COMPACTIFICATIONS OF HURWITZ SPACES

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ABSTRACT. We construct several modular compactifications of the Hurwitz space $H^d_{g/h}$ of genus g curves expressed as d-sheeted, simply branched covers of genus h curves. These compactifications are obtained by allowing the branch points of the covers to collide to a variable extent. They are very well-behaved if $d = 2, 3$, or if relatively few collisions are allowed. We recover as special cases the spaces of twisted admissible covers of Abramovich, Corti and Vistoli and the spaces of hyperelliptic curves of Fedorchuk.

1. INTRODUCTION

A fascinating aspect of the study of moduli spaces is the exploration of their birational geometry. By varying the moduli functor, one can construct a menagerie of birational models of a moduli space. These models are not only interesting in themselves, but also provide an unprecedented opportunity to explicitly study the Mori theory of some of the most interesting higher dimensional varieties. Pioneered by Hassett and Keel, such a study for the moduli space of curves continues to be a topic of intense current research [8].

We take up a similar study of a related moduli space, namely the Hurwitz space. The Hurwitz space H^d_g is the moduli space of genus g curves expressed as d-sheeted, simply branched covers of $\mathbf{P}^1$. These spaces have played a vital role in our understanding of the moduli of curves. They parametrize some of the most interesting loci, especially for small d, such as the hyperelliptic locus for $d = 2$ and the trigonal locus for $d = 3$. These loci in M_g are conjectured to play a crucial role in the Hassett–Keel program. Furthermore, in many ways, the Hurwitz spaces are easier to handle than M_g, and it is reasonable to aspire for a fruitful Hassett–Keel program in their context.

In this paper, we lay the groundwork for constructing a number of compactifications of H^d_g. The standard compactification due to Harris and Mumford [10] (further refined by Abramovich, Corti, and Vistoli [2]) parametrizes admissible covers, which are a particular kind of degenerations of simply branched covers where the branch points are forced to remain distinct. Our main idea is to explore compactifications where the branch points are allowed to coincide to a given extent. Although covers of $\mathbf{P}^1$ are our primary interest, we treat the case of covers of curves of arbitrary genus; this presents no additional difficulty.

We now describe our main results without diving into many technicalities. Fix a positive integer d and non-negative integers g, h and b related by the Riemann–Hurwitz formula

$$2g - 2 = d(2h - 2) + b.$$

Let $H^d_{g/h}$ be the space of smooth genus g curves expressed as d-sheeted, simply branched covers of smooth genus h curves. In symbols, $H^d_{g/h} = \{(\phi\colon C \to P)\}$, where C and P are smooth curves of genus g and h respectively, and ϕ is a simply branched cover of degree d. Let $M_{h;b}$ be the space of b distinct unordered points on smooth genus h curves. In symbols, $M_{h;b} = \{(P, \Sigma)\}$, where P

is a smooth curve of genus h and $\Sigma \subset P$ a reduced divisor of degree b. We have a morphism $\operatorname{br}: H^d_{g/h} \to M_{h;b}$ defined by

$$\operatorname{br}: (\phi\colon C \to P) \mapsto (P, \operatorname{br}\phi).$$

Our first technical result is the construction of an unscrupulous enlargement $\mathscr{H}^d_{g/h}$ of $H^d_{g/h}$ over a likewise unscrupulous enlargement of $\mathscr{M}_{h;b}$ of $M_{h;b}$; we now describe both. The non-separated Artin stack $\mathscr{M}_{h;b}$ is the stack of (P, Σ), where P is an at worst nodal curve of arithmetic genus h and $\Sigma \subset P$ a divisor of degree b supported in the smooth locus. The precise definition of $\mathscr{H}^d_{g/h}$ is slightly technical, but roughly speaking, it is the stack of $(\phi\colon C \to P)$, where P is an orbinodal curve of arithmetic genus h and ϕ a finite cover of degree d, étale over the nodes and the generic points of the components of P. There is no restriction on the singularities of C. The orbinodes serve to encode the admissibility criterion of Harris and Mumford [10], following the idea of Abramovich, Corti, and Vistoli [2]. The reader unfamiliar with this construction may imagine P to be simply a nodal curve and ϕ an admissible cover over the nodes of P. As said before, the stacks $\mathscr{M}_{h;b}$ and $\mathscr{H}^d_{g/h}$ are non-separated enlargements of $M_{h;b}$ and $H^d_{g/h}$, respectively. They continue to be related by the branch morphism $\operatorname{br}: \mathscr{H}^d_{g/h} \to \mathscr{M}_{h;b}$ given by

$$\operatorname{br}: (\phi\colon C \to P) \mapsto (P, \operatorname{br}\phi).$$

Theorem A (Theorem 3.8)**.** With the above notation, $\mathscr{H}^d_{g/h}$ and $\mathscr{M}_{h;b}$ are algebraic stacks, locally of finite type. The morphism $\operatorname{br}: \mathscr{H}^d_{g/h} \to \mathscr{M}_{h;b}$ is proper and of Deligne–Mumford type.

Theorem A gives a recipe to construct many compactifications of $H^d_{g/h}$. Indeed, let $\mathcal{X} \subset \mathscr{M}_{h;b}$ be a Deligne–Mumford substack containing $M_{h;b}$. If $\mathcal{X}$ is proper (over the base), then $\mathcal{X} \times_{\mathscr{M}_{h;b}} \mathscr{H}^d_{g/h}$ is a Deligne–Mumford stack containing $H^d_{g/h}$ that is also proper (over the base). In this sense, any suitable compactification of the space of branch divisors yields a corresponding compactification of the space of branched covers. Furthermore, we prove that if $\mathcal{X}$ has a projective coarse space, then so does $\mathcal{X} \times_{\mathscr{M}_{h;b}} \mathscr{H}^d_{g/h}$ (Theorem 6.1).

What makes the above recipe particularly fruitful is that we know several such $\mathcal{X}$'s, leading to several compactifications of $H^d_{g/h}$. These $\mathcal{X}$'s are the spaces of weighted pointed curves of Hassett [12], which we now recall. Let $\epsilon > 0$ be a rational number satisfying $b \cdot \epsilon + (2h - 2) > 0$. A point of $\mathscr{M}_{h;b}$ given by (P, Σ) is called *ϵ-stable* if $\epsilon \cdot \operatorname{mult}_p \Sigma \leq 1$ for all $p \in P$ and $\omega_P(\epsilon\Sigma)$ is ample. Let $\overline{\mathcal{M}}_{h;b}(\epsilon) \subset \mathscr{M}_{h;b}$ be the open substack consisting of ϵ-stable marked curves. Then $\overline{\mathcal{M}}_{h;b}(\epsilon)$ is a proper Deligne–Mumford stack that contains $M_{h;b}$ and admits a projective coarse space. Set

$$\overline{\mathcal{H}}^d_{g/h}(\epsilon) = \overline{\mathcal{M}}_{h;b}(\epsilon) \times_{\mathscr{M}_{h;b}} \mathscr{H}^d_{g/h}.$$

We call points of $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ *weighted admissible covers* or *ϵ-admissible covers*. Roughly speaking, these are admissible covers where $\lfloor 1/\epsilon \rfloor$ of the branch points can coincide.

Theorem B (Corollary 6.6)**.** With the above notation, the stack $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ of ϵ-admissible covers is a proper Deligne–Mumford stack that contains $H^d_{g/h}$ as an open substack. It admits a projective coarse space $\overline{H}^d_{g/h}(\epsilon)$ and a branch morphism to the stack $\overline{\mathcal{M}}_{h;b}(\epsilon)$ of ϵ-stable b-pointed genus h curves.

Theorem B recovers some spaces that have already appeared in literature. Plainly, the space $\overline{\mathcal{H}}^d_{g/h}(1)$ is the space of twisted admissible covers of Abramovich, Corti, and Vistoli [2]. In this space, the branch points are forced to remain distinct, and hence the only singularities of C are its nodes over the nodes of P. As ϵ decreases, $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ allows more and more branch points to coincide, and thus allows C to have progressively nastier singularities. We highlight that these singularities need not be Gorenstein (Example 6.9)! For $d = 2$ and $h = 0$, these spaces are the spaces of hyperelliptic curves constructed by Fedorchuk [7].

In general, the local structure of $\mathscr{H}^d_{g/h}$ is horrible. It may even have components other than the closure of $H^d_{g/h}$ (Example 6.11). For $d = 2$ and 3, however, $\mathscr{H}^d_{g/h}$ is smooth and irreducible (Theorem 5.5). The geometry of the resulting compactifications of the spaces of trigonal curves is the topic of forthcoming work [5].

The morphism $\mathrm{br} : \overline{\mathcal{H}}^d_{g/h}(\epsilon) \to \overline{\mathcal{M}}_{h;b}(\epsilon)$ is finite for ϵ close to 1, but not in general (Example 6.10). The fibers of br parametrize "crimps" of a fixed d-sheeted cover. We analyze these fibers in detail (Section 7).

Having described the main results, let us now describe our technical motivation. Our approach is inspired by Abramovich, Corti, and Vistoli [2]. We view a finite cover $\phi\colon C \to P$ as a family of length d schemes parametrized by P, or equivalently, as a map $\chi\colon P \to \mathscr{A}_d$, where $\mathscr{A}_d$ is the 'moduli stack of length d schemes.' This reinterpretation allows us to use the techniques from the well-studied topic of compactifications of spaces of maps into stacks. The stack $\mathscr{H}^d_{g/h}$ is thus constructed following Abramovich and Vistoli [1], which explains the central role played by orbinodal curves. Note, however, that their results cannot be used directly since they deal with maps into Deligne–Mumford stacks and $\mathscr{A}_d$ is not Deligne–Mumford. Nevertheless, the fact that $\mathscr{A}_d$ is the quotient of an affine scheme by the general linear group allows us to extend the essential arguments without much trouble.

We carry out our constructions in a slightly more general setting than that described above. It is useful in applications to have the flexibility to fix the ramification type of some fibers of the cover. Therefore, we work in the context of covers with arbitrary branching over a divisor and prescribed branching over distinct marked points on the base. Furthermore, it is notationally easier and conceptually no harder to refrain from fixing any numerical invariants as far as we can. Therefore, instead of $\mathscr{H}^d_{g/h}$ and $\mathscr{M}_{h;b}$, we simply have $\mathscr{H}^d$ and $\mathscr{M}$.

The paper is organized as follows. In Section 2, we introduce $\mathscr{A}_d$ and recall the notion of pointed orbinodal curves. In Section 3, we define $\mathscr{H}^d$ and state the main theorem (Theorem 3.8), which we prove in Section 4. In Section 5, we study the local structure of $\mathscr{H}^d$. In Section 6, we prove projectivity and describe the weighted admissible cover compactifications. In Section 7, we analyze the fibers of $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$. Section 4 is by far the most technical. The crucial geometric steps, namely the valuative criteria, are contained in Subsection 4.4.

Conventions. We work over a field $\mathbf{K}$ of characteristic zero. All schemes are understood to be locally Noetherian schemes over $\mathbf{K}$. We reserve the letter k for (variable) algebraically closed $\mathbf{K}$-fields. While working over an algebraically closed field k, "point" means "k-point," unless specified otherwise. An *algebraic stack* or an *algebraic space* is in the sense of Laumon and Moret-Bailly [16].

If X is an algebraic space, and $x \to X$ a geometric point, then $O_{X,x}$ denotes the stalk of O_X at x in the étale topology and we set $X_x = \operatorname{Spec} O_{X,x}$. The analytically inclined reader may imagine $O_{X,x}$ to be the ring of convergent power series around x and X_x to be a small simply-connected analytic neighborhood of x in X. For a local ring R, the symbol R^{sh} denotes its strict henselization and $\widehat{R}$ its completion.

The projectivization of a vector bundle E is denoted by $\mathbf{P}E$; this is the space of one-dimensional *quotients* of E. A morphism $X \to Y$ is *projective* if it factors as a closed embedding $X \hookrightarrow \mathbf{P}E$ followed by $\mathbf{P}E \to Y$ for some vector bundle E on Y.

A *curve* over a scheme S is a flat, proper morphism whose geometric fibers are purely one-dimensional. The source of this morphism could be a scheme, an algebraic space or a Deligne–Mumford stack; in the last case it is usually denoted by a curly letter. A curve over S is connected if its geometric fibers are connected. *Genus* always means arithmetic genus. By the genus of a stacky curve, we mean the genus of its coarse space. A *cover* is a representable, flat, surjective morphism. The symbol μ_n denotes the group of nth roots of unity; its elements are usually denoted by ζ.

2. Preliminaries

2.1. **The classifying stack of length d schemes.** Consider the category $\mathscr{A}_d$ fibered over **Schemes** whose objects over a scheme S are $(\phi \colon X \to S)$, where ϕ is a finite flat morphism of degree d. To prove that $\mathscr{A}_d$ is indeed an algebraic stack, we consider a more rigidified version. Since our schemes are assumed to be locally Noetherian, for a finite flat morphism $\phi \colon X \to S$, the sheaf $\phi_* O_X$ is a locally free O_S module of rank d. Therefore, the data of ϕ is equivalent to the data of an O_S algebra which is locally free of rank d as an O_S module. In the rigidified version of $\mathscr{A}_d$, we consider such algebras along with a marked O_S basis. Namely, we consider the contravariant functor $\mathscr{B}_d \colon \mathbf{Schemes} \to \mathbf{Sets}$ defined by

$$\mathscr{B}_d \colon S \mapsto \left\{ \begin{matrix} \text{Isomorphism classes of } (A, \tau) \text{, where } A \text{ is an } O_S \text{ algebra and } \tau \colon A \to \\ O_S^{\oplus d} \text{ an isomorphism of } O_S \text{ modules.} \end{matrix} \right\}.$$

Proposition 2.1. ([22, Proposition 1.1]) The functor $\mathscr{B}_d$ is representable by an affine scheme B_d of finite type.

Proof. Let $e_1, \dots, e_d$ be the standard basis of $O_S^{\oplus d}$. Then the data (A, τ) is equivalent to an O_S algebra structure on $O_S^{\oplus d}$. An O_S algebra structure is specified by maps of O_S modules

$$i \colon O_S \to O_S^{\oplus d}, \text{ say } 1 \mapsto \sum d_i e_i$$

and

$$m \colon O_S^{\oplus d} \otimes_S O_S^{\oplus d} \to O_S^{\oplus d}, \text{ say } e_i \otimes e_j \mapsto \sum c_{ij}^k e_k.$$

These maps make $O_S^{\oplus d}$ an O_S algebra with identity $i(1)$ and multiplication m if and only if the c_{ij}^k and the d_i satisfy certain polynomial conditions. Thus $\mathscr{B}_d$ is representable by a closed subscheme of $\mathbf{A}^{d^3+d} = \mathbf{A}\langle c_{ij}^k, d_i \rangle$. □

The scheme B_d admits a natural GL_d action, which is most easily described on the functor of points. A matrix $M \in \mathrm{GL}_d(S)$ acts on $B_d(S)$ by

$$M \colon (A, \tau) \mapsto (A, M \circ \tau). \tag{2.1}$$

Proposition 2.2. $\mathscr{A}_d$ is equivalent to the quotient $[B_d / \mathrm{GL}_d]$.

Proof. The proof is straightforward. Consider an object $\phi \colon X \to S$ in $\mathscr{A}_d(S)$. Let $A = \phi_* O_X$. Then A is an O_S algebra which is locally free of rank d as an O_S module. Set $P = \mathrm{Isom}_{O_S-\mathrm{mod}}(A, O_S^{\oplus d})$. Then $\pi \colon P \to S$ is a principal GL_d bundle, and we have a tautological isomorphism

$$\tau \colon \pi^* A \xrightarrow{\sim} O_P^{\oplus d}.$$

The data $(\pi^* A, \tau)$ gives a map $P \to B_d$, which is visibly GL_d equivariant. The assignment

$$(\phi \colon X \to S) \mapsto (\pi \colon P \to S, P \to B_d)$$

defines a morphism $\mathscr{A}_d \to [B_d / \mathrm{GL}_d]$ which is easily seen to be an isomorphism. □

We now recall the trace and discriminant of a finite cover. Let B be a ring and A a B algebra which is free of rank d as a B module. The *trace* of $a \in A$ is the trace of the B linear endomorphism on A given by multiplication by a. For the discriminant, consider the map $A \otimes_B A \to B$ obtained by composing the multiplication $A \otimes_B A \to A$ with the trace $A \to B$. Dualizing, we obtain a map $A \to A^\vee$. Taking the determinant of this map and dualizing once more, we obtain the *discriminant* $\delta \colon B \to (\det A)^{\otimes(-2)}$. Explicitly, if $\langle e_1, \dots, e_d \rangle$ is a B basis of A, then δ is given by the determinant of the matrix of traces:

$$\delta = \det[\mathrm{tr}(e_i e_j)]_{1 \leq i,j \leq d}. \tag{2.2}$$

The above construction globalizes: for a finite flat morphism $\phi \colon X \to S$, we get a trace $\mathrm{tr} : \phi_* O_X \to O_S$ and discriminant $\delta \colon O_S \to (\det \phi_* O_X)^{\otimes(-2)}$. Likewise, for the universal cover $\phi \colon \mathscr{X}_d \to \mathscr{A}_d$, we get the universal trace

$$\mathrm{tr} : \phi_* O_{\mathscr{X}_d} \to O_{\mathscr{A}_d},$$

and discriminant

$$\delta \colon O_{\mathscr{A}_d} \to (\det \phi_* O_{\mathscr{X}_d})^{\otimes(-2)}.$$

Let $\mathscr{E}_d \subset \mathscr{A}_d$ be the maximal open substack over which ϕ is étale. The following are well-known:

(1) $\mathscr{E}_d \subset \mathscr{A}_d$ is the locus where δ is invertible;
(2) $\mathscr{E}_d$ is equivalent to $B\mathbf{S}_d$, where $\mathbf{S}_d$ is the symmetric group on d letters.

We denote the zero locus of δ in $\mathscr{A}_d$ by Σ_d and call it the *universal branch locus*. For a map $\chi \colon S \to \mathscr{A}_d$, given by a cover $\phi \colon X \to S$, we set $\mathrm{br}\,\phi := S \times_\chi \Sigma_d$ and call it the *branch locus of* ϕ.

2.2. **Orbinodal curves.** We recall the notion of an orbinodal curve as introduced by Abramovich and Vistoli [1]. Our brief exposition is based on the work of Olsson [21]. Orbinodal curves are called "balanced twisted curves" in [1] and "twisted curves" in [21]. In short, a 'pointed orbinodal curve' is a stacky modification of a pointed nodal curve at the nodes and at the marked points. Étale locally near a node, it has the form

$$[\mathrm{Spec}\, k[u,v]/uv]/\mu_n,$$

where μ_n acts by $u \mapsto \zeta u, \quad v \mapsto \zeta^{-1} v$. Étale locally near a marked point, it has the form

$$[\mathrm{Spec}\, k[u]/\mu_n],$$

where μ_n acts by $u \mapsto \zeta u$. The formal definition follows.

Definition 2.3. Let S be a scheme. We say that the data

$$(\mathcal{C} \to C \to S; p_1, \dots, p_n : S \to C)$$

is a *pointed orbinodal curve* if the following are satisfied.

(1) $C \to S$ is a nodal curve and $p_i \colon S \to C$ pairwise disjoint sections.
(2) $\mathcal{C} \to S$ is a Deligne–Mumford stack with coarse space $\mathcal{C} \to C$. The coarse space map $\mathcal{C} \to C$ is an isomorphism over the open set $C^{\mathrm{gen}} \subset C$ which is the complement of the images of p_i and the singular locus of $C \to S$.

(3) Let $c \to C$ be a geometric point lying over $s \to S$. If c is a node of C_s, then there is an étale neighborhood $U \to C$ of c, an open set $T \subset S$ containing s, some $t \in O_T$, and $n \geq 1$ for which we have the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} \times_C U & \longrightarrow & U \\ {\scriptstyle \text{étale}}\big\downarrow & & \big\downarrow{\scriptstyle \text{étale}} \\ \left[\frac{\operatorname{Spec} O_T[u,v]}{(uv-t)}/\mu_n\right] & \longrightarrow & \frac{\operatorname{Spec} O_T[x,y]}{(xy-t^n)} \end{array},$$

Here μ_n acts by $u \mapsto \zeta u$ and $v \mapsto \zeta^{-1} v$, and the map on the bottom is given by $x \mapsto u^n$ and $y \mapsto v^n$.

(4) Let $s \to S$ be a geometric point and set $c = p_i(s)$. Then there is an étale neighborhood $U \to C$ of c and $n \geq 1$ for which we have the Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} \times_C U & \longrightarrow & U \\ {\scriptstyle \text{étale}}\big\downarrow & & \big\downarrow{\scriptstyle \text{étale}} \\ [\operatorname{Spec} O_S[u]/\mu_n] & \longrightarrow & \operatorname{Spec} O_S[x] \end{array},$$

Here μ_n acts by $u \mapsto \zeta u$, and the map on the bottom is given by $x \mapsto u^n$.

We abbreviate $(\mathcal{C} \to C \to S; p_1, \dots, p_n : S \to C)$ by $(\mathcal{C} \to C; p)$. A *morphism* between two pointed orbinodal curves $(\mathcal{C}_1 \to C_1; p_{1j})$ and $(\mathcal{C}_2 \to C_2; p_{2j})$ is a 1-morphism $F \colon \mathcal{C}_1 \to \mathcal{C}_2$ such that the induced map $F \colon C_1 \to C_2$ takes p_{1j} to p_{2j}.

Although the structure of $\mathcal{C}$ is specified for *some* étale neighborhood, it holds for any sufficiently small neighborhood. The precise statement from [21] follows.

Proposition 2.4. [21, Proposition 2.2, Definition 2.3] Let $(\mathcal{C} \to C; p)$ be a pointed orbinodal curve over S. For a geometric point $c \to C$, set

$$\mathcal{C}^{\text{sh}} = \mathcal{C} \times_C \operatorname{Spec} O_{C,c}.$$

Let $s \to S$ be the image of $c \to C$.

(1) Suppose c is a node of C_s and $t \in O_{S,s}$ and $x, y \in O_{C,c}$ are such that $O_{C,c}$ is isomorphic to the strict henselization of $O_{S,s}[x,y]/(xy - t^n)$ at the origin. Then, for some $n \geq 1$, we have

$$\mathcal{C}^{\text{sh}} \cong [\operatorname{Spec} O_{C,c}[u,v]/(uv - t, u^n - x, v^n - y)/\mu_n],$$

where μ_n acts by $u \mapsto \zeta u, \quad v \mapsto \zeta^{-1} v$.

(2) Suppose $c = p_i(s)$ and $x \in O_{C,c}$ is such that $O_{C,c}$ is isomorphic to the strict henselization of $O_{S,s}[x]$ at the origin. Then, for some $n \geq 1$, we have

$$\mathcal{C}^{\text{sh}} \cong [\operatorname{Spec} O_{C,c}[u]/(u^n - x)/\mu_n],$$

where μ_n acts by $u \mapsto \zeta u$.

3. The Big Hurwitz Stack $\mathscr{H}^d$

Fix a positive integer d. The goal of this section is to define the big Hurwitz stack $\mathscr{H}^d$. We first define the stack $\mathscr{M}$ of divisorially marked, pointed nodal curves.

Definition 3.1. Define the stack $\mathscr{M}$ of divisorially marked, pointed nodal curves as the category fibered over **Schemes** whose objects over S are

$$\mathscr{M}(S) = \{(P \to S; \Sigma; \sigma_1, \dots, \sigma_n)\},$$

where

(1) P is an algebraic space and $P \to S$ a connected nodal curve;
(2) $\Sigma \subset P$ is a Cartier divisor, flat over S, lying in the smooth locus of $P \to S$;
(3) $\sigma_j \colon S \to P$ are pairwise disjoint sections lying in the smooth locus of $P \to S$ and away from Σ.

Proposition 3.2. $\mathscr{M}$ is a smooth algebraic stack, locally of finite type.

Proof. Let $\mathscr{M}^{b,n} \subset \mathscr{M}$ be the subcategory where the degree of the marked divisor is b and the number of marked points is n. It suffices to prove the proposition for $\mathscr{M}^{b,n}$. Set $\mathscr{U} = \mathscr{M}^{0,0}$. By [, Theorem 1.1], $\mathscr{U}$ is an algebraic stack, locally of finite type. Since nodal curves have smooth deformation spaces, $\mathscr{U}$ is also smooth. Finally, it is clear that the forgetful morphism $\mathscr{M}^{b,n} \to \mathscr{U}$ is representable by smooth algebraic spaces of finite type. ☐

We now define $\mathscr{H}^d$. Recall our notation from Subsection 2.1:

$$\mathscr{A}_d \text{ is the classifying stack of schemes of length } d;$$
$$\mathscr{X}_d \to \mathscr{A}_d \text{ is the universal scheme of length } d;$$
$$\Sigma_d \subset \mathscr{A}_d \text{ is the universal branch locus};$$
$$\mathscr{E}_d = \mathscr{A}_d \setminus \Sigma_d \text{ is the locus of étale covers.}$$

Definition 3.3. Define the *big Hurwitz stack* $\mathscr{H}^d$ as the category fibered over **Schemes** whose objects over S are

$$\mathscr{H}^d(S) = \{(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \chi \colon \mathcal{P} \to \mathscr{A}_d)\}, \tag{3.1}$$

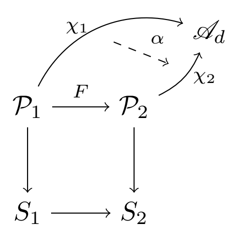
where

(1) $(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n)$ is a pointed orbinodal curve;
(2) $\chi \colon \mathcal{P} \to \mathscr{A}_d$ is a representable morphism that maps the following to $\mathscr{E}_d$: the generic points of the components of $\mathcal{P}_s$, the nodes of $\mathcal{P}_s$, and the preimages of the marked points in $\mathcal{P}_s$, for every fiber $\mathcal{P}_s$ of $\mathcal{P} \to S$.

A morphism between $(\mathcal{P}_1 \to P_1 \to S_1; \{\sigma_{1j}\}; \chi_1 \colon \mathcal{P}_1 \to \mathscr{A}_d)$ and $(\mathcal{P}_2 \to P_2 \to S_2; \{\sigma_{2j}\}; \chi_2 \colon \mathcal{P}_2 \to \mathscr{A}_d)$ over a morphism $S_1 \to S_2$ consists of two pieces of data: (F, α), where

(1) $F \colon \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of pointed orbinodal curves, and
(2) $\alpha \colon \chi_1 \to \chi_2 \circ F$ is a 2-morphism

such that (F, α) fits in a Cartesian diagram



(3.2)

We abbreviate $(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \chi \colon \mathcal{P} \to \mathscr{A}_d)$ by $(\mathcal{P} \to P; \sigma; \chi)$.

Remark 3.4. The careful reader may wonder what happened to the 2-morphisms between the 1-morphisms from $\mathcal{P}_1$ to $\mathcal{P}_2$. After all, the objects of $\mathscr{H}^d$ involve stacks, which makes it, *a priori*, a 2-category. However, by [1, Lemma 4.2.3], the 2-automorphism group of any 1-morphism $\mathcal{P}_1 \to \mathcal{P}_2$ is trivial. Thus, $\mathscr{H}^d$ is equivalent to a 1-category [1, Proposition 4.2.2]. What this means explicitly is that we treat two morphisms given by (F, α) and (F', α') as *the same* if they are related by a 2-morphism between F and F'.

Remark 3.5. Let us explain the condition of representability of χ (Definition 3.3 (2)). A morphism between two Deligne–Mumford stacks $f\colon \mathscr{X} \to \mathscr{Y}$ is representable if and only if for every geometric point $x \to \mathscr{X}$, the induced map of automorphism groups $\mathrm{Aut}_x(\mathscr{X}) \to \mathrm{Aut}_{f(x)}(\mathscr{Y})$ is injective [1, Lemma 4.4.3]. Thus the representability of χ means that the stack structure on $\mathcal{P}$ is the minimal one that affords a morphism to $\mathscr{A}_d$.

Remark 3.6. Let us explain the role played by the orbinodes. Consider an orbinodal curve near a node, say $\mathcal{U} = [\mathrm{Spec}\,(k[u,v]/uv)\,/\mu_n]$, and an étale cover $\mathcal{C} \to \mathcal{U}$. Observe that the induced map on the coarse spaces $C \to U$ is precisely an admissible cover in the sense of Harris and Mumford [10]. In this way, the orbinodes provide a way to deal with the admissibility condition.

Remark 3.7. Let us explain the role played by the marked points. Consider an orbinodal curve near a marked point, say $\mathcal{U} = [\mathrm{Spec}\,k[u]/\mu_n]$ with coarse space $U = \mathrm{Spec}\,k[t]$. The morphism χ maps $\mathcal{U}$ to $\mathscr{E}_d \cong B\mathbf{S}_d$; this corresponds to an étale cover $\mathcal{C} \to \mathcal{U}$. Note that in contrast to the fundamental group of the schematic curve U, the fundamental group of the stacky curve $\mathcal{U}$ is not trivial; it is precisely μ_n. Thus, $\mathcal{C} \to \mathcal{U}$ may be a non-trivial étale cover, specified by the monodromy

$$\mathrm{Aut}_0(\mathcal{U}) = \mu_n \to \mathrm{Aut}_0(B\mathbf{S}_d) = \mathbf{S}_d.$$

The condition of representability implies that this monodromy map is injective. On the level of coarse spaces, we thus get a cover $C \to U$ with monodromy around 0 given by a permutation of order n. By restricting to the open and closed substack where this permutation has a specific cycle structure, we can fully prescribe the ramification type of $C \to U$ over 0. In this way, the marked points provide a way to construct moduli spaces of covers with fibers of prescribed ramification.

It is useful to have a formulation of $\mathscr{H}^d$ purely in terms of finite covers. Since a map to $\mathscr{A}_d$ is nothing but a finite cover of degree d, we see that $\mathscr{H}^d$ may be equivalently described as the category whose objects over a scheme S are

$$\{(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \phi\colon \mathcal{C} \to \mathcal{P})\}, \tag{3.3}$$

where

(1) $(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n)$ is a pointed orbinodal curve;
(2) ϕ is a finite cover of degree d, étale over the following: the generic points of the components of $\mathcal{P}_s$, the nodes of $\mathcal{P}_s$, and the preimages of the marked points in $\mathcal{P}_s$, for every fiber $\mathcal{P}_s$ of $\mathcal{P} \to S$;
(3) Furthermore, for the open substack $\mathcal{U} := \mathcal{P} \setminus \mathrm{br}\,\phi$, the morphism $\mathcal{U} \to B\mathbf{S}_d$ corresponding to the étale cover $\mathcal{C}|_{\mathcal{U}} \to \mathcal{U}$ is representable.

In this form, a morphism from $(\mathcal{P}_1 \to P_1 \to S_1; \sigma_{1j}; \phi_1\colon \mathcal{C}_1 \to \mathcal{P}_1)$ to $(\mathcal{P}_2 \to P_2 \to S_2; \sigma_{2j}; \phi_2\colon \mathcal{C}_2 \to \mathcal{P}_2)$ is given by (F, G) where $F\colon \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of pointed orbinodal curves and $G\colon \mathcal{C}_1 \to \mathcal{C}_2$

a morphism over F fitting in a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{G} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{P}_1 & \xrightarrow{F} & \mathcal{P}_2. \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

We abbreviate $(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \phi\colon \mathcal{C} \to \mathcal{P})$ by $(\mathcal{P} \to P; \sigma; \phi)$. We use the formulation of $\mathscr{H}^d$ in terms of maps to $\mathscr{A}_d$ or in terms of finite covers depending on whichever is convenient.

The two stacks $\mathscr{H}^d$ and $\mathscr{M}$ are related by the branch morphism, which we now define. Consider an object $(\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \phi\colon \mathcal{C} \to \mathcal{P})$ in $\mathscr{H}^d(S)$. Identify $\operatorname{br}\phi$ with its image in P (it is anyway disjoint from the stacky points of $\mathcal{P}$). Then $\operatorname{br}\phi \subset P$ is an S-flat Cartier divisor. The branch morphism $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$ is defined by

$$\operatorname{br} : (\mathcal{P} \to P \to S; \sigma_1, \dots, \sigma_n; \phi\colon \mathcal{C} \to \mathcal{P}) \mapsto (P \to S; \operatorname{br}\phi; \sigma_1, \dots, \sigma_n).$$

Theorem 3.8 (Main)**.** $\mathscr{H}^d$ is an algebraic stack, locally of finite type. The morphism

$$\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$$

is proper and representable by Deligne–Mumford stacks.

Although Theorem 3.8 is motivated by the main theorem in [1], its proof is less involved, thanks to the advancement of technology related to stacks. There is a very general result for the existence of Hom stacks due to Aoki [3], but it is not suitable for our purpose because it does not yield the required finiteness properties. A generalization of Theorem 3.8 where $\mathscr{A}_d$ is replaced by a suitable global quotient $[U/G]$ seems plausible. This would also generalize the construction by Ciocan-Fontanine, Kim, and Maulik [4]. However, this is beyond the scope of the present work.

4. Proof of the Main Theorem

This section is devoted to proving Theorem 3.8. The proof is broken down into parts.

4.1. **That** $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$ **is an algebraic stack, locally of finite type.** We factor $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$ through a series of intermediate steps. The first step is the stack of pointed orbinodal curves. Let $\mathscr{M}^{\mathrm{orb}}$ be the category over **Schemes** whose objects over S are pointed orbinodal curves $(\mathcal{P} \to P \to S; \sigma)$. Denote by $\mathscr{M}^{\mathrm{orb}\le N}$ the subcategory of $\mathscr{M}^{\mathrm{orb}}$ where the order of the automorphism groups at the points of the orbinodal curve is bounded above by N. Denote by $\mathscr{M}^{b,*}$ (resp. $\mathscr{M}^{*,n}$, $\mathscr{M}^{b,n}$) the open substack of $\mathscr{M}$ where the marked divisor has degree b (resp. there are n marked points, degree b and n marked points). There is a morphism $\mathscr{M}^{\mathrm{orb}} \to \mathscr{M}^{0,*}$ given by

$$(\mathcal{P} \to P \to S; \sigma) \to (P \to S; \sigma).$$

We quote, without proof, a theorem of Olsson [21].

Theorem 4.1. [21, Theorem 1.9, Corollary 1.11] $\mathscr{M}^{\mathrm{orb}}$ and $\mathscr{M}^{\mathrm{orb}\le N}$ are smooth algebraic stacks, locally of finite type. $\mathscr{M}^{\mathrm{orb}\le N}$ is an open substack of $\mathscr{M}^{\mathrm{orb}}$. The morphism $\mathscr{M}^{\mathrm{orb}\le N} \to \mathscr{M}^{0,*}$ is representable by Deligne–Mumford stacks of finite type.

Define categories $\mathscr{F}inCov^d$ and $\mathscr{V}ect^d$ fibered over **Schemes** as follows

$$\mathscr{F}inCov^d(S) = \{(\mathcal{P} \to P \to S; \sigma; \phi\colon \mathcal{C} \to \mathcal{P}), \text{ where } \phi \text{ is finite, flat of degree } d\},$$
$$\mathscr{V}ect^d(S) = \{(\mathcal{P} \to P \to S; \sigma; \mathcal{F}), \text{ where } \mathcal{F} \text{ is locally free of rank } d \text{ on } \mathcal{P}\}.$$

In both definitions, $(\mathcal{P} \to P \to S; \sigma)$ is a pointed orbinodal curve. We have morphisms

$$\mathscr{H}^d \to \mathscr{F}inCov^d \to \mathscr{V}ect^d \to \mathscr{M}^{\text{orb}}. \tag{4.1}$$

Indeed, the first is obvious; the second is given by

$$(\mathcal{P} \to P; \sigma; \phi\colon \mathcal{C} \to \mathcal{P}) \mapsto (\mathcal{P} \to P; \sigma; \phi_* O_{\mathcal{C}});$$

and the last by

$$(\mathcal{P} \to P; \sigma; \mathcal{F}) \mapsto (\mathcal{P} \to P; \sigma).$$

We analyze each morphism in (4.1) one by one.

Before we proceed, we recall the notion of a *generating sheaf* on a Deligne–Mumford stack from [15, § 5.2]. Let $\mathcal{X}$ be a Deligne–Mumford stack with coarse space $\rho\colon \mathcal{X} \to X$. A locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ is a *generating sheaf* if for every quasi coherent sheaf $\mathcal{F}$, the morphism

$$\rho^* \rho_* (\mathscr{H}om_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) \otimes_{O_{\mathcal{X}}} \mathcal{E}) \to \mathcal{F}$$

is surjective. Equivalently, $\mathcal{E}$ is a generating sheaf if and only if for every point x of $\mathcal{X}$, the representation of $\operatorname{Aut}_x(\mathcal{X})$ on the fiber of $\mathcal{E}$ at x contains every irreducible representation of $\operatorname{Aut}_x(\mathcal{X})$.

We quote without proof a lemma from [9] that we use many times in the sequel.

Lemma 4.2. [9, Proposition 2.1] Let $P \to S$ be a nodal curve, where P is an algebraic space and S a scheme. Let $s \to S$ be a geometric point. Then there is an étale neighborhood $T \to S$ of s such that $P \times_S T \to T$ is projective.

Proposition 4.3. Let S be a scheme and $(\mathcal{P} \to P \to S; \sigma)$ a pointed orbinodal curve. There is a scheme T and a surjective étale morphism $T \to S$ such that

(1) $\mathcal{P}_T := \mathcal{P} \times_S T$ admits a finite, flat morphism from a projective scheme Z;
(2) $\mathcal{P}_T$ is the quotient of a quasi projective scheme by a linear algebraic group;
(3) $\mathcal{P}_T$ admits a generating sheaf.

Proof. The first statement is due to Olsson [21, Theorem 1.13]. The existence of a finite flat cover $Z \to \mathcal{P}_T$ implies that $\mathcal{P}_T$ is the quotient of an algebraic space Y by the action of a linear algebraic group by [6, Theorem 2.14]. We may assume that T is affine and, by Lemma 4.2, that P_T is projective over T. Then P_T is quasi-projective. In this case, Y can be proved to be quasi-projective [15, Remark 4.3]. Finally, since $\mathcal{P}$ is a quotient stack with a quasi projective coarse space, the third statement follows directly from [15, Theorem 5.3]. □

Proposition 4.4. $\mathscr{V}ect^d \to \mathscr{M}^{\text{orb}}$ is an algebraic stack, locally of finite type.

Proof. Let S be a scheme and $(\mathcal{P} \to P \to S; \sigma)$ an object of $\mathscr{M}^{\text{orb}}$. We must prove that the category of vector bundles of rank d on $\mathcal{P}$ is an algebraic stack, locally of finite type. It suffices to prove this after passing to an étale cover of S. By Proposition 4.3, we can assume that $\mathcal{P} \to S$ admits a generating sheaf and by Lemma 4.2, that $P \to S$ is projective. Now it can be shown that the stack $\mathscr{C}oh_{\mathcal{P}/S}$ of coherent sheaves on $\mathcal{P}$, flat over S, is an algebraic stack, locally of finite type. A smooth atlas is given by the Quot schemes of Olsson and Starr [20]. We omit the details; see the pre-print by Nironi [19, § 2.1] for a complete proof. Clearly, the stack of vector bundles of rank d on $\mathcal{P}$ is an open substack of $\mathscr{C}oh_{\mathcal{P}/S}$. □

Proposition 4.5. $\mathscr{F}inCov^d \to \mathscr{V}ect^d$ is representable by algebraic spaces of finite type.

For the proof, we need two easy lemmas.

Lemma 4.6. Let S be an affine scheme and $\mathcal{X} \to S$ be a proper Deligne–Mumford stack with coarse space $\rho\colon \mathcal{X} \to X$, where X is a scheme. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, flat over S. Then, there is a finite complex $M_\bullet$ of locally free sheaves on S:

$$M_0 \to M_1 \to \cdots \to M_n$$

such that for every $f\colon T \to S$, we have natural isomorphisms

$$H^i(f^*M_\bullet) \xrightarrow{\sim} H^i(\mathcal{X}_T, \mathcal{F}_T);$$

Proof. Let $F = \rho_*\mathcal{F}$. Then F is a coherent sheaf on X, flat over S. Since X is a proper scheme over S, the standard theorem on cohomology and base change for schemes [18, §II.5], gives a finite complex of locally free sheaves $M_\bullet$ with natural isomorphisms

$$H^i(f^*M_\bullet) \xrightarrow{\sim} H^i(X_T, F_T). \tag{4.2}$$

Now, the map $\rho_T\colon \mathcal{X}_T \to X_T$ is the map to the coarse space. Since maps to the coarse spaces are cohomologically trivial for quasi-coherent sheaves, we have $\rho_{T*}(\mathcal{F}_T) = F_T$ and a natural identification

$$H^i(X_T, F_T) = H^i(\mathcal{X}_T, \mathcal{F}_T). \tag{4.3}$$

Combining (4.2) and (4.3), we obtain the result. □

Lemma 4.7. Let $\mathcal{X} \to S$ and $\mathcal{F}$ be as in Lemma 4.6. Then the contravariant functor from $\mathbf{Schemes}_S$ to **Sets** defined by

$$(f\colon T \to S) \mapsto H^0(\mathcal{X}_T, \mathcal{F}_T)$$

is representable by an affine scheme $Sect_{\mathcal{F}/S}$ over S.

When no confusion is likely, we denote $Sect_{\mathcal{F}/S}$ by $Sect_{\mathcal{F}}$.

Proof. Let $M_\bullet$ be as in Lemma 4.6. Let $T_i = \operatorname{Spec}_S(\operatorname{Sym}^*(M_i^\vee))$ be the total spaces of the vector bundles M_i (we only care about $i = 0, 1$). Then T_i are vector bundles over S and we have a morphism $T_0 \to T_1$. Let $Sect_{\mathcal{F}} \subset T_0$ be the scheme theoretic preimage of the zero section of T_1. From the natural isomorphism

$$H^0(f^*M_\bullet) \xrightarrow{\sim} H^0(\mathcal{X}_T, \mathcal{F}_T),$$

it is easy to see that $Sect_{\mathcal{F}}$ represents the desired functor. □

We now have the tools to prove Proposition 4.5.

Proof of Proposition 4.5. Let S be a scheme and let $S \to \mathscr{V}ect^d$ be given by the object $(\mathcal{P} \to P \to S; \sigma; \mathcal{F})$ of $\mathscr{V}ect^d(S)$. We must prove that $\mathscr{F}inCov^d \times_{\mathscr{V}ect^d} S$ is an algebraic space of finite type. It suffices to prove this after passing to an étale cover of S. So, assume that S is affine and P is projective over S. By an *$O_{\mathcal{P}}$-algebra structure* on $\mathcal{F}$, we mean a pair (i, m), where $i\colon O_{\mathcal{P}} \to \mathcal{F}$ and $m\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ are morphisms of $O_{\mathcal{P}}$ modules that make $\mathcal{F}$ a sheaf of $O_{\mathcal{P}}$-algebras. Let $\mathscr{A}lg_{\mathcal{F}}$ be the stack of $O_{\mathcal{P}}$-algebra structures on $\mathcal{F}$. The operation of taking the spectrum gives an equivalence

$$\mathscr{A}lg_{\mathcal{F}} \xrightarrow{\sim} \mathscr{F}inCov^d \times_{\mathscr{V}ect^d} S.$$

Now, an algebra structure on $\mathcal{F}$ is determined by a global section (corresponding to i) of $\mathcal{F}$ and one (corresponding to m) of $\mathscr{H}om(\mathcal{F} \otimes \mathcal{F}, \mathcal{F})$ subject to the conditions

$$m \circ (i \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes 1) = \mathrm{id} \qquad \text{(multiplicative identity)}$$
$$m \circ \mathrm{sw} = m \qquad \text{(symmetry)}$$
$$m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes id) \qquad \text{(associativity)},$$

where $\mathrm{sw}\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ is the switch $x \otimes y \mapsto y \otimes x$. Each of these equations can be interpreted as the vanishing (agreeing with the zero section) of a morphism from $Sect_{\mathcal{F}} \times_S Sect_{\mathscr{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})}$ to a suitable $Sect$ space. For example, the equality

$$m \circ (\mathrm{id} \otimes 1) = \mathrm{id}$$

can be phrased as the vanishing of the morphism

$$Sect_{\mathcal{F}} \times_S Sect_{\mathscr{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})} \to Sect_{\mathscr{H}om(\mathcal{F}, \mathcal{F})}$$

defined by

$$(i, m) \mapsto m \circ (i \otimes \mathrm{id}) - \mathrm{id}\,.$$

Thus, $\mathscr{A}lg_{\mathcal{F}}$ is represented by the closed subscheme of $Sect_{\mathcal{F}} \times_S Sect_{\mathscr{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})}$ defined by vanishing of the equations given by the conditions above. □

We finish the final piece of (4.1).

Proposition 4.8. $\mathscr{H}^d \to \mathscr{F}inCov^d$ is an open immersion.

Proof. Let S be a scheme and $S \to \mathscr{F}inCov^d$ a morphism corresponding to $(\mathcal{P} \to P \to S; \sigma; \phi\colon \mathcal{C} \to \mathcal{P})$. Let $\pi\colon \mathcal{P} \to S$ be the projection. Denote by $\Sigma \subset P$ the image in P of the branch divisor of ϕ. Clearly, the locus $S_1 \subset S$ over which Σ is disjoint from the singular locus of $P \to S$ and the sections σ_i is an open subscheme. Over S_1, the Cartier divisor $\Sigma \subset P$ does not contain any components of the fibers and hence it is S_1-flat.

Let $\chi\colon \mathcal{P} \to \mathscr{A}_d$ be the morphism corresponding to the degree d cover $\mathcal{C} \to \mathcal{P}$. Let $\mathcal{I}_\chi \to \mathcal{P}$ be the inertia stack of χ. Then $\mathcal{I}_\chi \to \mathcal{P}$ is a representable finite morphism. The set $Z \subset \mathcal{P}$ over which $\mathcal{I}_\chi$ has a fiber of cardinality higher than one is a closed subset and its complement is exactly the locus where χ is representable. Let $S_2 = S_1 \cap (S \setminus \pi(Z))$.

Then, by definition, $\mathscr{H}^d \times_{\mathscr{F}inCov^d} S = S_2$, which is an open subscheme of S. □

We have finished the first part of the proof of Theorem 3.8.

Proposition 4.9. The morphism $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ is an algebraic stack, locally of finite type.

Proof. The forgetful morphism $\mathscr{M} \to \mathscr{M}^{0,*}$ is representable by algebraic spaces of finite type. Hence, it suffices to show that $\mathscr{H}^d \to \mathscr{M}^{0,*}$ is an algebraic stack, locally of finite type. We have the sequence of morphisms

$$\mathscr{H}^d \to \mathscr{F}inCov^d \to \mathscr{V}ect^d \to \mathscr{M}^{\mathrm{orb}} \to \mathscr{M}^{0,*}. \tag{4.4}$$

Starting from the right, Theorem 4.1, Proposition 4.4, Proposition 4.5 and Proposition 4.8 imply that each of the morphisms above is an algebraic stack, locally of finite type. Hence, so is their composite. □

4.2. **That** $\mathrm{br}: \mathscr{H}^d \to \mathscr{M}$ **is of finite type.** The strategy in this section is to study (4.4) more carefully and trim down the intermediate stacks so that they are of finite type.

Proposition 4.10. The morphism $\mathscr{H}^d \to \mathscr{M}^{\mathrm{orb}}$ factors through the open substack $\mathscr{M}^{\mathrm{orb} \leq N}$ for any $N \geq d!$.

Proof. Take an object $(\mathcal{P} \to P \to S; \sigma; \chi)$ of $\mathscr{H}^d$. Let p be a point of $\mathcal{P}$ which is either a node or a marked point in its fiber. Then χ maps a neighborhood of p into $\mathscr{E}_d \cong B\mathbf{S}_d$. Since χ is required to be representable, we have

$$\mathrm{Aut}_p(\mathcal{P}) \hookrightarrow \mathrm{Aut}_{\chi(p)}(B\mathbf{S}_d) = \mathbf{S}_d.$$

In particular, the size of $\mathrm{Aut}_p(\mathcal{P})$ is at most $d!$. ☐

Recall that $\mathscr{M}^{b,*} \subset \mathscr{M}$ is the open and closed substack where the marked divisor has degree b. Set

$$\mathscr{H}^d_b = \mathscr{M}^{b,*} \times_{\mathscr{M}} \mathscr{H}^d,$$

and denote by $\mathscr{V}ect^d_{l,N}$ the open substack of $\mathscr{V}ect^d$ parametrizing vector bundles of fiberwise degree l and $h^0 \leq N$.

Proposition 4.11. The morphism $\mathscr{H}^d_b \to \mathscr{V}ect^d$ factors through the open substack $\mathscr{V}ect^d_{l,N}$ for $l = -b/2$ and any $N \geq d$.

Proof. Consider a geometric point $(\mathcal{P} \to P; \sigma; \phi\colon \mathcal{C} \to \mathcal{P})$ of $\mathscr{H}^d_b$. Then, the branch divisor of ϕ, which is a section of $\det \phi_* O_{\mathcal{C}}^{\otimes(-2)}$, has degree b. Hence $\phi_* O_{\mathcal{C}}$ has degree l. Furthermore, since $\mathcal{C}$ is a reduced curve which is a degree d cover of the connected curve $\mathcal{P}$, we must have $h^0(\phi_* O_{\mathcal{C}}) \leq d$. ☐

Proposition 4.12. The morphism $\mathscr{V}ect^d_{l,N} \to \mathscr{M}^{\mathrm{orb}}$ is of finite type.

For the proof, we need some results about the boundedness of families of sheaves on Deligne–Mumford stacks. Let S be an affine scheme and $\mathcal{X} \to S$ a Deligne–Mumford stack with coarse space $\rho\colon \mathcal{X} \to X$, and a generating sheaf $\mathcal{E}$. Let $O_X(1)$ be an S-relatively ample line bundle on X. Let U be an S-scheme, not necessarily of finite type, and $\mathcal{F}$ a sheaf on $\mathcal{X}_U$. We say that the family of sheaves $(\mathcal{X}_U, \mathcal{F})$ is *bounded* if there is an S-scheme T of finite type and a sheaf $\mathcal{G}$ on $\mathcal{X}_T$ such that every geometric fiber $(\mathcal{X}_u, \mathcal{F}_u)$ appearing in $(\mathcal{X}_U, \mathcal{F})$ over U appears in $(\mathcal{X}_T, \mathcal{G})$ over T. In this case, we say that $(\mathcal{X}_T, \mathcal{G})$ *bounds* $(\mathcal{X}_U, \mathcal{F})$.

Set

$$F_{\mathcal{E}}(-) = \rho_* \mathscr{H}om_{\mathcal{X}}(\mathcal{E}, -).$$

Then $F_{\mathcal{E}}$ takes exact sequences of quasi-coherent sheaves on $\mathcal{X}$ to exact sequences of quasi-coherent sheaves on X, because ρ_* is cohomologically trivial.

Lemma 4.13. In the above setup, if the family $(X_U, F_{\mathcal{E}}(\mathcal{F}))$ is bounded, then the family $(\mathcal{X}_U, \mathcal{F})$ is also bounded.

Proof. Since $F_{\mathcal{E}}(\mathcal{F})$ is bounded, we have a surjection

$$O_X(-M)^{\oplus N} \otimes_S O_U \twoheadrightarrow F_{\mathcal{E}}(\mathcal{F})$$

for large enough M and N. Since $\mathcal{E}$ is a generating sheaf, this gives a surjection

$$\mathcal{E} \otimes_{\mathcal{X}} O_X(-M)^{\oplus N} \otimes_S O_U \twoheadrightarrow \mathcal{F}. \tag{4.5}$$

Let $\mathcal{K}$ be the kernel. Then, $(X_U, F_{\mathcal{E}}(\mathcal{K}))$ is also bounded, and by the same reasoning as above, we have a surjection

$$\mathcal{E} \otimes_{\mathcal{X}} O_X(-M')^{\oplus N'} \otimes_S O_U \twoheadrightarrow \mathcal{K} \tag{4.6}$$

for large enough M' and N'. Combining (4.5) and (4.6), $\mathcal{F}$ can be expressed as the cokernel

$$\mathcal{E} \otimes_{\mathcal{X}} O_X(-M')^{\oplus M'} \otimes_S O_U \to \mathcal{E} \otimes_{\mathcal{X}} O_X(-M)^{\oplus M} \otimes_S O_U \twoheadrightarrow \mathcal{F}.$$

Set

$$\mathcal{H} = \mathscr{H}om_{\mathcal{X}} \left(\mathcal{E} \otimes_{\mathcal{X}} O_X(-M')^{\oplus N'}, \mathcal{E} \otimes_{\mathcal{X}} O_X(-M)^{\oplus N} \right),$$

and $T = Sect_{\mathcal{H}/S}$. By Lemma 4.7, $T \to S$ is of finite type. Letting $\mathcal{G}$ be the cokernel of the universal homomorphism on $\mathcal{X}_T$, we see that $(\mathcal{X}_T, \mathcal{G})$ bounds $(\mathcal{X}_U, \mathcal{F})$. □

Remark 4.14. In the case of a curve $\mathcal{X} \to S$, the family $(X_U, F_{\mathcal{E}}(\mathcal{F}))$ is bounded if the degree, rank and h^0 of $F_{\mathcal{E}}(\mathcal{F})_u$ are bounded for $u \in U$.

We now have the tools to prove Proposition 4.12.

Proof of Proposition 4.12. Let S be a connected affine scheme and $S \to \mathscr{M}^{\mathrm{orb}}$ a morphism given by the pointed orbinodal curve $(\mathcal{P} \xrightarrow{\rho} P \to S; \sigma)$. We must prove that $\mathscr{V}ect^d_{l,N} \times_{\mathscr{M}^{\mathrm{orb}}} S \to S$ is of finite type. After passing to an étale cover of S if necessary, assume that

1. $P \to S$ is projective with relatively ample line bundle $O_P(1)$ (this is possible by Lemma 4.2),
2. We have a generating sheaf $\mathcal{E}$ on $\mathcal{P}$ (this is possible by Proposition 4.3).

Set $E = \rho_*\mathcal{E}$. Since $\mathcal{E} \otimes \rho^* O_P(-1)$ is also a generating sheaf, by twisting $\mathcal{E}$ by $\rho^* O_P(-1)$ enough times, assume that we have a surjection $O_P^{\oplus M} \to E$ for some M.

Let $U \to \mathscr{V}ect^d_{l,N} \times_{\mathscr{M}^{\mathrm{orb}}} S$ be a surjective map from a scheme (not necessarily of finite type), given by the family $(\mathcal{P}_U \to P_U \to U; \sigma; \mathcal{F})$. It suffices to prove that $(\mathcal{P}_U, \mathcal{F})$ is a bounded family of sheaves.

Set

$$F = F_{\mathcal{E}}(\mathcal{F}) = \rho_* \mathscr{H}om_{\mathcal{P}}(\mathcal{E}, \mathcal{F}).$$

By Remark 4.14, it suffices to show that the degree, rank and h^0 of F_u are bounded. The rank of F_u is constant; the degree of $\mathscr{H}om(\mathcal{E}, \mathcal{F})_u$ is constant. It is easy to see that the degree of $\mathscr{H}om(\mathcal{E}, \mathcal{F})_u$ and the degree of F_u differ by a bounded amount, depending only on $\mathcal{P} \to P$ and $\mathcal{E}$. Hence the degree of F_u is bounded. Likewise, it is easy to see that $h^0(F_u)$ and $h^0(\mathscr{H}om(\rho_*\mathcal{E}, \rho_*\mathcal{F})_u)$ differ by a bounded amount, depending only on $\mathcal{P} \to P$ and $\mathcal{E}$. On the other hand,

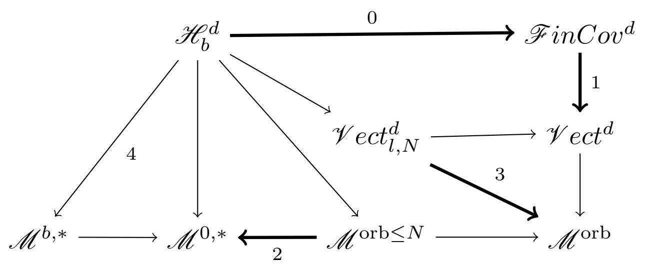
$$\begin{aligned} H^0(\mathscr{H}om(\rho_*\mathcal{E}, \rho_*\mathcal{F})_u) &= \mathrm{Hom}(E_u, \rho_*\mathcal{F}_u) \\ &\subset \mathrm{Hom}(O_{P_u}^{\oplus M}, \rho_*\mathcal{F}_u) \\ &= H^0(\mathcal{F}_u)^{\oplus M}. \end{aligned}$$

By hypothesis, the final vector space has dimension at most MN. It follows that $h^0(F_u)$ is bounded. We conclude that $(\mathcal{P}_U, \mathcal{F})$ is a bounded family of sheaves. □

We have now finished the second part of the proof of Theorem 3.8.

Proposition 4.15. $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ is of finite type.

Proof. Since the open substacks $\mathscr{M}^{b,*}$ cover $\mathscr{M}$, it suffices to show that $\mathrm{br} : \mathscr{H}_b^d = \mathscr{H}^d \times_{\mathscr{M}} \mathscr{M}^{b,*} \to \mathscr{M}^{b,*}$ is of finite type. With $l = -b/2$, and N large enough, we have the following diagram,



The thick arrows in the diagram are known to be of finite type: (0) is an open immersion, (1) is of finite type by Proposition 4.5, (2) by Theorem 4.1, and (3) by Proposition 4.12. Recall that for algebraic stacks $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, all locally of finite type, and morphisms $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$, we have the following:

(1) If $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ are of finite type, then $\mathcal{X} \to \mathcal{Z}$ is also of finite type;
(2) If $\mathcal{X} \to \mathcal{Z}$ is of finite type, then $\mathcal{X} \to \mathcal{Y}$ is also of finite type.

Using the two repeatedly reveals that (4) is also of finite type. □

4.3. **That** $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ **is Deligne–Mumford.**

Proposition 4.16. $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ is representable by Deligne–Mumford stacks.

Proof. The proof is straightforward. By Theorem 4.1, it suffices to check that $\mathscr{H}^d \to \mathscr{M}^{\mathrm{orb}}$ is representable by Deligne–Mumford stacks. In other words, we want this morphism to have unramified inertia. This can be checked on points. Let $(\mathcal{P} \to P \to \operatorname{Spec} k; \sigma; \mathcal{C} \to \mathcal{P})$ be a geometric point of $\mathscr{H}^d$. We must show that $\mathcal{C}$ has no infinitesimal automorphisms over the identity of $\mathcal{P}$. As $\mathcal{C} \to \mathcal{P}$ is a finite cover, these automorphisms are classified by $\operatorname{Hom}_{\mathcal{C}}(\Omega_{\mathcal{C}/\mathcal{P}}, O_{\mathcal{C}})$. Since $\mathcal{C} \to \mathcal{P}$ is unramified on the generic points of the components, $\Omega_{\mathcal{C},\mathcal{P}}$ is supported on a zero dimensional locus. Since $\mathcal{C}$ is reduced, it follows that $\operatorname{Hom}_{\mathcal{C}}(\Omega_{\mathcal{C}/\mathcal{P}}, O_{\mathcal{C}}) = 0$. □

4.4. **That** $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ **is proper.** We use the valuative criterion. Two pieces of notation will be helpful. If S is the spectrum of a local ring, denote by S° the punctured spectrum

$$S^\circ = S \setminus \{\text{closed point of } S\}.$$

For a Deligne–Mumford stack $\mathcal{X}$ with coarse space $\mathcal{X} \to X$ and a geometric point $x \to X$, set

$$\mathcal{X}_x = \mathcal{X} \times_X \operatorname{Spec} O_{X,x}.$$

It will be convenient to work with the spectrum of a *henselian* DVR. The reader unfamiliar with this notion should imagine it to be a small (in particular, simply-connected) complex disk.

We begin with a simple lemma about the following setup. Let r be a positive integer and G a finite group. Let R be a henselian DVR with residue field k and uniformizer t. Let O_S be the henselization of $R[x,y]/(xy - t^r)$ at the point corresponding to (t, x, y). For a positive integer a dividing r, define a finite extension $S_a \to S$ by

$$O_{S_a} = O_S[u,v]/(u^a - x, v^a - y, uv - t^{r/a}).$$

We have an action of μ_a on S_a over the identity of S by $u \mapsto \zeta u$ and $v \mapsto \zeta^{-1} v$.

Lemma 4.17. Let $\chi\colon S^\circ \to BG$ be a morphism given by a G torsor $E \to S^\circ$. Then χ extends to a morphism $[S_r/\mu_r] \to BG$. More generally, χ extends to a morphism $[S_a/\mu_a] \to BG$ if and only if the pullback of E to ${S_a}^\circ$ is trivial. Furthermore, in this case the extension of χ is representable if and only if a is the smallest with the above property.

Proof. To extend χ, we may work étale locally on the source. We use the étale cover $S_a \to [S_a/\mu_a]$. Note that S_a is simply connected (it is henselian). Hence the pullback of E to ${S_a}^\circ$ extends to S_a if and only if this pullback is trivial. Being trivial over ${S_r}^\circ$ is automatic, since ${S_r}^\circ$ is simply connected.

Note that ${S_r}^\circ \to S^\circ$ is the universal covering space—it is a μ_r-torsor where the source ${S_r}^\circ$ is simply connected. The G torsor $E \to S^\circ$ corresponds to a homomorphism $\mu_r \to G$. By the theory of covering spaces, the pullback of E along ${S_a}^\circ \to S^\circ$ is trivial if and only if $\mu_r \to G$ factors as

$$\mu_r \to \mu_a \to G, \tag{4.7}$$

where $\mu_r \to \mu_a$ is the map $\zeta \mapsto \zeta^{r/a}$. As we saw, in this case, we get a morphism $\chi\colon [S_a/\mu_a] \to BG$. Let $s \to [S_a/\mu_a]$ be the stacky point. Observe that the map on automorphism groups $\mathrm{Aut}_s([S_a/\mu_a]) \to \mathrm{Aut}_s(BG)$ is exactly the map $\mu_a \to G$ in (4.7). Since χ is representable precisely when $\mathrm{Aut}_s([S_a/\mu_a]) \to \mathrm{Aut}_s(BG)$ is injective, the result follows. □

Proposition 4.18. $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ is separated.

Proof. As br is of finite type, we may use the valuative criterion. Let R be a henselian DVR with residue field k, fraction field K and uniformizer t. Set $\Delta = \operatorname{Spec} R$. Denote the special, the general and a geometric general point of Δ by 0, η and $\overline{\eta}$ respectively. Let $(\mathcal{P}_i \to P_i \to \Delta; \sigma; \chi_i\colon \mathcal{P}_i \to \mathscr{A}_d)$, for $i = 1, 2$, be two objects of $\mathscr{H}^d(\Delta)$ over an object $(P; \Sigma; \sigma)$ of $\mathscr{M}(\Delta)$. Let $\phi_i\colon \mathcal{C}_i \to \mathcal{P}_i$ be the corresponding degree d covers and let

$$\psi\colon (\mathcal{C}_1 \to \mathcal{P}_1)|_\eta \to (\mathcal{C}_2 \to \mathcal{P}_2)|_\eta$$

be an isomorphism over the identity of P. We must show that ψ extends to an isomorphism of the orbinodal curves $\mathcal{P}_1 \to \mathcal{P}_2$ and the covers $\mathcal{C}_1 \to \mathcal{C}_2$ over all of Δ. Recall that P^{gen} is the complement of the markings σ_j in the smooth locus of $\mathcal{P} \to \Delta$.

Step 1: Extending $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$ over P^{gen}: Since $\mathcal{C}_i \to \mathcal{P}$ is étale over the generic points of the components of $\mathcal{P}|_0$, the map $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$ extends, except possibly at finitely many points on the central fiber. As a result, on P^{gen} we get an isomorphism of vector bundles

$$\psi^\#\colon {\phi_2}_* O_{\mathcal{C}_2}|_{P^{\mathrm{gen}}} \to {\phi_1}_* O_{\mathcal{C}_1}|_{P^{\mathrm{gen}}}$$

away from a locus of codimension two. Since P^{gen} is smooth, by Hartog's theorem, this isomorphism extends over all of P^{gen} and respects the $O_{P^{\mathrm{gen}}}$ algebra structures by continuity.

Step 2: Extending $\psi\colon \mathcal{P}_1 \to \mathcal{P}_2$ at the non-generic nodes: Let $p \to P|_0$ be a node not in the closure of $P|_\eta^{\mathrm{sing}}$. It suffices to extend ψ étale locally around p. The local ring $O_{P,p}$ must be the strict henselization of the ring $R[x,y]/(xy - t^r)$ at the point corresponding to (t, x, y) for some positive integer r. Recall that the χ_i are required to map the nodes to the substack $\mathscr{E}_d \cong B\mathbf{S}_d$ corresponding to étale covers. By the first step, the two maps $\chi_i\colon \operatorname{Spec} O_{P,p}^\circ \to B\mathbf{S}_d$ are isomorphic. Since both χ_i are representable, the structure of orbinodal curves (Proposition 2.4) and Lemma 4.17 imply that

$$(\mathcal{P}_1)_p \cong (\mathcal{P}_2)_p \cong \operatorname{Spec}[O_{P,p}[u,v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a],$$

for some divisor a of r. Thus, we can get an extension $\psi\colon \mathcal{P}_1 \to \mathcal{P}_2$.

Step 3: Extending $\psi\colon \mathcal{P}_1 \to \mathcal{P}_2$ at the marked points: Let $p \to P|_0$ be one of the marked points $\sigma_j(0)$. Then $O_{P,p}$ is the henselization of $R[x]$ at (t,x). Let $\overline{\sigma_j}$ be a geometric generic point of P over $\sigma_j\colon \eta \to P|_\eta$. By the structure of orbinodal curves (Proposition 2.4) for p and $\overline{\sigma_j}$, we have the picture for $i = 1, 2$:

$$\begin{array}{ccc} \mathcal{P}_{i,p} & \longrightarrow & \operatorname{Spec} O_{P,p} \\ \| & & \| \\ [\operatorname{Spec} R[v]^{\mathrm{sh}}/\mu_{r_i}] & \longrightarrow & \operatorname{Spec} R[x]^{\mathrm{sh}} \\ \uparrow & & \uparrow \\ [\operatorname{Spec} K[v]^{\mathrm{sh}}/\mu_{r_i}] & \longrightarrow & \operatorname{Spec} K[x]^{\mathrm{sh}} \\ \| & & \| \\ \mathcal{P}_{i,\overline{\sigma}_j} & \longrightarrow & \operatorname{Spec} O_{P,\overline{\sigma}_j} \end{array},$$

where μ_{r_i} acts by $v \mapsto \zeta v$. The isomorphism $\mathcal{P}_1|_\eta \to \mathcal{P}_2|_\eta$ gives an isomorphism $\mathcal{P}_{1,\overline{\sigma}_j} \to \mathcal{P}_{2,\overline{\sigma}_j}$. In particular, we get $r_1 = r_2 = r$. Furthermore, it is easy to see that an isomorphism $[\operatorname{Spec} K[v]^{\mathrm{sh}}/\mu_r] \to [\operatorname{Spec} K[v]^{\mathrm{sh}}/\mu_r]$ over the identity of coarse spaces $\operatorname{Spec} K[x]^{\mathrm{sh}} \to \operatorname{Spec} K[x]^{\mathrm{sh}}$ must be of the form $v \mapsto \zeta v$ for some rth root of unity ζ. Clearly, such an isomorphism can be extended to an isomorphism $[\operatorname{Spec} R[v]^{\mathrm{sh}}/\mu_r] \to [\operatorname{Spec} R[v]^{\mathrm{sh}}/\mu_r]$.

Step 4: Extending $\psi\colon \mathcal{P}_1 \to \mathcal{P}_2$ at the generic nodes: This step mirrors the previous step, with the appropriate change in the description of the orbinode. We give the details for completeness.

Let $p \to P|_0$ be a node in the closure of $P|_\eta^{\mathrm{sing}}$. Then $O_{P,p}$ is the henselization of $R[x,y]/xy$ at (t,x,y). Since Δ is henselian, we have a section $\sigma\colon \Delta \to P^{\mathrm{sing}}$ with $\sigma(0) = p$. Let $\overline{\sigma}$ be a geometric generic point of $P|_\eta$ over $\sigma\colon \eta \to P|_\eta$. By the structure of orbinodal curves (Proposition 2.4) for p and $\overline{\sigma}$, we have the picture for $i = 1, 2$:

$$\begin{array}{ccc} \mathcal{P}_{i,p} & \longrightarrow & \operatorname{Spec} O_{P,p} \\ \| & & \| \\ \left[\frac{\operatorname{Spec} R[u_i,v_i]^{\mathrm{sh}}}{(u_iv_i,\, u_i - x^{r_i},\, v_i - y^{r_i})}/\mu_{r_i}\right] & \longrightarrow & \operatorname{Spec} (R[x,y]/xy)^{\mathrm{sh}} \\ \uparrow & & \uparrow \\ \left[\frac{\operatorname{Spec} K[u_i,v_i]^{\mathrm{sh}}}{(u_iv_i,\, u_i - x^{r_i},\, v_i - y^{r_i})}/\mu_{r_i}\right] & \longrightarrow & \operatorname{Spec} (K[x,y]/xy)^{\mathrm{sh}} \\ \| & & \| \\ \mathcal{P}_{i,\overline{\sigma}} & \longrightarrow & \operatorname{Spec} O_{P,\overline{\sigma}} \end{array}.$$

The isomorphism $\psi\colon \mathcal{P}_1|_\eta \to \mathcal{P}_2|_\eta$ gives an isomorphism $\mathcal{P}_{1,\overline{\sigma}} \to \mathcal{P}_{2,\overline{\sigma}}$. In particular, we get $r_1 = r_2 = r$. Furthermore, see that an isomorphism $\psi\colon \mathcal{P}_{1,\overline{\sigma}} \to \mathcal{P}_{2,\overline{\sigma}}$ over the identity of coarse spaces $P_{1,\overline{\sigma}} \to P_{1,\overline{\sigma}}$ must be of the form $u_1 \mapsto \zeta_1 u_2$ and $v_1 \mapsto \zeta_2 v_2$ for some rth roots of unity ζ_1 and ζ_2. Such an isomorphism can be extended to an isomorphism $\mathcal{P}_{1,p} \to \mathcal{P}_{2,p}$.

Step 5: Extending $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$: By Step 2, Step 3 and Step 4 , we have an isomorphism $\psi\colon \mathcal{P}_1 \to \mathcal{P}_2$. By Step 1, we also have an isomorphism $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$ except over the node points and the marked points of $\mathcal{P}_i|_0$. However, $\mathcal{C}_i \to \mathcal{P}_i$ is étale over these points; hence ψ must extend to an isomorphism $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$.

□

Having proved separatedness, we turn to properness. The crucial ingredient is the following theorem of Horrocks [13].

Proposition 4.19. [13, Corollary 4.1.1] Let S be the spectrum of a regular local ring. If $\dim S = 2$, then every vector bundle on the punctured spectrum S° is trivial.

Proof. We only describe the main idea. See [13] for the full details.

Denote by $i\colon S^\circ \to S$ the inclusion map. Let E be a vector bundle on S°. If $\dim S \geq 2$, then i_*E can be shown to be a coherent sheaf on S with depth at least 2. If $\dim S = 2$, by the Auslander–Buchsbaum formula, we conclude that i_*E is projective, hence free. Therefore, E is free. □

Proposition 4.20. $\mathrm{br} : \mathscr{H}^d \to \mathscr{M}$ is proper.

Proof. A large chunk of the proof is identical to the proof in the paper of Abramovich and Vistoli [1, Proposition 6.0.4]. The final step is new; it uses Proposition 4.19 and the expression of $\mathscr{A}_d$ as the quotient of an affine scheme by GL_d.

As br is of finite type, we may use the valuative criterion. As before, let R be a henselian DVR with residue field k, fraction field K and uniformizer t. Set $\Delta = \operatorname{Spec} R$. Denote the special, the general and a geometric general point of Δ by 0, η and $\overline{\eta}$ respectively. Let $(P \to \Delta; \Sigma; \sigma)$ be an object of $\mathscr{M}(\Delta)$ and $(\mathcal{P}|_\eta \to P|_\eta; \sigma; \chi)$ an object of $\mathscr{H}^d(\eta)$. We want to extend it to an object over all of Δ, possibly after a base change.

Step 1. Extending χ at the generic points of the components: This step follows Step 2 in [1, Proposition 6.0.4]. We work étale locally. Let ζ be a geometric generic point of a component of $P|_0$. Then the local ring $O_{P,\zeta}$ is also a DVR. Since the branch divisor Σ does not contain any component of $P|_0$, the morphism χ sends the punctured spectrum P_ζ° to $\mathscr{E}_d$. We must extend it to a morphism $\chi\colon P_\zeta \to \mathscr{E}_d$. Since $\mathscr{E}_d \cong B\mathbf{S}_d$ is a proper Deligne–Mumford stack, such an extension is possible after passing to a finite cover $\widetilde{P}_\zeta \to P_\zeta$. By Abhyankar's Lemma [, Appendix I, Proposition 5.5] there is an n such that $\widetilde{P}_\zeta \to P_\zeta$ is isomorphic to $P_\zeta \times_{\operatorname{Spec} R} \operatorname{Spec} R[\sqrt[n]{t}] \to P_\zeta$. Thus, by passing to a sufficiently big cover $\operatorname{Spec} R[\sqrt[N]{t}] \to \operatorname{Spec} R = \Delta$, we can extend χ along the generic points of all the components of $P|_0$. Henceforth, replace R by $R[\sqrt[N]{t}]$. We thus have a morphism $\chi\colon P \to \mathscr{A}_d$ defined away from finitely many points on $P|_0$.

Step 2. Extending χ at the non-generic nodes: This step follows Step 3 in [1, Proposition 6.0.4]. Let $p \to P|_0$ be a node not in the closure of $P_\eta|^{\mathrm{sing}}$. We must describe an orbinodal structure at p and a representable extension of χ. It suffices to do both things in the étale topology. The stalk $O_{P,p}$ is isomorphic to $R[x,y]^{\mathrm{sh}}/(xy - t^r)$ for some $r \geq 1$. Since Σ is supported away from the nodes, the morphism χ sends the punctured spectrum P_p° to $\mathscr{E}_d \cong B\mathbf{S}_d$. As in Lemma 4.17, let a be the smallest integer dividing r such that χ extends to a morphism

$$\chi\colon [\operatorname{Spec} O_{P,p}[u,v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a] \to \mathscr{E}_d \cong B\mathbf{S}_d,$$

where μ_a, as usual, acts by $u \mapsto \zeta$ and $v \mapsto \zeta^{-1}v$. Construct $\mathcal{P}$ over P such that

$$\mathcal{P}_p = [\operatorname{Spec} O_{P,p}[u,v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a].$$

By Lemma 4.17, we have a representable extension $\chi\colon \mathcal{P}_p \to \mathscr{E}_d \cong B\mathbf{S}_d$..

Step 3: Extending χ at the generic nodes and marked points: This step follows Step 4 in [1, Proposition 6.0.4]. Let $p \to P|_0$ be in the closure of $P|_\eta^{\text{sing}}$. First, we extend the orbinode structure $\mathcal{P}|_\eta$ over p. Note that $O_{P,p}$ is isomorphic to the henselization of $R[x,y]/xy$ at (t,x,y). Since Δ is henselian, we have a section $\sigma\colon \Delta \to P^{\text{sing}}$ with $\sigma(0) = p$. Letting $\overline{\sigma}$ be a geometric generic point of this section, we get by Proposition 2.4

$$\mathcal{P}_{\overline{\sigma}} \cong [\operatorname{Spec} K[u,v]^{\text{sh}}/(uv, u^a - x, v^a - y)/\mu_a],$$

for some positive integer a. We extend $\mathcal{P}$ over P_p by the same formula

$$\mathcal{P}_p \cong [\operatorname{Spec} R[u,v]^{\text{sh}}/(uv, u^a - x, v^a - y)/\mu_a].$$

Having defined the orbinodal structure, we extend χ. Again, note that χ sends a neighborhood of p to the étale locus $\mathscr{E}_d \cong B\mathbf{S}_d$. We work étale locally on the source, on the étale cover $\operatorname{Spec} O_{P,p}[u,v]/(uv, u^a - x, v^a - y) \to \mathcal{P}_p$. We already have χ on the punctured spectrum $(\operatorname{Spec} O_{P,p}[u,v]/(uv, u^a - x, v^a - y))^\circ$. Since this punctured spectrum is simply connected, χ extends to a map $\chi\colon O_{P,p}[u,v]/(uv, u^a - x, v^a - y) \to \mathscr{E}_d$.

The case of marked points $p = \sigma_j(0)$ is entirely analogous, if not easier.

Step 4. Extending χ over all of $\mathcal{P}$: By the previous steps, we have a pointed orbinodal structure $\mathcal{P} \to P$ and an extension of χ on $\mathcal{P}$ away from finitely many smooth, non-stacky points of $\mathcal{P}|_0$. Let $p \to P|_0$ be such a point. Recall that $\mathscr{A}_d \cong [B_d/\operatorname{GL}_d]$, where B_d is an affine scheme (Proposition 2.2). The morphism $\chi\colon P_p^\circ \to \mathscr{A}_d$ is equivalent to a GL_d torsor $E^* \to P_p^\circ$ and a GL_d equivariant morphism $E^* \to B_d$. However, by Proposition 4.19, there are no nontrivial GL_d torsors on P_p°. In particular, E^* extends to a GL_d torsor $E \to P_p$. Next, $E^* \subset E$ is the complement of the codimension two locus $E|_p$. Since E is smooth and B_d affine, we have an extension $E \to B_d$ by Hartog's theorem. The extension is GL_d equivariant by continuity. Thus, we get an extension $\chi\colon P_p \to \mathscr{A}_d$.

Finally, note that the two divisors $\chi^*\Sigma_d$ and Σ are supported in the general locus P^{gen} and are equal, by construction, on the complement of a codimension two set. Hence, they must be equal. □

Remark 4.21. It may be helpful to recast Step 4 in terms of finite covers. Let $p \to P|_0$ be a smooth point. Assume that we have a finite cover $\phi\colon C \to U \setminus \{p\}$, where U is a neighborhood of p. We wish to extend it to a cover over all of U. By Proposition 4.19, the vector bundle $\phi_* O_C$ extends to a vector bundle over U. Next, we must extend the O_P algebra structure of $\phi_* O_C$. The algebra structure is specified by maps of vector bundles, which all extend over p by Hartog's theorem. The extensions continue to satisfy the identities to be an algebra by continuity. We thus get an extension of ϕ over all of U.

The proof of the main theorem is now complete. We recall the statement and collect the pieces of the proof.

Theorem (Theorem 3.8)**.** $\mathscr{H}^d$ is an algebraic stack, locally of finite type. The morphism

$$\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$$

is proper and representable by Deligne–Mumford stacks.

Proof. That br is an algebraic stack, locally of finite type is the content of Subsection 4.1, culminating in Proposition 4.9. That br is of finite type is done in Subsection 4.2, culminating in Proposition 4.15. That br is Deligne–Mumford is Proposition 4.16. Finally, the properness is checked in Subsection 4.4 in Proposition 4.18 and Proposition 4.20. □

5. The Local Structure of $\mathscr{H}^d$

In this section, we analyze the local structure of $\mathscr{H}^d$. The main consequence of our analysis is that $\mathscr{H}^d$ is smooth for $d = 2$ and 3 (Theorem 5.5). Throughout the section, we use the formulation of $\mathscr{H}^d$ in terms of finite covers instead of in terms of maps to $\mathscr{A}_d$.

We recall the standard setup of deformation theory. Let k be an algebraically closed field over $\mathbf{K}$. Denote by $\mathbf{Art}_k$ the category of local Artin rings with residue field k. For any object (A, m) of $\mathbf{Art}_k$, denote by 0 the special point of $\operatorname{Spec} A$. Let (A, m) and (A', m') be two objects of $\mathbf{Art}_k$ related by an exact sequence

$$0 \to J \to A' \to A \to 0.$$

Say that A' is a *small extension* of A by J if $m' \cdot J = 0$. Denote by Def_X the standard functor on $\mathbf{Art}_k$ classifying deformations of X, namely

$$\operatorname{Def}_X(A) = \{(X_A \to \operatorname{Spec} A, i)\},$$

where $X_A \to \operatorname{Spec} A$ is a flat morphism and $i\colon X_A|_0 \to X$ an isomorphism. We shorten $(X_A \to \operatorname{Spec} A, i)$ to just X_A, and call it a *deformation of X over A*. Likewise, for a morphism $\phi\colon X \to Y$, denote by Def_ϕ the functor classifying deformations of ϕ (allowing both X and Y to vary), namely

$$\operatorname{Def}_\phi(A) = \{(X_A \to \operatorname{Spec} A, Y_A \to \operatorname{Spec} A, \phi_A\colon X_A \to Y_A, i_X, i_Y)\},$$

where $X_A \to \operatorname{Spec} A$ and $Y_A \to \operatorname{Spec} A$ are flat morphisms and $i_X : X_A|_0 \to X$ and $i_Y : Y_A|_0 \to Y$ are isomorphisms making the obvious commutative diagram

$$\begin{array}{ccc} X_A|_0 & \xrightarrow{\phi_A|_0} & Y_A|_0 \\ \big\downarrow{\scriptstyle i_X} & & \big\downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{\phi} & Y. \end{array} \tag{5.1}$$

We shorten the unwieldy $(X_A \to \operatorname{Spec} A, Y_A \to \operatorname{Spec} A, \phi_A\colon X_A \to Y_A, i_X, i_Y)$ to just $(\phi_A\colon X_A \to Y_A)$ and call it a *deformation of ϕ over A*.

Let $\xi = (\mathcal{P} \to P; \sigma_1, \dots, \sigma_n; \phi\colon \mathcal{C} \to \mathcal{P})$ be such that $(\mathcal{P} \to P; \sigma_1, \dots, \sigma_n)$ is a (not necessarily proper) pointed orbinodal curve over k and $\phi\colon \mathcal{C} \to \mathcal{P}$ a finite cover, étale over the nodes and the marked points of P. Denote by Def_ξ the functor classifying deformations of ξ:

$$\operatorname{Def}_\xi(A) = \{(\mathcal{P}_A \to P_A \to \operatorname{Spec} A; \sigma_{i,A}; \phi_A : \mathcal{C}_A \to \mathcal{P}_A, i_C, i_P)\},$$

where $(\mathcal{P}_A \to P_A \to \operatorname{Spec} A; \sigma_{i,A})$ is a (not necessarily proper) pointed orbinodal curve, $\phi\colon \mathcal{C}_A \to \mathcal{P}_A$ a finite cover, and $i_P : \mathcal{P}_A|_0 \to \mathcal{P}$ and $i_C : \mathcal{C}_A|_0 \to \mathcal{C}$ isomorphisms commuting with ϕ_A and ϕ as in (5.1). If ξ corresponds to a point of $\mathscr{H}^d$, then we have a formally smooth morphism $\operatorname{Def}_\xi \to \mathscr{H}^d$. Our goal is to understand Def_ξ.

Following Fedorchuk [7, § 4.1], we first simplify the task of studying the deformations of ξ into the study of its deformations on Zariski local pieces. Following his terminology [7, § 4.1], let $\{U_i\}$ be an adapted affine open cover of P. This means that each U_i contains exactly one from the following: a node, a marked point or a point of $\operatorname{supp}(\operatorname{br}\phi)$. Set

$$\begin{aligned} \mathcal{U}_i &= U_i \times_P \mathcal{P} \\ \mathcal{V}_i &= \mathcal{C} \times_{\mathcal{P}} \mathcal{U}_i \\ \phi_i &= \phi|_{\mathcal{V}_i} : \mathcal{V}_i \to \mathcal{U}_i \\ \xi_i &= (\mathcal{U}_i \to U_i; \sigma_i; \phi_i : \mathcal{V}_i \to \mathcal{U}_i). \end{aligned}$$

In the last equation, σ_i is ignored if U_i does not contain any marked point. Set $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$, $\mathcal{V}_{ij} = \mathcal{V}_i \cap \mathcal{V}_j$, $\mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, and so on. Observe that $\mathcal{U}_{ij}$ does not contain orbinodes, marked points or branch points. To emphasize that these multiple intersections are schemes, we denote them by roman letters U_{ij}, V_{ij}, U_{ijk}, and so on.

We have restriction maps $\operatorname{Def}_\xi \to \operatorname{Def}_{\xi_i}$.

Proposition 5.1. With the above notation, the map $\operatorname{Def}_\xi \to \prod_i \operatorname{Def}_{\xi_i}$ is formally smooth.

Proof. Let $0 \to k \to A' \to A \to 0$ be a small extension. Assume that we are given a deformation ξ_A of ξ on A. Denote the restriction of ξ_A to $\mathcal{U}_i$ by $\xi_{i,A}$; it is a deformation of ξ_i. Suppose, furthermore, that we are given extensions $\xi_{i,A'}$ of $\xi_{i,A}$. We must prove that the $\xi_{i,A'}$ can be glued to get a global extension $\xi_{A'}$ of ξ_A.

Note that, by construction, U_{ij} is a nonsingular affine scheme. Therefore, its deformations are trivial. Let $p_{ij} \colon O_{\mathcal{U}_{i,A'}}|_{U_{ij}} \to O_{\mathcal{U}_{j,A'}}|_{U_{ij}}$ be an isomorphism over the identity

$$O_{\mathcal{P}_A}|_{U_{ij}} = O_{\mathcal{U}_{i,A}}|_{U_{ij}} \to O_{\mathcal{U}_{j,A}}|_{U_{ij}} = O_{\mathcal{P}_A}|_{U_{ij}}.$$

The choice of p_{ij} is given by an element of $\operatorname{Hom}(\Omega_{U_{ij}}, O_{U_{ij}})$. The isomorphisms p_{ij} may not be compatible on the triple overlaps U_{ijk}. However, since $H^2(\mathscr{H}om(\Omega_{\mathcal{P}}, O_{\mathcal{P}})) = 0$, the two co-cycle defined by $p_{ij} + p_{jk} - p_{ik}$ on U_{ijk} is in fact a co-boundary. As a result, by changing the choice of the p_{ij}, we can assure that they are compatible on triple overlaps. Thus, we obtain an orbinodal curve $(\mathcal{P}_{A'} \to P_{A'}; \sigma_{A'})$ over A' extending $(\mathcal{P}_A \to P_A; \sigma_A)$ over A. This takes care of one piece of an extension $\xi_{A'}$ of ξ_A.

Having constructed $\mathcal{P}_{A'}$, we construct $\mathcal{C}_{A'}$ similarly by choosing isomorphisms

$$c_{ij} \colon O_{\mathscr{V}_{i,A'}}|_{V_{ij}} \to O_{\mathscr{V}_{j,A'}}|_{V_{ij}}.$$

Since $\phi \colon V_{ij} \to U_{ij}$ is étale, we have an equality $\phi^*\Omega_{U_{ij}} = \Omega_{V_{ij}}$. Observe that if we wish to extend $\phi_A \colon \mathcal{C}_A \to \mathcal{P}_A$ to $\phi_{A'} \colon \mathcal{C}_{A'} \to \mathcal{P}_{A'}$, where $\mathcal{P}_{A'}$ is glued by the p_{ij} and $\mathcal{C}_{A'}$ by the c_{ij}, then $c_{ij} \in \operatorname{Hom}(\Omega_{V_{ij}}, O_{V_{ij}})$ must be the pullback of $p_{ij} \in \operatorname{Hom}(\Omega_{U_{ij}}, O_{U_{ij}})$. By choosing the c_{ij} in this way, we get the desired extension $\mathcal{C}_{A'}$ of $\mathcal{C}_A$ along with an extension $\phi_{A'} \colon \mathcal{C}_{A'} \to \mathcal{P}_{A'}$ of $\phi_A \colon \mathcal{C}_A \to \mathcal{P}_A$, completing the second piece of the extension $\xi_{A'}$ of ξ_A. ☐

Next, we analyze $\operatorname{Def}_{\xi_i}$. We use the forgetful morphisms $\operatorname{Def}_{\xi_i} \to \operatorname{Def}_{\mathcal{U}_i}$ and $\operatorname{Def}_{\xi_i} \to \operatorname{Def}_{\mathcal{V}_i}$.

Proposition 5.2. Retain the notation of Proposition 5.1.

(1) If $\mathcal{U}_i$ does not contain a point of $\operatorname{br}\phi$, then $\operatorname{Def}_{\xi_i}$ is formally smooth.
(2) If $\mathcal{U}_i$ contains a point of $\operatorname{br}\phi$, then $\operatorname{Def}_{\xi_i} \to \operatorname{Def}_{\mathcal{V}_i}$ is formally smooth.

Remark 5.3. In the second case, $\mathcal{U}_i$ does not contain any orbinode or marked point. Hence, it is a nonsingular scheme and $\operatorname{Def}_{\xi_i}$ is simply $\operatorname{Def}_{\phi_i}$.

Proof. In the first case, the map $\phi_i \colon \mathcal{V}_i \to \mathcal{U}_i$ is étale. Therefore, the forgetful map $\operatorname{Def}_{\xi_i} \to \operatorname{Def}_{(\mathcal{U}_i;\sigma_i)}$ is an isomorphism. We are thus reduced to showing that the deformations of the pointed orbinodal curve $(\mathcal{U}_i; \sigma_i)$ are unobstructed. This is shown in [2, § 3]. We briefly recall the argument. The obstructions to the deformations lie in $\mathscr{E}xt^2(\Omega_{\mathcal{U}_i}, O_{U_i})$. Étale locally, $\mathcal{U}_i$ is at worst a nodal curve; hence $\mathscr{E}xt^2(\Omega_{\mathcal{U}_i}, O_{U_i}) = 0$.

In the second case, $\mathcal{U}_i = U_i$ is a nonsingular affine scheme; its deformations are trivial. For the smoothness of $\operatorname{Def}_{\phi_i} \to \operatorname{Def}_{V_i}$, take an extension $A' \to A \to 0$ of rings in $\mathbf{Art}_k$, a deformation $\phi_{i,A} \colon V_{i,A} \to U_i \times \operatorname{Spec} A$ of ϕ_i over A and an extension $V_{i,A'} \to \operatorname{Spec} A'$ of $V_{i,A}$. We must construct an extension $\phi_{i,A'} \colon V_{i,A'} \to U_i \times \operatorname{Spec} A'$ of $\phi_{i,A}$. By the infinitesimal lifting property for U_i, the map $V_{i,A} \to U_i$ extends to a map $V_{i,A'} \to U_i$, yielding such an extension $\phi_{i,A'} \colon V_{i,A'} \to U_i \times \operatorname{Spec} A'$. ☐

Recall that a scheme (stack) is *smoothable* if it is the flat limit of non-singular schemes (stacks). Let $\mathcal{H}^d \subset \mathscr{H}^d$ be the open locus consisting of

$$(\mathcal{P} \to P; \sigma; \phi\colon \mathcal{C} \to \mathcal{P}),$$

where $\mathcal{C}$ and $\mathcal{P}$ are smooth and ϕ is simply branched.

Proposition 5.4. Retain the notation of Proposition 5.1. Let S be the set of indices i for which U_i contains a point of $\operatorname{br}\phi$.

(1) Def_ξ is smooth if and only if Def_{V_i} is smooth for all $i \in S$.
(2) The point of $\mathscr{H}^d$ given by ξ is in the closure of $\mathcal{H}^d$ if and only if V_i is smoothable for all $i \in S$.

Proof. Proposition 5.1 and Proposition 5.2 together give a smooth morphism $\operatorname{Def}_\xi \to \prod_{i\in S} \operatorname{Def}_{V_i}$, proving the first assertion. For the second, consider the smooth morphism

$$\operatorname{Def}_\xi \to \prod_{i \notin S} \operatorname{Def}_{\mathcal{U}_i} \times \prod_{i\in S} \operatorname{Def}_{V_i}. \tag{5.2}$$

For $i \notin S$, the $\mathcal{U}_i$ is either a smooth curve or an orbinodal curve. In either case, it is smoothable. By the smoothness of (5.2), if all the V_i are smoothable for $i \in S$ then ξ is in the closure of the locus of

$$(\mathcal{P} \to P; \sigma; \phi\colon \mathcal{C} \to \mathcal{P}),$$

with smooth $\mathcal{C}$ and $\mathcal{P}$. It is not hard to see that this locus is in the closure of $\mathcal{H}^d$, where the only additional constraint is that ϕ be simply branched. □

We record two important special cases.

Theorem 5.5. For $d = 2$ and 3, the stack $\mathscr{H}^d$ is smooth and contains $\mathcal{H}^d$ as a dense open substack.

Proof. We begin with a general observation. For a finite (flat) cover $\phi\colon X \to Y$ of degree d, we have an exact sequence

$$0 \to O_Y \to \phi_* O_X \to F \to 0,$$

split by $1/d$ times the trace map $\operatorname{tr} : \phi_*O_X \to O_Y$. Therefore, the vector bundle F admits a map $F \to \phi_* O_X$. Since $\phi_* O_X$ is a sheaf of O_Y algebras, we get a map $\operatorname{Sym}^*(F) \to \phi_* O_X$, which is clearly surjective. In other words, $\phi\colon X \to Y$ naturally factors as an embedding

$$\iota\colon X \hookrightarrow \operatorname{Spec}_Y \operatorname{Sym}^*(F) \tag{5.3}$$

followed by the projection $\operatorname{Spec}_Y \operatorname{Sym}^*(F) \to Y$.

We now prove the theorem. By Proposition 5.4, it suffices to prove that Def_{V_i} is smooth and V_i is smoothable for all i for which $\phi_i\colon V_i \to U_i$ is ramified. In the case of $d = 2$, the embedding ι in (5.3) exhibits V_i as a divisor in a nonsingular affine surface. It is now well-known that Def_{V_i} is smooth and V_i is smoothable. In the case of $d = 3$, the embedding ι exhibits V_i as a subscheme of a nonsingular affine threefold. Since V_i is a reduced curve, it is Cohen–Macaulay. Thus V_i is a Cohen–Macaulay subscheme of codimension two in a nonsingular affine variety. This lets us conclude that Def_{V_i} is smooth [11, § 2.8] and V_i is smoothable [23, Theorem 2]. □

6. Projectivity

In this section, we prove that the branch morphism is projective on coarse spaces by showing that the Hodge line bundle is relatively anti-ample. We begin by defining the Hodge bundle.

Let $(\mathcal{P} \to P; \sigma; \phi \colon \mathcal{C} \to \mathcal{P})$ be the universal object over $\mathscr{H}^d$. Let $\pi_{\mathcal{P}} \colon \mathcal{P} \to \mathscr{H}^d$ and $\pi_{\mathcal{C}} \colon \mathcal{C} \to \mathscr{H}^d$ be the projections. When no confusion is likely, we denote both by π. Define the *Hodge bundle* Λ on $\mathscr{H}^d$ by

$$\Lambda = (R^1\pi_* O_{\mathcal{C}})^\vee.$$

Then Λ is a locally free sheaf on $\mathscr{H}^d$. Define the line bundle λ by

$$\lambda = \det \Lambda.$$

We use additive notation for λ. So, $-\lambda$ denotes the dual of λ.

Throughout, we use without explicit reference that separated Deligne–Mumford stacks have coarse spaces [14, Corollary 1.3]. We also repeatedly use that Deligne–Mumford stacks admit a finite surjective map from a scheme [26, Proposition 2.6]. This is typically used in the following guise: if we have a map from X to the coarse space Y of a Deligne–Mumford stack $\mathcal{Y}$, then there is a scheme $\widetilde{X}$ with a finite and surjective morphism $\widetilde{X} \to X$ such that $\widetilde{X} \to Y$ lifts to $\widetilde{X} \to \mathcal{Y}$.

Theorem 6.1. Let $\mathcal{M}$ be a Deligne–Mumford stack separated over $\mathbf{K}$ and let $\mathcal{M} \to \mathscr{M}$ be a morphism. Set $\mathcal{H} = \mathcal{M} \times_{\mathscr{M}} \mathscr{H}^d$. Denote by H and M the coarse spaces of $\mathcal{H}$ and $\mathcal{M}$ respectively. Then the induced morphism

$$\mathrm{br} : H \to M$$

is projective. In particular, if M is projective, so is H.

The essential ingredient in the proof is the following lemma.

Lemma 6.2. Let $s \colon \operatorname{Spec} k \to \mathscr{M}$ be a geometric point, and X a scheme with a quasi-finite morphism $X \to s \times_{\mathscr{M}} \mathscr{H}^d$. Then the pullback of $-\lambda$ to X is ample.

Proof. Without loss of generality, X is reduced and connected. By replacing X by its normalization $X^\nu \to X$ if necessary, assume further that X is normal. Let $(P; \Sigma; \sigma)$ be the marked nodal curve over k corresponding to the point s and $(\mathcal{P} \to P \times X; \sigma \times X; \phi \colon \mathcal{C} \to \mathcal{P})$ the family over X giving the map to $s \times_{\mathscr{M}} \mathscr{H}^d$. Construct $\widetilde{\mathcal{C}} \to \mathcal{C}$ by normalizing $\mathcal{C}$ over $\mathcal{P}^{\mathrm{sm}}$. Explicitly, $\widetilde{\mathcal{C}}$ is such that we have

$$\widetilde{\mathcal{C}} \times_P (P \setminus \Sigma) = \mathcal{C} \times_P (P \setminus \Sigma), \text{ and}$$
$$\widetilde{\mathcal{C}} \times_P \mathcal{P}^{\mathrm{sm}} = (\mathcal{C} \times_P \mathcal{P}^{\mathrm{sm}})^\nu.$$

It is easy to see using the result of Teissier [24, Theorem 1] that the fibers of $\widetilde{\mathcal{C}} \times_P P^{\mathrm{sm}} \to X$ are the normalizations of the corresponding fibers of $\mathcal{C} \times_P P^{\mathrm{sm}} \to X$.

Consider the family of finite covers $\widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P}$ over X. Let $t \to X$ be a k-point. Then $\widetilde{\mathcal{C}}_t$ is smooth except over the nodes of $\mathcal{P}_t$ and $\widetilde{\mathcal{C}}_t \to \mathcal{P}_t$ is étale over the nodes of $\mathcal{P}_t$. This implies that there are only finitely many isomorphism types for the cover $\widetilde{\mathcal{C}}_t \to \mathcal{P}_t$. Since X is connected, the fibers over X of $\widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P}$ must all be isomorphic as finite covers. By replacing X by a finite cover if necessary, we can make $\widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P}$ a constant family. In other words, we get $\widetilde{\phi_0} \colon \mathcal{C}_0 \to \mathcal{P}_0$ over k such that

$$\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_0 \times X, \quad \mathcal{P} = \mathcal{P}_0 \times X, \text{ and } \widetilde{\phi} = \widetilde{\phi_0} \times X.$$

In the rest of the proof, we treat $O_{\mathcal{C}}$ and $O_{\widetilde{\mathcal{C}}}$ as bundles on $\mathcal{P}$, omitting ϕ_* and $\widetilde{\phi}_*$ to lighten notation. Denote by I_Σ the ideal of Σ in $\mathcal{P}$. The inclusion $O_{\mathcal{C}} \subset O_{\widetilde{\mathcal{C}}}$ is an isomorphism except over

$\Sigma \times X$. Hence, the quotient $O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}}$ is annihilated by $I^N_{\Sigma_0 \times X}$ for N large enough. In other words, for every point t of X, we have

$$I^N_{\Sigma} \cdot O_{\widetilde{\mathcal{C}}_t} \subset O_{\mathcal{C}_t}. \tag{6.1}$$

As a result, $O_{\mathcal{C}_t}$ is determined by the subspace $H^0(O_{\mathcal{C}_t}/I^N_\Sigma \cdot O_{\widetilde{\mathcal{C}}_t})$ of $H^0(O_{\widetilde{\mathcal{C}}_t}/I^N_\Sigma \cdot O_{\widetilde{\mathcal{C}}_t})$.

Consider the following sequence on $\mathcal{P}$:

$$0 \to O_{\mathcal{C}}/(I^N_{\Sigma \times X} \cdot O_{\widetilde{\mathcal{C}}}) \to O_{\widetilde{\mathcal{C}}}/(I^N_{\Sigma \times X} \cdot O_{\widetilde{\mathcal{C}}}) \to O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}} \to 0.$$

Applying π_*, we obtain a sequence of vector bundles on X:

$$0 \to \pi_* \left(O_{\mathcal{C}}/(I^N_{\Sigma \times X} \cdot O_{\widetilde{\mathcal{C}}})\right) \to \pi_* \left(O_{\widetilde{\mathcal{C}}}/(I^N_{\Sigma \times X} \cdot O_{\widetilde{\mathcal{C}}})\right) \to \pi_* \left(O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}}\right) \to 0. \tag{6.2}$$

Since $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_0 \times X$, the middle vector bundle is in fact trivial:

$$\pi_* \left(O_{\widetilde{\mathcal{C}}}/(I^N_{\Sigma_0 \times X} \cdot O_{\widetilde{\mathcal{C}}})\right) = V \otimes O_X, \text{ where } V = H^0\left(O_{\widetilde{\mathcal{C}_0}}/(I^N_{\Sigma_0} \cdot O_{\widetilde{\mathcal{C}_0}})\right).$$

The sequence (6.2) gives us a morphism $\mu\colon X \to \mathbf{G}$, where $\mathbf{G}$ is the Grassmannian of quotients of V of the appropriate dimension. Moreover, by our discussion above, for every point t of X, the fiber $\phi_t\colon \mathcal{C}_t \to \mathcal{P}_t$ is determined by $\mu(t)$. Since $X \to s \times_{\mathscr{M}} \mathscr{H}^d$ is quasi-finite, μ must also be quasi-finite. We conclude that the pullback to X of the Plücker line bundle on $\mathbf{G}$ is ample. By (6.2), this pullback is simply $\det \pi_* \left(O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}}\right)$. On the other hand, applying π_* to the exact sequence

$$0 \to O_{\mathcal{C}} \to O_{\widetilde{\mathcal{C}}} \to O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}} \to 0,$$

and keeping in mind that $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_0 \times X$ is a constant family, we get

$$\det \pi_* \left(O_{\widetilde{\mathcal{C}}}/O_{\mathcal{C}}\right) \cong \det R^1\pi_* O_{\mathcal{C}}.$$

We deduce that the right hand side, which is the pullback of $-\lambda$ to X, is ample. □

Proof of Theorem 6.1. We want to show that $\operatorname{br} : H \to M$ is projective. Denote also by λ the pullback to $\mathcal{H}$ of λ on $\mathscr{H}^d$. Since $\operatorname{Pic}(\mathcal{H}) \otimes \mathbf{Q} = \operatorname{Pic}(H) \otimes \mathbf{Q}$, we may treat λ as a $\mathbf{Q}$ line bundle on H. We claim that $-\lambda$ is br-ample. It suffices to check this on the fibers of $\operatorname{br} : H \to M$. Let $s \to M$ be a k-point and set $H_s = \operatorname{br}^{-1}(s)$. Choose a lift $\overline{s} \to \mathcal{M}$ of $s \to M$. Then H_s is the coarse space of $\overline{s} \times_{\mathcal{M}} \mathcal{H}$. There is a scheme X and a finite surjective map $X \to \overline{s} \times_{\mathcal{M}} \mathcal{H}$. Lemma 6.2 implies that $-\lambda$ is ample on X. Since $X \to H_s$ is finite and surjective, we deduce that $-\lambda$ is ample on H_s. □

6.1. **Spaces of weighted admissible covers.** The proper morphism $\mathscr{H} \to \mathscr{M}$ lets us construct several compactifications of different variants of the Hurwitz spaces. Some of these have appeared in literature in different guises. Fix non-negative integers g, h, and b related by

$$2g - 2 = d(2h - 2) + b.$$

Let $\mathscr{M}_{h;b} \subset \mathscr{M}$ be the open and closed substack whose k points correspond to $(P; \Sigma)$, where P is a connected curve of arithmetic genus h and $\Sigma \subset P$ a divisor of degree b. Let $\mathcal{M}_{h;b} \subset \mathscr{M}_{h;b}$ be the open substack where P is smooth and Σ is reduced. Then $\mathscr{M}_{h;b}$ is a smooth stack and it contains $\mathcal{M}_{h;b}$ as a dense open substack.

Let $\mathscr{H}^d_{g/h} \subset \mathscr{M}_{h;b} \times_{\mathscr{M}} \mathscr{H}^d$ be the open and closed substack whose k points are $(\mathcal{P} \to P; \phi\colon \mathcal{C} \to \mathcal{P})$, where $\mathcal{C}$ is connected. By the Riemann–Hurwitz formula, $\mathcal{C}$ has (arithmetic) genus g. Note that the classical Hurwitz space $H^d_{g/h}$ mentioned in Section 1 is the coarse space of the open substack $\mathcal{H}^d_{g/h}$ of $\mathscr{H}^d_{g/h}$ defined by

$$\mathcal{H}^d_{g/h} = \mathcal{M}_{h;b} \times_{\mathscr{M}_{h;b}} \mathscr{H}^d_{g/h}.$$

It parametrizes ($\phi\colon C \to P$) where P and C are smooth and ϕ is simply branched. The stack $\mathcal{H}^d_{g/h}$ is often called the *small Hurwitz stack*.

We recall a sequence of open substacks of $\mathscr{M}_{h;b}$ that contain $\mathcal{M}_{h;b}$ and are proper over the base field. These are the spaces of *weighted pointed stable curves* constructed by Hassett [12].

Definition 6.3. Let ϵ be a rational number. Let P be a nodal curve over k and $\Sigma \subset P$ a divisor supported in the smooth locus. We say that (P, Σ) is *ϵ-stable* if

(1) for every point p of P, we have

$$\epsilon \cdot \operatorname{mult}_p(\Sigma) \leq 1;$$

(2) the $\mathbf{Q}$ line bundle $\omega_P \otimes O_P(\epsilon\Sigma)$ is ample, where ω_P is the dualizing line bundle of P.

Denote by $\overline{\mathcal{M}}_{h;b}(\epsilon) \subset \mathscr{M}_{h;b}$ the open substack parametrizing ϵ-stable marked curves.

Recall the main theorem from [12].

Theorem 6.4. [12, Theorem 2.1, Variation 2.1.3] $\overline{\mathcal{M}}_{h;b}(\epsilon)$ is a Deligne–Mumford stack, proper over $\mathbf{K}$. It admits a projective coarse space $\overline{M}_{h;b}(\epsilon)$.

If $\deg(\omega_P(\epsilon\Sigma)) = \epsilon \cdot b + 2h - 2 \leq 0$, then $\overline{\mathcal{M}}_{h;b}(\epsilon)$ is empty. Otherwise, it contains $\mathcal{M}_{h;b}$ as a dense open substack.

Definition 6.5. Define the stack $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ of *ϵ-admissible covers* by the formula

$$\overline{\mathcal{H}}^d_{g/h}(\epsilon) = \overline{\mathcal{M}}_{h;b}(\epsilon) \times_{\mathscr{M}_{h;b}} \mathscr{H}^d_{g/h}.$$

We sometimes call ϵ-admissible covers *weighted admissible covers*.

Corollary 6.6. $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ is a Deligne–Mumford stack, proper over $\mathbf{K}$. It admits a projective coarse space $\overline{H}^d_{g/h}(\epsilon)$ and a morphism

$$\operatorname{br} : \overline{\mathcal{H}}^d_{g/h}(\epsilon) \to \overline{\mathcal{M}}_{h;b}(\epsilon).$$

Proof. Follows directly from Theorem 3.8 and Theorem 6.1. □

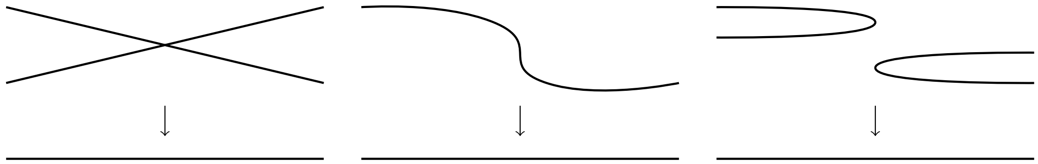
As before, if $\epsilon \cdot b + 2h - 2 \leq 0$, then $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ is empty. Otherwise, it contains $\mathcal{H}^d_{g/h}$ as an open substack (but it may not be dense; see Example 6.11).

6.2. **Examples.** We describe the geometry of the spaces of weighted admissible covers by some illustrative examples.

These spaces generalize some known compactifications of Hurwitz spaces, mentioned in the following two examples.

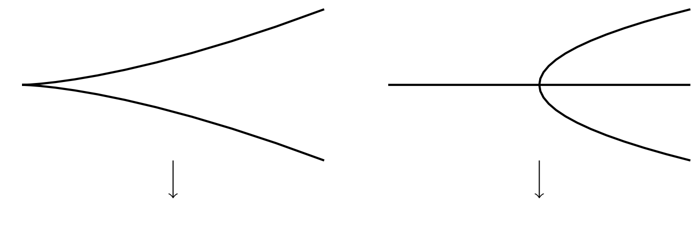
Example 6.7 (Twisted admissible covers)**.** Consider the case $\epsilon = 1$ and the resulting stack of 1-admissible covers $\overline{\mathcal{H}}^d_{g/h}(1)$. It parametrizes $(\mathcal{P} \to P; \phi\colon \mathcal{C} \to \mathcal{P})$, where $\operatorname{br}\phi \subset P$ is étale over the base. The induced morphism on coarse spaces $C \to P$ is an admissible cover in the sense of Harris and Mumford [10] (but with unordered branch points).

By Proposition 5.4, the stack $\overline{\mathcal{H}}^d_{g/h}(1)$ is smooth and contains the small Hurwitz stack $\mathcal{H}_{g/h}$ as a dense open substack. In fact, $\overline{\mathcal{H}}^d_{g/h}(1)$ is essentially the stack of *twisted admissible covers* of Abramovich, Corti, and Vistoli [2]; the only difference is that in [2], the branch points are ordered, whereas in $\overline{\mathcal{H}}^d_{g/h}(1)$, they are unordered.



(a) Node (b) Triple ramification (c) Stacked simple ramification

FIGURE 1. Possible local pictures of ϕ for $1/3 < \epsilon \leq 1/2$



(a) Cusp (b) Ramified node

FIGURE 2. Some of the possible local pictures of ϕ for $1/4 < \epsilon \leq 1/3$

Example 6.8 (Spaces of hyperelliptic curves)**.** Consider the case $h = 0$ and $d = 2$, and the resulting stacks $\overline{\mathcal{H}}^2_g(\epsilon)$ of ϵ-admissible covers. Consider a k-point of $\overline{\mathcal{H}}^2_g(\epsilon)$, given by a cover $(\mathcal{P} \to P; \phi\colon \mathcal{C} \to \mathcal{P})$. Say $\lfloor 1/\epsilon \rfloor = n$. Away from over the nodes of $\mathcal{P}$, the singularities of $\mathcal{C}$ are (étale) locally of the form

$$y^2 - x^m,$$

for $m \leq n$. Thus, the spaces $\overline{H}^2_g(\epsilon)$ are just the spaces of hyperelliptic curves with A_{n-1} singularities constructed by Fedorchuk [7].

The singularities of $\mathcal{C}$ get much more interesting for higher degrees, as illustrated in the next example.

Example 6.9 (Singularities of $\mathcal{C}$)**.** Let $(\mathcal{P} \to P; \phi\colon \mathcal{C} \to \mathcal{P})$ be a k-point of $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$. Notice that we do not explicitly restrict the singularities of $\mathcal{C}$; the restrictions are imposed indirectly by the allowed multiplicity of the branch divisor. We list some examples of the singularities that appear on $\mathcal{C}$ for small values of $1/\epsilon$ and $d \geq 3$.

(1) $1/2 < \epsilon \leq 1$

In this case, $\mathcal{C}$ is smooth (except, of course, over the nodes of $\mathcal{P}$) and simply branched over $\mathcal{P}$.

(2) $1/3 < \epsilon \leq 1/2$

In this case, $\mathcal{C}$ can have only nodal singularities. Also, the branches of the nodes must be individually unramified over $\mathcal{P}$ as in Figure 1(a). This case also allows certain kinds of multiple ramification in ϕ: it can be triply ramified as in Figure 1(b) or it can have two simple ramification points lying over the same point of $\mathcal{P}$ as in Figure 1(c).

(3) $1/4 < \epsilon \leq 1/3$

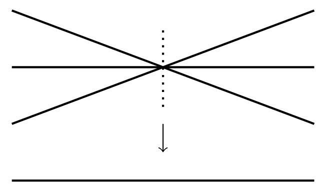


FIGURE 3. Planar triple points are allowed for $\epsilon \leq 1/6$

In this case, $\mathcal{C}$ can have nodal and cuspidal (formally $k[\![x, y]\!]/(y^2 - x^3)$) singularities as in Figure 2(a). This case also allows even more multiple ramification in ϕ; for example, it is possible to have ramification types (4), or (3, 2) or (2, 2, 2) in a fiber of ϕ. Another interesting possibility is a *ramified node* (Figure 2(b)). It is a combination of multiple ramification and the development of a singularity. This is a node on $\mathcal{C}$, one of whose branches is simply ramified over $\mathcal{P}$, formally expressed by

$$k[\![t]\!] \to k[\![t, x]\!]/x(x^2 - t).$$

(4) $\epsilon \leq 1/4$

In this case, $\mathcal{C}$ can have non-Gorenstein singularities. Indeed, the spatial triple point (formally the union of the coordinate axes in $\mathbf{A}^3$) is a branched cover of a line with branch divisor of multiplicity four. Since multiplicity four is allowed in the branch divisor for $\epsilon \leq 1/4$, the cover $\mathcal{C} \to \mathcal{P}$ can have formal local picture of a spatial triple point:

$$k[\![t]\!] \to k[\![t, x, y]\!]/(xy, y(x - t), x(y - t)).$$

In the case of admissible covers ($\epsilon = 1$) and in the case of hyperelliptic curves ($d = 2$), the branch morphism is finite. This is no longer the case if $d \geq 3$ and ϵ is sufficiently small. In fact, as soon as $\epsilon \leq 1/6$, we have positive dimensional fibers, as illustrated in the next example.

Example 6.10 (Non-finiteness of the branch morphism)**.** For every $c \in k$, consider the the planar triple point expressed as a triple cover of a smooth curve (Figure 3) by the formal description:

$$k[\![t]\!] \to k[\![t, x]\!]/x(x - t)(x - ct). \tag{6.3}$$

Although the rings $k[\![t, x]\!]/x(x - t)(x - ct)$ are isomorphic for different choices of c, they are *not* necessarily isomorphic as $k[\![t]\!]$ algebras. Said differently, although the singularities $\operatorname{Spec} k[\![t, x]\!]/x(x-t)(x-ct)$ are abstractly isomorphic, they are *not* necessarily isomorphic as triple covers of $\operatorname{Spec} k[\![t]\!]$. One way to see this is the following. Consider the tangent space to $\operatorname{Spec} k[\![t, x]\!]/x(x - t)(x - ct)$ at $(0, 0)$. In this two dimensional vector space, there are four distinguished one dimensional subspaces: the three tangent spaces of the branches and the kernel of the projection to the tangent space of $\operatorname{Spec} k[\![t]\!]$. The moduli of the configuration of these four subspaces depends on c. Up to a finite ambiguity, different choices of c give non-isomorphic triple covers.

Using the basis $\langle 1, x, x^2 \rangle$ of $k[\![t, x]\!]/x(x - t)(x - ct)$ as a $k[\![t]\!]$ module, and recalling from (2.2) that the discriminant δ is the determinant of the 3×3 matrix $\operatorname{tr}(x^i \cdot x^j)$, we get $\langle \delta \rangle = \langle t^6 \rangle$. That is, planar triple points appear in our moduli spaces if six or more branch points are allowed to coincide.

For $d \geq 3$, $\epsilon \leq 1/6$ and h, b large enough to allow $\epsilon \cdot b + 2h - 2 \geq 0$, the formal descriptions in (6.3) are realizable in covers of a fixed genus h curve with a fixed branch divisor. In that case, we get infinitely many points in a fiber of $\operatorname{br} : \overline{\mathcal{H}}^d_{g/h}(\epsilon) \to \overline{\mathcal{M}}_{h;b}(\epsilon)$.

In the case of admissible covers ($\epsilon = 1$), the small Hurwitz stack $\mathcal{H}^d_{g/h}$ is dense in $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$. By Theorem 5.5, this remains the case for arbitrary ϵ if $d \leq 3$. However, this is not true in general, as illustrated by the following example.

Example 6.11 (Extraneous components in $\overline{\mathcal{H}}^d_{g;h}(\epsilon)$)**.** For a sufficiently large d and a sufficiently small ϵ, we exhibit a point in $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ that is not in the closure of $\mathcal{H}^d_{g/h}$. For simplicity, take $h = 0$; the phenomenon is local, so the case of $h = 0$ can be used to construct examples for any h.

Let C be a reduced, connected curve that is not a flat limit of smooth curves (see the article by Mumford [17] for the existence of such curves). For sufficiently large d, we have a finite map $\phi\colon C \to \mathbf{P}^1$ of degree d. Let ϵ be so small that $\epsilon \cdot \operatorname{mult}_p(\operatorname{br}\phi) \leq 1$ for all $p \in \mathbf{P}^1$. Then $(\mathbf{P}^1; \phi\colon C \to \mathbf{P}^1)$ is a point in $\overline{\mathcal{H}}^d_g(\epsilon)$ which, by construction, is not in the closure of $\mathcal{H}^d_g$. Thanks to Theorem 5.5, there are no extraneous components for $d = 2$ or 3. By Proposition 5.4, unsmoothable singularities are the only reason for extraneous components.

We end the section with a question prompted by Example 6.11.

Question 6.12. For which d, g and h is $\mathscr{H}^d_{g/h}$ irreducible? More generally, for which d, g, h and ϵ is $\overline{\mathcal{H}}^d_{g/h}(\epsilon)$ irreducible?

7. Moduli of d-gonal Singularities and Crimping

The goal of this section is to understand the fibers of $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$. Consider a point $s\colon \operatorname{Spec} k \to \mathscr{M}$. For simplicity, assume that it corresponds to a smooth curve P with a marked divisor Σ. The fiber of br over s consists precisely of degree d covers $\phi\colon C \to P$ with $\operatorname{br}\phi = \Sigma$. Let $\widetilde{C} \to C$ be the normalization. Since $\widetilde{C}$ is smooth, the cover $\widetilde{C} \to P$ is determined by its restriction $\widetilde{C}|_{P\setminus\Sigma} \to P \setminus \Sigma$, which is étale. Since there are only finitely many étale covers of degree d of a smooth curve, there are only finitely many possibilities for $\widetilde{\phi}\colon \widetilde{C} \to P$. The fiber of br over s thus decomposes into finitely many (open and closed) components corresponding to the choice of $\widetilde{\phi}\colon \widetilde{C} \to P$. Within each component, $C \to P$ is obtained by *crimping* a fixed $\widetilde{C} \to P$ over the points of Σ. The crimping can be described formally locally around the points of Σ in P. In this way, the description of the fibers of br includes the discrete global data of the normalization and the continuous local data of the crimping.

Moduli of singular curves and the phenomenon of crimping have been studied extensively by van der Wyck [25]. Our study of crimping in the context of finite covers, however, is much more elementary.

7.1. The space of crimps of a finite cover.

Let $\mathcal{Y}$ be a reduced, purely one dimensional Deligne–Mumford stack over k and $\Sigma \subset \mathcal{Y}$ a Cartier divisor. Let $\widetilde{\phi}\colon \widetilde{\mathcal{X}} \to \mathcal{Y}$ a finite cover of degree d, étale over $\mathcal{Y} \setminus \Sigma$. In all the cases we consider, $\mathcal{Y}$ is either a (pointed) orbinodal curve or the spectrum of a DVR. Define the functor $\operatorname{Crimp}_{\widetilde{\phi},\Sigma} : \mathbf{Schemes}_k \to \mathbf{Sets}$ of *crimps of* $\widetilde{\phi}$ *over* Σ by

$$\operatorname{Crimp}_{\widetilde{\phi},\Sigma}(T) = \{(\widetilde{\mathcal{X}} \times T \to \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T)\}/\text{Isomorphism},$$

where $\phi\colon \mathcal{X} \to \mathcal{Y} \times T$ is a finite cover of degree d with $\operatorname{br}(\phi) = \Sigma \times T$. Two such crimps $\widetilde{\mathcal{X}} \times T \to \mathcal{X}_i \to \mathcal{Y} \times T$, for $i = 1, 2$, are isomorphic if there is an isomorphism $\mathcal{X}_1 \to \mathcal{X}_2$ that commutes with the relevant maps

$$\begin{array}{ccccc} \widetilde{\mathcal{X}} \times T & \longrightarrow & \mathcal{X}_1 & \longrightarrow & \mathcal{Y} \times T \\ \| & & \downarrow & & \| \\ \widetilde{\mathcal{X}} \times T & \longrightarrow & \mathcal{X}_2 & \longrightarrow & \mathcal{Y} \times T \end{array}.$$

We sometimes write $\mathrm{Crimp}(\widetilde{\phi}, \Sigma)$ instead of $\mathrm{Crimp}_{\widetilde{\phi},\Sigma}$ for better readability.

If $\mathcal{Z} \to \mathcal{Y}$ is a morphism such that $\Sigma_{\mathcal{Z}} \subset \mathcal{Z}$ is also a divisor, then we have a natural transformation

$$\mathrm{Crimp}(\widetilde{\phi}, \Sigma) \to \mathrm{Crimp}(\widetilde{\phi}_{\mathcal{Z}}, \Sigma_{\mathcal{Z}})$$

defined by

$$(\widetilde{\mathcal{X}} \times T \to \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T) \mapsto (\widetilde{\mathcal{X}}_{\mathcal{Z}} \times T \to \mathcal{X}_{\mathcal{Z}} \xrightarrow{\phi_{\mathcal{Z}}} \mathcal{Z} \times T).$$

Let $G = \mathrm{Aut}(\widetilde{\phi})$ be the group of automorphisms of $\widetilde{\mathcal{X}}$ over the identity of $\mathcal{Y}$. This is a finite group, which acts on $\mathrm{Crimp}(\widetilde{\phi}, \Sigma)$ as follows:

$$G \ni \alpha \colon (\widetilde{\mathcal{X}} \times T \xrightarrow{\nu} \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T) \mapsto (\widetilde{\mathcal{X}} \times T \xrightarrow{\nu \circ \alpha^{-1}} \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T).$$

Remark 7.1. A crimp may be equivalently thought of as a suitable subalgebra $\phi_* O_{\mathcal{X}}$ of the algebra $\widetilde{\phi}_* O_{\widetilde{\mathcal{X}} \times T}$ on $\mathcal{Y} \times T$. Then isomorphism of crimps simply becomes equality of subalgebras. The action of G is induced by the action of G on $\widetilde{\phi}_* O_{\widetilde{\mathcal{X}}}$.

Throughout, we view $O_{\widetilde{\mathcal{X}} \times T}$ and $O_{\mathcal{X}}$ as sheaves of algebras on $\mathcal{Y} \times T$, omitting $\widetilde{\phi}_*$ and ϕ_* to lighten notation. Observe that the quotient $O_{\widetilde{\mathcal{X}} \times T}/O_{\mathcal{X}}$ is an $O_{\mathcal{Y} \times T}$ module supported entirely on $\Sigma \times T$. In other words, $\widetilde{\mathcal{X}} \times T \to \mathcal{X}$ is an isomorphism away from $\Sigma \times T$.

Having defined $\mathrm{Crimp}(\widetilde{\phi}, \Sigma)$ in wide generality, we turn to the case of interest. Let $(\mathcal{P} \to P; \sigma_1, \dots, \sigma_n)$ be a pointed orbinodal curve and $\Sigma \subset \mathcal{P}$ a divisor supported in the general locus $P^{\mathrm{gen}} = \mathcal{P}^{\mathrm{sm}} \setminus \sigma_1, \dots, \sigma_n$. Let $\widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P}$ be a finite cover, étale over $\mathcal{P} \setminus \Sigma$. We begin by making precise our remark that crimps can be described formally locally around the points of Σ.

Proposition 7.2. Let $\widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P}$ and Σ be as above.

(1) Let $U \subset \mathcal{P}$ be an open set containing Σ. Then the transformation

$$\mathrm{Crimp}(\widetilde{\phi}, \Sigma) \to \mathrm{Crimp}(\widetilde{\phi}_U, \Sigma)$$

is an isomorphism.

(2) The transformation

$$\mathrm{Crimp}(\widetilde{\phi}, \Sigma) \to \prod_{s \in \mathrm{supp}(\Sigma)} \mathrm{Crimp}(\widetilde{\phi} \times_{\mathcal{P}} \mathrm{Spec}\, O_{P,s}, \Sigma \times_{\mathcal{P}} \mathrm{Spec}\, O_{P,s})$$

is an isomorphism.

(3) The transformation

$$\mathrm{Crimp}(\widetilde{\phi}, \Sigma) \to \prod_{s \in \mathrm{supp}(\Sigma)} \mathrm{Crimp}(\widetilde{\phi} \times_{\mathcal{P}} \mathrm{Spec}\, \widehat{O_{P,s}}, \Sigma \times_{\mathcal{P}} \mathrm{Spec}\, \widehat{O_{P,s}})$$

is an isomorphism.

Proof. The last assertion is the strongest, so we prove it. Following Remark 7.1, we treat crimps as subalgebras. For brevity, we set

$$\widehat{P}_s = \operatorname{Spec} \widehat{O_{P,s}}, \quad \Sigma_s = \Sigma \times_P \widehat{P}_s, \text{ and } \widetilde{C}_s = \widetilde{\mathcal{C}} \times_{\mathcal{P}} \widehat{P}_s,$$

Given crimps $\widetilde{C}_s \times T \to C_s \to \widehat{P}_s \times T$ for $s \in \operatorname{supp}(\Sigma)$, construct a subalgebra $O_{\mathcal{C}}$ of $O_{\widetilde{\mathcal{C}} \times T}$ as the fiber product of algebras

$$\begin{array}{ccc} O_{\mathcal{C}} & \longrightarrow & O_{\widetilde{\mathcal{C}} \times T} \\ \downarrow & & \downarrow \\ \prod_s O_{C_s} & \longrightarrow & \prod_s O_{\widetilde{C}_s \times T} \end{array}.$$

We thus get a natural transformation

$$\prod_{s \in \operatorname{supp}(\Sigma)} \operatorname{Crimp}(\widetilde{\phi} \times_{\mathcal{P}} \operatorname{Spec} \widehat{O_{P,s}}, \Sigma \times_{\mathcal{P}} \operatorname{Spec} \widehat{O_{P,s}}) \to \operatorname{Crimp}(\widetilde{\phi}, \Sigma).$$

It is easy to check that it is inverse to the transformation in (3). □

7.2. **Crimps over a disk.** Thanks to Proposition 7.2, we now focus on the crimps of covers of the formal disk. Set $R = k[\![t]\!]$ and $\Delta = \operatorname{Spec} R$. Let Δ° be the punctured disk $\Delta \setminus \{0\}$. Fix a finite cover $\widetilde{\phi} \colon \widetilde{C} \to \Delta$ of degree d, étale over Δ°, with $\operatorname{br}(\widetilde{\phi})$ given by $\langle t^a \rangle$. Fix a divisor $\Sigma \subset \Delta$ given by $\langle t^b \rangle$ and set $\delta = (b - a)/2$.

Proposition 7.3. Let $\widetilde{C} \times T \to C \xrightarrow{\phi} \Delta \times T$ be a crimp with $\operatorname{br}(\phi) = \Sigma \times T$. Set $Q = O_{\widetilde{C} \times T}/O_C$. Then Q is a T-flat sheaf on $\Delta \times T$ annihilated by t^b. The restriction of Q to the fibers of $\Delta \times T \to T$ has length δ.

Proof. In the proof, all the linear-algebraic operations are over $O_{\Delta \times T}$. First, Q is T-flat simply because the inclusion $i \colon O_C \hookrightarrow O_{\widetilde{C} \times T}$ remains an inclusion when restricted to the fibers of $\Delta \times T \to T$. For the rest, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_{\Delta \times T} & \xrightarrow{\widetilde{\delta}} & (\det O^\vee_{\widetilde{C} \times T})^{\otimes 2} & \longrightarrow & \widetilde{B} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \det(i^\vee)^2} & & \downarrow & & \\ 0 & \longrightarrow & O_{\Delta \times T} & \xrightarrow{\delta} & (\det O^\vee_C)^{\otimes 2} & \longrightarrow & B & \longrightarrow & 0 \end{array}.$$

The horizontal maps $\widetilde{\delta}$ and δ define the respective branch divisors as in Subsection 2.1. In particular, B is annihilated by $\langle t^b \rangle$. The snake lemma yields the sequence

$$0 \to \widetilde{B} \to B \to \operatorname{coker}(\det(i^\vee)^2) \to 0. \tag{7.1}$$

Since t^b annihilates B, it annihilates $\operatorname{coker}(\det(i^\vee)^2)$, hence $\operatorname{coker}(\det(i^\vee))$, hence $\operatorname{coker}(i^\vee)$ and hence $\operatorname{coker} i = Q$.

To compute the length of Q on the fibers, replace T by a field. By (7.1), we get

$$\begin{aligned} 2 \operatorname{length} Q &= \operatorname{length}(\operatorname{coker}(\det(i^\vee)^2)) \\ &= \operatorname{length} B - \operatorname{length} \widetilde{B} \\ &= b - a = 2\delta. \end{aligned}$$

□

Remark 7.4. Recall that for a curve singularity C with normalization C^ν, the *δ-invariant* is the length of O_{C^ν}/O_C. In Proposition 7.3, if $\widetilde{C}$ is smooth, then δ is indeed the δ invariant of C. In this case, we get the relationship $2\delta = b - a$ between the delta invariant δ, and the multiplicities b and a of the branch divisors of the cover and its normalization, respectively.

We now exhibit the space of crimps over a disk explicitly as a projective scheme. Set $F = O_{\widetilde{C}}/t^b O_{\widetilde{C}}$ and denote by $\mathrm{Quot} = \mathrm{Quot}(F, \delta)$ the Quot scheme of length δ quotients of the Δ module F. Since $\operatorname{supp} F$ is projective (it is finite!), Quot is a projective scheme. The idea is to identify quotients which arise as $O_{\widetilde{C}}/O_C$. For this to be true, the quotient must satisfy the following two properties:

(1) The kernel must be closed under multiplication, to get a subalgebra O_C of $O_{\widetilde{C}}$;
(2) The resulting $C \to \Delta$ must have the right branch divisor.

We now formalize both conditions. Let $\pi \colon \Delta \times \mathrm{Quot} \to \Delta$ be the projection. On $\Delta \times \mathrm{Quot}$ we have the universal sequence

$$0 \to S \to F \otimes_k O_{\mathrm{Quot}} \to Q \to 0.$$

The multiplication $F \otimes_\Delta F \to F$ induces maps

$$S \otimes_{\Delta \times \mathrm{Quot}} S \to (F \otimes_\Delta F) \otimes_k O_{\mathrm{Quot}} \to F \otimes_k O_{\mathrm{Quot}} \to Q.$$

Define the closed subscheme $X \subset \mathrm{Quot}$ as the annihilator of the composite map $\pi_*(S \otimes_{\Delta \times \mathrm{Quot}} S) \to \pi_* Q$ on Quot. This takes care of (1).

On $\Delta \times X$, the sheaf S inherits the structure of an $O_{\Delta \times X}$ algebra. Form the subalgebra O_C of $O_{\widetilde{C} \times X}$ as the fiber product

$$\begin{array}{ccc} O_C & \longrightarrow & O_{\widetilde{C} \times X} \\ \Big\downarrow & & \Big\downarrow \\ S & \longrightarrow & F \otimes_k O_X \end{array},$$

and set $C = \operatorname{Spec} O_C$.

Claim. In the above setup, $C \to \Delta \times X$ is flat.

Proof. By the definition of O_C, we have the sequence

$$0 \to O_C \to O_{\widetilde{C} \times X} \to Q \to 0.$$

Since Q is X-flat, we conclude that O_C is X-flat and $O_C \to O_{\widetilde{C} \times X}$ remains an inclusion when restricted to the fibers of $\Delta \times X \to X$. For every point $x \in X$, the sheaf O_{C_x} is a subsheaf of the free sheaf $O_{\widetilde{C}}$ and hence is free. It follows that O_C is a locally free $\Delta \times X$ module. □

We currently have $\widetilde{C} \times X \to C \xrightarrow{\phi} \Delta \times X$, where $\widetilde{C} \to C$ is an isomorphism over $\Delta^\circ \times X$ and $C \to \Delta \times X$ is finite and flat. We now enforce (2). Define B by

$$0 \to O_{\Delta \times X} \xrightarrow{\delta} (\det O_C^\vee)^{\otimes 2} \to B \to 0,$$

where the linear algebraic operations are over $O_{\Delta \times X}$, and δ is the usual discriminant as in Subsection 2.1. Observe that δ remains an injection when restricted to the fibers of $\pi \colon \Delta \times X \to X$, and hence B is X-flat. See that B has fiberwise length b. Define the closed subscheme $Y \subset X$ as the annihilator of

$$\pi_* B \xrightarrow{t^b} \pi_* B.$$

This condition would be superfluous if X were reduced. However, it appropriately restricts the non-reduced structure on X, taking care of (2).

By construction, we have a crimp $\widetilde{C} \times Y \to C \xrightarrow{\phi} \Delta \times Y$ with $\operatorname{br} \phi = \Sigma \times Y$. We thus get a morphism

$$Y \to \operatorname{Crimp}(\widetilde{\phi}, \Sigma). \tag{7.2}$$

Proposition 7.5. The morphism $Y \to \operatorname{Crimp}(\widetilde{\phi}, \Sigma)$ in (7.2) is an isomorphism. In particular, $\operatorname{Crimp}(\widetilde{\phi}, \Sigma)$ is a projective scheme.

Proof. We construct the inverse $\operatorname{Crimp}(\widetilde{\phi}, \Sigma) \to Y$ to (7.2). Let T be a scheme and $\widetilde{C} \times T \to C \xrightarrow{\phi} \Delta \times T$ a crimp with branch divisor $\Sigma \times T$. Define the quotient $Q = O_{\widetilde{C} \times T}/O_C$. By Proposition 7.3, Q is a T-flat quotient of $O_{\widetilde{C} \times T}/t^b O_{\widetilde{C} \times T} = F \otimes_k O_T$, fiberwise of length δ. This gives a map $T \to \operatorname{Quot}(F, \delta)$. Since the kernel of $F \otimes_k O_T \to Q$ is the image of O_C, it is closed under multiplication. Hence $T \to \operatorname{Quot}$ factors through $T \to X$. Since $\operatorname{br}(\phi) = \Sigma \times T$, the cokernel of

$$O_{\Delta \times T} \xrightarrow{\delta} (\det O_C^{\vee})^{\otimes 2}$$

is annihilated by t^b. Therefore, the map $T \to X$ factors through $T \to Y$. In this way, we get a morphism $\operatorname{Crimp}(\widetilde{\phi}, \Sigma) \to Y$, which is clearly inverse to (7.2). □

Corollary 7.6. Let $\widetilde{\mathcal{C}} \to \mathcal{P}$ be a finite cover of an orbinodal curve and $\Sigma \subset P^{\mathrm{gen}}$ a divisor. Then the functor $\operatorname{Crimp}(\widetilde{\mathcal{C}} \to \mathcal{P}, \Sigma)$ is representable by a projective scheme.

Proof. Follows immediately from Proposition 7.2 and Proposition 7.5. □

Finally, we relate the spaces of crimps with the fibers of $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$. Let $p \colon \operatorname{Spec} k \to \mathscr{M}$ be a point corresponding to a divisorially marked, pointed curve $(P; \Sigma; \sigma_1, \dots, \sigma_n)$. As usual, we abbreviate $\sigma_1, \dots, \sigma_n$ by σ. Let Γ be the set of $(\mathcal{P} \to P; \sigma; \widetilde{\phi} \colon \widetilde{\mathcal{C}} \to \mathcal{P})$, where $(\mathcal{P} \to P; \sigma)$ is a pointed orbinodal curve and $\widetilde{\phi}$ a finite cover of degree d such that

(1) $\widetilde{\mathcal{C}} \times_{\mathcal{P}} \mathcal{P}^{\mathrm{sm}}$ is smooth;
(2) $\widetilde{\phi}$ is étale over $\mathcal{P} \setminus \Sigma$; and
(3) $\widetilde{\phi}$ corresponds to a representable classifying map $\mathcal{P} \to \mathscr{A}_d$.

Assume that no two elements of Γ are isomorphic over the identity of P. Then Γ is a finite set. We have a morphism

$$\bigsqcup_{\Gamma} \operatorname{Crimp}(\widetilde{\phi}, \Sigma) \to p \times_{\mathscr{M}} \mathscr{H}^d. \tag{7.3}$$

given by

$$(\widetilde{\mathcal{C}} \times T \to \mathcal{C} \xrightarrow{\phi} \mathcal{P} \times T) \mapsto (\mathcal{P} \times T \to P \times T; \sigma \times T; \mathcal{C} \xrightarrow{\phi} \mathcal{P} \times T).$$

Recall that we have an action of $\operatorname{Aut}(\widetilde{\phi})$ on $\operatorname{Crimp}(\widetilde{\phi}, \Sigma)$. The morphism above clearly descends to a morphism

$$\bigsqcup_{\Gamma} [\operatorname{Crimp}(\widetilde{\phi}, \Sigma)/\operatorname{Aut}(\widetilde{\phi})] \to p \times_{\mathscr{M}} \mathscr{H}^d. \tag{7.4}$$

Proposition 7.7. The morphism in (7.3) is finite and surjective. The morphism in (7.4) is representable and a bijection on k-points.

Proof. The statement is true almost by design. Nevertheless, here are the details. Let $s \colon \operatorname{Spec} k \to p \times_{\mathscr{M}} \mathscr{H}^d$ be a point given by $(\mathcal{P}' \to P; \sigma; \mathcal{C} \to \mathcal{P}')$. We first check that the fiber of (7.3) over s is nonempty and forms one orbit under the group action. Let $\mathcal{C}' \to \mathcal{C}$ be the partial normalization obtained by normalizing $\mathcal{C}$ away from its nodes over the nodes of $\mathcal{P}'$. Then $(\mathcal{P}' \to P; \sigma; \mathcal{C}' \to \mathcal{P}')$ is isomorphic to some $(\mathcal{P} \to P; \sigma; \widetilde{\mathcal{C}} \to \mathcal{P})$ in Γ. Identify $\mathcal{P}'$ and $\mathcal{P}$ via an isomorphism $\mathcal{P} \xrightarrow{\sim} \mathcal{P}'$ over the identity of P.

For every choice of isomorphism $\widetilde{\mathcal{C}} \to \mathcal{C}'$ over $\mathcal{P}$, we have a point $\widetilde{\mathcal{C}} \to \mathcal{C} \to \mathcal{P}$ of $\operatorname{Crimp}(\widetilde{\mathcal{C}} \to \mathcal{P}, \Sigma)$ lying over s. Conversely, it is clear these are exactly the points in the fiber of (7.3) over s. We conclude that (7.3) is finite, surjective and (7.4) is a bijection on k points.

Finally, suppose we have a non-trivial automorphism $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ over the identity of $\mathcal{P}$ that induces an automorphism $\phi\colon \mathcal{C} \to \mathcal{C}$. Then, clearly, ϕ is non-trivial. Hence (7.4) is representable. □

Proposition 7.7 is as close as we can come to explicitly identifying the fibers of $\operatorname{br} : \mathscr{H}^d \to \mathscr{M}$. This is good enough for determining many crude properties like dimension.

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