THE THURSTON COMPACTIFICATION OF THE STABILITY MANIFOLD OF A GENERIC ANALYTIC K3 SURFACE

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Abstract. Let X be an analytic K3 surface with $\operatorname{Pic} X = 0$. We describe the closure of the Bridgeland stability manifold of X obtained using the masses of semi-rigid objects.

1. Introduction

Associated to a triangulated category $\mathcal{C}$ is the complex manifold $\operatorname{Stab}(\mathcal{C})$ whose points are the Bridgeland stability conditions on $\mathcal{C}$ [9]. Understanding the global geometry of $\operatorname{Stab}(\mathcal{C})$ is an important question with far-reaching applications. For example, when $\mathcal{C}$ is the derived category of coherent sheaves on a K3 surface, the simple connectedness of $\operatorname{Stab}(\mathcal{C})$ allows us to recover the group of auto-equivalences of $\mathcal{C}$ [7]. When $\mathcal{C}$ is the 2-Calabi–Yau category associated to a quiver, the topology of $\operatorname{Stab}(\mathcal{C})$ has implications for the word/conjugacy problems and the $K(\pi, 1)$-conjecture for the associated Artin group [11, 16].

To better understand the global geometry of a non-compact space like $\operatorname{Stab}(\mathcal{C})$, it is useful to have a compactification. There have been several (partial) compactifications in the literature; see, for example, [3, 5, 8, 10]. The goal of this paper is to completely describe the compactification constructed in [3] when $\mathcal{C}$ is the derived category of coherent sheaves on a generic analytic K3 surface.

The compactification in [3] is motivated by viewing a stability condition as a metric, and in particular by Thurston's compactification of the Teichmüller space of hyperbolic metrics on a surface. We recall the main idea. Given a stability condition σ on $\mathcal{C}$ and an object $x \in \mathcal{C}$, the *mass* of x with respect to σ, denoted by $m_\sigma(x)$, is the sum $m_\sigma(x) = \sum_i |Z_\sigma(x_i)|$, where the x_i are the σ-Harder–Narasimhan (HN) factors of x and Z_σ is the central charge of σ. To construct the compactification, we fix a set of objects S, and consider the map $m \colon \mathbf{P}\operatorname{Stab}(\mathcal{C}) = \operatorname{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^S$ given by $\sigma \mapsto [m_\sigma]$. The proposed compactification is the closure of the image of m.

Theorem 1.1. *Let X be an analytic K3 surface with $\operatorname{Pic}(X) = 0$. Let $S \subset D^b\operatorname{Coh}(X)$ be the set of semi-rigid objects. The map $m \colon \mathbf{P}\operatorname{Stab}(D^b\operatorname{Coh}(X)) \to \mathbf{P}^S$ is a homeomorphism onto its image. The image is a 2-dimensional open ball and its closure is a 2-dimensional closed ball.*

See Figure 1 for an illustration of the compactified stability space. The boundary contains a distinguished point represented by the function $\hom(\mathcal{O}_X, -)$ (red point in Figure 1). This is the mass functions of a lax stability condition in the sense of [10]. The other vertices are mass functions of lax pre-stability conditions. The other boundary points do not have such interpretation.

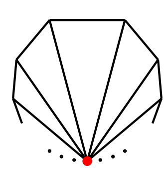

Figure 1. For an analytic K3 surface X with $\operatorname{Pic}(X) = 0$, the compactified $\mathbf{P}\operatorname{Stab}(X)$ is a closed 2-ball, tiled by the translates of a triangle by the action of the spherical twist in $\mathcal{O}_X$. A distinguished point (red) in the boundary corresponds to the function $\hom(\mathcal{O}_X, -)$.

Theorem 1.1 is a combination of Theorem 4.6 and Theorem 4.7 in the main text. The discussion of the points in the boundary is in Section 4.4.

For a positive real number q, the mass map has a natural q-analogue m_q. The closure of the image of the stability manifold under m_q is also a closed disk. However, in its boundary, the red point in Figure 1 is replaced by a closed interval (see Figure 2).

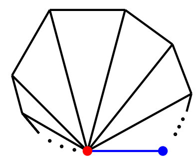

FIGURE 2. The closure of $m_q(\mathbf{P}\,\mathrm{Stab}(X))$ is also a closed disk. The boundary has an additional interval, whose blue end-point is the q-hom functional $\hom_q(\mathcal{O}_X, -)$.

For $q = 1$, the distinguished red point in the boundary has two interpretations: one as the hom function $\hom(\mathcal{O}_X, -)$ and the second as the mass function of a lax stability condition σ in which $\mathcal{O}_X$ is massless. For $q \neq 1$, the two interpretations diverge. The q-hom function $\hom_q(\mathcal{O}_X, -)$ yields the blue end-point in Figure 2 and the q-mass function $m_q(\sigma)$ yields the red end-point.

We can reconcile the two pictures (Figure 1 and Figure 2) by drawing them in the upper half plane instead of the disk (see Figure 3). The $q = 1$ picture (Figure 1) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto z + 1$. The only additional point in the closure (in the closed disk) is the point at infinity. The $q \neq 1$ picture (Figure 2) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto qz + 1$. In this case, the closure (in the closed disk) contains an additional interval. This q-deformation is a simpler version of the q-deformed Farey tesselation observed in [2].

FIGURE 3. The tiling of the disk by triangles in the $q = 1$ case (left) versus the $q \neq 1$ case (right).

In the course of the proof of the main theorem, we also characterise all semi-rigid objects of $D^b\,\mathrm{Coh}(X)$. Up to twists by $\mathcal{O}_X$ and homological shifts, the only such objects are the skyscraper sheaves $\mathbf{k}_x$ (Proposition 3.1).

There are a few other cases where the Thurston compactification of the stability manifold has been completely described. These include: the 2-Calabi–Yau categories associated to quivers of rank 2 [3] and the derived categories of coherent sheaves on algebraic curves [14]. In [15] the authors prove that for any (algebraic) K3 surface X, taking S to be the set of spherical objects gives an injective map $m\colon \mathbf{P}\,\mathrm{Stab}(X) \to \mathbf{P}^S$. Understanding its image and its closure is an important goal. The case of non-algebraic K3s treated here is a step towards it.

1.1. **Conventions.** An *analytic K3 surface* is a connected, simply-connected, and compact complex manifold X of dimension 2 with $h^1(\mathcal{O}_X) = 0$. By $D^b(X)$ we mean the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X, as studied in [12]. For a point $x \in X$, we denote by $\mathbf{k}_x$ the push-forward to X of the structure sheaf of x, and call it the *skyscraper sheaf* at x. By $\mathrm{Stab}(X)$, we denote the set of (locally finite) Bridgeland stability conditions on $D^b(X)$ with a numerical central charge; that is, where the central charge $Z\colon K(D^b(X)) \to \mathbf{C}$ factors through the Chern character $\mathrm{ch}\colon K(D^b(X)) \to H^*(X, \mathbf{Q})$. We let $\mathbf{P}\,\mathrm{Stab}(X)$ be the quotient of $\mathrm{Stab}(X)$ by the standard action of $\mathbf{C}$, in which $z = x + i\pi y$ acts by scaling the central charge by e^z and shifting the slicing by y. Given a set S, we let $\mathbf{R}^S$ be the space of functions $S \to \mathbf{R}$ with the product topology and $\mathbf{P}^S$ the projective space $\left(\mathbf{R}^S - \{0\}\right)/\text{scaling}$.

1.2. **Outline.** In Section 2, we recall the description of stability conditions on an analytic K3 surface X with $\operatorname{Pic} X = 0$. In Section 3, we identify the semi-rigid objects of $D^b(X)$. The bulk of the paper is Section 4, in which we study the embedding of $\mathbf{P}\operatorname{Stab}(X)$ given by the masses of semi-rigid objects. In Section 5, we study the q-analogue of the mass embedding. We do not include the definitions and the basic properties of stability conditions, and refer the reader to the original source [9] or exposition [6].

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2. Stability conditions on generic K3 surfaces

Throughout, fix an analytic K3 surface X with $\operatorname{Pic} X = 0$. Since X is a K3 surface, $D^b(X)$ is a 2-Calabi–Yau category. That is, for $x, y \in D^b(X)$, we have a natural isomorphism

$$\operatorname{Hom}(x, y) \cong \operatorname{Hom}(y, x[2]).$$

2.1. **The Mukai lattice.** The Mukai lattice $\mathcal{N}(X)$ of X is given by

$$\mathcal{N}(X) = (H^0 \oplus H^4)(X, \mathbf{Z}).$$

Taking the class of X as a generator of the H^0 summand and the class of a point $x \in X$ as a generator of the H^4 summand, we get an identification

$$\mathcal{N}(X) = \mathbf{Z} \oplus \mathbf{Z}.$$

The Mukai pairing is then given by

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) = \alpha_1\beta_2 + \alpha_2\beta_1.$$

Given $F \in D^b(X)$, we let $[F] = (\operatorname{ch}_0 F, \operatorname{ch}_0 F - \operatorname{ch}_2 F) \in \mathcal{N}(X)$ be its Mukai vector. Then we have

$$[\mathcal{O}_X] = (1, 1) \text{ and } [\mathbf{k}_x] = (0, 1).$$

In particular, $[\mathcal{O}_X]$ and $[\mathbf{k}_x]$ form a basis of $\mathcal{N}(X)$.

2.2. **Standard stability conditions.** We recall basic facts about stability conditions on X from [12, § 4]. Let $\mathcal{F}$ and $\mathcal{T}$ be the full-subcategories of $\operatorname{Coh}(X)$ consisting of torsion free and torsion sheaves, respectively. Then $(\mathcal{F}, \mathcal{T})$ forms a torsion pair. Let $\mathcal{A}$ be the tilt of $\operatorname{Coh}(X)$ in this torsion pair. Explicitly,

$$\mathcal{A} = \{E \in D^b(X) \mid H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \text{ and for all } i \notin \{0, 1\} : H^i(E) = 0\}.$$

Then $\mathcal{A}$ is the heart of a bounded t-structure on $D^b(X)$.

Let $\mathbf{H} \subset \mathbf{C}$ be the (open) upper half plane. As proved in [12, § 4.2], for every $z \in \mathbf{H} \cup \mathbf{R}_{<0}$, we have a stability condition σ_z on $D^b(X)$ whose $(0, 1]$ heart is $\mathcal{A}$ and whose central charge is given by

$$Z\colon [\mathbf{k}_x] \mapsto -1 \text{ and } Z\colon [\mathcal{O}_X] \mapsto -z.$$

For every $w \in -\mathbf{H}$, we have a stability condition σ_w on $D^b(X)$ whose $(0, 1]$ heart is $\operatorname{Coh}(X)$ and whose central charge is given by

$$Z\colon [\mathbf{k}_x] \mapsto -1 \text{ and } Z\colon [\mathcal{O}_X] \mapsto -w.$$

See Figure 4 for a sketch of the two central charges.

Remark 2.1. The combined domain of the parameters z and w in [12, § 4.2] is $\mathbf{C} - \mathbf{R}_{\geq -1}$. For us, it is $\mathbf{C} - \mathbf{R}_{\geq 0}$. The difference is due to a slight change in parametrisation. The central charge of σ_z in [12, § 4.2] sends $\mathbf{k}_x$ to -1 (same as ours) and $\mathcal{O}_X$ to $-z-1$ (we send it to $-z$). So our parametrisation and the parametrisation in [12, § 4.2] are related by $z \mapsto z + 1$.

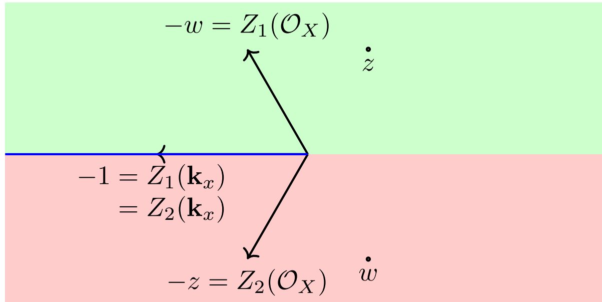

FIGURE 4. For $w \in -\mathbf{H}$ (red), a central charge Z_1 as above defines a stability condition with heart $\operatorname{Coh}(X)$. For $z \in \mathbf{H}$ (green) and $z \in \mathbf{R}_{<0}$ (blue), a central charge Z_2 as above defines a stability condition whose heart is the tilt of $\operatorname{Coh}(X)$ with respect to torsion and torsion-free sheaves.

We call the stability conditions σ_z for $z \in \mathbf{H} \cup -\mathbf{H} \cup \mathbf{R}_{<0}$ the *standard stability conditions*. We say that the stability conditions σ_z for $z \in \mathbf{R}_{<0}$ are *on the wall*, and the rest are *off the wall*.

Let W_+ (resp. W_- and W_0) be the union of the $\mathbf{C}$-orbits of the stability conditions σ_z for $z \in \mathbf{H}$ (resp. $-\mathbf{H}$ and $\mathbf{R}_{<0}$). By definition, the sets W_+, W_-, and W_0 are invariant under the $\mathbf{C}$-action. It is easy to check that they are also invariant under the $\widehat{\mathrm{GL}}_2^+(\mathbf{R})$-action, and hence coincide with the sets with the same name defined in the proof of [12, Theorem 4.8]. Set $W = W_+ \cup W_- \cup W_0$.

2.3. **All stability conditions.** Recall that the only spherical objects in $D^b(X)$ are the shifts of $\mathcal{O}_X$ (see [12, Proposition 2.15]). Let $T \colon D^b(X) \to D^b(X)$ be the spherical twist in $\mathcal{O}_X$.

Proposition 2.2. *The set $W \subset \operatorname{Stab}(X)$ is open and the union of its translates T^nW, for $n \in \mathbf{Z}$, is* $\operatorname{Stab}(X)$.

Proof. That W is open is proved in [12, Theorem 4.8]. That $\operatorname{Stab}(X) = \bigcup T^n W$ is [12, Corollary 4.7]. $\square$

The following proposition allows us to identify the stability conditions in W_+, W_-, and W_0. Recall that since, up to shifts, $\mathcal{O}_X$ is the only spherical object, it must be stable in any stability condition [12, Proposition 2.15].

Proposition 2.3. *Let σ be a stability condition and let ϕ be the phase of $\mathcal{O}_X$. Then σ is in W if and only if all the skyscraper sheaves $\mathbf{k}_x$ are σ-stable of the same phase ψ. In this case, we have*

(1) $\sigma \in W_-$ *if* $\psi \in (\phi, \phi + 1)$,
(2) $\sigma \in W_+$ *if* $\psi \in (\phi + 1, \phi + 2)$,
(3) $\sigma \in W_0$ *if* $\psi = \phi + 1$.

Proof. Since all skyscraper sheaves $\mathbf{k}_x$ are σ-stable of the same phase for a standard stability condition, the same is true for any $\sigma \in W$. Conversely, suppose all $\mathbf{k}_x$ are σ-stable of the same phase. Using the $\mathbf{C}$-action, assume that their phase ψ is 1 and their central charge is -1. By [12, Proposition 4.6], we conclude that σ is standard.

Suppose $\sigma = \sigma_z$ for $z \in -\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$. Whether $z \in -\mathbf{H}$ or $\mathbf{H}$ or $\mathbf{R}_{<0}$ is distinguished by the phase ϕ of $\mathcal{O}_X$. For $z \in -\mathbf{H}$, we have $\phi \in (0, 1)$; for $z \in \mathbf{H}$, we have $\phi \in (-1, 0)$; and for $z \in \mathbf{R}_{<0}$, we have $\phi = 0$. $\square$

Proposition 2.4. *We have $TW_+ = W_-$ and $T^{-1}W_- = W_+$.*

Proof. We prove that for a standard $\sigma \in W_-$, we have $T(\sigma) \in W_+$, and for a standard $\sigma \in W_+$, we have $T^{-1}(\sigma) \in W_-$. Then the proposition follows.

Take a standard $\sigma \in W_-$ and let us prove that $T^{-1}(\sigma) \in W_+$. Let $\phi \in (0, 1)$ be the phase of $\mathcal{O}_X$. It is easy to check that the ideal sheaves I_x of points $x \in X$ are σ-stable of the same phase $\psi \in (0, \phi)$.

Let $x \in X$ be any point. Since $\mathrm{Hom}^*(\mathcal{O}_X, \mathbf{k}_x) = \mathbf{C}$, we have the exact triangle

$$\mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathbf{k}_x \to T\mathbf{k}_x \xrightarrow{+1} .$$

Therefore, $T\mathbf{k}_x = I_x[1]$. So $T\mathbf{k}_x$ is σ-stable of phase $\psi + 1$. Therefore, $T^{-1}I_x[1] = \mathbf{k}_x$ is $T^{-1}(\sigma)$-stable of phase $\psi + 1 \in (1, \phi + 1)$. On the other hand, $T^{-1}\mathcal{O}_X = \mathcal{O}_X[1]$ is $T^{-1}(\sigma)$-stable of phase ϕ, so $\mathcal{O}_X$ is $T^{-1}(\sigma)$-stable of phase $\phi - 1$. We now apply Proposition 2.3.

Now take a standard $\sigma \in W_+$ and let us prove that $T(\sigma) \in W_-$. Let $\phi \in (-1, 0)$ be the phase of $\mathcal{O}_X$. The objects $T^{-1}\mathbf{k}_x$ are σ-stable of phase $\psi \in (\phi + 1, 1)$ (see [12, Remark 4.3 (i)]). Therefore, the skyscraper sheaves $\mathbf{k}_x$ are $T(\sigma)$-stable of phase $\psi \in (\phi + 1, 1)$. Since $\mathcal{O}_X$ is σ-stable of phase ϕ, it is $T(\sigma)$-stable of phase $\phi + 1$. We again apply Proposition 2.3. □

We now turn to the topology of the set of standard stability conditions and the stability conditios in W. Let $H \subset \mathrm{Stab}(X)$ be the set of standard stability conditions. Let $R = \mathbf{C} \setminus \mathbf{R}_{\geq 0}$. We have a map $R \to H$ given by $z \mapsto \sigma_z$. We also have the projection map $H \to \mathbf{P}W = W/\mathbf{C}$.

Proposition 2.5. *The maps $R \to H$ and $H \to \mathbf{P}W$ are homeomorphisms.*

Proof. By definition, the map $R \to H$ is a bijection. By the proof of [12, Theorem 4.8] (part (ii)), the map $R \to H$ is continuous. Its inverse is given by $\sigma \mapsto -Z_\sigma(\mathcal{O}_X)$, which is also continuous. So $R \to H$ is a homeomorphism.

By Proposition 2.3, the map $H \to \mathbf{P}W$ is surjective. Owing to the normalisation of the phase and mass of $\mathbf{k}_x$, it is also injective. It remains to prove that the inverse is continuous. We know that W is an open subset of $\mathrm{Stab}(X)$. It is also $\mathbf{C}$-invariant, so $\mathbf{P}W$ is an open subset of $\mathbf{P}\,\mathrm{Stab}(X)$. Thus, the map $\mathbf{P}W \to \mathbf{P}\,\mathrm{Hom}(\mathcal{N}(X), \mathbf{C})$ is a local homeomorphism. We have the commutative diagram

$$\begin{tikzcd} R & H \arrow[l, "\sim"'] & \mathbf{P}W \arrow[l] \arrow[d] \\ R \arrow[u, equal] & & \mathbf{P}\,\mathrm{Hom}(\mathcal{N}(X), \mathbf{C}), \arrow[ll] \end{tikzcd}$$

where the bottom map is given by $Z \mapsto Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$. Since this map is continuous, it follows that $\mathbf{P}W \to H$ is continuous. □

3. Semi-rigid objects

Recall that an object F in $D^b(X)$ is *semi-rigid* if

$$\hom^i(F, F) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $x \in X$, the skyscraper sheaf $F = \mathbf{k}_x$ and the ideal sheaf $F = I_x$ are semi-rigid. We now characterise the semi-rigid objects of $D^b(X)$. Recall that $T\colon D^b(X) \to D^b(X)$ is the spherical twist in $\mathcal{O}_X$.

Proposition 3.1. *Let X be a K3 surface with* $\mathrm{Pic}\,X = 0$. *Let $F \in D^b(X)$ be semi-rigid. Then there exists $x \in X$ and integers m, n such that $F \cong T^n\mathbf{k}_x[m]$.*

We split the proof in two lemmas.

Lemma 3.2. *Fix a standard stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be semi-rigid and σ-semi-stable. Then there exists $x \in X$ such that F or $T^{-1}F$ is a shift of $\mathbf{k}_x$.*

Proof. Since F is semi-rigid, $[F] \cdot [F] = 0$ in $\mathcal{N}(X)$. So $[F]$ is an integer multiple of $(0, 1)$ or $(1, 0)$.

Suppose $[F]$ is a multiple of $(0, 1)$. Since $[\mathbf{k}_x] = (0, 1)$, after applying a shift, we may assume that F is σ-semi-stable of the same phase as $\mathbf{k}_x$, namely 1. It is easy to check that the abelian category of σ-semi-stable objects of phase 1 is $\mathcal{F}$, the category of torsion sheaves on X. It is a finite length category whose simple objects are the skyscraper sheaves $\mathbf{k}_x$. So F is an iterated extension of skyscraper sheaves. Since $\hom^1(F,F)=2$, the Mukai lemma [12, Lemma 2.7] implies that F must simply be a skyscraper sheaf.

Suppose $[F]$ is a multiple of $(1,0)$. Then $[T^{-1}F]$ is a multiple of $(0,1)$ and $T^{-1}F$ is semi-stable with respect to $\tau = T^{-1}\sigma$. By Proposition 2.4, we have $\tau \in W_+$. By applying a rotation, assume that τ is standard. Then, after applying a shift, we may assume that $T^{-1}F$ is τ-semi-stable of the same phase as $\mathbf{k}_x$, namely 1. Again, it is easy to check that the abelian category of τ-semi-stable objects of phase 1 is $\mathcal{F}$. We now proceed as before. □

Given a stability condition σ, denote by ϕ^+_σ and ϕ^-_σ the highest and lowest phases of the factors in the σ-HN filtration. If σ is clear from the context, we omit the subscript.

Lemma 3.3. *Fix a standard stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be a semi-rigid object. There exists a non-negative integer n such that T^nF is σ-semi-stable.*

Proof. Since F is semi-rigid, all stable factors of F are either spherical or semi-rigid, and only one stable factor is semi-rigid [12, Proposition 2.9]. The only spherical object, up to shift, is $\mathcal{O}_X$. By Lemma 3.2, the only semi-stable semi-rigid objects, up to shift, are $\mathbf{k}_x$ and $T^{-1}\mathbf{k}_x$. In particular, the phases of the HN factors of F lie in the discrete subset of $\mathbf{R}$ given by

$$(\phi_\sigma(\mathcal{O}_X)+\mathbf{Z}) \cup (\phi_\sigma(\mathbf{k}_x)+\mathbf{Z}) \cup \left(\phi_\sigma(T^{-1}\mathbf{k}_x)+\mathbf{Z}\right).$$

Therefore, there exists a discrete $\Phi \subset \mathbf{R}$ such that for every semi-rigid object F, we have

$$\phi^+(F) - \phi^-(F) \in \Phi.$$

If F itself is semi-stable, we simply take $n=0$. Otherwise, up to shift, a stable HN factor of F of highest or lowest phase must be $\mathcal{O}_X$. We apply [4, Theorem 3.5] with $Y=F$ and $X=\mathcal{O}_X$. Then for $F' = TF$ or $F' = T^{-1}F$, we have

$$\phi^+(F') - \phi^-(F') < \phi^+(F) - \phi^{-1}(F).$$

By repeated applications of [4, Theorem 3.5] and using that $\phi^+ - \phi^-$ lies in the discrete set $\Phi \subset \mathbf{R}$, we conclude that there exists an integer n such that T^nF is semi-stable. □

Having proved the two lemmas, we are ready to prove Proposition 3.1—the only semi-rigid objects of $D^b(X)$, up to twisting by $\mathcal{O}_X$ and shifting, are the skyscraper sheaves $\mathbf{k}_x$.

Proof of Proposition 3.1. Combine Lemma 3.2 and Lemma 3.3. □

4. The mass embedding

Recall that X is an analytic K3 surface with $\operatorname{Pic} X = 0$. Let S be the set of isomorphism classes of semi-rigid objects of $D^b(X)$. In this section, we describe the mass embedding

$$m\colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S$$

and the closure of its image.

4.1. HN filtration of semi-rigid objects.

To understand the mass embedding, we must understand the HN filtrations of the objects of S. By Proposition 3.1, the objects of S, up to shift, are $T^n\mathbf{k}_x$ for $x \in X$ and $n \in \mathbf{Z}$. For points $x, y \in X$, the behaviour of $T^n\mathbf{k}_x$ and $T^n\mathbf{k}_y$ is entirely analogous to each other. So we lose nothing by fixing a particular point $x \in X$ and taking

$$S = \{T^n\mathbf{k}_x \mid n \in \mathbf{Z}\}.$$

We may then write the points of $\mathbf{P}^S$ as homogeneous vectors $[x_n \mid n \in \mathbf{Z}] = [\cdots : x_{-1} : x_0 : x_1 : \cdots]$. In these coordinates, the spherical twist T acts as a shift.

We first treat HN filtrations with respect to off the wall stability conditions.

Proposition 4.1. *Let $\sigma \in W_-$. Then the σ-HN factors of $F = T^n\mathbf{k}_x$, in decreasing order of phase, are as follows.*

(1) *For $n = 0$ and 1, the object F is stable.*
(2) *For $n \geq 2$, the semi-stable (= stable) factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$.*
(3) *For $n \leq -1$, the semi-stable (= stable) factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and $\mathbf{k}_x$.*

Proof. Recall that $\mathbf{k}_x$ and $T\mathbf{k}_x = I_x[1]$ are stable for stability conditions in W_-. So (1) follows.

Consider the triangle

$$\operatorname{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) \otimes \mathcal{O}_X \to T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x \xrightarrow{+1} . \tag{1}$$

We have

$$\begin{aligned}\operatorname{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) &= \operatorname{Hom}^*(T^{-n+1}\mathcal{O}_X, \mathbf{k}_x)\\ &= \operatorname{Hom}^*(\mathcal{O}_X[n-1], \mathbf{k}_x)\\ &= \mathbf{C}[-n+1].\end{aligned}$$

By substituting in (1) and shifting, we get

$$T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x \to \mathcal{O}_X[-n+2] \xrightarrow{+1} . \tag{2}$$

Let us assume $n \geq 2$, and induct on n. Assume we know that the HN factors of $T^{n-1}\mathbf{k}_x$ (in decreasing order of phase) are $T\mathbf{k}_x$ followed by $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+3$. Concatenating the HN filtration of $T^{n-1}\mathbf{k}_x$ and the map $T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$. Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete.

Now let us assume $n \leq -1$, and induct on $-n$. Consider the triangle

$$\mathcal{O}_X[-n] \to T^n\mathbf{k}_x \to T^{n+1}\mathbf{k}_x \xrightarrow{+1}, \tag{3}$$

obtained by replacing n by $n+1$ in (2) and shifting. Assume we know that the HN factors of $T^{n+1}\mathbf{k}_x$ (in decreasing order of phase) are $\mathcal{O}_X[i]$ for $-n-1 \geq i \geq 1$ and $\mathbf{k}_x$. By augmenting the HN filtration of $T^{n+1}\mathbf{k}_x$ by the map $\mathcal{O}_X[-n] \to T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and $\mathbf{k}_x$. Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete. □

For stability conditions on the wall, the HN filtration degenerates as expected.

Proposition 4.2. *Let $\sigma \in W_0$. Then the σ-HN factors of $F = T^n\mathbf{k}_x$, in decreasing order of phase, are as follows.*

(1) *For $n = -1$, 0 and 1, the object F is semi-stable.*
(2) *For $n \geq 2$, the semi-stable factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$.*
(3) *For $n \leq -2$, the semi-stable factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 2$ and $T^{-1}\mathbf{k}_x$.*

Proof. The proof is analogous to the proof of Proposition 4.1. □

4.2. **The mass map.** We now have the tools to describe the mass map

$$m \colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S.$$

Proposition 4.3. *Let $\sigma \in \mathbf{P}W_-$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $b = |Z_\sigma(T\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$.*

(1) *The numbers a, b, c are positive real numbers satisfying*

$$b < a+c, \quad a < b+c, \quad c < a+b.$$

(2) *We have*

$$m_\sigma \colon T^n\mathbf{k}_x \mapsto \begin{cases} a - nc & \text{if } n \leq 0,\\ b + (n-1)c & \text{if } n \geq 1.\end{cases}$$

(3) *Let $\Delta_0 \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form*

$$[\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots],$$

where a is at index 0 and b is at index 1, and where a, b, c are positive real numbers satisfying the inequalities in (1). Then $m \colon \mathbf{P}W_- \to \Delta_0$ is a homeomorphism.

Proof. Part (1) follows from the fact that the classes of $\mathcal{O}_X$, $\mathbf{k}_x$, and $T\mathbf{k}_x$ satisfy

$$[\mathcal{O}_X] = [\mathbf{k}_x] - [T\mathbf{k}_x].$$

Part (2) follows from Proposition 4.1.

For part (3), let $\Delta \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ that satisfy the conditions in (1). Then we have a homeomorphism $\Delta \to \Delta_0$ given by

$$[a : b : c] \mapsto [\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots].$$

We use $[a : b : c] \in \Delta$ as coordinates on Δ_0. By Proposition 2.5, the map $w \mapsto \sigma_w$ gives a homeomorphism $-\mathbf{H} \to \mathbf{P}W_-$. We use $w \in -\mathbf{H}$ as a coordinate on $\mathbf{P}W_-$. In these coordinates, writing down the inverse map $\omega \colon \Delta \to \mathbf{P}W_-$ amounts to re-constructing the central charge given a, b, c. This can be done using the cosine rule (see Figure 5). Precisely, we have

$$\omega([a : b : c]) = -(b/a \exp(i\theta) - 1), \text{ where } \theta = \arccos\left(\frac{a^2+b^2-c^2}{2ab}\right) \in (0, \pi), \tag{4}$$

which is continuous. $\square$

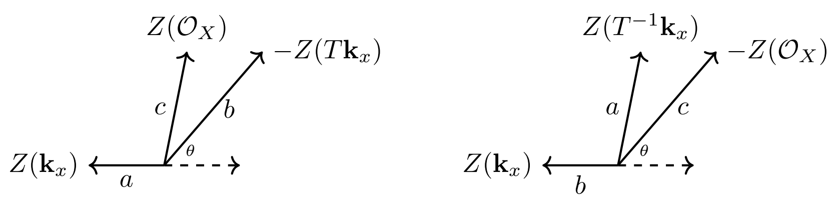

FIGURE 5. We can use the cosine rule to reconstruct the central charge of a standard $\sigma \in W_-$ from the masses a, b, c of $\mathbf{k}_x, T\mathbf{k}_x, \mathcal{O}_X$ (left) and of $\sigma \in W_+$ from the masses a, b, c of $T^{-1}\mathbf{k}_x, \mathbf{k}_x, \mathcal{O}_X$ (right).

For $n \in \mathbf{Z}$, let $\Delta_n \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form

$$[\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots],$$

where a is at index n, and where a, b, c are positive real numbers satisfying the (strict) triangle inequalities. Denote by $T \colon \mathbf{P}^S \to \mathbf{P}^S$ the map that shifts the homogeneous coordinates rightwards by 1, so that $\Delta_n = T^n \Delta_0$. Recall that we also denote by $T \colon \operatorname{Stab}(X) \to \operatorname{Stab}(X)$ the action of the spherical twist by $\mathcal{O}_X$. We have

$$m(T(\sigma)) = T(m(\sigma)).$$

Proposition 4.3 implies that the mass map $T^n\mathbf{P}W_- \to \Delta_n$ is a homeomorphism. In particular, the mass map $T^{-1}\mathbf{P}W_- = \mathbf{P}W_+ \to \Delta_{-1}$ is a homeomorphism. It is useful to write the inverse $\Delta_{-1} \to \mathbf{P}W_+$ using coordinates $[a : b : c]$ on Δ_{-1} as in the proof of Proposition 4.3 and the coordinate on W_+ given by $z \in \mathbf{H}$. Recall that the $[a : b : c]$ coordinates represent $a = m(T^{-1}\mathbf{k}_x)$ and $b = m(\mathbf{k}_x)$ and $c = m(\mathrm{O}_X)$. Then the map $[a : b : c] \mapsto z$ is (see Figure 5):

$$[a : b : c] \mapsto c/b \exp(i\theta), \text{ where } \theta = \arccos\left(\frac{b^2+c^2-a^2}{2bc}\right) \in (0, \pi). \tag{5}$$

Let $I_0 \subset \mathbf{P}^S$ be the set of points of the form

$$[\cdots : a+2c : a+c : a : a+c : a+2c : \cdots],$$

where a is at index 0 and a, c are positive real numbers.

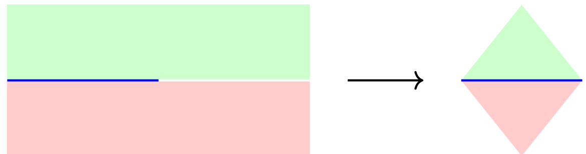

FIGURE 6. The mass map gives a homeomorphism from the set of standard stability conditions parametrised by $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$ and the union of two open triangles and a segment that forms a wall between them.

Proposition 4.4. *Let $\sigma \in \mathbf{P}W_0$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$. Then*

$$m_\sigma \colon T^n\mathbf{k}_x \mapsto a + |n|c.$$

Furthermore, the map $m\colon \mathbf{P}W_0 \to I_0$ is a homeomorphism.

Proof. The description of m_σ follows from Proposition 4.2. The inverse of $m\colon \mathbf{P}W_0 \to I_0$ is given using the central charge $Z(\mathbf{k}_x) = -1$ and $Z(\mathcal{O}_X) = c/a$. □

Proposition 4.5. *The map $m\colon \mathbf{P}W \to \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a homeomorphism.*

See Figure 6 for a sketch.

Proof. The set $\mathbf{P}W$ is the disjoint union of $\mathbf{P}W_-$, $\mathbf{P}W_+$, and $\mathbf{P}W_0$. The sets Δ_0, I_0, and Δ_{-1} are also disjoint. Furthermore, the maps $\mathbf{P}W_- \to \Delta_0$, $\mathbf{P}W_+ \to \Delta_{-1}$, and $\mathbf{P}W_0 \to I_0$ are homeomorphisms. So $m\colon \mathbf{P}W \to \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a continuous bijection.

We check that the inverse is continuous. Since $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0} \to \mathbf{P}W$ is a homeomorphism, we use the former as local coordinates for $\mathbf{P}W$. Let $\overline{\Delta} \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b \le a + c, \quad a < b + c, \quad c < a + b.$$

It is easy to check that the map $\overline{\Delta} \to \Delta_0 \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_0 \cup I_0$. Using (4), we see that the inverse map $\Delta_0 \cup I_0 \to -\mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto -b/a \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \in [0, \pi),$$

which is continuous.

Let $\overline{\Delta}' \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b < a + c, \quad a \le b + c, \quad c < a + b.$$

Then the map $\overline{\Delta}' \to \Delta_{-1} \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_{-1} \cup I_0$. Using (5), we see that the inverse map $\Delta_{-1} \cup I_0 \to \mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto c/b \exp(i\theta), \text{ where } \theta = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \in (0, \pi],$$

which is continuous.

Since the inverse is continuous on $\Delta_0 \cup I_0$ and $\Delta_{-1} \cup I_0$, we conclude that it is continuous on $\Delta_0 \cup \Delta_{-1} \cup I_0$. □

Let $D \subset \mathbf{P}^S$ be the union of the triangles Δ_n for $n \in \mathbf{Z}$ and the intervals I_n for $n \in \mathbf{Z}$.

Theorem 4.6. *The mass map gives a homeomorphism $m\colon \mathbf{P}\operatorname{Stab}(X) \to D$.*

Proof. By Proposition 2.2 and Proposition 2.4, we see that $\mathbf{P}\operatorname{Stab}(X)$ is the union of $T^n\mathbf{P}W_-$ for $n \in \mathbf{Z}$ and $T^n\mathbf{P}W_0$ for $n \in \mathbf{Z}$. From Proposition 2.3, it follows that this is a disjoint union. Likewise, D is the disjoint union of Δ_n for $n \in \mathbf{Z}$ and I_n for $n \in \mathbf{Z}$. Since $m\colon \mathbf{P}W_- \to \Delta_0$ and $m\colon \mathbf{P}W_0 \to I_0$ are bijections, we conclude that $m\colon \mathbf{P}\operatorname{Stab}(X) \to D$ is a bijection. It is also continuous. It remains to prove that the inverse is continuous.

Let $U = \Delta_0 \cup I_0 \cup \Delta_{-1}$. Observe that

$$U = \{[a_n] \in D \mid 2a_0 < a_1 + a_{-1}\}.$$

So $U \subset D$ is open. From Proposition 4.5, we know that the inverse of m is continuous on U. But T^nU for $n \in \mathbf{Z}$ form an open cover of D. So the inverse of m is continuous on D. □

4.3. **Identifying the image and its closure.** Let $\overline{D} \subset \mathbf{P}^S$ be the closure of D. Our next goal is to identify the homeomorphism classes of $\overline{D}$ and D. To do so, it will be useful to work with an auxiliary space, which we now define.

Let $I \subset \mathbf{P}^\mathbf{1}$ be the set of $[v : w]$ where v, w are non-negative real numbers. Then I is homeomorphic to a closed interval. Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ be the two point compactification of $\mathbf{R}$, also homeomorphic to a closed interval. Our auxiliary space will be $\overline{\mathbf{R}} \times I$.

Define the transformation T on $\overline{\mathbf{R}} \times I$ by

$$T\colon (u, [v, w]) \mapsto (u + 1, [v : w]).$$

Recall that we also denote by T the action of the spherical twist by $\mathcal{O}_X$ on $\mathbf{P}\operatorname{Stab}(X)$ and the rightward shift by 1 on $\mathbf{P}^S$. (We intentionally use the same letter T to denote these maps, which are related.) Our eventual goal is to understand $\overline{D}$ via a T-equivariant parametrisation

$$\pi\colon \overline{\mathbf{R}} \times I \to \overline{D}.$$

For $n \in \mathbf{Z}$, set

$$P_n = (\cdots, 2, 1, 0, 1, 2, \cdots) \in \mathbf{R}^S,$$

where the 0 is at index n. Note that $P_n = T^nP_0$. Set

$$Q = (\cdots, 1, 1, 1, \cdots) \in \mathbf{R}^S.$$

Observe that P_{-1}, P_0, and Q are the three vertices of the closure $\overline{\Delta}_0$ of the triangle $\Delta_0 \subset \mathbf{P}^S$, which is the homeomorphic image of $\mathbf{P}W_-$. The three sides of $\overline{\Delta}_0$ are the line segments $P_{-1}P_0$, $P_{-1}Q$, and P_0Q. The open line segment P_0Q is the homeomorphic image of $\mathbf{P}W_0$. The entire picture is T-invariant, so the discussion above holds with $-1, 0, 1$ replaced by $n - 1, n, n + 1$ for any $n \in \mathbf{Z}$.

Consider the map $\pi\colon [0, 1] \times I \to \mathbf{P}^S$ defined by

$$\pi(u, [v : w]) = (1 - u)(wQ + vP_0) + u(wQ + vP_1).$$

Note that for $u = 0$ (resp. $u = 1$), the map π is a homeomorphism onto the closure of I_0 (resp. I_1), which are the two sides P_0Q and P_1Q of the triangle $\overline{\Delta}_0$. For $0 < u < 1$, the map π linearly interpolates between the two end-points $\pi(0, [v : w])$ and $\pi(1, [v : w])$, and hence its image is $\overline{\Delta}_0$. In fact, it is easy to check that the map

$$\pi\colon [0, 1] \times (I - \{[0 : 1]\}) \to \overline{\Delta}_0 - \{[\cdots 1 : 1 : 1 : \cdots]\}$$

is a homeomorphism, and π sends the entire segment $[0, 1] \times [0 : 1]$ to the point $[\cdots 1 : 1 : 1 : \cdots]$. Note that, with the T actions as before, we have

$$T\pi(0, [v : w]) = \pi T(0, [v : w]).$$

Thus, π extends to a unique T-equivariant continuous map

$$\pi\colon \mathbf{R} \times I \to \mathbf{P}^S.$$

Explicitly, for $x = n + u$, where $n \in \mathbf{Z}$ and $u \in [0, 1)$, we have

$$\pi(x, [v : w]) = wQ + (1 - u)vP_n + uvP_{n+1}.$$

Note that in $\mathbf{R}^S$ we have

$$\lim_{n\to\pm\infty} \frac{1}{n} P_n = (\cdots, 1, 1, 1, \cdots).$$

Extend π to $\{\pm\infty\} \times I$ by setting

$$\pi(\pm\infty, [v : w]) = [\cdots : 1 : 1 : 1 : \cdots].$$

Then, using the limit computation above, it is easy to check that π_q is continuous.

Theorem 4.7. *The map $\pi\colon \overline{\mathbf{R}} \times I \to \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{\pm\infty\} \times I \cup \overline{\mathbf{R}} \times \{[0 : 1]\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0,1] \to B$ be the contraction of C to a point. Then the induced map $\pi\colon B \to \mathbf{P}^S$ is a homeomorphism onto $\overline{D} = \overline{m(\mathbf{P}\,\mathrm{Stab}(X))}$.

Note that B is homeomorphic to a closed disk. See Figure 7 for a sketch.

Proof. We have seen that π is continuous. It is easy to check that it is injective on the complement of C, and its image on the complement of C does not include the point $[\cdots : 1 : 1 : 1 : \cdots]$. It evidently sends all points of C to $[\cdots : 1 : 1 : 1 : \cdots]$. So it induces a continuous injective map $\pi\colon B \to \mathbf{P}^S$. Since B is compact and $\mathbf{P}^S$ is Hausdorff, π maps B homeomorphically onto its image. By construction, π maps the interior of $\overline{\mathbf{R}} \times I$ to D. So $\pi(B)$ must be $\overline{D}$. $\square$

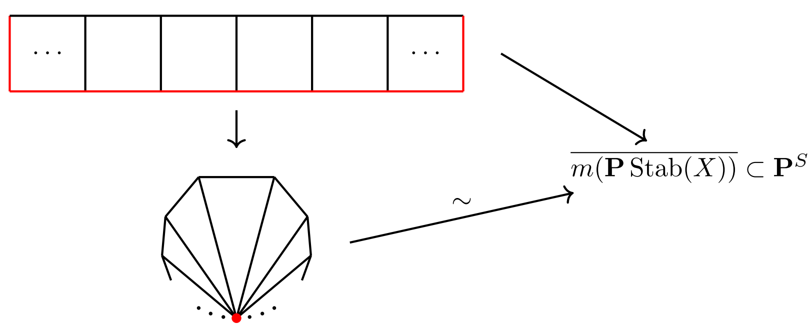

FIGURE 7. The map $\pi\colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\mathrm{Stab}(X)$. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0,1]$ by collapsing three sides (red). The $\mathbf{Z}$-indexed decomposition of the image into triangles corresponds to the translates of a fundamental domain of $\mathbf{P}\,\mathrm{Stab}(X)$ by the spherical twist T.

4.4. **Points of the boundary.** Observe that $\overline{D}$ contains the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. This is the common vertex (drawn in red in Figure 7) of all the triangles that tile $\overline{D}$. It is the unique T-invariant point of $\overline{D}$. This point is precisely the projectivised hom function $\hom(\mathcal{O}_X, -)$, whose value on $T^n\mathbf{k}_x$ for any $n \in \mathbf{Z}$ is

$$\dim \mathrm{Hom}^*(\mathcal{O}_X, T^n\mathbf{k}_x) = 1.$$

The fact that $\bullet$ is in the boundary is a reflection of the following more general fact.

Theorem 4.8 ([3, Corollary 4.13])**.** *Let a be a spherical object of a triangulated category $\mathcal{C}$, and assume that it is a stable object of a stability condition σ. Let S be a set of objects of $\mathcal{C}$ such that no object in S has an endomorphism of negative degree. For simplicity, also assume that no shift of a is in S. Let T be the spherical twist in a. Then, in $\mathbf{P}^S$, we have the equality*

$$\lim_{n\to\pm\infty} T^n[m_\sigma] = [\hom(a, -)].$$

The point $\bullet$ also has an interpretation as the mass function of a lax stability condition in the sense of Broomhead, Pauksztello, Ploog, and Woolf [10]. We quickly recall the main features of the definition. A *lax stability condition* is a slicing P and a compatible central charge Z. The central charge is allowed to vanish on the classes of non-zero semi-stable objects (such objects are called "massless"). The pair (P, Z) must satisfy the following two finiteness conditions:

(1) The slicing P is locally finite.
(2) The central charge satisfies the support property. That is, for a choice of a norm $\| - \|$ on $\mathcal{N}(X)$, there exists a positive constant c such that for every massive stable object s, we have $|Z(s)|/\|s\| > c$.

A pair (P, Z) that satisfies the first condition is called a *lax pre-stability condition.*

Recall that $\mathcal{A}$ is the tilt of $\operatorname{Coh} X$ in the torsion pair defined by torsion and torsion-free sheaves. We let P to be the slicing defined by $P(1) = \mathcal{A}$ and $P(\phi) = 0$ for $\phi \in (0, 1)$. The simple objects of $P(1)$ are the skyscraper sheaves $\mathbf{k}_x$ and the objects $E[1]$, where E is a vector bundle of rank on X with no non-trivial sub-bundles (see [12, Remark 4.3 (iii)]). We let $Z(\mathcal{O}_X) = 0$ and $Z(\mathbf{k}_x) = -1$.

Proposition 4.9. *The pair (P, Z) as above defines a lax stability condition σ that is a limit of standard stability conditions. Furthermore, $m(\sigma) = [\cdots : 1 : 1 : 1 : \cdots]$.*

Proof. It is easy to check that the abelian category $\mathcal{A}$ is of finite length (Noetherian and Artinian). So the slicing is locally finite. Let E be a vector bundle with no non-trivial sub-bundles, and let $[E] = r[\mathcal{O}_X] + m[\mathbf{k}_x]$. Then $r = \operatorname{rk} E$ and $Z(E) = -m$. Assume that E is not isomorphic to $\mathcal{O}_X$. Then $\operatorname{Hom}(\mathcal{O}_X, E) = \operatorname{Hom}(E, \mathcal{O}_X) = 0$. So

$$0 \geq \chi(\mathcal{O}_X, E) = 2r + m,$$

and hence $m \leq -2r$. As a result, with the norm on $\mathcal{N}(X)$ in which $[\mathcal{O}_X]$ and $[\mathbf{k}_x]$ form an orthonormal basis, we see that

$$|Z(E)|/\|E\| \geq \frac{|m|}{\sqrt{r^2 + m^2}} \geq \frac{2}{\sqrt{5}}.$$

So the support property holds.

Finally, note that σ is the limit of the stability conditions in $\mathbf{P}W_0$ as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches 0. Since $m_\sigma(T^n \mathbf{k}_x) = 1$, the last equality follows. $\square$

Consider the points P_n of $\overline{D}$. These are the vertices of the tiling triangles other than the vertex $\bullet$. They form a single T-orbit, so it suffices to focus on one of them, say $P_0 = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$, with the 0 at index 0. Note that this is the common vertex, other than $\bullet$, of the triangles $\mathbf{P}W_+ \cong \Delta_{-1}$ and $\mathbf{P}W_- = \Delta_0$. This is the mass function of a lax pre-stability condition, which does not satisfy the support property. Let P be the same slicing as before, and set $Z(\mathcal{O}_X) = 1$ and $Z(\mathbf{k}_x) = 0$.

Proposition 4.10. *The pair (P, Z) as above defines a lax pre-stability condition τ that is a limit of standard stability conditions and $m(\tau) = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$. However, τ does not satisfy the support property.*

Proof. Since $\mathcal{A}$ is of finite length, the slicing is locally finite. So τ is a lax pre-stability condition. It is easy to check that $m_\tau(T^n \mathbf{k}_x) = |n|$. Note that τ is the limit of stability conditions in $\mathbf{P}W_0$ as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches $-\infty$.

To see that the support property fails for τ, recall that the simple objects of $\mathcal{A}$ are the skyscraper sheaves $\mathbf{k}_x$ and shifts by 1 of vector bundles with non-trivial sub-bundles. The skyscraper sheaves are massless, and hence do not obstruct the support property. On the other hand, for a fixed integer $r \geq 2$ and sufficiently large c_2 (depending on r), there exist vector bundles E of rank r on X with no-nontrivial sub-bundles and $c_2(E) = c_2$ (see [1, Théoème 5.3]). For such a vector bundle E, we have $Z_\tau(E) = r$ but the norm of $[E] = (r, r - c_2)$ may be arbitrarily large. So $|Z(E)|/\|E\|$ is not bounded below by any positive constant. $\square$

Finally, consider a point on the open line segment joining v_0 and v_1. This point is in the closure of $\mathbf{P}W_- = \Delta_0$. We claim that it is *not* the mass function of a lax pre-stability condition arising as a limit of stability conditions W_-.

To see this, it is helpful to consider a handful of other semi-stable objects. Let $n \geq m$ be positive integers. Let $x_1, \ldots, x_n \in X$ be distinct points, and set $S = \{x_1, \ldots, x_n\}$. We say that a morphism $\pi\colon \mathcal{O}_X^{\oplus m} \to \mathcal{O}_S$ is *generic* if for every subset $T \subset S$, the induced map on global sections

$$H^0(\mathcal{O}_X^{\oplus m}) \to H^0(\mathcal{O}_T)$$

has maximal rank, namely $\min(m, |T|)$.

For some $w \in -\mathbf{H}$, let $\sigma = \sigma_w$ be the corresponding standard stability condition. Let $I_{m,n}$ be the kernel of a generic morphism from $\mathcal{O}_X^{\oplus m}$ to the structure sheaf of n-points. Then it is easy to check that $I_{m,n}$ is σ-stable.

Fix a point $p \in \overline{D}$ on the line segment joining v_0 and v_1. Then, for some $t > 0$, we can write

$$p = [\cdots : 2 + t : 1 : t : 1 + 2t : \cdots].$$

If we take a sequence of standard stability conditions in W_- whose mass function approaches p, their slicings do not converge. Therefore, there is no limiting lax pre-stability condition with the mass function p. We now make this precise.

Recall that the topology on the space of slicings is induced by the metric d defined as follows. For a slicing P and non-zero object c, let $\phi_P^{\pm}(c)$ denote the highest/lowest phase of the P-HN factors of c. Then the distance $d(P, Q)$ between two slicings P and Q is

$$d(P, Q) = \sup_{c \neq 0} \left\{ \max(|\phi_P^+(c) - \phi_Q^+(c)|, |\phi_P^-(c) - \phi_Q^-(c)|) \right\}.$$

Suppose σ is a lax stability condition that is a limit of a sequence of standard stability conditions σ_w for $w \in -\mathbf{H}$ with $m(\sigma) = p$. Then, possibly after a rotation and scaling, the central charge of σ must send $\mathbf{k}_x$ to -1 and $\mathcal{O}_X$ to $-1 - t$. But then

$$Z(I_{m,n}) = mZ(\mathcal{O}_X) - nZ(\mathbf{k}_x) = n - m(1 + t).$$

It follows that for for every (n, m) with $n/m > (1 + t)$, the sheaf $I_{m,n}$ is σ-semi-stable of phase 0 and for $n/m < (1 + t)$, it is σ-semi-stable of phase 1. But this is absurd. Indeed, for a standard stability condition σ_w, we have

$$\inf_{n/m > 1+t} \phi_\sigma(I_{n,m}) = \sup_{n/m < 1+t} \phi_\sigma(I_{n,m}),$$

so the same equality must hold in the limit.

In summary, we see three distinct kinds of limit points in the boundary from the point of view of lax stability conditions:

(1) The object $\mathcal{O}_X$ can become massless in a lax stability condition, leading to the limit mass function Q.
(2) The objects $\mathbf{k}_x$ and $I_x = T\mathbf{k}_x[-1]$ can become massless in a lax pre-stability condition, leading to the limit mass functions P_0 and P_1.
(3) Other semi-stable sheaves (for example, $I_{m,n}$) cannot become massless in lax pre-stability conditions.

This trichotomy is consistent with the density of the phase diagram (see the discussion in [10, § 12]). Let $\sigma \in W_-$ be a standard stability condition. It is easy to check that the classes $r[\mathcal{O}_X] + n[\mathbf{k}_x]$ that support semi-stable sheaves are precisely $r = 0$ and $n \geq 1$; or $r \geq 1$ and $n = 0$; or $r \geq 1$ and $-n \geq r$ (see Figure 8). Consider the phase diagram—the possible phases of semi-stable objects plotted on the unit circle. There, $\mathcal{O}_X$ is an isolated point, $\mathbf{k}_x$ is a right accumulation point, and I_x is a left accumulation point. At all points on the arc from $\mathbf{k}_x[-1]$ to I_x (and its negative), the phase diagram is dense in the circle. As $\mathcal{O}_X$ becomes massless, the stability conditions converge preserving the support property. As $\mathbf{k}_x$ or I_x become massless, the slicings converge, but the support property is lost. But if the central charge vanishes on a point on the open arc from $\mathbf{k}_x[-1]$ to I_x, even the slicings do not converge.

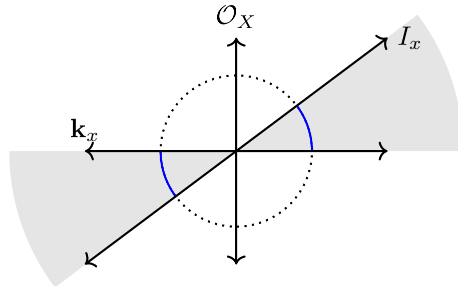

FIGURE 8. The central charges of semi-stable objects in a standard stability condition with heart $\operatorname{Coh} X$ are the lattice points in the shaded region. As a result, the phases are dense in the blue region of the unit circle.

5. The q-mass embedding

Fix a positive real number q. Given a stability condition σ and an object x, recall that the q-mass of x with respect to σ is defined by

$$m_{q,\sigma}(x) = \sum |Z_\sigma(x_i)| q^{\phi(x_i)},$$

where the sum is taken over the σ-HN factors x_i of x, and $\phi(x_i)$ is the phase of x_i. We have the map

$$m_q \colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S$$

given by $\sigma \mapsto m_{q,\sigma}$. We describe the image of m_q and its closure for $q \neq 1$. Most of the arguments are direct analogues of the arguments for $q = 1$, so we will be brief.

Let $\sigma \in \mathbf{P}W_-$. Set $a = m_{q,\sigma}(\mathbf{k}_x)$ and $b = m_{q,\sigma}(T\mathbf{k}_x)$ and $c = m_{q,\sigma}(\mathcal{O}_X)$. Owing to the triangle

$$\mathcal{O}_X \to \mathbf{k}_x \to T\mathbf{k}_x \xrightarrow{+1},$$

the positive real numbers a, b, c satisfy the q-triangle inequalities

$$b < a + qc, \quad a < b + c, \quad c < a + q^{-1}b. \tag{6}$$

(See [13, Proposition 3.3] for a proof of the q-triangle inequalities). From the σ-HN filtration of $T^n\mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma} \colon T^n\mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ b & \text{for } n = 1, \\ b + cq^0 + \cdots + cq^{-n+2} & \text{for } n \geq 2. \end{cases}$$

So, in homogeneous coordinates, the q-mass map is

$$m_q \colon \sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots]$$

Let $\Delta \subset \mathbf{P}^2$ be the set consisting of $[a : b : c]$ where a, b, c are positive real numbers satisfying (6). Then the map $\mathbf{P}W^- \to \Delta$ that takes σ to $[m_{q,\sigma}(\mathbf{k}_x) : m_{q,\sigma}(T\mathbf{k}_x) : m_{q,\sigma}(\mathcal{O}_X)]$ is a homeomorphism. The proof is analogous to the proof of Proposition 4.3 (3), but uses the q-analogue of the cosine rule [2, Lemma 5.2].

Consider $\sigma \in \mathbf{P}W_0$. With a, b, c as before, we have $b = a + qc$. From the σ-HN filtration of $T^n\mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma} \colon T^n\mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ a + cq + \cdots + cq^{-n+2} & \text{for } n \geq 1. \end{cases}$$

So, in homogeneous coordinates, the q-mass map is

$$\sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : a + cq : a + cq + c : \cdots].$$

Set $I_0 = m_q(\mathbf{P}W_0)$ and $I_n = T^n I_0$. Then $m_q \colon T^n \mathbf{P}W_0 \to I_n$ is a homeomorphism.

Let $D_q \in \mathbf{P}^S$ be the union of Δ_n and I_n for $n \in \mathbf{Z}$.

Theorem 5.1. *The q-mass map*

$$m_q \colon \mathbf{P}\,\mathrm{Stab}(X) \to D_q$$

is a homeomorphism.

The proof is analogous to the proof of Theorem 4.6.

We now identify the homeomorphism type of D_q and its closure $\overline{D}_q$. The basic technique is as before—by parametrising $\overline{D}_q$ by a compactified infinite strip of squares. But the resulting picture is slightly different. Without loss of generality, assume $q > 1$.

Recall that $I \subset \mathbf{P^1}$ is the set of $[v : w]$ where v, w are non-negative real numbers. Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ be the two point compactification of $\mathbf{R}$. Define the transformation T on $\overline{\mathbf{R}} \times I$ by

$$T \colon (u, [v, w]) \mapsto (u + 1, [qv : w]).$$

Recall that we also denote by T the action of the spherical twist by $\mathcal{O}_X$ on $\mathbf{P}\,\mathrm{Stab}(X)$ and the rightward shift by 1 on $\mathbf{P}^S$.

We define a T-equivariant parametrisation

$$\pi_q \colon \overline{\mathbf{R}} \times I \to \overline{D}_q,$$

which is a q-analogue of the parametrisation π from Theorem 4.7. For $n \in \mathbf{Z}$, set

$$P_n = (\cdots, 1 + q, 1, 0, 1, 1 + q^{-1}, \cdots) \in \mathbf{R}^S,$$

where the 0 is at index n. Note that $P_n = T^n P_0$. Set

$$Q = (\cdots, 1, 1, 1, \cdots) \in \mathbf{R}^S.$$

Consider the map $\pi_q \colon [0, 1] \times I \to \mathbf{P}^S$ defined by

$$\pi_q(u, [v : w]) = (1 - u)(wQ + vP_0) + u(wQ + q^{-1}vP_1).$$

Note that for $u = 0$ (resp. $u = 1$), the map π_q is a homeomorphism onto the closure of I_0 (resp. I_1), which are the two sides of the triangle Δ_0. For $0 < u < 1$, the map π_q linearly interpolates between the two end-points $\pi_q(0, [v : w])$ and $\pi_q(1, [v : w])$, and hence its image is $\overline{\Delta}_0$. Also observe that $\pi_q(u, [0 : w]) = Q$. Furthermore, with the T actions as before, we have

$$T\pi_q(0, [v : w]) = \pi_q T(0, [v : w]).$$

Thus, π_q extends to a unique T-equivariant continuous map

$$\pi_q \colon \mathbf{R} \times I \to \mathbf{P}^S.$$

Explicitly, for $x = n + u$, where $n \in \mathbf{Z}$ and $u \in [0, 1)$, we have

$$\pi_q(x, [v : w]) = wQ + (1 - u)q^{-n}vP_n + uq^{-n-1}vP_{n+1}.$$

Let $\delta = 1 + q^{-1} + q^{-2} + \cdots$. Then, in $\mathbf{R}^S$ we have

$$\lim_{n\to-\infty} P_n = (\dots, \delta, \delta, \delta, \dots) \text{ and } \lim_{n\to\infty} q^{-n}P_n = (\dots, q\delta, \delta, q^{-1}\delta, \dots).$$

(On the right hand side of the last equation, the δ is at index -1.) Extend $\overline{\pi}_q$ to $\{\pm\infty\} \times I$ by setting

$$\pi_q(-\infty, [v : w]) = [\cdots : 1 : 1 : 1 : \cdots],$$

and

$$\pi_q(+\infty, [v : w]) = w[\cdots : 1 : 1 : 1 : \cdots] + v[\cdots : q : 1 : q^{-1} : \cdots].$$

Using the limit computation above, it follows that this extension is continuous.

Theorem 5.2. *The map $\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{-\infty\} \times [0,1] \cup \overline{\mathbf{R}} \times \{0\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0,1] \to B$ be the contraction of C to a point. Then the induced map $\pi_q \colon B \to \mathbf{P}^S$ is a homeomorphism onto $\overline{D}_q = \overline{m_q(\mathbf{P}\operatorname{Stab}(X))}$.

The proof is analogous to that of Theorem 4.7. See Figure 9 for a sketch.

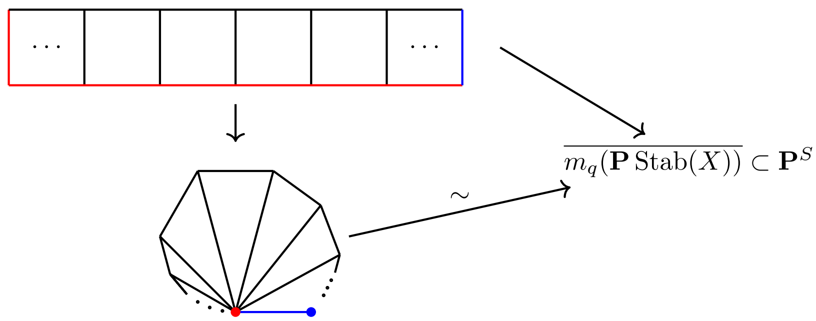

Figure 9. The map $\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\operatorname{Stab}(X)$ under the q-mass map. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0,1]$ by collapsing two sides (red).

Instead of a unique T-fixed point of $\overline{D}_q$, as was the case for $q = 1$, for $q \neq 1$ we have two such points. These are the blue and red end-points of the blue interval in Figure 9. The blue end-point is the point $\bullet = [\cdots : q : 1 : q^{-1} : \cdots :]$. It is the q-hom function $\hom_q(\mathcal{O}_X, -)$, whose value on $T^n\mathbf{k}_x$ is

$$\dim_q \operatorname{Hom}^*(\mathcal{O}_X, T^n\mathbf{k}_x) = q^{-n}.$$

(By definition, $\dim_q$ of the graded vector space $\mathbf{C}[m]$ is q^m). Again, the fact that $\hom_q(\mathcal{O}_X, -)$ appears in the closure of the q-mass embedding of the stability manifold is a reflection of a general theorem—the q-analogue of Theorem 4.8 (see [3, Corollary 4.13]).

Note that $\bullet$ is not in the closure of the standard stability conditions $\mathbf{P}W$, nor is it in the closure of $T^n\mathbf{P}W$ for any fixed n. To reach $\bullet$, we must traverse an infinite sequence of hearts. It is easy to see that it is not the q-mass function of a lax stability condition.

The red end-point is the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. It is the q-mass function of the lax stability condition σ from Proposition 4.9.

The other vertices of the triangles form one orbit, and are q-mass functions of lax pre-stability conditions. For example, the vertex $v_0 = [\cdots : 1+q : 1 : 0 : 1 : 1+q^{-1} : \cdots]$ is the q-mass function of the lax pre-stability condition $q^{-1} \cdot \tau$ where τ is as in Proposition 4.10.

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