APPENDIX TO "CYCLIC COVERING MORPHISMS ON $\overline{M}_{0,n}$" BY MAKSYM FEDORCHUK

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Appendix A. Class of the Hodge eigenbundle using orbifold Riemann–Roch

In [2, Section 6], the class of the Hodge eigenbundle $\mathbb{E}_j$ was computed indirectly by calculating it on the F-curves. We describe a more direct method using the Grothendieck–Riemann–Roch theorem for Deligne–Mumford stacks. Throughout, μ_r denotes the group of rth roots of unity and $\mathbf{C}(i)$ the line bundle on $B\mu_r$ associated to the character $\zeta \mapsto \zeta^i$.

A.1. **Expressing $\mathbb{E}_j$ as a pushforward.** Let $p \geq 2$ be an integer and $(d_1, \dots, d_n)$ a sequence of non-negative integers such that $\sum d_j = mp$ for some integer m. Consider the cyclic cover $\phi\colon C \to \mathbf{P}^1$ given by the regular projective model of

$$y^p = (x - x_1)^{d_1} \dots (x - x_n)^{d_n}.$$

In these coordinates, O_C is generated as an $O_{\mathbf{P}^1}$ module by the functions 1, y/f_1, $\dots$, y^{p-1}/f_{p-1} where $f_j = (x - x_1)^{\lfloor jd_1/p \rfloor} \dots (x - x_n)^{\lfloor jd_n/p \rfloor}$. Note that these generators are μ_p eigenvectors. Letting D be the $\mathbf{Q}$ divisor $D = \sum (d_i/p) x_i$, we thus have the μ_p-equivariant decomposition

$$\phi_* O_C = \bigoplus_{j=0}^{p-1} O_{\mathbf{P}^1}(-jm) \otimes O_{\mathbf{P}^1}(\lfloor jD \rfloor),$$

where the local generator of the jth summand is y^j/f_j. We then get the μ_p-eigenspace decomposition

$$H^1(C, O_C) = \bigoplus_{j=0}^{p-1} H^1\left(\mathbf{P}^1, O_{\mathbf{P}^1}(-jm) \otimes O_{\mathbf{P}^1}(\lfloor jD \rfloor)\right).$$

The above analysis for an individual cover goes through over the moduli stack. Let $\mathcal{M}$ be the stack of p-divisible n-pointed orbicurves with the universal curve $\pi\colon \mathcal{P} \to \mathcal{M}$ with n sections σ_i and let $\mathcal{L}$ be a line bundle on $\mathcal{P}$ with $\mathcal{L}^p = O_{\mathcal{P}}(\sum d_i \sigma_i)$. On $\mathcal{M}$ lives the universal cyclic p cover $\phi\colon \mathcal{C} \to \mathcal{P}$ whose fibers are as above. Setting $D = \sum (d_i/p) \sigma_i$, we get the the μ_p-equivariant decomposition

$$\phi_* O_{\mathcal{C}} = \bigoplus_{j=0}^{p-1} \mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor).$$

Applying $R\pi_*$ gives the μ_p-eigenbundle decomposition

$$R^1\pi_*(O_{\mathcal{C}}) = \bigoplus_{j=0}^{p-1} R^1\pi_*\left(\mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor)\right).$$

Since the left hand side is the dual of the Hodge bundle, we get

$$\mathbb{E}_j = R^1\pi_*(\mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor))^*. \tag{1}$$

A.2. **Applying Grothendieck–Riemann–Roch.** Equation 1 reduces the computation of $c_1(\mathbb{E}_j)$ to the familiar problem of computing the Chern classes of a (derived) push-forward. Since the push forward is from a stack, we have to use a bit of technology. For less cluttered notation, we calculate in the following setting: Let B be a smooth projective curve, $\pi\colon \mathcal{P} \to B$ a generically smooth family of rational orbinodal curves with $\mathcal{P}$ smooth, and $\mathcal{F}$ a line bundle on $\mathcal{P}$. For an orbinode $x \in \mathcal{P}$ with stabilizer μ_{r_x}, let $\mathcal{F}|_x \cong \mathbf{C}(i_x)$, where $0 \leq i_x < r_x$. Denote by ω the class of the relative dualizing sheaf of $\mathcal{P} \to B$.

Proposition A.1. *In the above setup, we have*

$$c_1(R\pi_*\mathcal{F}) = \frac{c_1(\mathcal{F})^2}{2} - \frac{c_1(\mathcal{F})\cdot\omega}{2} - \sum_{\text{Orbi } x \in \mathcal{P}} \frac{i_x(r_x - i_x)}{2r_x}$$

Proof. By the GRR for Deligne–Mumford stacks [1, Corollary 5.3], we have

$$\operatorname{ch} R\pi_*\mathcal{F} = \pi_*\left(\operatorname{ch}\left(t\left(\frac{f^*\mathcal{F}}{\lambda_{-1}N_f^*}\right)\right)\frac{\operatorname{td}(I\mathcal{P})}{\operatorname{td}(B)}\right), \tag{2}$$

where $f\colon I\mathcal{P} \to \mathcal{P}$ is the inertia, N_f^* is the conormal bundle of f, the operator λ_{-1} on $K(I\mathcal{P})$ is the alternating sum of wedge powers $1 - \Lambda^1 + \Lambda^2 + \dots$ and the operator t on $K(I\mathcal{P}) \otimes \mathbf{C}$ is the 'twisting operator' [1, § 4]. We evaluate these objects in our setup. First, note that the coarse space map $\mathcal{P} \to P$ is an isomorphism except possibly at the singular points of $P \to B$, where the local picture is

$$[\operatorname{Spec} k[u,v]/(uv - t)/\mu_r] \to \operatorname{Spec} k[x,y]/(xy - t^r), \tag{3}$$

with μ_r acting by $u \mapsto \zeta u$ and $v \mapsto \zeta^{-1}v$. Since the orbistructure is at isolated points, the inertia stack is

$$I\mathcal{P} = \mathcal{P} \sqcup \bigsqcup_{\text{Orbi } x\in\mathcal{P}} (\mu_{r_x} \setminus \{\text{id}\}) \times B\mu_{r_x}.$$

The normal bundle N_f is trivial on the $\mathcal{P}$ component and is $\mathbf{C}(1)\oplus\mathbf{C}(-1)$ on the $\{\zeta\}\times B\mu_{r_x}$ component. The twisting operator acts trivially on the $\mathcal{P}$ component and sends $\mathbf{C}(i)$ to $\zeta^i\mathbf{C}(i)$ on the $\{\zeta\}\times B\mu_{r_x}$ component. Thus, Equation 2 becomes

$$\operatorname{ch} R\pi_*\mathcal{F} = \pi_*\left(1 + c_1(\mathcal{F}) + \frac{c_1^2(\mathcal{F})}{2}\right)\left(1 - \frac{c_1(\Omega_{\mathcal{P}/B})}{2} + \frac{c_1^2(\Omega_{\mathcal{P}/B}) + c_2(\Omega_{\mathcal{P}/B})}{12}\right) + \sum_{\substack{\text{Orbi } x\\ 1\neq\zeta\in\mu_{r_x}}} \frac{\zeta^{i_x}}{r_x(2-\zeta-\zeta^{-1})}.$$

Let $\omega_{\mathcal{P}/B}$ be the relative dualizing sheaf. Let $\delta_{\mathcal{P}/B}$ the class of the support of $\operatorname{coker}(\Omega_{\mathcal{P}/B} \to \omega_{\mathcal{P}/B})$ (with multiplicities) so that $c_2(\Omega_{\mathcal{P}/B}) = \delta_{\mathcal{P}/B}$. Define $\omega_{P/B}$ and $\delta_{P/B}$ analogously. Identifying the rational Chow groups of $\mathcal{P}$ and P, we have $c_1(\omega_{\mathcal{P}/B}) = c_1(\omega_{P/B}) = \omega$. The value of δ, however, is different for $\mathcal{P} \to B$ and $P \to B$: the numerical contribution of the orbinode in Equation 3 is $1/r$ whereas that of the node is r. With these simplifications, we get

$$c_1(R\pi_*\mathcal{F}) = \frac{c_1(\mathcal{F})^2}{2} - \frac{c_1(\mathcal{F})\cdot\omega}{p} + \frac{\omega^2 + \delta_{P/B}}{12} + \sum_{\text{Orbi } x} \frac{1/r_x - r_x}{12} + \left(\sum_{1\neq\zeta\in\mu_{r_x}} \frac{\zeta^{i_x}}{r_x(2-\zeta-\zeta^{-1})}\right). \tag{4}$$

Since $P \to B$ is a family of rational curves, we have $\omega^2 + \delta_{P/B} = 0$. The following is a nice exercise

$$\sum_{1\neq\zeta\in\mu_r} \frac{\zeta^i}{2-\zeta-\zeta^{-1}} = \frac{i(i-r)}{2} + \frac{r^2-1}{12} \quad (\text{for } 0 \le i < r).$$

Substituting in Equation 4 gives the result. □

Proposition A.2. *The class of the Hodge eigenbundle is given by*

$$c_1(\mathbb{E}_j) = \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p\langle p - jd_i\rangle_p\psi_i - \sum_{I,J}\langle jd(I)\rangle_p\langle jd(J)\rangle_p\Delta_{I,J}\right).$$

Proof. Take a smooth curve B and a map $B \to \mathcal{M}$ transverse to the boundary. Let $\pi\colon \mathcal{P} \to B$ and $\mathcal{L}$ be pullbacks from $\mathcal{M}$. Set $\mathcal{F} = \mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD\rfloor)$. Then $c_1(\mathcal{F}) = -\sum(\langle jd_i\rangle_p/p)\cdot\sigma_i$. Let $x \in \mathcal{P}$ be an orbinode corresponding to the boundary divisor $\Delta_{I,J}$. Set $d = \gcd(p, jd(I))$. Then the stabilizer at x has order $r = p/d$ and $\mathcal{F}|_x \cong \mathbf{C}(i)$ for $i = \langle jd(I)/d\rangle_r$. Also, x contributes r towards $\Delta_{I,J}$. Using Equation 1 and

Proposition A.1, we conclude that

$$c_1(\mathbb{E}_j) = c_1(R\pi_*\mathcal{F}) = \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p^2 \sigma_i^2 + p\langle jd_i\rangle_p \sigma_i \cdot \omega - \sum_{\text{Orbi } x} \frac{p^2\langle jd(I)/d\rangle_r \langle jd(J)/d\rangle_r}{r}\right)$$

$$= \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p \langle p - jd_i\rangle_p \psi_i - \sum_{I,J} \langle jd(I)\rangle_p \langle jd(J)\rangle_p \Delta_{I,J}\right).$$

□

References

[1] Dan Edidin, Riemann-Roch for Deligne-Mumford stacks, *arXiv:1205.4742 [math.AG]*, November 2012.

[2] Maksym Fedorchuk, Cyclic covering morphisms on $\overline{M}_{0,n}$, *arXiv:1105.0655 [math.AG]*, May 2011.