of C_t is a smooth quartic curve.

Now let

$$\mu : V \dashrightarrow \mathbb{P}^2$$

denote the quadratic Cremona transformation with indeterminacy set R, well-defined up to post-composition with elements of PGL(3). By applying Goppa's theorem to the divisor $q_1 + \dots + q_{13}$ on any smooth member of the pencil C_t, we conclude that the tuple $(\mu(q_1), \dots, \mu(q_{13}))$ is associated to $(q_1, \dots, q_{13})$. Let $R' \subset \mathbb{P}^2$ denote the three points which are the images of the three lines contracted under μ. Since $(p_1, \dots, p_{13})$ is associated to $(q_1, \dots, q_{13})$ by assumption, and since the latter tuple is general, it follows that there is a unique element $g \in \mathrm{PGL}(3)$ which takes $(\mu(q_1), \dots, \mu(q_{13}))$ to $(p_1, \dots, p_{13})$.

Set $T := g(R')$. We will verify the membership

$$([T], p_1, \dots, p_{13}) \in B,$$

and then declare

$$\Phi([V], q_1, \dots, q_{13}) := ([T], p_1, \dots, p_{13}).$$

To that end, we must show $([T], p_1, \dots, p_{13})$ satisfies the requirements for membership in $\mathcal{Y}$. First, T is a reduced, non-collinear set of three points because the same was true for R.

Second, if some point p_i was contained in the triangle spanned by T, then by applying an appropriate reverse Cremona transformation, it would follow that the corresponding point q_i was contained in the triangle spanned by R, contrary to assumption. Third, the points p_i are distinct because no two q_j's are contained in a line spanned by two of the points of R. Fourth and finally, by considering the curves $\mu(C_t)$ for $t \in \mathbb{P}^1$ general, we find at least two irreducible quintic curves singular at T and passing through $(p_1, \ldots, p_{13})$.

The function $\Phi : A \to B$ having been defined, we now define a mapping $\Psi : B \to A$ which is readily seen to be its inverse. To start, choose $([T], p_1, \ldots, p_{13}) \in B$. Let $\widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at the three points of T. On $\widetilde{\mathbb{P}}^2$, let L and E denote the divisor classes of a general line in $\mathbb{P}^2$ (pulled back to the blow-up) and the sum of the three exceptional curves, respectively, and let $Q, Q' \in |5L - 2E|$ be the strict transforms of two of the quintic curves mentioned in the membership requirements for $\mathcal{Y}$. Observe that $Q \cap Q' = \{p_1, \ldots, p_{13}\}$. By applying Goppa's theorem to the divisor $p_1 + \cdots + p_{13}$ on Q, it follows that the map

$$\gamma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^9$$

given by the complete linear series $|6L - 3E|$ is such that the tuple $(\gamma(p_1), \ldots, \gamma(p_{13})) \in (\mathbb{P}^9)^{13}$ is associated to $(p_1, \ldots, p_{13}) \in (\mathbb{P}^2)^{13}$. Much as before, there is a unique element $h \in \mathrm{PGL}(10)$ which sends $(\gamma(p_1), \ldots, \gamma(p_{13}))$ to $(q_1, \ldots, q_{13})$. The map

$$h \circ \gamma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^9$$

has as image a 3-uple Veronese surface V containing the points q_i for all i. Define Ψ by $\Psi([T], p_1, \ldots, p_{13}) := ([V], q_1, \ldots, q_{13})$. Φ and Ψ are inverses, proving the theorem. □

2.3. **Returning to the original objective.** The stated objective in this paper is to determine $\#\pi^{-1}(\{(q_1, \ldots, q_{13})\})$ for general points $q_i \in \mathbb{P}^9$. Using Theorem 2.3, we instead will try to compute

$$\#\eta^{-1}(\{p_1, \ldots, p_{13}\})$$

for general points $p_i \in \mathbb{P}^2$. We will still face several obstacles.

The first obstacle arises when we want to refer to the rank 9 "vector bundle" E on $\mathrm{Hilb}_3 \mathbb{P}^2$ whose fiber over a point $[T]$ is the vector space

$$H^0\left(\mathbb{P}^2, \left(\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_T^2\right) \otimes \mathcal{O}_{\mathbb{P}^2}(5)\right).$$

With such a bundle we can apply the Porteous formula to access the locus where the natural morphism

$$\underline{H^0\left(\mathbb{P}^2, \mathcal{I}_{\{p_1,\ldots,p_{13}\}}(5)\right)} \to E$$

has at least a 2-dimensional kernel – the key membership condition defining the set $\eta^{-1}(\{(p_1, \ldots, p_{13})\})$. Unfortunately, E is not actually a vector bundle because its rank jumps from 9 to 10 over the locus parametrizing fat schemes. And so our attention turns to replacing $\mathrm{Hilb}_3 \mathbb{P}^2$ with a better parameter space, the space of *complete triangles* CT, which resolves this jumping wrinkle. We do all this and more in the next section, which is dedicated to the geometry and construction of CT.

3. The space of complete triangles

Let $\mathrm{Hilb}_3\,\mathbb{P}^2$ and $\mathrm{Hilb}_3\,\check{\mathbb{P}}^2$ denote the Hilbert schemes of 0-dimensional, length 3 subschemes of a projective plane and its dual plane, respectively. Upon choosing coordinates $[X : Y : Z]$ for $\mathbb{P}^2$ and $[\check{X} : \check{Y} : \check{Z}]$ for $\check{\mathbb{P}}^2$, the group PGL(3) acts on both Hilbert schemes naturally. Under this action, $\mathrm{Hilb}_3\,\mathbb{P}^2$ decomposes into 7 orbits – (A) three non-collinear points, (B) three collinear points, (C) a length two point and a reduced point, noncollinear, (D) a length two point and a collinear point, (E) A length 3 nonreduced subscheme of a conic, (F) a length 3 nonreduced subscheme of a line, (G) a fat point, given by the square of a maximal ideal. Of these, (F) and (G) are closed. Our objective in this section is to construct and analyze a PGL(3)-equivariant modification of $\mathrm{Hilb}_3\,\mathbb{P}^2$ which we call the space of *complete triangles*, and which we denote by CT.

3.1. Nets of conics, Jacobian spaces, and constructing CT.

Definition 3.1. (1) *Let $T \in \mathrm{Hilb}_3\,\mathbb{P}^2$ be a length three scheme. T's* **net of conics** *is the vector space $\Lambda_T := H^0(\mathcal{I}_T(2)) \subset H^0(\mathcal{O}_{\mathbb{P}^2}(2))$.*

(2) *The* **Jacobian matrix** *of three homogeneous quadratic forms Q_1, Q_2, Q_3 in the variables X, Y, Z is:*

$$\begin{bmatrix} \frac{\partial Q_1}{\partial X} & \frac{\partial Q_1}{\partial Y} & \frac{\partial Q_1}{\partial Z} \\ \frac{\partial Q_2}{\partial X} & \frac{\partial Q_2}{\partial Y} & \frac{\partial Q_2}{\partial Z} \\ \frac{\partial Q_3}{\partial X} & \frac{\partial Q_3}{\partial Y} & \frac{\partial Q_3}{\partial Z} \end{bmatrix} \tag{3}$$

(3) *The* **Jacobian space** *of $V \subset H^0(\mathbb{P}^2, \mathcal{O}(2))$ is the vector space spanned by all 2×2 minors of the Jacobian matrix of any basis $\langle Q_1, Q_2, Q_3\rangle$ of V.*

(4) *If $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is 3-dimensional, we let $V^* \subset H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2))$ denote its apolar 3-dimensional space.*

(5) *A scheme $T \subset \mathbb{P}^2$ is* **fat** *if it is isomorphic to $\operatorname{Spec} k[x, y]/(x^2, xy, y^2)$. A scheme $T \subset \mathbb{P}^2$ is* **thin** *if it is contained in a line. We define $\mathsf{Fat}, \mathsf{Thin} \subset \mathrm{Hilb}_3\,\mathbb{P}^2$ to be the closed loci of fat and thin schemes, respectively.*

(6) *A subscheme $T \subset \mathbb{P}^2$ (or $\check{\mathbb{P}}^2$) consisting of three distinct non-collinear points is called an* **honest triangle**

Remark 3.2. (1) Λ_T is a three dimensional vector space, no matter the scheme T, as is easily checked for each of the 7 PGL(3) orbits separately.

(2) Apolarity is the natural pairing between $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2))$, where the latter space is viewed as homogeneous second order differential operators on the former space. So, for instance, the pairing outputs 2 for the pair $X^2, \check{X}^2$.

Proposition 3.3. *Let $T \subset \mathbb{P}^2$ be any length three scheme, Λ_T its net of conics. Then the space $\mathsf{Jac}(\Lambda_T^*)$ is 3-dimensional.*

Proof. By projective equivariance of the assignment $T \mapsto \mathsf{Jac}(\Lambda_T^*)$, it suffices to check the proposition on seven representatives of the PGL(3)-orbits, which we omit. □

Definition 3.4. *Let $T \subset \mathbb{P}^2$ be a length 3 scheme. We set $\Lambda_T^\dagger := \mathsf{Jac}(\Lambda_T^*)$.*

3.2. **Examples.** Let us give some calculations related to some of the things we've introduced. As a reminder, $[X : Y : Z]$ are homogeneous coordinates in $\mathbb{P}^2$, and $[\check{X} : \check{Y} : \check{Z}]$ are dual coordinates. Thus, the point $[\check{X} : \check{Y} : \check{Z}]$ represents the line in $\mathbb{P}^2$ defined by the equation $\check{X}X + \check{Y}Y + \check{Z}Z = 0$. Brackets $\langle\rangle$ will denote "$\Bbbk$-linear span".

Example 3.5. *Let $T \subset \mathbb{P}^2$ be the three coordinate points (orbit (A)), so that $\Lambda_T = \langle XY, YZ, XZ\rangle$. Then $\Lambda_T^* = \langle \check{X}^2, \check{Y}^2, \check{Z}^2\rangle$, whose Jacobian space is $\langle \check{X}\check{Y}, \check{Y}\check{Z}, \check{X}\check{Z}\rangle$. Therefore,*

$$\Lambda_T^\dagger = \langle \check{X}\check{Y}, \check{Y}\check{Z}, \check{X}\check{Z}\rangle,$$

which is the net of conics for the coordinate points in $\check{\mathbb{P}}^2$.

Example 3.6. *Let T be a length three non-reduced subscheme of a conic – orbit (E). T is given by the vanishing scheme of the net $\Lambda_T = \langle XY, X^2, YZ\rangle$. Therefore, $\Lambda_T^* = \langle \check{Y}^2, \check{Z}^2, \check{X}\check{Z}\rangle$.*

Computing the Jacobian space, we get

$$\Lambda_T^\dagger = \langle \check{Y}\check{Z}, \check{Y}\check{X}, \check{Z}^2\rangle$$

which is the net of conics of a length three subscheme of a smooth conic in $\check{\mathbb{P}}^2$.

Example 3.7. *Now suppose T is a length three scheme contained in the line L given by $Z = 0$ (orbit (F)). Then $\Lambda_T = \langle XZ, YZ, Z^2\rangle$. Note that this net depends only on L, regardless of the particular scheme $T \subset L$.*

The dual space Λ_T^* is $\langle \check{X}^2, \check{X}\check{Y}, \check{Y}^2\rangle$, which is also easily checked to be its own Jacobian. Therefore,

$$\Lambda_T^\dagger = \langle \check{X}^2, \check{X}\check{Y}, \check{Y}^2\rangle.$$

Example 3.8. *Let T be the fat point supported at $[0 : 0 : 1]$ (orbit (G)). Its net of conics is $\Lambda_T = \langle X^2, XY, Y^2\rangle$. The dual space Λ_T^* is $\langle \check{X}\check{Z}, \check{Y}\check{Z}, \check{Z}^2\rangle$. This latter space is its own Jacobian. Therefore*

$$\Lambda_T^\dagger = \langle \check{X}\check{Z}, \check{Y}\check{Z}, \check{Z}^2\rangle.$$

Proposition 3.9. *Let $T \in \operatorname{Hilb}_3 \mathbb{P}^2$ be arbitrary. Then $\Lambda_T^\dagger$ is the net of conics for some (not necessarily unique) $T^* \in \operatorname{Hilb}_3 \check{\mathbb{P}}^2$.*

Proof. One checks this orbit by orbit – we have done the most interesting examples above. □

We can now state our main definition:

Definition 3.10. *The moduli space of* **complete triangles** *is the closed subscheme*

$$\mathsf{CT} \subset \operatorname{Hilb}_3 \mathbb{P}^2 \times \operatorname{Hilb}_3 \check{\mathbb{P}}^2$$

parametrizing pairs (T, T^*) satisfying

$$\Lambda_T^\dagger = \Lambda_{T^*}.$$

Remark 3.11. The scheme structure on CT is induced by the condition $\Lambda_T^\dagger = \Lambda_{T^*}$. Indeed, by Theorem 3.3 and Theorem 3.2, the assignments $(T, T^*) \mapsto \Lambda_T^\dagger$ and $(T, T^*) \mapsto \Lambda_{T^*}$ yield two vector subbundles of the trivial bundle $\underline{H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2))}$ over $\operatorname{Hilb}_3 \mathbb{P}^2 \times \operatorname{Hilb}_3 \check{\mathbb{P}}^2$. Requiring these two subbundles to be equal yields the scheme structure on CT. CT's functor of points is inherent in this description.

The diagonal action of PGL(3) on $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ induces an action on CT thanks to the projective equivariance of the assignment

$$\Lambda_T \mapsto \Lambda_T^\dagger.$$

From these calculations and projective equivariance of the construction $\Lambda_T \mapsto \Lambda_T^\dagger$, we conclude:

Proposition 3.12. *Suppose* $(T, T^*) \in \mathsf{CT}$. *Then:*

1. *T is an honest triangle if and only if T^* is an honest triangle. In this case, the points of T^* correspond to the lines of the triangle spanned by pairs of points of T.*
2. *If T is a nonreduced scheme which is neither fat nor thin, then T^* is unique, and vice versa.*
3. *T is a fat scheme if and only if T^* is a thin scheme, and T^* is contained in the line dual to the support of T. The same statement holds with T and T^* interchanged.*
4. *T^* is uniquely determined by T if and only T is not fat.*

Proof. Follows from calculations similar to those in Theorem 3.5, Theorem 3.8 and Theorem 3.7 – we leave the details to the reader. □

Remark 3.13. From Theorem 3.7, notice that if T is thin and $(T, T^*) \in \mathsf{CT}$, then T^* is the fat scheme supported on the point dual to the line containing T.

Proposition 3.14. *The reduction* CT_{red} *is* 6*-dimensional and irreducible.*

Proof. From Theorem 3.12, it suffices to show: Given a pair $(T, T^*) \in \mathsf{CT}$ with T fat, there exists an irreducible pointed curve $(B, 0)$ and a map $f\colon B \to \mathsf{CT}$ such that $f(0) = (T, T^*)$ and for all $b \in B \setminus \{0\}$, $f(b)$ is an honest triangle. This is sufficient because the open locus of honest triangles is clearly 6-dimensional and irreducible. Note that by symmetry, we need not consider the case where T is thin.

Since T is fat, Theorem 3.12 says T^* is thin. In any case, since the open subset of $\mathrm{Hilb}_3(\check{\mathbb{P}}^2)$ parametrizing triples of three distinct, non-collinear points is Zariski dense, there exists a pointed curve $(B, 0)$ and a map

$$f\colon B \to \mathrm{Hilb}_3 \check{\mathbb{P}}^2$$

with $f(0) = T^*$, and $f(b)$ a triple of three non-collinear points, for all $b \neq 0$. For all points $b \in B$, the space $\Lambda_{f(b)}^\dagger$ determines a unique length three scheme $T_b \in \mathrm{Hilb}_3(\mathbb{P}^2)$ such that $\Lambda_{f(b)}^\dagger = \Lambda_{T_b}$. Therefore, the map f lifts to a map $f\colon B \to \mathsf{CT}$, and this lift has the desired properties, namely that $f(0) = (T, T^*)$ and $f(b)$ is an honest triangle for $b \neq 0$. □

3.3. **Smoothness.** Our next major objective is to show that CT is smooth. We do so by establishing smoothness at a particular point $(T, T^*) \in \mathsf{CT}$, and then concluding by exploiting the PGL(3) action, upper semi-continuity and Theorem 3.14.

Proposition 3.15. *Let $(T, T^*) \in \mathsf{CT}$ be the complete triangle with T given by the homogeneous ideal (X^2, XY, Y^2), and T^* given by the ideal $(\check{Z}, \check{X}^3)$. The scheme CT is smooth at (T, T^*).*

Proof. We will calculate the space of first order deformations of $(T, T^*) \in \mathsf{CT}$, and demonstrate that it is 6 dimensional. This is enough by Theorem 3.14.

Let us pass to affine coordinates; we let $x = X/Z, y = Y/Z, a = \check{X}/\check{Y}, c = \check{Z}/\check{Y}$. The general first order deformation of the ideal $I = (x^2, xy, y^2)$ is given by

$$I_\varepsilon := (x^2 + \varepsilon(\alpha_1 x + \beta_1 y), xy + \varepsilon(\alpha_2 x + \beta_2 y), y^2 + \varepsilon(\alpha_3 x + \beta_3 y)), \tag{4}$$

for free choices of constants $\alpha_i, \beta_j \in \Bbbk$, while the general first order deformation of $J = (a^3, c)$ is

$$J_\varepsilon = (a^3 + \varepsilon(\gamma_1 a^2 + \gamma_2 a + \gamma_3), c + \varepsilon(\delta_1 a^2 + \delta_2 a + \delta_3)) \tag{5}$$

where γ_i and δ_i vary freely in $\Bbbk$.

From (4), the corresponding first order deformation of the induced net of conics $\Lambda_T = \langle X^2, XY, Y^2 \rangle$ is given by

$$\Lambda_T(\varepsilon) := \langle X^2 + \varepsilon(\alpha_1 XZ + \beta_1 YZ), XY + \varepsilon(\alpha_2 XZ + \beta_2 YZ), Y^2 + \varepsilon(\alpha_3 XZ + \beta_3 YZ) \rangle. \tag{6}$$

Next, we must identify the first order deformation of the net of conics $\Lambda_{T^*} = \langle \check{X}\check{Z}, \check{Y}\check{Z}, \check{Z}^2 \rangle$ induced by the deformation (5). First, note that $c^2 \in J_\varepsilon$. Therefore, we must determine how the conics $\check{X}\check{Z}$ and $\check{Y}\check{Z}$ must be deformed. By homogenizing the element $c + \varepsilon(\delta_1 a^2 + \delta_2 a + \delta_3) \in J_\varepsilon$, we get the deformation:

$$\check{Y}\check{Z} + \varepsilon(\delta_1 \check{X}^2 + \delta_2 \check{X}\check{Y} + \delta_3 \check{Y}^2).$$

Finally, by multiplying $c + \varepsilon(\delta_1 a^2 + \delta_2 a + \delta_3)$ by a, subtracting $\varepsilon\delta_1 a^3$, then homogenizing we deduce the following deformation:

$$\check{X}\check{Z} + \varepsilon(\delta_2 \check{X}^2 + \delta_3 \check{X}\check{Y})$$

Altogether, the corresponding first order deformation of the net Λ_{T^*} is given by

$$\Lambda_{T^*}(\varepsilon) := \langle \check{X}\check{Z} + \varepsilon(\delta_2 \check{X}^2 + \delta_3 \check{X}\check{Y}), \check{Y}\check{Z} + \varepsilon(\delta_1 \check{X}^2 + \delta_2 \check{X}\check{Y} + \delta_3 \check{Y}^2), \check{Z}^2 \rangle \tag{7}$$

Now, we must impose the condition that the pair of deformations $(I_\varepsilon, J_\varepsilon)$ belong in CT. By definition, this is the condition

$$\Lambda_T(\varepsilon)^\dagger = \Lambda_{T^*} \tag{8}$$

The left hand side of (8) is simple to calculate. First, the deformation $\Lambda_T(\varepsilon)^*$ is given by:

$$\Lambda_T(\varepsilon)^* = \langle \check{X}\check{Z} + \varepsilon(-\frac{\alpha_1}{2}\check{X}^2 - \alpha_2 \check{X}\check{Y} - \frac{\alpha_3}{2}\check{Y}^2), \check{Y}\check{Z} + \varepsilon(-\frac{\beta_1}{2}\check{X}^2 - \beta_2 \check{X}\check{Y} - \frac{\beta_3}{2}\check{Y}^2), \check{Z}^2 \rangle$$

and so applying the Jacobian we get

$$\Lambda_T(\varepsilon)^\dagger = \langle \check{X}\check{Z} + \varepsilon(\beta_2 \check{X}^2 + (\alpha_2 - \beta_3)\check{X}\check{Y} + \alpha_3 \check{Y}^2), \check{Y}\check{Z} + \varepsilon(\beta_1 \check{X}^2 + (\beta_2 - \alpha_1)\check{X}\check{Y} - \alpha_2 \check{Y}^2), \check{Z}^2 \rangle$$

Equating term by term with (7) we get the following system of equations:

$$\begin{aligned}\delta_2 &= \beta_2\\ \delta_3 &= \alpha_2 - \beta_3\\ 0 &= \alpha_3\\ \delta_1 &= \beta_1\\ \delta_2 &= \beta_2 - \alpha_1\\ \delta_3 &= -\alpha_2\end{aligned}$$

We note that this system of equations clearly cuts out a 6-dimensional space of solutions in the vector space $\Bbbk\langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3\rangle$ – indeed, we may freely choose $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3$, after which the α and β variables are uniquely determined, concluding the proof. ◻

Remark 3.16. In the setting of the proof of Theorem 3.15, we note that the extra single condition $\delta_1 = 0$ is equivalent to the condition that the 1'st order deformation of the ideal J_ε continues to contain a linear element. But then the allowable parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ determine a deformation of the ideal I of the form $((x-\varepsilon\alpha)^2, (x-\varepsilon\alpha)(y-\varepsilon\beta), (y-\varepsilon\beta)^2)$. This is evidently the tangent space to Fat at the point T.

Furthermore, a deformation of J with $\delta_1 \neq 0$ induces a non-zero tangent vector of T which is a nonzero normal vector to Fat at T.

Theorem 3.17. *The scheme* CT *is smooth.*

Proof. We use upper semi-continuity, Theorem 3.15, and Theorem 3.14. The group PGL(3) acts on the scheme CT. Under this action, it is easy to check that either the point (T, T^*) from Theorem 3.15 or its symmetric flip (where we interchange X, Y, Z with $\check{X}, \check{Y}, \check{Z}$ everywhere) is contained in the closure of every orbit. We conclude that the tangent space is 6-dimensional at every point in CT by semi-continuity, and conclude by Theorem 3.14. ◻

Corollary 3.18. *The space of complete triangles* CT *is the closure (with reduced, induced scheme structure) of the graph of the triangle map*

$$\tau\colon \operatorname{Hilb}_3 \mathbb{P}^2 \dashrightarrow \operatorname{Hilb}_3 \check{\mathbb{P}}^2$$

which assigns to a set of three distinct, non-collinear points $T \subset \mathbb{P}^2$, the set of lines spanned by pairs in T.

Proof. This follows from part (1) of Theorem 3.12, Theorem 3.14, and Theorem 3.17. (The latter is needed to show that CT is reduced.) ◻

Let $\phi_1\colon \mathsf{CT} \to \operatorname{Hilb}_3 \mathbb{P}^2$ and $\phi_2\colon \mathsf{CT} \to \operatorname{Hilb}_3 \check{\mathbb{P}}^2$ denote the forgetful maps.

Corollary 3.19. *The space* CT *is isomorphic to the blow up* $\mathrm{Bl}_{\mathsf{Fat}} \mathrm{Hilb}_3 \mathbb{P}^2$ *and to the blow up* $\mathrm{Bl}_{\mathsf{Fat}} \mathrm{Hilb}_3 \check{\mathbb{P}}^2$*. Under these isomorphisms, the forgetful maps* ϕ_i *are the respective blow down maps.*

Proof. We prove that $\mathsf{CT} \simeq \mathrm{Bl}_{\mathsf{Fat}} \mathrm{Hilb}_3 \mathbb{P}^2$ – the argument for $\mathrm{Bl}_{\mathsf{Fat}} \mathrm{Hilb}_3 \check{\mathbb{P}}^2$ is the same.

The projection $\phi_1 \colon \mathsf{CT} \to \mathrm{Hilb}_3 \mathbb{P}^2$ is such that $\phi^{-1}(\mathsf{Fat})$ has codimension 1 in CT, from the set-theoretic statement (3) in Theorem 3.12. This preimage is also smooth, thanks to the observation in Theorem 3.16, combining the PGL(3) action with upper-semicontinuity.

The universal property of blow ups gives us an induced morphism $\iota \colon \mathsf{CT} \to \mathrm{Bl}_{\mathsf{Fat}} \mathrm{Hilb}_3 \mathbb{P}^2$ factoring ϕ_1. We conclude by applying Zariski's Main Theorem to the morphism ι. □

3.4. **Resolving powers.** Let $n \geq 2$ be an integer, and let

$$\mu_n \colon \mathrm{Hilb}_3 \mathbb{P}^2 \dashrightarrow \mathrm{Hilb}_{3\binom{n+1}{2}} \mathbb{P}^2$$

denote the rational map defined by sending an ideal sheaf $\mathcal{I}$ to the ideal sheaf $\mathcal{I}^n$. The maps μ_n fail to be defined along $\mathsf{Fat} \subset \mathrm{Hilb}_3 \mathbb{P}^2$, since the colength of $\mathcal{I}^n$ is $\binom{2n+1}{2}$ rather than $3\binom{n+1}{2}$ when $\mathcal{I} = \mathfrak{m}_p^2$. Here $\mathfrak{m}_p \subset \mathcal{O}_{\mathbb{P}^2}$ denotes the ideal sheaf of the point $p \in \mathbb{P}^2$. The excess in colength is $\binom{n}{2}$. Conveniently, μ_n extends to a morphism on CT for all n, as we now explain. Let

$$\gamma \colon \mathrm{Hilb}_3 \check{\mathbb{P}}^2 \to \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$$

denote the composition of the Hilbert-Chow morphism $\mathrm{Hilb}_3 \check{\mathbb{P}}^2 \to \mathrm{Sym}^3 \check{\mathbb{P}}^2$ with the natural "multiplication" morphism

$$m \colon \mathrm{Sym}^3 \check{\mathbb{P}}^2 \to \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$$

induced by the S_3-equivariant "multiplication of linear forms" map

$$\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \to \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3)).$$

We view points of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$ as cubic curves in $\mathbb{P}^2$.

Thus we see that every complete triangle $(T, T^*) \in \mathsf{CT}$ comes naturally equipped with a cubic curve $\gamma(T^*) \subset \mathbb{P}^2$ which is a union of lines, possibly with multiplicities. If (T, T^*) is an honest triangle, then $\gamma(T^*)$ is simply the triangle spanned by T – in particular, $\gamma(T^*)$ gives a global section of $\mathcal{I}_\Gamma^2(3)$. If $(T, T^*) \in \mathsf{CT}$ is such that T is a fat scheme, then $\gamma(T^*)$ is an "asterisk" consisting of three lines passing through the support point of T.

Theorem 3.20. *The rational map* μ_n *extends to a morphism on* CT *for all* n*.*

Proof. For simplicity of notation, we let γ denote the ideal sheaf generated by the cubic $\gamma(T^*)$. Consider the assignment

$$(T, T^*) \mapsto \mathcal{I}_T^n + \gamma\, \mathcal{I}_T^{n-2} + \gamma^2\, \mathcal{I}_T^{n-4} + \ldots \tag{9}$$

where the sum ends at $\gamma^{n/2}$ or $\gamma^{(n-1)/2}\, \mathcal{I}_T$ depending on the parity of n. We leave it to the reader to check that the ideal on the right side of Equation (9) has the correct colength $3\binom{n+1}{2}$, even if T is a fat scheme. Hence, this assignment induces a morphism $\mathsf{CT} \to \mathrm{Hilb}_{3\binom{n+1}{2}} \mathbb{P}^2$.

Finally, if (T, T^*) is an honest triangle, then γ is already contained in $\mathcal{I}_T^2$ and therefore the above assignment reduces to $\mathcal{I}_T \mapsto \mathcal{I}_T^n$, the nth power map, concluding the proof. □

Corollary 3.21. *Let S be any smooth surface. Then the nth power map*

$$\mu_n\colon \operatorname{Hilb}_3 S \dashrightarrow \operatorname{Hilb}_{3\binom{n+1}{2}} S$$

extends to a regular map on $\operatorname{Bl}_{\mathsf{Fat}} \operatorname{Hilb}_3 S$.

Proof. It suffices to prove the statement étale locally on S. However, étale locally S is isomorphic to $\mathbb{P}^2$. The result follows from Theorem 3.19 and Theorem 3.20. □

Returning to the overarching narrative, in order to count the number of points in

$$\eta^{-1}(\{p_1, \ldots, p_{13}\}) \subset \operatorname{Hilb}_3 \mathbb{P}^2$$

we will use the squaring map μ_2 to produce a map between two vector bundles on CT. An appropriate degeneracy scheme of this map will include $\eta^{-1}(\{p_1, \ldots, p_{13}\})$ along with undesirable excess. We will remove the excess in Section 4 – for now, we content ourselves with identifying the relevant vector bundles.

3.5. **Some vector bundles on CT.** We simplify notation and let

$$\mathsf{sq} : \mathsf{CT} \to \operatorname{Hilb}_9 \mathbb{P}^2$$

denote the squaring map μ_2 from the previous section. We let

$$\mathcal{Z} \subset \mathsf{CT} \times \mathbb{P}^2$$

denote the universal length 9 subscheme induced by sq, and write p_1, p_2 for the projections of $\mathsf{CT} \times \mathbb{P}^2$ to first and second factors, respectively. Finally, we denote by $\mathcal{V}_n$ the sheaf

$$p_{1*}\left(\mathcal{I}_{\mathcal{Z}} \otimes p_2^*\mathcal{O}(n)\right). \tag{10}$$

Proposition 3.22. *$\mathcal{V}_n$ is a vector bundle on CT of rank $\binom{n+2}{2} - 9$ for all $n \geq 5$.*

Proof. The claim being made is that every length 9 scheme of the form $\mathsf{sq}(T, T^*)$ imposes independent conditions on degree $n \geq 5$ homogeneous forms. This can either be checked on a case-by-case basis, or can be checked for example on the special complete triangle (T, T^*) in Theorem 3.15, as well as its dual. Then an appeal to semi-continuity gives the conclusion for any complete triangle (T, T^*) where T is not contained in a line. For those complete triangles (T, T^*) with T contained in a line, the claim can be checked directly. □

Lemma 3.23. *Let $(T, T^*) \in \mathsf{CT}$ be any point such that $T \notin \mathsf{Thin}$. Then the sheaf $\mathcal{I}_{\mathsf{sq}(T,T^*)}(n)$ is generated by global sections for $n \geq 4$, i.e. the scheme defined by the common vanishing of all forms in $H^0(\mathbb{P}^2, \mathcal{I}_{\mathsf{sq}(T,T^*)}(n))$ is precisely $\mathsf{sq}(T, T^*)$ when $n \geq 4$.*

Proof. By combining an isotrivial specialization using the PGL(3) action with semi-continuity, we need only verify this for the pair (T, T^*) where, in affine coordinates (x, y), T is given by $V(x^2, xy, y^2)$ and T^* defines the “three” concurrent lines consisting of $V(x)$ reckoned 3 times. Here, the ideal of $\mathsf{sq}(T, T^*)$ is

$$(x^4, x^3y, x^2y^2, xy^3, y^4, x^3),$$

which is clearly generated by polynomials of degree $\leq n$ once $n \geq 4$, proving the result. □

3.6. **A Chern class calculation using localization on CT.** Maintain the notation in Section 3.5 and define

$$E = p_{1*}\left(\mathcal{O}_{\mathcal{Z}} \otimes p_2^*\mathcal{O}(5)\right).$$

Then E is a vector bundle of rank 9 on CT – indeed, the whole point of introducing CT in the first place was to be able to invoke the bundle E.

The objective of this section is to evaluate the integer

$$c_3^2(E) - c_2(E)c_4(E) \in H^6(\mathsf{CT}, \mathbb{Z}) = \mathbb{Z}$$

using the technique of localization.

We recall the localization formula in the context of enumerative geometry from [ES96]. Let X be a smooth projective variety of dimension n with a $\mathbb{G}_{\mathrm{m}}$ action. Let $F \subset X$ be the set of fixed points, and assume that F is finite. Let E be an equivariant vector bundle on X. Let $p(c_1, c_2, \dots)$ be a polynomial in formal variables $c_1, c_2, \dots$. Assume that p is weighted homogeneous of degree n when the variable c_i is given weight i. The goal generally is to compute

$$p(c_1(E), c_2(E), \dots) \in H^{2n}(X, \mathbb{Z}) = \mathbb{Z}.$$

For $x \in F$, let $\sigma_i(E, x)$ be the value of the ith elementary symmetric polynomial in the weights of the $\mathbb{G}_{\mathrm{m}}$ acting on $E|_x$. Set $f(E, x) = p(\sigma_1(E, x), \sigma_2(E, x), \dots)$.

Theorem 3.24 (Bott's localization formula [ES96, Theorem 2.2])**.** *We have the equality*

$$p(c_1(E), c_2(E), \dots) = \sum_{x \in F} \frac{f(E, x)}{\sigma_n(T_X, x)}.$$

We now compute the ingredients of the right hand side in Theorem 3.24 for $X = \mathsf{CT}$. Put homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$. Consider the $\mathbb{G}_{\mathrm{m}}$ action on the three dimensional vector space $\langle X, Y, Z \rangle$ by

$$t \cdot (X, Y, Z) = (t^a X, t^b Y, t^c Z),$$

where $(a, b, c) \in \mathbb{Z}^3$ are distinct, general integers. This action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{P}^2$ induces compatible actions on $\check{\mathbb{P}}^2$, $\mathcal{O}_{\mathbb{P}^2}(n)$, $\mathrm{Hilb}_3\,\mathbb{P}^2$, CT, and E. Observe that S_3 acts on (X, Y, Z) by permutations, and accordingly on a, b, c.

3.7. **The 31 fixed points of the $\mathbb{G}_{\mathrm{m}}$ action on CT.** We now list the S_3-orbits of the 31 fixed points of the $\mathbb{G}_{\mathrm{m}}$ action on CT. The format is $(I; J)$, where $I \subset k[X, Y, Z]$ is the homogeneous ideal of a length 3 scheme and $J \subset k[\check{X}, \check{Y}, \check{Z}]$ is a an ideal describing a length 3 subscheme of $\check{\mathbb{P}}^2$.

(1) $(XY, YZ, XZ; \check{X}\check{Y}, \check{X}\check{Z}, \check{Y}\check{Z})$ — the unique 1+1+1 configuration,
(2) $(XY, XZ, Y^2; \check{X}\check{Y}, \check{X}\check{Z}, \check{Z}^2)$ and its 6 permutations – non-linear 1+2 configuration,
(3) $(X, Y^2Z; \check{Y}^2, \check{Y}\check{Z}, \check{Z}^2)$ and its 6 permutations – linear 1+2 configuration,
(4) $(X, Y^3; \check{Y}^2, \check{Y}\check{Z}, \check{Z}^2)$ and its 6 permutations – linear 3 configuration,
(5) $(X^2, XY, Y^2; \check{Z}, \check{Y}^3)$ and its 6 permutations – fat point (non-linear 3 configuration),
(6) $(X^2, XY, Y^2; \check{Z}, \check{Y}^2\check{X})$ and its 6 permutations – fat point (non-linear 3 configuration).

This (1)-(6) ordering of S_3 orbit representatives will be systematically adhered to for all the following weight computations, including throughout the `sage` code in §7.

Proposition 3.25. *The weights of the $\mathbb{G}_m$ action on E at the fixed points of type (1)-(6) are:*

(1) $(5a, 5b, 5c, 4a+b, 4a+c, 4b+a, 4b+c, 4c+a, 4c+b)$,
(2) $(5a, 4a+b, 4a+c, 5c, 4c+a, 4c+b, 3c+a+b, 3c+2b, 2c+3b)$ *and its 6 permutations,*
(3) $(5b, 4b+a, 4b+c, 5c, 4c+a, 4c+b, 3c+a+b, 3c+2b, 2c+3b)$ *and its 6 permutations,*
(4) $(5c, 4c+b, 3c+2b, 2c+3b, c+4b, 5b, 4c+a, 3c+a+b, 2c+a+2b)$ *and its 6 permutations,*
(5) $(5c, 4c+a, 3c+2a, 4c+b, 3c+a+b, 2c+2a+b, 3c+2b, 2c+2b+a, 2c+3b)$ *and its 6 permutations,*
(6) $(5c, 4c+a, 3c+2a, 2c+3a, 4c+b, 3c+a+b, 3c+2b, 2c+2b+a, 2c+3b)$ *and its 6 permutations.*

Proof. In order to indicate how to perform these computations, we will do the case (2) as an example – the reader can then check that no new complications arise in the general calculation.

Note that in the case (2), the image in $\mathrm{Hilb}_9\,\mathbb{P}^2$ is cut out simply by the square of the ideal; the cubic XY^2 is redundant information. The length 3 scheme T cut out by $\langle XY, XZ, Y^2\rangle$ is supported at $[1:0:0]$ and $[0:0:1]$. Near $[1:0:0]$, we can use affine coordinates $y = Y/X$ and $z = Z/X$. In these coordinates, the ideal is (y, z), and its square is (y^2, yz, z^2). The section X^5 of $\mathcal{O}_{\mathbb{P}^2}(5)$ is $\mathbb{G}_m$ equivariant and non-vanishing at $[1:0:0]$. We thus get a $\mathbb{G}_m$ equivariant basis of $H^0(T \cap \{X \neq 0\}, \mathcal{O}_{\mathsf{sq}(T,T^*)}(5))$ given by

$$X^5\langle 1, y, z\rangle.$$

The corresponding weights are $5a, 4a+b, 4a+c$. Near $[0:0:1]$, we can use affine coordinates $x = X/Z$ and $y = Y/Z$. In these coordinates, the ideal is (x, y^2), and its square is (x^2, xy^2, y^4). The section Z^5 of $\mathcal{O}_{\mathbb{P}^2}(5)$ is $\mathbb{G}_m$ equivariant section and non-vanishing at $[0:0:1]$. We thus get a $\mathbb{G}_m$ equivariant basis of $H^0(T \cap \{Z \neq 0\}, \mathcal{O}_{\mathsf{sq}(T,T^*)}(5))$ given by

$$Z^5 \cdot \langle 1, x, y, xy, y^2, y^3\rangle.$$

The corresponding weights are $5c, 4c+a, 4c+b, 3c+a+b, 3c+2b, 2c+3b$. Combining the contributions from $[1:0:0]$ and $[0:0:1]$, we get the full set of weights. □

Proposition 3.26. *The weights of the $\mathbb{G}_m$ action on the tangent bundle at the fixed points of type (1) - (6) are:*

(1) $(c-a, c-b, b-c, b-a, a-b, a-c)$,
(2) $(a-c, a-b, c-a, 2c-2b, b-a, c-b)$, *and its six permutations,*
(3) $(c-a, 2c-2b, b-a, c-b, b-c, b-a)$, *and its six permutations,*
(4) $(c-a, 3c-3b, b-a, 2c-2b, 2b-c-a, c-b)$, *and its six permutations,*
(5) $(3b-3a, 2b-2a, b-a, c-a, c-b, a-2b+c)$, *and its six permutations,*
(6) $(a-b, c-b, c-a, 2b-2a, c-b, b-a)$, *and its six permutations.*

Proof. To see how these are calculated, we consider two representative examples: (3) and (6).

At a complete triangle of type (3), the map $\varphi_1 : \mathsf{CT} \to \mathrm{Hilb}_3\,\mathbb{P}^2$ is a local isomorphism. Therefore, we have

$$T_p\,\mathsf{CT} \cong T_p\,\mathrm{Hilb}_3\,\mathbb{P}^2 \cong \mathrm{Hom}_{\mathbb{P}^2}(I, \mathcal{O}_{\mathbb{P}^2}/I).$$

At $[1:0:0]$, we have $I = (y, z)$ where $y = Y/X$ and $z = Z/X$. So we get $\operatorname{Hom}(I, \mathcal{O}/I) = \Bbbk\langle \widehat{y}, \widehat{z}\rangle$ where $\widehat{y}$ and $\widehat{z}$ are the dual variables to y and z; their weights are $a-b$ and $a-c$, respectively. At $[0:0:1]$, we have $I = (x, y^2)$ where $x = X/Z$ and $y = Y/Z$. So we get

$$\operatorname{Hom}(I, \mathcal{O}/I) = \operatorname{Hom}_k(\langle x, y^2\rangle, \langle 1, y\rangle) = \Bbbk\langle \widehat{x} \otimes 1, \widehat{x} \otimes y, \widehat{y^2} \otimes 1, \widehat{y^2} \otimes y\rangle.$$

The weights of these elements are $c-a, b-a, 2c-2b, c-b$. Combining the contributions from $[1:0:0]$ and $[0:0:1]$, we get the full set of weights.

Now we shift focus to a fixed point of type (6). At this point, the map $\mathsf{CT} \to \operatorname{Hilb}_3 \check{\mathbb{P}}^2$ is a local isomorphism. Denote by $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$ the variables dual to X, Y, and Z. Then the corresponding point in $\operatorname{Hilb}_3 \check{\mathbb{P}}^2$ is the point

$$\langle \widehat{Z}, \widehat{Y}^2\widehat{X}\rangle.$$

As in the third case, we get the weights $(c-a, 2b-2a, c-b, b-a, a-b, c-b)$.

□

We can now compute top degree Chern expressions of the vector bundle E. In particular, using Theorem 3.24, we get

$$c_3(E)^2 - c_2(E)c_4(E) = 2^7 \times 11 \times 41 = 57728. \tag{11}$$

The computation is carried out in `sage` (see `localization.sage`).

Before continuing with our story, we take a moment to collect other weight calculations which will be needed in the last section. We maintain the 1-6 ordering of the S_3 orbits of fixed points throughout.

Proposition 3.27. *Let $\mathcal{O}^{[3]}$, $\mathcal{O}(1)^{[3]}$, and $\mathcal{O}(2)^{[3]}$ denote the tautological rank 3 bundles pulled back to CT, and let $\mathcal{O}(H)$ denote the line bundle on CT corresponding to the divisor of those (T, T^*) such that T meets a fixed line. The torus weights of these bundles at the fixed points in CT are listed below. (One then applies the action of S_3 on the letters $\{a, b, c\}$.)*

$\mathcal{O}^{[3]}$: (1) $(0,0,0)$
(2) $(0, b-c, 0)$,
(3) $(0, b-c, 0)$,
(4) $(0, b-c, 2b-2c)$,
(5) $(0, a-c, b-c)$,
(6) $(0, a-c, b-c)$.

$\mathcal{O}(1)^{[3]}$: (1) (a, b, c),
(2) (a, b, c),
(3) (b, b, c),
(4) $(c, b, 2b-c)$,
(5) (a, b, c),
(6) (a, b, c).

$\mathcal{O}(2)^{[3]}$: (1) $(2a, 2b, 2c)$,
(2) $(2a, b+c, 2c)$,
(3) $(2b, b+c, 2c)$,

(4) $(2b, b+c, 2c)$,
(5) $(a+c, b+c, 2c)$,
(6) $(a+c, b+c, 2c)$.

$\mathcal{O}(3)^{[3]}$: (1) $(3a, 3b, 3c)$,
(2) $(3a, b+2c, 3c)$,
(3) $(3b, b+2c, 3c)$,
(4) $(b+2c, 2b+c, 3c)$,
(5) $(a+2c, b+2c, 3c)$,
(6) $(a+2c, b+2c, 3c)$.

$\mathcal{O}(H)$: (1) $(a+b+c)$,
(2) $(a+2c)$,
(3) $(b+2c)$,
(4) $(3c)$,
(5) $(3c)$,
(6) $(3c)$.

4. Circumventing excess

4.1. **Why 57728 is wrong.** The calculation (11) done in the previous section is unfortunately not the number $\nu_{3,2}$. The problem is that the evaluation map

$$\underline{H^0(\mathbb{P}^2, \mathcal{I}_{\Gamma_{13}}(5))} \to E, \tag{12}$$

where $\Gamma_{13} \subset \mathbb{P}^2$ is a general set of 13 points (see Section 2.3), has 2-dimensional kernel over certain points $(T, T^*) \in \mathsf{CT}$ which should not be counted for our enumerative problem – there is excess in the degeneracy scheme. For a simple example, observe that if T consists of three of the points of Γ_{13}, then the evaluation mapping over (T, T^*) automatically has at least a 2-dimensional kernel. There are more complicated contributions to the excess. (Ultimately this is explained by Theorem 4.10 below.) To dodge the excess, we change our viewpoint and work in a Grassmannian bundle over CT which we denote by SQP and call *the space of singular quintic pencils*. Recall that the vector bundle $\mathcal{V}_5$ (see Theorem 3.22) is related to E by an exact sequence

$$0 \to \mathcal{V}_5 \to \underline{H^0(\mathbb{P}^2, \mathcal{O}(5))} \to E \to 0$$

over CT.

Definition 4.1. *The* **space of singular quintic pencils**, *denoted* SQP, *is the smooth variety* $\mathrm{Gr}(2, \mathcal{V}_5)$ *representing* 2*-dimensional subspaces in the fibers of* $\mathcal{V}_5$.

A point of SQP corresponds to a triple (T, T^*, Λ) where $(T, T^*) \in \mathsf{CT}$ is a complete triangle and Λ is a pencil of quintic curves, all containing the length 9 scheme $\mathsf{sq}(T, T^*)$. We let

$$\varphi : \mathsf{SQP} \to \mathsf{CT}$$

denote the map sending (T, T^*, Λ) to (T, T^*); φ is a $\mathrm{Gr}(2, 12)$-bundle. The variety SQP also affords a natural map

$$\eta : \mathsf{SQP} \to \mathrm{Gr}\left(2, H^0(\mathbb{P}^2, \mathcal{O}(5))\right)$$

with formula $\eta(T, T^*, \Lambda) = \Lambda$.

Let us state the main objective of this section from the outset, to better orient the reader:

Theorem 4.2. *Let $p \in \mathbb{P}^2$ be a point, and let $\mathsf{Dom}(p) \subset \mathsf{SQP}$ be as in Theorem 4.8 below. Then*

$$\nu_{3,2} = \int_{\mathsf{SQP}} [\mathsf{Dom}(p)]^{13}.$$

We begin by establishing some definitions.

4.2. First definitions.

Definition 4.3. *Let $W \subset H^0(\mathbb{P}^2, \mathcal{O}(n))$ be a subspace. The* **base-scheme** *of W, denoted $\mathsf{Base}(W)$ is the subscheme of $\mathbb{P}^2$ which is the common vanishing scheme of all elements of W.*

Definition 4.4. *We let*

$$\mathsf{Inf} \subset \mathsf{SQP}$$

denote the closed subset consisting of triples (T, T^*, Λ) such that $\mathsf{Base}(\Lambda)$ is infinite. We let

$$\mathsf{Fin} \subset \mathsf{SQP}$$

denote the open complement of Inf.

4.3. **Point conditions on singular quintic pencils.** Fix a point $p \in \mathbb{P}^2$. The point p determines the hyperplane

$$H_p \subset H^0(\mathbb{P}^2, \mathcal{O}(5))$$

consisting of quintic forms vanishing at p, and therefore also determines a codimension 2 sub-Grassmannian

$$\mathrm{Gr}\left(2, H_p\right) \subset \mathrm{Gr}\left(2, H^0\left(\mathbb{P}^2, \mathcal{O}(5)\right)\right).$$

Definition 4.5. *Maintain the setting immediately prior. We define*

$$\mathsf{Bpt}(p) \subset \mathsf{SQP}$$

to be the subscheme $\eta^{-1}\left(\mathrm{Gr}(2, H_p)\right)$.

As a set, $\mathsf{Bpt}(p)$ consists of those triples $(T, T^*, \Lambda) \in \mathsf{SQP}$ having the property that $p \in \mathsf{Base}(\Lambda)$.

Proposition 4.6. *$\mathsf{Bpt}(p)$ is a codimension 2 local complete intersection subscheme of SQP.*

Proof. Since $\mathrm{Gr}(2, H_p)$ is smooth and is a codimension 2 subvariety of $\mathrm{Gr}\left(2, H^0(\mathbb{P}^2, \mathcal{O}(5))\right)$, and since SQP is also smooth, it follows that $\operatorname{codim} \mathsf{Bpt}(p) \leq 2$, and it suffices to show $\operatorname{codim} \mathsf{Bpt}(p) = 2$. It is clear that $\mathsf{Bpt}(p) \neq \mathsf{SQP}$, so let us assume for sake of contradiction that $C \subset \mathsf{Bpt}(p)$ is an irreducible component which is a divisor in SQP.

Dimension constraints yield two possibilities. The first possibility is $\varphi(C) = \mathsf{CT}$, and the second possibility is that $\varphi(C) \subset \mathsf{CT}$ is a divisor. Before continuing the proof, we prove a general claim:

Lemma 4.7. *Let* $(T, T^*) \in \mathsf{CT}$ *be arbitrary, and let* $W = H^0(\mathbb{P}^2, \mathcal{I}_{\mathsf{sq}(T,T^*)}(5))$. *Then*

$$\operatorname{Supp} \mathsf{Base}(W) = \operatorname{Supp} T$$

if and only if $T \notin \mathsf{Thin}$ *and otherwise* $\operatorname{Supp} \mathsf{Base}(W)$ *is the line* $\langle T \rangle$.

Proof of 4.7. The claim rests on the observation that $\operatorname{Supp} T = \operatorname{Supp} \mathsf{Base}(\Lambda_T)$ if and only if T is not contained in a line. Here Λ_T is the net of conics (Theorem 3.1) containing T. If $T \in \mathsf{Thin}$, then the quintic curves in W are those which contain the line spanned by T as a component, and whose residual quartic curve contains T. From this description it is clear that $\operatorname{Supp} \mathsf{Base}(W) = \langle T \rangle$ as claimed.

So we can and will assume $T \notin \mathsf{Thin}$. Then the inclusion

$$\operatorname{Supp} \mathsf{Base}(H^0(\mathcal{I}_T^2(4))) \subset \operatorname{Supp} T$$

is seen by considering the pairwise products of three quadratic polynomials spanning Λ_T. Therefore, $\operatorname{Supp} \mathsf{Base}(H^0(\mathcal{I}_T^2(5))) \subset \operatorname{Supp} T$ as well. Since $\mathsf{Base}(W) \subset \mathsf{Base}(H^0(\mathcal{I}_T^2(5)))$, we conclude that $\operatorname{Supp} \mathsf{Base}(W) \subset \operatorname{Supp} T$. The opposite inclusion is trivial, concluding the proof. □

Returning to the proof of Theorem 4.6, we consider the case $\varphi(C) = \mathsf{CT}$. Choose a general honest triangle (T, T^*) – in particular, $p \notin T$. Then, from Theorem 4.7 we know that $T = \operatorname{Supp} \mathsf{Base}(W)$. Thus, the point p imposes a non-trivial condition on the elements of W, i.e. the vector space $V \subset W$ consisting of those $w \in W$ satisfying $w(p) = 0$ has codimension 1. Therefore $\varphi^{-1}(\{(T, T^*)\}) \cap C \subset \operatorname{Gr}(2, V)$ has codimension at least 2 in $\varphi^{-1}(\{(T, T^*)\})$, and hence C cannot be a divisor in SQP, our desired contradiction.

It remains to deal with the possibility that $\varphi(C)$ is a (irreducible) divisor in CT. Let (T, T^*) be a general element of $\varphi(C)$, assuming it is a divisor. In particular, $p \notin T$ and if $(T, T^*) \in \mathsf{Thin}$ then $p \notin \langle T \rangle$, as otherwise $\varphi(C)$ would have codimension strictly larger than 1. By Theorem 4.7, it follows that the space of sections of W vanishing at p is a proper subspace $V \subset W$. Since $\varphi^{-1}(\{(T, T^*)\}) \cap C \subset \operatorname{Gr}(2, V)$, and $\operatorname{Gr}(2, V)$ has codimension 2 in $\operatorname{Gr}(2, W)$, it follows that C has codimension at least 3 in SQP, a contradiction. The proposition is now proved. □

The proof of Theorem 4.6 justifies the following definition.

Definition 4.8. (1) *We define*

$$\mathsf{Dom}(p) \subset \mathsf{Bpt}(p)$$

to be the unique irreducible component (with reduced, induced scheme structure) whose general point (T, T^*, Λ) *satisfies*

(a) (T, T^*) *is an honest triangle, and*

(b) $p \notin T$.

(2) *We define*

$$\mathsf{Inc}(p) \subset \mathsf{Bpt}(p)$$

to be the irreducible component (with reduced, induced scheme structure) consisting of triples (T, T^*, Λ) *such that* $p \in T$.

(3) *We define*

$$\mathsf{Lin}(p) \subset \mathsf{Bpt}(p)$$

to be the irreducible component (with reduced, induced scheme structure) whose general point corresponds to a triple (T, T^*, Λ) *satisfying*
(a) $T \in \mathsf{Thin}$
(b) $p \notin T$, *and*
(c) T *and* p *are collinear.*

Remark 4.9. Observe that if (T, T^*) is such that $p \notin T$ and if $\mathsf{sq}(T, T^*) \cup \{p\}$ imposes ten independent conditions on quintic forms, then $\mathsf{Dom}(p)$ is the unique irreducible component of $\mathsf{Bpt}(p)$ lying over a sufficiently small neighborhood of (T, T^*). Indeed, over a neighborhood of $(T, T^*) \in \mathsf{CT}$ the map $\eta|_{\mathsf{Bpt}(p)} : \mathsf{Bpt}(p) \to \mathsf{CT}$ is then a $\mathrm{Gr}(2, 11)$-bundle.

Lemma 4.10. $\mathsf{Dom}(p), \mathsf{Inc}(p)$, *and* $\mathsf{Lin}(p)$ *are the irreducible components of* $\mathsf{Bpt}(p)$*.*

Proof. Let $Z \subset \mathsf{Bpt}(p)$ be an irreducible component, and let (T, T^*, Λ) be a general element of Z. We must show Z is one of the three listed sets.

If $p \in T$, then $Z = \mathsf{Inc}(p)$ (for dimension reasons) and we are done. So we may and will assume $p \notin T$ from here.

If $T \in \mathsf{Thin}$, then Λ must be a pencil of quintics which has the line $\langle T \rangle$ in its base scheme. If $p \in \langle T \rangle$ then again by counting dimensions we conclude $Z = \mathsf{Lin}(p)$. If $p \notin \langle T \rangle$ on the other hand, then the set of all such (T, T^*, Λ) does not have large enough dimension to contribute an irreducible component of $\mathsf{Bpt}(p)$.

If (T, T^*) is such that $T \notin \mathsf{Thin}$ then by Theorem 3.23 the twisted ideal sheaf $\mathcal{I}_{\mathsf{sq}(T,T^*)}(5)$ is globally generated, and therefore $\mathsf{sq}(T, T^*) \cup \{p\}$ imposes ten independent conditions on quintic forms. Therefore, $Z = \mathsf{Dom}(p)$ by Theorem 4.9, finishing the proof. ◻

Our next objective is to determine the multiplicities of $\mathsf{Bpt}(p)$ along its three irreducible components $\mathsf{Dom}(p), \mathsf{Inc}(p)$, and $\mathsf{Lin}(p)$ (Theorem 4.10). We need a bit of preparation before continuing. If we let $S \to \mathsf{SQP}$ denote the universal rank 2 vector bundle, then we can let

$$S^\dagger \subset S \times_{\mathsf{SQP}} S \tag{13}$$

denote the bundle of frames for S/SQP. A point of $S^\dagger$ is a tuple (F, G, T, T^*) where F and G are linearly independent quintic forms which are both elements of $H^0\left(\mathbb{P}^2, \mathcal{I}_{\mathsf{sq}(T,T^*)}(5)\right)$. The natural morphism $S^\dagger \to \mathsf{SQP}$ is smooth and faithfully flat (it is a GL_2-torsor), and therefore, if we use the † superscript in the obvious way, it suffices to study the multiplicities of $\mathsf{Dom}(p)^\dagger, \mathsf{Inc}(p)^\dagger$, and $\mathsf{Lin}(p)^\dagger$ as irreducible components of $\mathsf{Bpt}(p)^\dagger$.

The reason for passing to $S^\dagger$ is that $\mathsf{Bpt}(p)^\dagger$ is a global complete intersection of two divisors: If we define

$$H_1(p) := \left\{ \begin{array}{c} (F,G,T,T^*) \in S^\dagger \text{ such} \\ \text{that } F(p)=0. \end{array} \right\} \text{ and}$$
$$H_2(p) := \left\{ \begin{array}{c} (F,G,T,T^*) \in S^\dagger \text{ such} \\ \text{that } G(p)=0. \end{array} \right\},$$

then $\mathsf{Bpt}(p)^\dagger = H_1(p) \cap H_2(p)$ as schemes. For transversality arguments, it behooves us to better illuminate the three tangent spaces of $S^\dagger, H_1(p)$, and $H_2(p)$ at a point (F,G,T,T^*). We will only need to analyze the situation where $T \subset \mathbb{P}^2$ consists of three distinct points $\{a,b,c\}$, and where F and G possess only ordinary nodes at a,b,c. We will use the language of deformation theory, working over the dual numbers $\Bbbk[\varepsilon]/(\varepsilon^2)$.

A first order deformation of F (resp. G) is given by $F+\varepsilon F'$ (resp. $G+\varepsilon G'$) where F' (resp. G') is a quintic form. A first order deformation of F (resp. G) which continues to have three nodes and which *allows their locations* a,b *and* c *to deform* is of the form $F+\varepsilon F'$ where $F'(a)=F'(b)=F'(c)=0$. Therefore, the tangent space $T_{(F,G,T,T^*)}S^\dagger$ is a linear subspace of the product vector space $H^0\left(\mathbb{P}^2, \mathcal{I}_T(5)\right) \times H^0\left(\mathbb{P}^2, \mathcal{I}_T(5)\right)$, one which we identify next.

The essential question we must answer is: Given a form F' vanishing at a,b, and c, how do we determine the deformation of the nodes a,b,c induced by $F+\varepsilon F'$? A local calculation reveals a clean answer. We will focus on just the point a. F is a global section of the sheaf $\mathfrak{m}_a^2(5)$, and therefore induces an element

$$\mathsf{H}_F \in \left(\mathfrak{m}_a^2/\mathfrak{m}_a^3\right) \otimes \mathcal{O}(5).$$

By the natural isomorphism

$$\mathfrak{m}_a^2/\mathfrak{m}_a^3 \simeq \mathrm{Sym}^2(\mathfrak{m}_a/\mathfrak{m}_a^2),$$

and in light of the inclusion (char. $\Bbbk \neq 2$)

$$\mathrm{Sym}^2(\mathfrak{m}_a/\mathfrak{m}_a^2) \subset \mathrm{Hom}_{\Bbbk}\left(\left(\mathfrak{m}_a/\mathfrak{m}_a^2\right)^\vee, \mathfrak{m}_a/\mathfrak{m}_a^2\right),$$

we may view H_F as an element of

$$\mathrm{Hom}\left(\left(\mathfrak{m}_a/\mathfrak{m}_a^2\right)^\vee, \left(\mathfrak{m}_a/\mathfrak{m}_a^2\right)\otimes \mathcal{O}(5)\right).$$

The notation is chosen because in local coordinates if $f(x,y)$ is an affine quintic obtained by dehomogenizing F then H_F is represented by the 2×2 *Hessian matrix*

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

As a consequence of F having an ordinary node at a, we see that H_F is invertible.

Now, suppose $F+\varepsilon F'$ is a deformation satisfying $F'(a)=0$. Then F' determines its differential

$$dF'_a \in \left(\mathfrak{m}_a/\mathfrak{m}_a^2\right) \otimes \mathcal{O}(5),$$

and therefore

$$\tau_{F,a}(F') := \mathsf{H}_F^{-1}(dF'_a)$$

is a well-defined element of the tangent space

$$\left(\mathfrak{m}_a/\mathfrak{m}_a^2\right)^\vee = T_a\mathbb{P}^2.$$

This vector is the deformation of the node at a induced by the deformation $F + \varepsilon F'$, as can easily be checked in local coordinates. We are prepared to express the tangent space $T_{(F,G,T,T^*)}S^\dagger$:

$$T_{(F,G,T,T^*)}S^\dagger = \left\{ \begin{array}{c} \text{Pairs of quintics } (F', G') \text{ vanishing} \\ \text{at } T \text{ satisfying } \tau_{F,t}(F') = \tau_{G,t}(G') \text{ for} \\ \text{all three points } t \in T \end{array} \right\}. \tag{14}$$

Lemma 4.11. *The divisors $H_1(p)$ and $H_2(p)$ intersect transversely at a general point of* $\mathsf{Lin}(p)^\dagger$.

Proof. We will exhibit a single point

$$(F, G, T, T^*) \in \mathsf{Lin}(p)^\dagger$$

where the two tangent spaces $T_{(F,G,T,T^*)}H_1(p)$ and $T_{(F,G,T,T^*)}H_2(p)$ are distinct, codimension 1 spaces of the ambient tangent space $T_{(F,G,T,T^*)}S^\dagger$.

Consider then, the following tuple (F, G, T, T^*) and point p:

$$\begin{aligned} F &= XY(Y - Z)(Y - \lambda Z)Z, \\ G &= XY(Y - Z)(Y - \mu Z)Z, \\ T &= \{a = [0 : 0 : 1], b = [0 : 1 : 1], c = [0 : 1 : 0]\}, \\ p &= [0 : \lambda : 1], \end{aligned}$$

where $\lambda, \mu \in \Bbbk$ are to be chosen generally. (F, G, T, T^*) is evidently contained in $\mathsf{Lin}(p)^\dagger$: all points a, b, c, p lie on the line $X = 0$. Additionally, F and G have only simple nodes at the points a, b, c, and so the description in (14) of $T_{(F,G,T,T^*)}S^\dagger$ as a vector space of certain pairs (F', G') applies. And so, consider the pair

$$\begin{aligned} F' &= Y(Y - Z)(Y - \lambda Z)Z^2, \\ G' &= Y(Y - Z)(Y - \mu Z)Z^2. \end{aligned}$$

A local calculation (omitted) then shows that the three membership conditions of (14), namely

$$\tau_{F,t}(F') = \tau_{G,t}(G'), \ \forall t \in T,$$

are met. Furthermore, (F', G') is evidently contained in $T_{(F,G,T,T^*)}H_1(p)$ (because $F'(p) = 0$) and is not contained in $T_{(F,G,T,T^*)}H_2(p)$ (because $G'(p) \neq 0$). The lemma follows. □

Theorem 4.12. *As codimension* 2 *cycles in* SQP,

$$[\mathsf{Bpt}(p)] = [\mathsf{Dom}(p)] + 4\,[\mathsf{Inc}(p)] + [\mathsf{Lin}(p)]\,. \tag{15}$$

Proof. We must explain the multiplicities. Since the natural map $S^\dagger \to \mathsf{SQP}$ is smooth and surjective, it suffices to prove

$$\left[\mathsf{Bpt}(p)^\dagger\right] = \left[\mathsf{Dom}(p)^\dagger\right] + 4\left[\mathsf{Inc}(p)^\dagger\right] + \left[\mathsf{Lin}(p)^\dagger\right] \tag{16}$$

as codimension 2 cycles in the frame bundle $S^\dagger$. The coefficient of $\mathsf{Lin}(p)^\dagger$ is explained by Theorem 4.11. We will only focus on the coefficient 4 of $\mathsf{Inc}(p)^\dagger$, as the ideas apply equally well (and with fewer complications) to the coefficient of $\mathsf{Dom}(p)^\dagger$.

We will choose a sufficiently general 2-dimensional étale-local slice of $S^\dagger$ at a general point $(F, G, T, T^*) \in \mathsf{Inc}(p)^\dagger$. So, let $(U, q) \subset S^\dagger$ be a smooth pointed surface with étale-local coordinates s, t at q, and suppose $q = (F, G, T, T^*)$ is a general point of $\mathsf{Inc}(p)^\dagger$. Observe that $T = \{p, t_2, t_3\}$ is an honest triangle, so there is an étale neighborhood V of $(T, T^*) \in \mathsf{CT}$ which is isomorphic to an étale neighborhood the point $(p, t_2, t_3) \in (\mathbb{P}^2)^3$. Let $\alpha_1 : V \to \mathbb{P}^2$ denote projection onto the first factor. Finally, as part of the generic hypotheses on U, after possibly shrinking U, suppose the composite $U \to V \to \mathbb{P}^2$ is unramified at q.

Having made the choice of the general slice U, our objective is to understand the two curves $H_i(p) \cap U, i = 1, 2$ locally near q. Dehomogenizing the family of forms parametrized by U, and letting $p = (0, 0) \in \mathbb{A}^2$, we obtain a pair of varying polynomials dependent on (s, t):

$$f_{(s,t)}(x, y) = c_{11}(x - u)^2 + c_{12}(x - u)(y - v) + c_{22}(y - v)^2 + \dots$$
$$g_{(s,t)}(x, y) = d_{11}(x - u)^2 + d_{12}(x - u)(y - v) + d_{22}(y - v)^2 + \dots,$$

where $u = u(s, t)$ and $v = v(s, t)$ are the coordinates of the particular node which coincides with the point p at $s = t = 0$. The coefficients c_{ij}, d_{ij} are also functions of (s, t), and when $s = t = 0$ we may assume that $f_{(0,0)}$ and $g_{(0,0)}$ are tri-nodal quintics whose tangent cones at $p = (0, 0)$ do not share a line.

Now, by introducing the condition "$H_1(p)$" we are simply plugging in $x = y = 0$ into $f_{(s,t)}$ and requesting vanishing. This gives the equation

$$c_{11}u^2 + c_{12}uv + c_{22}v^2 + \cdots = 0,$$

where the excluded terms lie in $(s, t)^3$. We obtain a similar local equation for $H_2(p) \cap U$ with c_{ij}'s replaced with d_{ij}'s. The two functions u, v can be taken to be local coordinates at q – this is due to the genericity assumptions on the slice U. It follows that the local equations $H_i(p) \cap U$ inherit the property of having ordinary nodes at q from the fact that $f_{(0,0)}$ and $g_{(0,0)}$ both had ordinary nodes. Thus, $H_1(p) \cap U$ and $H_2(p) \cap U$ are two curves nodal at q. The tangent cones of $H_1 \cap U$ and $H_2 \cap U$ at q do not share a line because the same was true for the curves $f_{(0,0)}$ and $g_{(0,0)}$. Thus, the multiplicity 4 occurring in (16) is explained. □

4.4. **Thirteen point conditions.** From here on, let $\Gamma_{13} = \{p_1, \dots, p_{13}\}$ be 13 general points in $\mathbb{P}^2$, and set

$$\Theta := \bigcap_{i=1,\dots,13} \mathsf{Dom}(p_i). \tag{17}$$

Our next major objective is Theorem 4.25, which states that Θ is indeed the set we must enumerate. The strategy is of course to argue that certain possible and undesirable types of points do not occur in the intersection. We first deal with possibilities inside the open set Fin using dimension counts in Theorem 4.13, and Theorem 4.14. Then we deal with undesirable possibilities inside Inf – this is more difficult, occupying Theorem 4.16, Theorem 4.17, Theorem 4.18, and Theorem 4.19. Theorem 4.19 in particular uses a limit-linear series argument.

4.4.1. *Finite base schemes.*

Lemma 4.13. *Every point* $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Fin}$ *satisfies:* $(T, T^*, \Lambda) \notin \mathsf{Inc}(p_i)$ *and* $(T, T^*, \Lambda) \notin \mathsf{Lin}(p_i)$ *for all* $p_i \in \Gamma_{13}$*.*

Proof. This follows from combining a dimension count with the condition that Γ_{13} is a general set. We only prove the Inc statement – the other case proceeds mutatis mutandis.

Without losing generality, we need only prove the statement $(T, T^*, \Lambda) \notin \mathsf{Inc}(p_1)$. As $\mathsf{Dom}(p_1)$ is irreducible and 24-dimensional,

$$\dim \mathsf{Dom}(p_1) \cap \mathsf{Inc}(p_1) \leq 23.$$

And so, the locally closed set $U := \mathsf{Dom}(p_1) \cap \mathsf{Inc}(p_1) \cap \mathsf{Fin}$ is at most 23-dimensional.

Every $(T, T^*, \Lambda) \in U$ determines the 0-dimensional, length 25 scheme $\mathsf{Base}(\Lambda) \subset \mathbb{P}^2$. Let

$$U' \subset U \times (\mathbb{P}^2)^{12}$$

denote the set parametrizing quadruples (T, T^*, Λ, W) where $(T, T^*, \Lambda) \in U$ and where $W = (w_1, \ldots, w_{12})$ is a 12-tuple of points in $\mathbb{P}^2$ satisfying $w_i \in \mathsf{Base}(\Lambda)$ for all i. Then the forgetful map $U' \to U$ is quasi-finite, and hence

$$\dim U' \leq 23.$$

The map $U' \to (\mathbb{P}^2)^{12}$ defined by $(T, T^*, \Lambda, W) \mapsto W$ therefore cannot dominate $(\mathbb{P}^2)^{12}$, and hence the set of 13 *general* points $\{p_1, \ldots, p_{13}\}$ cannot be contained in $\mathsf{Base}(\Lambda)$ for $(T, T^*, \Lambda) \in \bigcap_{i=1}^{13} \mathsf{Dom}(p_i) \cap \mathsf{Inc}(p_1) \cap \mathsf{Fin}$, implying the lemma. □

Lemma 4.14. *Let* $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Fin}$*. Then* T *is an honest triangle, and* $T \cap \Gamma_{13} = \varnothing$*.*

Proof. The statement "$T \cap \Gamma_{13} = \varnothing$" follows immediately from Theorem 4.13, so we will assume it and show that T is an honest triangle.

Pick $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Fin}$. As $\mathsf{Base}(\Lambda)$ is finite, it follows that $T \notin \mathsf{Thin}$ (as otherwise $\mathsf{Base}(\Lambda)$ contains the line spanned by T). Therefore, we need only show that T is reduced.

As in the proof of Theorem 4.13, we perform a dimension count. The locus $V \subset \mathsf{Fin}$ consisting of triples (T, T^*, Λ) where T is non-reduced is a Cartier divisor. And so, $\dim V = 25$. Let

$$V' \subset V \times (\mathbb{P}^2)^{13}$$

denote the scheme parametrizing quadruples (T, T^*, Λ, W), where $(T, T^*, \Lambda) \in V$ and $W = (w_1, \ldots, w_{13})$ is a 13-tuple of points satisfying $w_i \in \mathsf{Base}(\Lambda)$ for all i. Since the forgetful map $V' \to V$ is quasi-finite, it follows that $\dim V' = 25$. Therefore the second projection $V' \to (\mathbb{P}^2)^{13}$ cannot be dominant, and in particular cannot contain the general tuple $(p_1, \ldots, p_{13})$ in its image, implying the lemma. □

4.4.2. *Dealing with infinite base schemes.* We now take on the challenge of showing emptiness of $\Theta \cap \mathsf{Inf}$.

Definition 4.15. (1) *If $\Lambda \subset H^0(\mathbb{P}^2, \mathcal{O}(d))$ is a pencil of degree d curves, with base scheme $\mathsf{Base}(\Lambda)$, we define the* **fixed curve** *of $\mathsf{Base}(\Lambda)$ to be the Cartier divisor on $\mathbb{P}^2$ defined by the greatest common factor of any two general elements of Λ.*

(2) *If the fixed curve of Λ has degree e, we define the* **moving part** *of Λ, denoted Λ', to be the pencil of degree $d-e$ curves obtained by dividing the equations of the members of Λ by the equation of the fixed curve.*
(3) *If Λ is a pencil of curves with fixed curve C, then a* **fixed point** *of Λ will mean a point in $C \setminus \mathsf{Base}(\Lambda')$.*
(4) *If Λ is a pencil of curves with fixed curve C, then an* **isolated point** *of Λ will mean a point in $\mathsf{Base}(\Lambda') \setminus C$.*
(5) *If Λ is a pencil of curves with fixed curve C, then an* **embedded point** *of Λ will mean a point in $C \cap \mathsf{Base}(\Lambda')$.*

Lemma 4.16. *Suppose Λ is a pencil of quintic curves whose fixed curve is a quartic Q. Suppose furthermore that Λ satisfies one of the following:*

(a) $\mathsf{Base}(\Lambda)$ *has no embedded points and* $\operatorname{Sing} Q$ *consists of at most two points, each a node,*
(b) $\mathsf{Base}(\Lambda)$ *has no embedded points and* $\operatorname{Sing} Q$ *consists of a single ordinary cusp,*
(c) $\mathsf{Base}(\Lambda)$ *has one embedded point at a smooth point of Q while* $\operatorname{Sing} Q$ *has at most one singular point which is a node,*
(d) $\operatorname{Sing} Q$ *consists of a single node and this node is the embedded point of* $\mathsf{Base}(\Lambda)$.

Then $\mathsf{Base}(\Lambda)$ does not contain any subscheme of the form $\mathsf{sq}(T, T^*)$ for any $(T, T^*) \in \mathsf{CT}$.

Proof. The moving part Λ' of Λ is a pencil of lines; we let $b \in \mathbb{P}^2$ denote the base point Λ'. Then, $\mathsf{Base}(\Lambda)$ contains an embedded point if and only if $b \in Q$, in which case b is the sole embedded point. Before taking each case up in turn, observe that as $\mathsf{sq}(T, T^*)$ has 2-dimensional Zariski tangent space at each of its points, if $\mathsf{sq}(T, T^*) \subset \mathsf{Base}(\Lambda)$ then T must be supported on the singular points of Q or on the embedded point of $\mathsf{Base}(\Lambda)$, if it exists (or both).

(a) In this scenario, one of the two nodes, call it $n \in Q$, must support a length ≥ 2 connected component $T_n \subset T$ of T.

If T_n has length 2, then in affine coordinates (x, y) around n the ideal $\mathcal{I}_{T_n}$ is (y, x^2) and so the ideal $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ is given by (y^2, yx^2, x^4). The ideal $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ does not contain an element with non-degenerate quadratic part, and so Q's local equation cannot be contained it, eliminating this case.

If T_n has length 3 and is curvilinear, then in suitable *analytic* local coordinates (u, v) around n, the ideal $\mathcal{I}_{T_n}$ can be taken to be (u, v^3), and so $\mathcal{I}_{\mathsf{sq}(T,T^*)} = (u^2, uv^3, v^6)$. Once again, an analytic local equation for Q cannot be contained in $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ because it would have a non-degenerate quadratic term.

Finally, if T_n is a fat point, then $\mathsf{sq}(T,T^*)$ (for any T^*) is contained in the cube of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{\mathbb{P}^2,n}$, while Q's local equation lies in $\mathfrak{m}^2 \setminus \mathfrak{m}^3$. And so in all possible scenarios, Q's local equation cannot be contained in $\mathcal{I}_{\mathsf{sq}(T,T^*)}$, which is what we needed to show.

(b) Let $c \in Q$ denote the cusp. If $\mathsf{sq}(T,T^*) \subset \mathsf{Base}(\Lambda)$ then T is entirely supported on c, and hence T is either fat or curvilinear.

If T is a fat point, then $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ is contained in the cube of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{\mathbb{P}^2,c}$, yet Q's local equation is contained in $\mathfrak{m}^2 \setminus \mathfrak{m}^3$ (a cusp is a double point) – so the fat possibility is eliminated.

If T is curvilinear, then in suitable analytic local coordinates (u,v) around c, we have $\mathcal{I}_T = (u, v^3)$ and so $\mathcal{I}_{\mathsf{sq}(T,T^*)} = (u^2, uv^3, v^6)$. Suppose $g(u,v)$ is an analytic local equation for Q. If $g \in \mathcal{I}_{\mathsf{sq}(T,T^*)}$ then, because a cusp is a double point, after scaling by an element in $\Bbbk^\times$ we must have

$$g = u^2 + h_1(u,v) \cdot u^2 + h_2(u,v) \cdot uv^3 + h_3(u,v) \cdot v^6,$$

where the h_i are power series and where h_1 has no constant term. Now observe that there are no power series

$$u(t), v(t) \in \Bbbk[\![t]\!]$$

with vanishing constant terms such that

$$\operatorname{ord} g(u(t), v(t)) = 3,$$

a necessary condition for the germ g to define a cusp. (Here the order ord of a power series with variable t is the degree of the first non-zero term.) Thus the germ of a defining equation of Q at c cannot be contained in $\mathcal{I}_{\mathsf{sq}(T,T^*)}$, eliminating this curvilinear possibility.

(c) Hypothetically, if $\mathsf{sq}(T,T^*) \subset \mathsf{Base}(\Lambda)$ then our first claim is that T must be entirely supported on the embedded point $b \in Q$: Let $n \in Q$ be the node of Q if it exists. By what is written immediately prior to the proof of part (a), T is supported somewhere in the set $\{b, n\}$. Let T_b, T_n denote the connected components of T supported on the respective points. If $\operatorname{length} T_n \geq 2$, we argue as in part (a) to conclude that $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ cannot contain a defining equation for Q. If $\operatorname{length} T_b = 2$, then in local coordinates (x,y) near b the ideal $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ is (x^2, xy^2, y^4). If f is a local defining equation of Q near b, then the local ideal of $\mathsf{Base}(\Lambda)$ is given by

$$\mathcal{J} := (xf, yf).$$

This ideal $\mathcal{J}$ has two linearly independent quadratic elements, modulo $(x,y)^3$, while $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ does not. Thus $\mathsf{sq}(T,T^*) \not\subset \mathsf{Base}(\Lambda)$ in this case.

Therefore, we may assume that T is entirely supported at the point b. There are now two cases to investigate: either T is a fat point, or T is curvilinear.

In the fat case, $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ is contained in the cube of the maximal ideal $\mathfrak{m} = (x,y) \subset \mathcal{O}_{\mathbb{P}^2,b}$, while the two generators of $\mathcal{J}$ are not (Q is smooth at b). So $\mathsf{sq}(T,T^*) \not\subset \mathsf{Base}(\Lambda)$ in this case.

On the other hand, if T is curvilinear a calculation as in the proof of part (b) implies that, modulo $\mathfrak{m}^3$, $\mathcal{I}_{\mathsf{sq}(T,T^*)} \cap \mathfrak{m}^2$ consists of a single quadratic form on $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ up to scaling. However, $\mathcal{J} \cap \mathfrak{m}^2$ has two linearly independent quadratic forms modulo $\mathfrak{m}^3$, and thus again $\mathsf{sq}(T,T^*) \not\subset \mathsf{Base}(\Lambda)$. So our hypothetical situation is impossible, as we needed to show.

(d) Let $n \in Q$ denote the node, let $x, y \in \mathcal{O}_{\mathbb{P}^2,n}$ be local affine coordinates of $\mathbb{P}^2$ near n, and let $f(x,y) \in \mathcal{O}_{\mathbb{P}^2,n}$ denote a local defining equation of Q. If, hypothetically, $\mathsf{sq}(T,T^*) \subset \mathsf{Base}(\Lambda)$, then T must be entirely supported at n. As in the proof of part (c), we consider the two possibilities: T is either a fat point or is curvilinear with length 3. If T is fat, then $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ is contained in $\mathfrak{m}^3$, where $\mathfrak{m} = (x,y)$ is the maximal ideal. Furthermore, $\mathcal{I}_{\mathsf{sq}(T,T^*)}$ contains exactly one element (up to scale) in $\mathfrak{m}^3$, modulo $\mathfrak{m}^4$. However, the ideal $\mathcal{J} := (xf, yf)$ contains two $\Bbbk$-linearly independent such elements. Thus, T cannot be fat.

If T is curvilinear we observe that, modulo $\mathfrak{m}^4$, the elements of $\mathcal{I}_{\mathsf{sq}(T,T^*)} \cap \mathfrak{m}^3$ are binary cubic forms (on $(\mathfrak{m}/\mathfrak{m}^2)^\vee$) all sharing a common factor which is a perfect square. This can be checked after passing to (any) analytic local coordinates (u,v) – we can choose convenient coordinates where $\mathcal{I}_T = (u, v^3)$ and so $\mathcal{I}_{\mathsf{sq}(T,T^*)} = (u^2, uv^3, v^6)$. Indeed, modulo $(u,v)^4$ all elements in $\mathcal{I}_{\mathsf{sq}(t,T^*)} \cap (u,v)^3$ are multiples of u^2. This property of having a perfect square common factor is not shared by the ideal $\mathcal{J}$, and hence $\mathsf{sq}(T,T^*) \not\subset \mathsf{Base}(\Lambda)$, eliminating this curvilinear possibility.

□

Proposition 4.17. *If* (T, T^*, Λ) *is an element of* $\Theta \cap \mathsf{Inf}$ *then the fixed curve of* $\mathsf{Base}(\Lambda)$ *consists of a reduced line.*

Proof. We proceed by considering one by one the possible degrees of the fixed curve $C \subset \mathsf{Base}(\Lambda)$. Each case is resolved by the tension between generality of $\Gamma_{13} = \{p_1, \ldots, p_{13}\}$ (which by assumption is contained in $\mathsf{Base}(\Lambda)$) on the one hand, and the number of points in the intersection $\Gamma_{13} \cap C$ on the other. For ease of reading, let $\Gamma_C := \Gamma_{13} \cap C$.

(1) *Assuming* $\deg(C) = 4$, the moving part of Λ is a pencil of lines. Furthermore, C must be reduced, as at most 5 of the points of Γ_{13} can lie on a conic. There are two possibilities for the number $\#\Gamma_C$: either 12 or 13, depending on whether the basepoint of Λ' is in Γ_{13} or not. Suppose first that $\#\Gamma_C = 13$. The unique, general, pencil of quartics determined by Γ_C has finitely many singular members, each with a *single* node located away from $\Gamma_C = \Gamma_{13}$. However, this prevents any length 9 scheme of the form $\mathsf{sq}(T,T^*)$ from being contained in $\mathsf{Base}(\Lambda)$, by parts (a), (c) and (d) of Theorem 4.16.

If $\#\Gamma_C = 12$, the argument is similar: the net of quartics determined by Γ_C has singular members of only three types: (1) a nodal curve with a unique node, (2) a curve with exactly two nodes for singularities, and (3) a curve with a unique ordinary cusp. Accordingly, C is either smooth or has qualities (1), (2), or (3) just listed.

The thirteenth point not contained in C must be the unique basepoint of the moving part of Λ. So, by virtue of tangent space considerations, the only way for the length

9 scheme $\mathsf{sq}(T, T^*)$ to be contained in $\mathsf{Base}(\Lambda)$ is if $\mathsf{sq}(T, T^*) \subset C$. However, this is precluded by parts (a) and (b) of Theorem 4.16.

(2) *Assuming* $\deg(C) = 3$, the moving part Λ' of Λ is a pencil of conics. As before, C must be reduced, as Λ' has at most 4 of the points of Γ_{13} in its base scheme, and so the remaining points of Γ_{13} cannot be contained in a line. In fact, this type of reasoning shows there is only one *a priori* possibility: $\#\Gamma_C = 9$ and C is the unique (smooth) cubic curve determined by Γ_C. The moving part Λ' of Λ is then a pencil of conics with base-scheme consisting of the four points $\Gamma_{13} \setminus \Gamma_C$. The pencil Λ does not contain a subscheme of the form $\mathsf{sq}(T, T^*)$ in its base scheme $\mathsf{Base}(\Lambda)$ again by tangent space considerations, because $\mathsf{Base}(\Lambda)$ is the union of the smooth curve C and the four reduced points $\Gamma_{13} \setminus \Gamma_C$.

(3) *Assuming* $\deg(C) = 2$, then once again C must be reduced as is seen by an argument similar to that found in the previous paragraph. There are only two *a priori* possibilities for $\#\Gamma_C$: $\#\Gamma_C = 4$ or 5. Now, $\#\Gamma_C$ cannot be 4, because then the moving part of Λ, a pencil of cubics, must contain all 9 of the points $\Gamma_{13} \setminus \Gamma_C$ in its base scheme, contrary to the general nature of Γ_{13}. Thus $\#\Gamma_C = 5$ and C is the unique smooth conic determined by Γ_C. The remaining 8 points of $\Gamma_{13} \setminus \Gamma_C$ define a general pencil of cubics, and this pencil of cubics is then the moving part Λ' of Λ. Λ' has a 9-th basepoint not contained in C (again because Γ_{13} is general). Thus $\mathsf{Base}(\Lambda)$ is the smooth scheme consisting of the smooth conic C and 9 reduced points not contained in C. This base scheme evidently does not contain any subscheme of the form $\mathsf{sq}(T, T^*)$, again by tangent space considerations.

We've finished the analysis of all cases, and the proposition follows. □

Proposition 4.18. *Suppose* $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Inf}$. *Then the moving part* Λ' *of* Λ *is a pencil of quartics whose general member is a smooth quartic curve.*

Proof. We argue by counting dimensions, keeping in mind that $\Gamma_{13} = (p_1, \ldots, p_{13})$ is a general tuple. Λ' must necessarily be a pencil of quartics thanks to Theorem 4.17.

Let W denote the quasi-projective variety parametrizing pairs (L, Π) where L is a line in $\mathbb{P}^2$ and Π is a pencil of quartics all sharing a singular point, and such that $\mathsf{Base}(\Pi)$ is finite. Such Π's vary in a 22-dimensional family, while a choice of L provides 2 more dimensions, and so $\dim W = 24$.

For contradiction's sake, suppose $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Inf}$ is such that $\mathsf{Base}(\Lambda)$ has a line L as fixed part and such that $\Pi := \Lambda'$, the moving part of Λ, satisfies $(L, \Pi) \in W$. By assumption, Γ_{13} is contained in $\mathsf{Base}(\Lambda)$, and at most 2 of the points of Γ_{13} may lie on L by generality of Γ_{13}. On the other hand, at most 11 of the points of Γ_{13} may lie in the (finite) set $\mathsf{Base}(\Pi)$ because Π varies in a 22-dimensional family. Therefore, *exactly* 2 points of Γ_{13} must lie on L and the remaining 11 points of Γ_{13} must lie inside $\mathsf{Base}(\Pi)$. Furthermore, dimension considerations force (L, Π) to be a general point of W.

The contradiction will come from the fact that the base scheme $\mathsf{Base}(\Lambda)$ is the disjoint union of L (without embedded points) and $\mathsf{Base}(\Pi)$. This is true because $(L, \Pi) \in W$ is general, and so $\mathsf{Base}(\Pi)$ consists of one point of multiplicity 4 along with 12 other reduced points, none lying on L. It is impossible for any scheme of the form $\mathsf{sq}(T, T^*)$ (which is everywhere non-reduced, has 2-dimensional tangent space at all points, and has total length 9) to be contained in $\mathsf{Base}(\Lambda) = L \cup \mathsf{Base}(\Pi)$, providing our contradiction. □

The next proposition is needed to remove an *a priori* possible situation in $\Theta \cap \mathsf{Inf}$.

Proposition 4.19. *Suppose* (T, T^*, Λ) *is an element of* $\mathsf{Dom}(p) \cap \mathsf{Lin}(p)$ *satisfying:*

(1) *The fixed part of* Λ *is the line* L *spanned by* T*, and*
(2) $p \notin T$.

Then the moving part Λ' *of* Λ *satisfies*

$$\text{length}\,(\mathsf{Base}(\Lambda') \cap L) = 4.$$

Proof. The proof uses a limit linear series style argument. First observe that since the fixed part of Λ is the reduced line L, the moving part Λ' is a pencil of quartics and the length of $\mathsf{Base}(\Lambda') \cap L$ cannot exceed 4. We will approach the point (T, T^*, Λ) along a general 1-parameter family contained in $\mathsf{Dom}(p)$.

So, let $(B, 0)$ be a smooth, irreducible pointed affine curve, and let $\pi : \mathbb{P}^2_B \to B$ denote the natural projection. Suppose t_1, t_2, t_3 are sections of π satisfying:

(a) For $b \neq 0$, the three points $t_i(b)$ form an honest triangle, denoted $\mathcal{T}_b \subset \mathbb{P}^2_b$.
(b) The $b \to 0$ flat limit of $\mathcal{T}_b$, denoted $\mathcal{T}_0$, is thin. We let $L \subset \mathbb{P}^2_{b=0}$ denote the line containing $\mathcal{T}_0$. ($\mathcal{T}_0$ is simply T from the statement of the lemma, but we use calligraphic font for consistency in the following paragraphs.)
(c) $\mathcal{T}_0$ does not contain the point p. (This is the second assumption in the proposition.)
(d) The line L contains p.
(e) For $b \neq 0$, the four points $t_1(b), t_2(b), t_3(b)$, and p are in linear general position.

If $b \neq 0$, then the vector space

$$V_b := H^0\left(\mathbb{P}^2_b, \mathcal{I}^2_{\mathcal{T}_b}\,\mathcal{I}_p(5)\right)$$

is 11-dimensional because $\mathcal{T}_b$ and p are in linear general position. Yet, when $b = 0$, the vector space V_0 has dimension 12. Indeed, a quintic curve in $\mathbb{P}^2_0$ passing through p and singular along $\mathcal{T}_0$ necessarily contains the line L as an irreducible component and secondly its residual quartic curve Q must contain $\mathcal{T}_0$. The equations of such quartic curves form a 12-dimensional vector space. Still, as B is a smooth curve, the $b \to 0$ limit of V_b is a well-defined 11-dimensional vector space inside V_0 – the following claim is equivalent to the conclusion of the proposition:

Claim 4.20. *There exists a point* $q \in L$ *such that*

$$\lim_{b \to 0} V_b = \left\{ \begin{array}{c} \textit{Quintic forms which are a product} \\ L \cdot Q \textit{ where } Q \textit{ is a quartic} \\ \textit{containing the divisor } \mathcal{T}_0 + q \textit{ on } L \end{array} \right\}.$$

Going forward we focus on Theorem 4.20. We blow up $\mathbb{P}^2_B$ along the line $L \subset \mathbb{P}^2_{b=0}$. Let

$$\widetilde{\mathbb{P}^2_B}$$

denote the blow up $\mathrm{Bl}_L \mathbb{P}^2_B$, and denote by

$$\beta : \widetilde{\mathbb{P}^2_B} \to \mathbb{P}^2_B$$

the blowdown map. The structural morphism $\pi : \widetilde{\mathbb{P}^2_B} \to B$ has fiber over $b \neq 0$ simply equal to $\mathbb{P}^2_b$, while the fiber $\pi^{-1}(\{0\})$ is the transverse union of two smooth surfaces: the exceptional divisor of β, denoted E, and the proper transform of $\mathbb{P}^2_0 \subset \mathbb{P}^2_B$ in the blow up, denoted P. P maps isomorphically onto $\mathbb{P}^2_0$ under β.

The current geometric circumstance has certain features we want to emphasize:

- β expresses E as a $\mathbb{P}^1$-bundle over L. As such, E is isomorphic to the Hirzebruch surface $\mathbb{F}_1$, with the intersection $E \cap P$ being the directrix curve which we denote by $D \subset E$. We let F denote the divisor class of a fiber of the bundle $\beta|_E : E \to L$.
- Since $L \cap \mathcal{T} = \mathcal{T}_0 = \mathcal{T} \cap \mathbb{P}^2_0$ is a Cartier divisor on $\mathcal{T}$, it follows that $\mathcal{T}$ lifts isomorphically to $\widetilde{\mathbb{P}^2_B}$. Let $\widetilde{\mathcal{T}} \subset \widetilde{\mathbb{P}^2_B}$ denote this lift, which is unique. (We've used here the assumption that $\mathcal{T}_0$ is thin.)
- $E \cap \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_0$. Furthermore, $\widetilde{\mathcal{T}} \cap P = \varnothing$ because the sections t_i comprising $\mathcal{T}$, by virtue of being sections, are not tangent to $\mathbb{P}^2_0$ from the start.
- Under β, the proper transform of the constant section $\{p\} \times B \subset \mathbb{P}^2_B$ becomes a section of π which intersects E at a point $\widetilde{p} \in E$ not on the directrix $D \subset E$.
- Clearly $\beta(\widetilde{p}) \notin \mathcal{T}_0$ because $p \notin \mathcal{T}_0$. And therefore $\widetilde{p} \notin \widetilde{\mathcal{T}}_0$.
- Define $\mathcal{L}$ to be the invertible sheaf $\beta^*\mathcal{O}(5)(-E)$ on $\widetilde{\mathbb{P}^2_B}$. Then $\mathcal{L}|_P \simeq \mathcal{O}_P(4)$ and $\mathcal{L}|_E \simeq \mathcal{O}_E(D + 5F)$.

Now we consider the finite, length 4 subscheme $Z := \widetilde{p} \cup \widetilde{\mathcal{T}}_0 \subset E$. On E, the term *line* refers to any *irreducible* curve in the linear series $|D + F|$ on E. In particular, lines do not intersect the directrix D.

Lemma 4.21. *Maintain the setting immediately prior. Then*

$$h^0(E, \mathcal{I}_Z(D + 2F)) \geq 1$$

and $h^0(E, \mathcal{I}_Z(D + 2F)) > 1$ *if and only if Z is contained in a line, in which case* $h^0(E, \mathcal{I}_Z(D + 2F)) = 2$.

Proof of Theorem 4.21. This follows easily after translating the statement into a claim about length 5 subschemes in the plane imposing independent conditions on conics. To get to this translation, simply blow down the Hirzebruch surface E to a plane, contracting D. We omit the details. ◻

At this point, for clarity of exposition, we make the following simplifying assumption:

Assumption 4.22. *Assume that $Z \subset E$ is* **not** *contained in a line.*

Operating under this assumption, we can finish the argument. On the reducible surface $P \cup E$, the global sections of

$$\mathcal{I}^2_{\widetilde{\mathcal{T}}_0} \mathcal{I}_{\widetilde{p}} \otimes \mathcal{L}|_{P \cup E}$$

consist of the data of

- A global section Q of $\mathcal{O}_P(4)$, and
- a global section of $\mathcal{O}_E(D+5F)$ which, if nonzero, defines a reducible divisor of the form

$$\beta^{-1}(\mathcal{T}_0) \cup C$$

where C is the unique curve in the linear series $|\mathcal{I}_Z(D+2F)|$ (Theorem 4.21)

satisfying the compatibility condition that both sections agree on the curve $D = E \cap P$. Now we observe that under our simplifying Theorem 4.22, the curve C is irreducible and meets D at a single point q. Therefore, on D we obtain a particular degree 4 divisor, namely $\mathcal{T}_0 + q$, which the quartic Q is forced to contain in its zero scheme if Q is to contribute to a global section of

$$\mathcal{I}^2_{\widetilde{\mathcal{T}}_0} \mathcal{I}_{\widetilde{p}} \otimes \mathcal{L}|_{P\cup E}.$$

Unwinding what this means before blowing up, we arrive at the conclusion of the theorem.

Finally, we will explain how to finish the proof of Theorem 4.19 in the case where Theorem 4.22 does not hold. Suppose $Z \subset E$ is contained in a line, and call the line $L_1 \subset E$. We then blow up the threefold $\widetilde{\mathbb{P}}^2_B$ along L_1. Once again, $\widetilde{\mathcal{T}}$ will lift to the new blow up, as will the section which for general $b \in B$ selects the point p. Blowing up more and more in a similar fashion if necessary, after finitely many blow ups the strict transform of $\mathcal{T}$ and $\{p\} \times B$ are no longer collinear over $b = 0$. Then, we proceed as we did under the simplifying assumption, except now the special fiber is a chain of surfaces $E_0 \cup E_1 \cup \cdots \cup P$, the E_i's being $\mathbb{F}_1$'s, glued one to the next along directrices on one side and lines on the other. Then the line bundle $\beta^*\mathcal{O}(5)(-E_0 - E_1 - \ldots E_k)$ serves the same role as $\mathcal{L}$ did in the simplified situation, and the argument runs parallel to the simple case. □

Remark 4.23. The conclusion of Theorem 4.19 implies in particular that the pencil Λ contains an element which is the union of the doubled line $2L$ with a cubic curve.

Proposition 4.24. *Recall the set* Θ *from Equation* (17)*. Then* $\Theta \cap \mathsf{Inf} = \emptyset$.

Proof. Suppose for contradiction's sake that $(T, T^*, \Lambda) \in \Theta \cap \mathsf{Inf}$. By Theorem 4.18, Λ is a pencil of the form $\{L \cdot Q_t\}, t \in \mathbb{P}^1$, where L is a linear form (defining a line of the same name) and where Q_t is a pencil of quartics *with smooth general member*. Considering tangent space dimensions, the condition $\mathsf{sq}(T, T^*) \subset \mathsf{Base}(\Lambda)$ implies that T is supported entirely on the set of embedded points $L \cap \mathsf{Base}\{Q_t\}$.

The first claim we make is that T must in fact be a subscheme of L. This is seen by checking all alternative possibilities – we will investigate the most challenging case, and leave the rest to the reader. Let x, y denote affine coordinates, and consider the situation where $T = V(x - y^2, y^3)$ and $L = V(x)$. We claim that if $x \cdot q(x, y) \in (x - y^2, y^3)^2$ then the constant and linear terms of $q(x, y)$ must both vanish. It is clear that q cannot have a constant term. To prove that q has no linear term, it suffices to prove the same after applying the automorphism $x \mapsto x + y^2, y \mapsto y$. Thus, we must show that if

$$(x + y^2) \cdot h(x, y) \in (x^2, xy^3, y^6)$$

then h has no linear term. Let $h_1, h_2, \dots$ be the linear, quadratic, etc... terms of h, so that $h = h_1 + h_2 + \dots$ First, by noting that the y^3-term of $(x + y^2) \cdot h$ must vanish, we conclude that h_1 must be a multiple of x. Write $h_1 = \lambda x$ for some $\lambda \in \Bbbk$. Next, looking at the xy^2-term we see that the y^2-term of h_2 must be $-\lambda y^2$. Finally, by considering the y^4-term we also conclude that the y^2-term of h_2 must be 0. Thus $\lambda = 0$ and, as claimed, $h_1 = 0$. The conclusion is that such a T is eliminated from consideration from the case-by-case analysis because it would contradict the requirement that the general element of the pencil Q_t is a *smooth* quartic. The remaining cases proceed in a similar fashion.

We therefore assume $T \subset L$. From the containment $\mathsf{sq}(T, T^*) \subset \mathsf{Base}\{L \cdot Q_t\}$, it follows that

$$T \subset \mathsf{Base}\{Q_t\}.$$

Now we bring in the assumption that the 13 *general* points $p_1, \dots, p_{13}$ are contained in $\mathsf{Base}\,\Lambda$. Not all p_i can be contained in $\mathsf{Base}\{Q_t\}$. Indeed, Q_t would then be uniquely determined and a general pencil. But it is not general: $\{Q_t\}$ contains a thin length 3 scheme (namely T) in its base scheme. It follows that at least one point, say p_{13} is contained in $L \setminus \mathsf{Base}\{Q_t\}$. Therefore, $(T, T^*, \Lambda) \in \mathsf{Dom}(p_{13}) \cap \mathsf{Lin}(p_{13})$, and Theorem 4.19 applies. Theorem 4.23 then says there is an element of the pencil $\{Q_t\}$ of the form $L \cdot C$ where C is a cubic form. However, then at least 11 of the remaining 12 points $p_1, \dots, p_{12}$ are contained in the cubic defined by C, contradicting generality of $\{p_1, \dots, p_{13}\}$. The proposition follows. □

Theorem 4.25. *Let $p_1, \dots, p_{13}$ be 13 general points in $\mathbb{P}^2$, and let Θ be as in Equation* (17). *Then*

$$\Theta = \left\{ \begin{array}{c} (T, T^*) \in \mathsf{CT} \textit{ such that} \\ T \textit{ is a singular triad} \\ \textit{for } p_1, \dots, p_{13} \end{array} \right\}.$$

Proof. It is clear that the mentioned set of singular triads is contained in Θ – the content of the theorem lies in the reverse inclusion. Theorem 4.24 gives the inclusion $\Theta \subset \mathsf{Fin}$. Then the theorem immediately follows from Theorem 4.14 and Bézout's theorem. □

5. Intersections in SQP

Let $p \in \mathbb{P}^2$ be a point and let $\ell \subset \mathbb{P}^2$ be a line. In this section we let

$$\mathsf{inc}(p), \mathsf{lin}(p)$$

be the cycles on CT consisting of points (T, T^*) in CT where T contains the point p or where T is collinear with p, respectively. Observe that both cycles are pulled back from $\mathrm{Hilb}_3 \mathbb{P}^2$ under the blowdown map $\mathsf{CT} \to \mathrm{Hilb}_3 \mathbb{P}^2$. For brevity, we will write $\mathcal{O}(d)^{[3]}$ for the same-named tautological bundle on $\mathrm{Hilb}_3 \mathbb{P}^2$, but pulled back to CT. Finally we let $H \subset \mathsf{CT}$ denote the divisor class of the locus of complete triangles (T, T^*) such that T intersects the line ℓ non-trivially - H is also evidently pulled back from $\mathrm{Hilb}^3 \mathbb{P}^2$.

Definition 5.1. *Define* $c_i(j) \in \mathrm{CH}^i\,\mathsf{CT}$ *to be the Chern class* $c_i\left(\mathcal{O}(j)^{[3]}\right)$*. Define* $e_k \in \mathrm{CH}^k\,\mathsf{CT}$ *to be the Chern class* $c_k(E)$*, where* E *is the rank* 9 *vector bundle from Section 3.6. Set*

$$\Delta_0 := e_3^2 - e_2e_4, \tag{18}$$

$$\Delta_2 := e_2^2 - e_1e_3 \tag{19}$$

$$\Delta_4 := e_1^2 - e_2, \tag{20}$$

$$\Delta_6 := [\mathsf{CT}]. \tag{21}$$

Lemma 5.2. *For each* $i = 0, 1, 2, 3$ *we have*

$$\Delta_{2i} = \varphi_*\left([\mathsf{Bpt}]^{13-i}\right).$$

Proof. Let $i = 0, 1, 2$, or 3. Fix $13 - i$ general points on $\mathbb{P}^2$ and let V denote the vector space of quintic forms vanishing at those $13-i$ points. Then the class $[\mathsf{Bpt}]^{13-i} \in \mathrm{CH}^{26-2i}\,\mathsf{SQP}$ can be represented by the cycle which parametrizes triples (T, T^*, Λ) where the $\Lambda \subset V$.

Pushing this cycle down to CT via φ, we get the degeneracy scheme of the natural evaluation map of vector bundles

$$ev : \underline{V} \to E$$

consisting of points in CT where ev has at least a 2-dimensional kernel. The lemma follows from applying Porteous's formula to ev. □

Recall the action of $\mathbb{G}_\mathrm{m}$ on CT from Section 3.6. In order to use localization, we need to express the classes $[\mathsf{lin}(p)]$ and $[\mathsf{inc}(p)]$ as combinations of Chern classes of $\mathbb{G}_\mathrm{m}$-equivariant bundles:

Lemma 5.3. *In the Chow ring* $\mathrm{CH}^\bullet\,\mathsf{CT}$*, we have*

$$[\mathsf{lin}(p)] = c_2(1), \tag{22}$$

$$[\mathsf{inc}(p)] = H^2 - c_2(1) + 2c_2(2) - c_2(3) \tag{23}$$

Proof. The first equation follows from the definition of the the second Chern class – the subschemes lying in some member of the pencil of lines through p are precisely those comprising the cycle $\mathsf{lin}(p)$. The second equation is a consequence of general formulas for $c_2(d)$ for all d found on page 93 of [ELB06]. In *loc. cit.* the authors refer to $c_2(d)$ by the symbol $\mathcal{P}_d$. We get the expression for $\mathsf{inc}(p)$ by combining the $d = 2$ and $d = 3$ cases of the formulas in [ELB06]. (Observe that, although [ELB06] concerns cycles on the Hilbert scheme $\mathrm{Hilb}_3\,\mathbb{P}^2$, we may pull them back to CT via the forgetful map $\mathsf{CT} \to \mathrm{Hilb}_3\,\mathbb{P}^2$.) □

Proof of Theorem 1.2. By Theorem 4.25 our task is to compute

$$\int_{\mathsf{SQP}} [\mathsf{Dom}(p)]^{13}.$$

We will do so by instead computing

$$\int_{\mathsf{CT}} \varphi_*\left([\mathsf{Dom}(p)]^{13}\right),$$

which by Theorem 4.12 equals

$$\int_{\mathsf{CT}} \varphi_* \left(([\mathsf{Bpt}(p)] - 4[\mathsf{Inc}(p)] - [\mathsf{Lin}(p)])^{13} \right).$$

Using the push-pull formula for φ and Theorem 5.2, we must then compute:

$$\int_{\mathsf{CT}} \sum_{i=0}^{3} (-1)^i \binom{13}{i} \cdot \Delta_{2i} \cdot (4[\mathsf{inc}(p)] + [\mathsf{lin}(p)])^i . \tag{24}$$

By expressing all terms of (24) in terms of the Chern classes e_k of E and using Theorem 5.3, we then use the fixed-point analysis for E in §3.6 and the calculations in Theorem 3.27 and evaluate the Atiyah-Bott localization expression. We perform this computation in `sage` in the file `localization`.

□

6. Lingering questions

Countless questions remain unanswered – we highlight some below.

Question 6.1. *Association's presence in all known cases of the Veronese counting problem is hard to ignore – what is its role in the general problem?*

In fact, a closer look shows an intriguing possibility, which we now explain. In all known instances, it appears as though association is a composite of a Cremona transformation followed by a Veronese embedding. Specifically, let $m = \binom{n+d}{d} + n + 1$, and consider a general tuple of points $(p_1, \dots, p_m) \in (\mathbb{P}^n)^m$. Let $(q_1, \dots, q_m) \in (\mathbb{P}^N)^m$ be associated to $(p_1, \dots, p_m)$, where $N = \binom{n+d}{d} - 1$. Finally let $\mathrm{ver}_d : \mathbb{P}^n \to \mathbb{P}^N$ denote the standard d-uple Veronese embedding (after choosing coordinates).

In every understood case of the general Veronese counting problem, there exists a Cremona transformation $\gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ and an automorphism $g : \mathbb{P}^N \to \mathbb{P}^N$ such that the composite $g \circ \mathrm{ver}_d \circ \gamma$ sends each point p_i to its corresponding point q_i. The Cremona transformation γ need not be unique, but the fact that it exists in the first place is not obvious to us. And so a follow-up to Theorem 6.1 would be to determine whether a "$g \circ \mathrm{ver}_d \circ \gamma$" mapping always exists which outputs an associated set for $p_1, \dots, p_m$ in every instance of the enumerative problem.

Remark 6.2. In this direction, a dimension count suggests that for a fixed set of 14 general points $a_1, \dots, a_{14}$ in $\mathbb{P}^3$, there should exist finitely many *quadro-quartic* Cremona transformations $\gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that the composition of γ with a 2-uple Veronese embedding sends the tuple $(a_1, \dots, a_{14})$ to an associated tuple $(b_1, \dots, b_{14}) \in (\mathbb{P}^9)^{14}$. This may hint at an approach, admittedly ambitious, to finding the number $\nu_{2,3}$ of 2-Veronese 3-folds through 14 general points.

Question 6.3. *Is it possible to confirm Coble's example using `homotopycontinuation.jl`?*

Question 6.4. *It can be shown that the monodromy group of Coble's enumerative problem $\nu_{2,2} = 4$ is the full symmetric group S_4. Is the monodromy group for $\nu_{3,2}$ also the full symmetric group?*

Question 6.5. *Is there a straightforward degeneration argument reproving* $v_{2,2} = 4$*?*

Question 6.6. *Association is a characteristic independent theory. By following Coble's reasoning in characteristic* p*, we find that when* $p = 2$ *and only when* $p = 2$*, the number* $v_{2,2}$ *drops – it changes from* 4 *to* 2 *because the* 2*-torsion of a general elliptic curve in characteristic* 2 *consists of only* 2 *points. Modulo which primes* p *(if any) does our* $v_{3,2} = 4246$ *calculation change?*

Question 6.7. *Are there other integral expressions of the form*

$$v_{d,n} = \int_X \alpha^m$$

where X *is a relatively tractable smooth projective variety and* α *is some cycle on* X*?*

7. Sage code

We provide the sage code we used to compute the number $v_{3,2}$, as well as several checks we performed to acquire confidence that there are no mistakes in the computation of weights of relevant bundles at fixed points. One thing to note is that we plug in specific values for a, b, c into the equivariant expressions arising in the Atiyah-Bott localization formula. These are secretly constant, so the reader can verify that by changing the inputs of a, b, c, the various calculations below do not change. This is a further check on the integrity of the calculations.

```
var('a','b','c')

def symmetrize(p):
    return p(a=a,b=b,c=c) + p(a=a,b=c,c=b) + p(a=b,b=a,c=c) + p(a=b,b=c,c=a) + p(a=c,b=a,c=b) + p(a=c,b=b,c=a)

# The ith elementary symmetric polynomial
def sigma(i, L):
    ind = Set(range(0,len(L)))

    return sum([ prod([L[x] for x in S]) for S in ind.subsets(i) ])

# BUNDLE WEIGHTS AT FIXED POINTS:
# The first entry corresponds to an honest triangle, which is unchanged under the action of S_3 permuting
      homogeneous coordinates, and the rest have orbits of size 6.


# Weight data for the bundle E on CT.
E = [
    (5*a, 5*b, 5*c, 4*a+b, 4*a+c, 4*b+c, 4*b+a, 4*c+a, 4*c+b),
    (5*a, 4*a+b, 4*a+c, 5*c, 4*c+a, 4*c+b, 3*c+a+b, 3*c+2*b, 2*c+3*b),
    (5*b, 4*b+a, 4*b+c, 5*c, 4*c+a, 4*c+b, 3*c+a+b, 3*c+2*b, 2*c+3*b),
    (5*c, 4*c+b, 3*c+2*b, 2*c+3*b, c+4*b, 5*b, 4*c+a, 3*c+a+b, 2*c+a+2*b),
    (5*c, 4*c+a, 3*c+2*a, 4*c+b, 3*c+a+b, 2*c+2*a+b, 3*c+2*b, 2*c+2*b+a, 2*c+3*b),
    (5*c, 4*c+a, 3*c+2*a, 2*c+3*a, 4*c+b, 3*c+a+b, 3*c+2*b, 2*c+2*b+a, 2*c+3*b)
    ]

# Weight data for the tangent bundle T of CT
T = [
    (c-a, c-b, b-c, b-a, a-b, a-c),
    (a-c, a-b, c-a, 2*c-2*b, b-a, c-b),
    (c-a, 2*c-2*b, b-a, c-b, b-c, b-a),
    (c-a, 3*c-3*b, b-a, 2*c-2*b, 2*b-c-a, c-b),
    (3*b-3*a, 2*b-2*a, b-a, c-a, c-b, a-2*b+c),
```

```
    (a-b, c-b, c-a, 2*b-2*a, c-b, b-a)
    ]

# Some Chern class expressions of E at the six representative fixed points.
c1E = [ expand(sigma(1, x)) for x in E ]
c2E = [ expand(sigma(2, x)) for x in E ]
c3E = [ expand(sigma(3, x)) for x in E ]
c4E = [ expand(sigma(4, x)) for x in E ]
c6T = [ expand(sigma(6, x)) for x in T ]

# This is the Atiyah-Bott localization formula for computing -c2(E) * c4(E) + c3(E)^2. It is the wrong answer, due
     to excess.
Wrong = expand(-c2E[0]*c4E[0] + c3E[0]^2)/expand(c6T[0]) + sum([symmetrize(expand(-c2E[i]*c4E[i] + c3E[i]^2)/expand(
     c6T[i])) for i in range(1,6)])

# O's tautological rank three bundle's weights at the six fixed representative fixed points
OOO = [
    (0, 0, 0),
    (0, b-c, 0),
    (0, b-c, 0),
    (0, b-c, 2*b-2*c),
    (0, a-c, b-c),
    (0, a-c, b-c)
    ]

c1OOO = [ expand(sigma(1, x)) for x in OOO ]
c2OOO = [ expand(sigma(2, x)) for x in OOO ]
c3OOO = [ expand(sigma(3, x)) for x in OOO ]

# O(1)'s tautological rank 3 bundle's weights at the 6 representative fixed points, followed by its Chern classes at
      those points.
Oone = [
    (a, b, c),
    (a, b, c),
    (b, b, c),
    (c, b, 2*b-c),
    (a, b, c),
    (a, b, c)
    ]

c1Oone = [ expand(sigma(1, x)) for x in Oone ]
c2Oone = [ expand(sigma(2, x)) for x in Oone ]
c3Oone = [ expand(sigma(3, x)) for x in Oone ]

# O(2)'s tautological rank 3 bundle's weights at the six representative fixed points, followed by its Chern classes
     at those points.
Otwo = [
    (2*a, 2*b, 2*c),
    (2*a, b+c, 2*c),
    (2*b, b+c, 2*c),
    (2*b, b+c, 2*c),
    (a+c, b+c, 2*c),
    (a+c, b+c, 2*c)
    ]

c1Otwo = [ expand(sigma(1, x)) for x in Otwo ]
c2Otwo = [ expand(sigma(2, x)) for x in Otwo ]
c3Otwo = [ expand(sigma(3, x)) for x in Otwo ]
```

```
# O(3)'s tautological bundles weights at the six representative fixed points, followed by its Chern classes at those
      points.
Othree = [
   (3*a, 3*b, 3*c),
   (3*a, b+2*c, 3*c),
   (3*b, b+2*c, 3*c),
   (b+2*c, 2*b+c, 3*c),
   (a+2*c, b+2*c, 3*c),
   (a+2*c, b+2*c, 3*c)
   ]

c1Othree = [ expand(sigma(1, x)) for x in Othree ]
c2Othree = [ expand(sigma(2, x)) for x in Othree ]
c3Othree = [ expand(sigma(3, x)) for x in Othree ]

# The line bundle H's weights at the six representative fixed points
H = [
   a+b+c,
   a+2*c,
   b+2*c,
   3*c,
   3*c,
   3*c
   ]
# We will just use the symbol H for its first Chern class c1H

# Delta_0's expression at the six representative fixed points
Delta0 = [ c3E[i]^2-c2E[i]*c4E[i] for i in range(0,6)]


# Delta_2's expression at the six representative fixed points
Delta2 = [ c2E[i]^2-c1E[i]*c3E[i] for i in range (0,6)]

# Delta_4's expression at the six representative fixed points
Delta4 = [ c1E[i]^2-c2E[i] for i in range(0,6)]



# THE CLASSES INC, LIN, ETC:
# The class Inc to be used in localization formula (tuple of 6 degree two expressions in abc)

Inc = [H[i]^2 - c2Oone[i] + 2*c2Otwo[i] - c2Othree[i] for i in range(0,6)]

# the class of Lin to be used in localization formula (tuple of 6 degre two expressions in abc)

Lin = [c2Oone[i] for i in range(0,6)]

# The class 4Inc + Lin. This is relevant because of the relation BP(p) = Dom(p) + 4Inc(p) + Lin(p).

FourIncPlusLin = [4*Inc[i] + Lin[i] for i in range(0,6)]

# The expression we wish to integrate, an expression in a,b,c at each of the six representative fixed points.

INTEGRAND = [Delta0[i] - 13*Delta2[i]*FourIncPlusLin[i] + 78*Delta4[i]*FourIncPlusLin[i]^2 - 286*FourIncPlusLin[i]^3
      for i in range(0,6)]

# Application of Atiyah-Bott to integrate INTEGRAND, summing over all 31 fixed points, remembering that the honest
     triangle is its own S3 orbit, so we do not symmetrize it.
```

```
Answer = expand(INTEGRAND[0])/expand(c6T[0]) + sum([symmetrize(expand(INTEGRAND[i])/expand(c6T[i])) for i in range
     (1,6)])



# TESTS:
# Let's compute H^6. This should be 15.
Hsixth = expand(H[0]^6)/expand(c6T[0]) + sum([symmetrize(expand(H[i]^6)/expand(c6T[i])) for i in range(1,6)])

#print(Hsixth(a=2, b=5, c=-9)) This indeed yields 15.

# Let's compute H^4*Inc. This should be 3.
HfourthInc = expand(Inc[0]*H[0]^4)/expand(c6T[0]) + sum([symmetrize(expand(Inc[i]*H[i]^4)/expand(c6T[i])) for i in
     range(1,6)])

# print(HfourthInc(a=2,b=3,c=-2)) #This indeed gives 3.

# Let's compute c_{3}(2)^2, which should be (4 choose 3) = 4.
Otwocheck = expand(c3Otwo[0]^2)/expand(c6T[0]) + sum([symmetrize(expand(c3Otwo[i]^2)/expand(c6T[i])) for i in range
     (1,6)])

# print(Otwocheck(a=5,b=2,c=7)) #Indeed this gives 4.

# Let's compute c_3(3)^{2}, which should be (9 choose 3).
Othreecheck = expand(c3Othree[0]^2)/expand(c6T[0]) + sum([symmetrize(expand(c3Othree[i]^2)/expand(c6T[i])) for i in
     range(1,6)])

# print(Othreecheck(a=-3, b=5, c=4)) #This indeed gives (9 choose 3) = 84.

# Let's compute c_3(2)*c_3(3), which should be (6 choose 3) = 20.
Otwothreecheck = expand(c3Othree[0]*c3Otwo[0])/expand(c6T[0]) + sum([symmetrize(expand(c3Othree[i]*c3Otwo[i])/expand
     (c6T[i])) for i in range(1,6)])

# print(Otwothreecheck(a=3,b=2,c=-5)) #Indeed this gives 20.

Lincubed = expand(Lin[0]^3)/expand(c6T[0]) + sum([symmetrize(expand(Lin[i]^3)/expand(c6T[i])) for i in range(1,6)])

# print(Lincubed(a=-2,b=13,c=4)) yields 0 as is should because Lin cubed is indeed 0

IncsquaredLin = expand(Inc[0]^2 * Lin[0])/expand(c6T[0]) + sum([symmetrize(expand(Inc[i]^2 * Lin[i])/expand(c6T[i]))
      for i in range(1,6)])

# print(IncsquaredLin(a=-1,b=12,c=13)) yields 0, which it should.

Inccubed = expand(Inc[0]^3)/expand(c6T[0]) + sum([symmetrize(expand(Inc[i]^3)/expand(c6T[i])) for i in range(1,6)])

# print(Inccubed(a=12,b=-3,c=5)) yields 1, as it should because Inc^3 = 1 for simple geometric reasons.

#FINAL COMPUTATIONS:
print(Wrong(a=45,b=3,c=10)) # This is the wrong answer 57728, where we apply the Porteous formula to the bundle E.
print(Answer(a=-20,b=9,c=7)) #This gives 4246.
```

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