Syzygy divisors on Hurwitz spaces

Anand Deopurkar and Anand Patel

ABSTRACT. We describe a sequence of effective divisors on the Hurwitz space $H_{d,g}$ for d dividing $g - 1$ and compute their cycle classes on a partial compactification. These divisors arise from vector bundles of syzygies canonically associated to a branched cover. We find that the cycle classes are all proportional to each other.

1. Introduction

The Hurwitz space $H_{d,g}$ is the moduli space of maps $\alpha\colon C \to \mathbf{P}^1$, where C is a smooth algebraic curve of genus g and α is finite of degree d. It is one of the oldest moduli spaces studied in algebraic geometry. Indeed, its idea goes back to the time of Riemann—a time when algebraic curves were thought of primarily as branched covers of the Riemann sphere. It was put on a rigorous modern algebraic footing by Fulton [11]. It was compactified by Harris and Mumford [13], whose construction was then refined by Mochizuki [16] and Abramovich, Corti, and Vistoli [1]. We refer the reader to [20] for an introduction to Hurwitz spaces.

The Hurwitz spaces have attracted mathematical attention not only because of their intrinsic appeal, but also because of their role in illuminating the geometry of the moduli space M_g. Indeed, it was through the Hurwitz spaces that Riemann [19] computed the dimension of M_g and Severi [21], building on work of Clebsch [5] and Hurwitz [14], showed that M_g is irreducible. In more recent times, Harris and Mumford [13] used the compactified Hurwitz spaces to carry out a divisor class computation to show that $\overline{M}_g$ is of general type for large g. Hurwitz spaces and their variants have also been of interest outside of algebraic geometry. For example, spaces of branched covers of $\mathbf{P}^1$ with a given Galois group feature prominently in inverse Galois theory [9, 10], and spaces of covers of $\mathbf{P}^1$ by $\mathbf{P}^1$ play a key role in dynamics [22].

Although Hurwitz spaces have been around for centuries, there are still many basic open questions about them. One such question is the question of placing them in the birational classification of varieties: For which d and g is $H_{d,g}$ rational, unirational, uniruled, rationally connected, or of general type? As with many other questions about them, the answer is known only for very small or very large d. For $d \leq 5$, the space $H_{d,g}$ is known to be unirational, thanks to a determinantal description of covers of degree up to 5. For $d > \lfloor g/2 \rfloor$, the space $H_{d,g}$ dominates M_g. Since M_g has non-negative Kodaira dimension for $g \geq 22$, in this case $H_{d,g}$ cannot be uniruled. The intermediate cases are rather mysterious, but remain an active area of research. See, for example, the recent

work [12] on unirationality results for $d = 6$ and all $g \leq 28$ (and several more, up to $g = 45$).

At the heart of determining the birational type of $\overline{H}_{d,g}$ is the question of understanding its cone of effective divisors. Indeed, modulo an obstruction coming from singularities, saying that $\overline{H}_{d,g}$ is of general type is equivalent to saying that its canonical class lies in the interior of its effective cone. To show this, we need not know the full effective cone; it would suffice to know enough effective divisors, whose classes contain the canonical class in their convex span.

One way to get effective divisors is by using topology. A general point of $H_{d,g}$ parametrizes simply branched covers. In codimension 1, this simple topological picture can specialize in two ways: the cover can develop a ramification point of index 2 or can have two ramification points of index 1 over the same branch point. The two possibilities give two effective divisors on $H_{d,g}$.

The goal of this paper is to describe a number of other effective divisors on $H_{d,g}$ (for d dividing $g - 1$) and to compute their classes on a partial compactification $\widetilde{H}_{d,g}$. Their origin is distinctly algebraic, orthogonal to any topological considerations. They are in the spirit of the classical Maroni divisor, and generalize the Casnati–Ekedahl divisors studied by the second author [17].

Before describing the divisors, we recall the Maroni divisor. A finite map $\alpha \colon C \to \mathbf{P}^1$ canonically factors as an embedding $\iota \colon C \to \mathbf{P}E$ followed by a projection $\pi \colon \mathbf{P}E \to \mathbf{P}^1$, where E is a vector bundle of rank $d - 1$ and degree $g + d - 1$ on $\mathbf{P}^1$. In the cases where the rank divides the degree, the bundle E is balanced for generic α—it is a twist of the trivial bundle. The *Maroni divisor* is the locus of α for which it is unbalanced.

Our divisors $\mu_1, \ldots, \mu_{d-3}$, which we call *syzygy divisors*, are defined analogously for a sequence of other vector bundles $N_1, \ldots, N_{d-3}$ associated with α. Roughly, N_i is the bundle whose fiber at $t \in \mathbf{P}^1$ is the vector space of $(i - 1)$th syzygies among the generators of the homogeneous ideal of $C_t \subset \mathbf{P}E_t$. If d divides $g - 1$, then N_i has rank dividing the degree, and for generic α, it is balanced. The divisor μ_i is the locus where it is unbalanced.

THEOREM 1.1 (Main). *Suppose d divides $g - 1$. Let i be an integer with $1 \leq i \leq d - 3$. The locus $\mu_i \subset \widetilde{H}_{d,g}$ is an effective divisor whose class in $\mathrm{Pic}_{\mathbf{Q}} \widetilde{H}_{d,g}$ is given by*

$$[\mu_i] = A_i \left(6(gd - 6g + d + 6) \cdot \zeta - d(d - 12) \cdot \kappa - d^2 \cdot \delta \right),$$

where

$$A_i = \binom{d - 4}{i - 1}^2 \frac{(d - 2)(d - 3)}{6(i + 1)(d - i - 1)}.$$

In the theorem, $\widetilde{H}_{d,g}$ is the coarse moduli space of $\alpha \colon C \to \mathbf{P}^1$, where C is an irreducible curve of arithmetic genus g with at worst nodal singularities and α is a finite map of degree d. The classes ζ, κ, and δ are certain tautological divisor classes in $\mathrm{Pic}_{\mathbf{Q}} \widetilde{H}_{d,g}$; they are conjectured to generate $\mathrm{Pic}_{\mathbf{Q}} \widetilde{H}_{d,g}$. See § 2 for definitions.

The most surprising feature of the divisor class is that (up to scaling) it is independent of i. However, we do not expect the divisors μ_i themselves to be supported on the same set (see § 4.4). This is reminiscent of a similar phenomenon with the Brill–Noether divisors on $\overline{M}_g$—the classes of all the divisorial Brill–Noether loci are proportional, although the loci themselves are different.

Theorem 1.1 gives the class of μ_i on a *partial* compactification $\widetilde{H}_{d,g}$ of $H_{d,g}$. It is an interesting (and challenging) problem to compute the class of the closure of μ_i on a full

compactification. This was carried out for the Maroni divisor for $d = 3$ in [**6**] and for higher d in [**23**]. It would also be interesting to find replacements for μ_i when d does not divide $g - 1$. This would be analogous to the replacement of the Maroni divisor in the case of odd genus trigonal curves found in [**6**].

The paper is organized as follows. In § 2, we recall the (largely conjectural) description of the Picard group of $\widetilde{H}_{d,g}$ and describe a number of divisor classes on $\widetilde{H}_{d,g}$. These include the syzygy divisors $\mu_1, \dots, \mu_{d-3}$, whose existence is contingent on the fact that the syzygy bundles N_i are balanced for a generic cover. In § 3, we discuss the generic splitting type of the syzygy bundles N_i. In § 4, we carry out the main computation of the divisor class of μ_i.

We work over an algebraically closed field of characteristic zero. All schemes and stacks are locally of finite type over this field. A *point* is a closed point, unless mentioned otherwise. The projectivization $\mathbf{P}U$ of a vector bundle U denotes the space of one-dimensional *quotients*. The Hurwitz spaces $\widetilde{H}_{d,g}$ and $H_{d,g}$ have *unparametrized* source and target. That is, $\alpha_1 \colon C_1 \to \mathbf{P}^1$ and $\alpha_2 \colon C_2 \to \mathbf{P}^1$ are considered isomorphic if there are isomorphisms $\phi \colon C_1 \to C_2$ and $\psi \colon \mathbf{P}^1 \to \mathbf{P}^1$ with $\alpha_2 \circ \phi = \psi \circ \alpha_1$.

2. Divisors on Hurwitz spaces

The goal of this section is to recall the rational Picard group of the Hurwitz space and some divisor classes in it. Since we are only interested in the rational Picard group, we may work either with the Deligne–Mumford stack $\widetilde{\mathcal{H}}_{d,g}$ or the coarse moduli space $\widetilde{H}_{d,g}$. We will pass from one to the other without comment.

2.1. The rational Picard group. Denote by

$$\alpha \colon \mathcal{C} \to \mathcal{P}$$

the universal object over $\widetilde{\mathcal{H}}_{d,g}$. Here $\pi \colon \mathcal{C} \to \widetilde{\mathcal{H}}_{d,g}$ is a family of irreducible genus g curves with at worst nodal singularities, $p \colon \mathcal{P} \to \widetilde{\mathcal{H}}_{d,g}$ is a family of smooth genus 0 curves, and α a finite morphism of degree d over $\widetilde{\mathcal{H}}_{d,g}$. The universal family allows us to write the following two 'tautological' divisor classes on $\mathcal{H}_{d,g}$:

$$\pi_* \left(c_1(\omega_\pi^2) \right) \quad \text{and} \quad p_* \left(c_1(\omega_\pi) \cdot c_1(\alpha^* \omega_p) \right).$$

(The third natural product $c_1(\omega_p)^2$ vanishes.) Set

$$\kappa = \pi_* \left(c_1(\omega_\pi^2) \right) \quad \text{and} \quad \zeta = \frac{-1}{2} p_* \left(c_1(\omega_\pi) \cdot c_1(\alpha^* \omega_p) \right).$$

Together with the divisor Δ—the locus of $\alpha \colon C \to \mathbf{P}^1$ where C is singular—we get three divisor classes on $\widetilde{H}_{d,g}$. Conjecturally, these exhaust the Picard group, at least modulo torsion.

CONJECTURE 2.1 (See [**8**]). *The divisor classes κ, ξ, and Δ generate* $\mathrm{Pic}_{\mathbf{Q}}(\widetilde{H}_{d,g})$.

The conjecture has been proved for $d \leq 5$ and for $d > 2g - 2$. For $d \leq 5$, the proof uses the unirational parametrization of $\widetilde{H}_{d,g}$ [**7**]. For $d > 2g - 2$, the proof uses the fibration $\widetilde{H}_{d,g} \to M_g$ and the deep result of Harer that $\mathrm{Pic}_{\mathbf{Q}} M_g$ has rank 1 [**8**, **16**]. The intermediate cases are still open. At any rate, all the divisors we consider in this paper can be written explicitly as linear combinations of κ, ζ, and Δ.

2.2. Divisors from the topology of covers. We have three natural divisors on $\widetilde{H}_{d,g}$ arising from topological considerations. A generic point of $\widetilde{H}_{d,g}$ represents a cover $\alpha\colon C \to \mathbf{P}^1$ that has simple branching. That is, α has $b = 2g + 2d - 2$ distinct branch points and over each branch point, there is a unique ramification point at which the local degree of α is 2.

A simply branched cover specializes in three topologically distinct ways in codimension 1; each possibility gives a divisor on $\widetilde{H}_{d,g}$. The divisor T is the locus of α that have a point of higher ramification—a point $x \in C$ at which the local degree of α is at least 3. The divisor D is the locus of α that have at least two distinct ramification points over the same branch point. The divisor δ is the locus of α whose domain C is singular. It is easy to see that T, D, and δ are irreducible divisors in $\widetilde{H}_{d,g}$.

REMARK 2.2. We can use the topological considerations above to obtain locally closed subsets of $\widetilde{H}_{d,g}$ of higher codimension. Doing so gives a stratification of $\widetilde{H}_{d,g}$ according to the topological type of $\alpha\colon C \to \mathbf{P}^1$. A complete specification of the topological type of α is rather intricate. It includes, for example, the types of ramification profiles for α, the number of singularities of C, and the location of the singularities relative to the ramification profiles.

2.3. Divisors from the algebra of covers. Just as we get special loci in $\widetilde{H}_{d,g}$ from non-generic topological behavior, we get special loci in $\widetilde{H}_{d,g}$ from non-generic algebraic behavior. We make this precise using a structure theorem for finite morphisms due to Casnati and Ekedahl [4], which we first recall.

Let X and Y be integral schemes and $\alpha\colon X \to Y$ a finite flat Gorenstein morphism of degree $d \geq 3$. The map α gives an exact sequence

$$(2.1) \qquad\qquad 0 \to \mathcal{O}_Y \to \alpha_* \mathcal{O}_X \to E_\alpha^\vee \to 0,$$

where $E = E_\alpha$ is a vector bundle of rank $(d-1)$ on Y, called the *Tschirnhausen bundle* of α. Denote by ω_α the dualizing sheaf of α. Applying $\mathrm{Hom}_Y(-, \mathcal{O}_Y)$ to (2.1), we get

$$(2.2) \qquad\qquad 0 \to E \to \alpha_* \omega_\alpha \to \mathcal{O}_Y \to 0.$$

The map $E \to \alpha_* \omega_\alpha$ induces a map $\alpha^* E \to \omega_\alpha$.

THEOREM 2.3 (See [4, Theorem 2.1]). *In the above setup, $\alpha^* E \to \omega_\alpha$ gives an embedding $\iota\colon X \to \mathbf{P}E$ with $\alpha = \pi \circ \iota$, where $\pi\colon \mathbf{P}E \to Y$ is the projection. Moreover, the following hold.*

(1) The resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathbf{P}E}$ module has the form

$$(2.3) \qquad \begin{aligned} 0 \to \pi^* N_{d-2}(-d) &\to \pi^* N_{d-3}(-d+2) \to \pi^* N_{d-4}(-d+3) \to \dots \\ &\dots \to \pi^* N_2(-3) \to \pi^* N_1(-2) \to \mathcal{O}_{\mathbf{P}E} \to \mathcal{O}_X \to 0, \end{aligned}$$

where the N_i are vector bundles on Y. Restricted to a point $y \in Y$, this sequence is the minimal free resolution of $X_y \subset \mathbf{P}E_y$.

(2) The ranks of the N_i are given by

$$\mathrm{rk}\, N_i = \frac{i(d-2-i)}{d-1} \binom{d}{i+1},$$

(3) We have $N_{d-2} \cong \pi^* \det E$. Furthermore, the resolution is symmetric, that is, isomorphic to the resolution obtained by applying $\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}E}}(-, N_{d-2}(-d))$.

We call the resolution in (2.3) the *Casnati–Ekedahl* resolution of α.

Let us take $Y = \mathbf{P}^1$. Every vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles. The multi-set of degrees of the line bundles appearing in the direct sum decomposition is unique. We refer to this multi-set as the *splitting type* of the bundle. We say that a bundle V is *balanced* if the splitting type is $\{a, \ldots, a\}$ for some a.

PROPOSITION 2.4. *Let $\alpha\colon C \to \mathbf{P}^1$ be a point of $\widetilde{H}_{d,g}$. Denote by E the Tschirnhausen bundle and by N_i the syzygy bundles in the Casnati–Ekedahl resolution of α. Then*

$$\deg E = (g + d - 1), \text{ and}$$

$$\deg N_i = (d - 2 - i)(g + d - 1)\binom{d-2}{i-1}.$$

PROOF. The branch divisor of α is cut out by a section of $(\det E)^{\otimes 2}$. Therefore, we get $2 \deg E = 2g + 2d - 2$, from which the first equation follows. We postpone the proof of the second equation to § 4 (See Corollary 4.3). □

Suppose d divides $g - 1$. Then the rank of N_i divides its degree.

PROPOSITION 2.5. *If d divides $g - 1$, then for a generic $\alpha\colon C \to \mathbf{P}^1$ in $\widetilde{H}_{d,g}$ and $i = 1, \ldots, d - 2$, the bundle N_i is balanced.*

We postpone the proof to § 3.

DEFINITION 2.6. Suppose d divides $g - 1$. Define the *ith syzygy divisor* $\mu_i \subset \widetilde{H}_{d,g}$ as the locus of $\alpha\colon C \to \mathbf{P}^1$ for which the bundle N_i is unbalanced.

There is a natural scheme structure on $\mu_i \subset \widetilde{\mathcal{H}}_{d,g}$, defined as follows. Let $U \to \widetilde{\mathcal{H}}_{d,g}$ be an étale local chart for the moduli stack over which the conic bundle $\mathcal{P}_U \to U$ admits a relative $\mathcal{O}(1)$. Consider the bundle $\mathrm{End}(N_i) \otimes \mathcal{O}(-1)$ on $\mathcal{P}_U$. Note that $\chi\left(\mathrm{End}(N_i) \otimes \mathcal{O}(-1)\right) = 0$ and $h^1(\mathrm{End}(N_i) \otimes \mathcal{O}(-1) \geq 1$ if and only if N_i is unbalanced. The divisor μ_i is the zero locus of the first Fitting ideal of $R^1 p_* \left(\mathrm{End}(N_i) \otimes \mathcal{O}(-1)\right)$. Henceforth, μ_i is understood to have this scheme structure.

REMARK 2.7. We can use the splitting types of E and N_i to define locally closed subsets of $\widetilde{H}_{d,g}$ of higher codimensions. Doing so gives a stratification of $\widetilde{H}_{d,g}$ according to the isomorphism types of the bundles appearing in the Casnati–Ekedahl resolution. This stratification has a distinctly algebro-geometric favor, and it should be in some sense orthogonal to the topological stratification discussed in Remark 2.2. See [17] for more on this stratification.

2.4. Relations between various divisor classes. Assuming Conjecture 2.1, the divisors defined in § 2.2 and § 2.3 ought to be expressible as linear combinations of the tautological divisors κ, ζ, and δ. Such an expression for the higher syzygy divisors μ_i is the content of § 4. In this section, we give the expressions for all the other divisors.

Denote by E the Tschirnhausen bundle of the universal cover $\alpha\colon \mathcal{C} \to \mathcal{P}$. In addition to the divisors disused so far, it will be useful to also consider the following three auxiliary divisors:

$$p_* c_1(E)^2, \quad p_* \mathrm{ch}_2(E), \quad \pi_* c_1(\omega_\alpha)^2.$$

Lastly, denote by $\lambda = c_1\left(\pi_* \omega_\pi\right)$ the class of the Hodge line bundle on $\widetilde{H}_{d,g}$ and by K the canonical divisor class of $\widetilde{H}_{d,g}$. Set

$$b = 2g + 2d - 2.$$

This is the degree of the branch divisor of the covers in $\widetilde{H}_{d,g}$.

PROPOSITION 2.8. *The following identities hold in* $\mathrm{Pic}_{\mathbf{Q}}(\widetilde{H}_{d,g})$:

(1) $12\lambda = \kappa + \delta$
(2) $p_* c_1(E)^2 = \frac{b}{2} \cdot \zeta$
(3) $p_* \mathrm{ch}_2(E) = \frac{1}{12} \cdot \kappa + \frac{1}{2} \cdot \zeta + \frac{1}{12} \cdot \delta$
(4) $\pi_* c_1(\omega_\alpha)^2 = \kappa + 4 \cdot \zeta$
(5) $T = 2 \cdot \kappa + 6 \cdot \zeta - \delta$
(6) $D = -3 \cdot \kappa + (b - 10)\zeta + \delta$
(7) $\mu = -\frac{d}{6} \cdot \kappa + \frac{b-2d}{2} \cdot \zeta + \frac{d}{6} \cdot \delta$
(8) $K = \kappa + \zeta - \delta$

PROOF. We compute all the divisor classes on a generic one parameter family $B \to \widetilde{\mathcal{H}}_{d,g}$. Let $\alpha \colon C \to P$ be the pull-back of the universal family to B with the two projections $\pi \colon C \to B$ and $p \colon P \to B$. Set $\sigma = -c_1(\omega_p)/2$.

(1) This is the well-known Mumford relation.

(2) Let $\beta \subset P$ be the branch divisor of α. Since β is cut out by a section of $(\det E)^{\otimes 2}$, we have
$$[\beta] = 2c_1(E).$$
Since $p \colon P \to B$ is a $\mathbf{P}^1$ bundle, we have a relation
$$[\beta] = a\sigma + p^* D$$
for some $a \in \mathbf{Z}$ and $D \in \mathrm{Pic}(B)$. Since $[\beta]$ has degree b on the fibers of p, we get $a = b$. By comparing $\sigma \cdot [\beta]$ and $[\beta]^2$, we get

(2.4) $$c_1(E)^2 = bc_1(E) \cdot \sigma.$$

Since β is the push-forward of the ramification divisor of α, which has class $c_1(\omega_\alpha)$, we have
$$\alpha_* \left(c_1(\omega_\alpha) \right) = 2c_1(E).$$
Multiplying the above by σ, noting that $\omega_\alpha \cdot \sigma = \omega_\pi \cdot \sigma$, and using (2.4) yields the second relation.

(3) Applying Rpi_* to both sides of the equation
$$\alpha_* \mathcal{O}_C = \mathcal{O}_P \oplus E^\vee$$
and using Grothendieck–Riemann–Roch for the right hand side yields the third relation.

(4) Using $c_1(\omega_\alpha) = c_1(\omega_\pi) + 2\sigma$ and (2.4) yields the fourth relation.

(5, 6) To get T and D, we sketch the argument from [**17**, Proposition 3.2]. Assuming B is sufficiently generic, the only singularities of β will be nodes and cusps, and the map from the ramification divisor ρ to the branch divisor β will be the normalization. A simple local computation of the branch divisor of a cover specializing to a point of D or T shows that the nodes correspond to intersections of B with D, and the cusps with the intersections of B with T. Therefore, we get
$$p_a(\beta) - p_a(\rho) = T + D.$$
By adjunction on C and P, this leads to
$$(\beta^2 - 2\rho^2)/2 = T + D.$$
The branch points of $\rho \to B$ correspond to the intersections of B with δ or with T. From adjunction on C and Riemann–Hurwitz, we get
$$2\rho^2 + \beta \cdot c_1(\omega_\pi) = T + \delta.$$

Solving for T and D, and using the previous relations yields the fifth and the sixth relations.

(7) The class of μ is given by the Bogomolov expression $c_1(E)^2 - 2d\,\mathrm{ch}_2(E)$, which yields the seventh relation (See § 4.1 for the Bogomolov expression).

(8) We sketch two ways to compute the canonical divisor. Note that the map $\widetilde{\mathcal{H}}_{d,g} \to \widetilde{H}_{d,g}$ is unramified in codimension 1, so the canonical class of the stack is the same as that of the coarse space.

First, consider the morphism $\mathrm{br}: U \to V$, where $V \subset \mathbf{P}(\mathrm{Sym}^b \mathbf{P}^1) /\!\!/ \mathrm{SL}(2)$ is the open locus where at most two of the b marked points coincide, $U \subset \widetilde{H}_{d,g}$ is the locus of covers where at most two branch points coincide, and br is the morphism that assigns to a cover its branch divisor. It is easy to check that the complements of V and U have codimension 2, and hence it suffices to work on V and U for divisor calculations. Let $\Delta \subset U$ be the complement of the locus of b distinct points. A simple local calculation shows that

$$\mathrm{br}^{-1}\Delta = 3T + 2D + \delta.$$

The canonical divisor of U is

$$K_U = -\frac{(b+1)}{2b-2} \cdot \Delta.$$

By Riemann–Hurwitz, we get

$$K_W = \mathrm{br}^* K_U + 2T + D,$$

which combined with the previous relations yield the eighth relation.

Another way is to use the deformation theory of maps developed in [**18**]. We can identify the tangent space to $\widetilde{\mathcal{H}}_{d,g}$ at $\alpha: C \to \mathbf{P}^1$ as the the kernel of the induced map

$$\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega_{\mathbf{P}^1}, \alpha_* \mathcal{O}_C).$$

We can compute the Chern classes of the bundles on $\widetilde{\mathcal{H}}_{d,g}$ defined by both terms, and their difference yields the Chern class of the tangent bundle of $\widetilde{\mathcal{H}}_{d,g}$.

$\square$

3. The generic splitting type

The goal of this section is to discuss the splitting type of the syzygy bundle N_i for a generic cover, and to prove that it is balanced when d divides $g - 1$. Note, however, that the degree of N_i may be divisible by its rank even when d does not divide $g - 1$. One may expect N_i to be generically balanced even in this setting. This is not quite true, as the following example shows for the first bundle N_1.

EXAMPLE 3.1. Consider a general degree 6, genus 4 cover $\alpha: C \to \mathbf{P}^1$. We will show that the splitting of N_1 is $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(3)^{\oplus 7} \oplus \mathcal{O}_{\mathbf{P}^1}(4)$. The degree of N_1 is 27, and its rank is 9, so N_1 is balanced if and only if it has a summand of degree ≥ 4.

Let h denote the divisor class of the relative $\mathcal{O}(1)$ on $\mathbf{P}E$, and let f be the class of a fiber of $\mathbf{P}E \to \mathbf{P}^1$. Then the linear system $|h - 2f|$ restricts to the complete canonical system on $C \subset \mathbf{P}E$, and furthermore, every element of the linear system $|2h - 4f|$ is obtained as a sum of products of elements in $|h - 2f|$. Since the canonical model of C lies on a unique quadric Q, we see that there is a unique element of $|2h - 4f|$ containing C. This, in turn, translates into an $\mathcal{O}(4)$ summand in N_1.

The example above can be generalized, provided the genus is small compared to the degree. For large g, however, we expect that all bundles in the Casnati–Ekedahl resolution will be balanced. Evidence for this is given by the next theorem.

THEOREM 3.2 (See [**3**]). *The bundle N_1 is balanced for a general branched cover provided g is much larger than d. When d divides $g-1$, all syzygy bundles N_i are balanced for a general branched cover.*

The statement for N_1 is the main result of [**3**]; the statement for d dividing $g-1$ is [**3**, Proposition 2.4].

We now give a brief overview of the proof that the syzygy bundles are generically balanced when this divisibility constraint holds. Since the Hurwitz space is irreducible, and the condition of being balanced is open, it suffices to provide one example of a cover where it holds.

Consider the surface $S = E \times \mathbf{P}^1$, where E is any elliptic curve. Let D be any smooth curve on S with $D \cdot (\{e\} \times \mathbf{P}^1) = k$ and $D \cdot (E \times \{t\}) = d$. We will argue that the projection $D \to \mathbf{P}^1$ has the property that every syzygy bundle N_i is balanced.

The surface S embeds in $\mathbf{P}^{d-1} \times \mathbf{P}^1$ so that the projection to $\mathbf{P}^{d-1}$ is the projection $S \to E$ composed with the embedding of E as an elliptic normal curve of degree d. The curve D is then the intersection of S with a divisor $H \subset \mathbf{P}^{d-1} \times \mathbf{P}^1$ which restricts to a hyperplane in every $\mathbf{P}^{d-1}$.

The main point is that the minimal free resolution of the elliptic normal curve $E \subset \mathbf{P}^{d-1}$ (embedded by any complete linear system of degree d) has the same shape as the Casnati–Ekedahl resolution of a degree d branched cover. This is equivalent to saying that elliptic normal curves are arithmetically Gorenstein. The minimal free resolution of $E \subset \mathbf{P}^{d-1}$ pulls back to a relative minimal free resolution of $\mathcal{O}_S$ as an $\mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}$-module. More precisely, we get a resolution

(3.1)
$$0 \to \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}(-d) \to V_{d-3} \otimes \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}(-d+2) \to V_{d-4} \otimes \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}(-d+3) \to \cdots$$
$$\cdots \to V_2 \otimes \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}(-3) \to V_1 \otimes \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1}(-2)^{\oplus r_1} \to \mathcal{O}_{\mathbf{P}^{d-1} \times \mathbf{P}^1} \to \mathcal{O}_S \to 0,$$

where the V_i are vector spaces of the same dimension as the rank of the bundles N_i in the Casnati–Ekedahl resolution of a degree d branched cover, and the twists refer to twists by the pullback of $\mathcal{O}_{\mathbf{P}^{d-1}}(1)$. The restriction of this resolution to the relative hyperplane H yields the Casnati–Ekedahl resolution of $D = H \cap S$. Note that the pullback of $\mathcal{O}_{\mathbf{P}^{d-1}}(1)$ to H is $\mathcal{O}_H(1) \otimes \pi^* L$ where $\pi \colon H \to \mathbf{P}^1$ is the projection, and L is a line bundle on $\mathbf{P}^1$. Therefore, the terms in the resolution (3.1) restrict to $\pi^*(V_i \otimes L^{-i-1}) \otimes \mathcal{O}_H(-i-1)$. We thus get $N_i = V_i \otimes L^{-i-1}$, which is balanced.

Since D is a curve of type (d,k) on $E \times \mathbf{P}^1$, its genus g is $d(k-1)+1$. This is where we get the degree-genus restriction $g \equiv 1 \pmod{d}$.

REMARK 3.3. The strategy above required understanding the relative resolution of the (trivial) genus one fibration $S \to \mathbf{P}^1$. In general, if $f : X \to \mathbf{P}^1$ is a genus one fibration with simple nodes as singularities, then a relative degree d divisor $D \subset X$ yields a relative embedding

$$X \hookrightarrow \mathbf{P}(f_* \mathcal{O}_X(D)) \to \mathbf{P}^1$$

and X enjoys a relative resolution with exactly the same form as the Casnati–Ekedahl resolution of a degree d branched cover. The bundles appearing in the relative resolution of X and the Casnati–Ekedahl resolution for $D \to \mathbf{P}^1$ are determined by each other,

and one is balanced if and only if the other is. In this way, the study of Casnati–Ekedahl resolutions is intimately related to the study of relative resolutions of genus one fibrations.

REMARK 3.4. One might be able to deduce that N_i is as balanced as possible (that is, $h^1(\operatorname{End} E(-1)) = 0$) even when $g \not\equiv 1 \pmod d$ as follows. Notice that for a singular $D \subset E \times \mathbf{P}^1$, the argument sketched above still holds without change. If one understands how the syzygy bundles N_i are related for D and its normalization $\widetilde{D}$, one might be able to handle the cases where $g \not\equiv 1 \pmod d$.

The strategies outlined in Remark 3.3 and Remark 3.4 have not been fully explored. The authors intend to investigate them in the future. Notice that the idea of using branched covers on elliptic fibrations parallels the idea of using curves on K3 surfaces aprés [15].

4. The divisor class of μ_i

The goal of this section is obtain the divisor class of the higher syzygy divisors μ_i.

4.1. The Bogomolov expression. Let B be a smooth curve and $p \colon P \to B$ a $\mathbf{P}^1$ bundle. Let E be a vector bundle of rank r on P which is balanced on the generic fiber of p. Denote by $\mu(E)$ the locus of points in B over which E is unbalanced with the scheme structure given by the first Fitting ideal of $R^1 p_*(\operatorname{End} E \otimes \mathcal{O}(-1))$.

PROPOSITION 4.1. *In the above setup, we have*

$$[\mu(E)] = c_1^2(E) - 2r \operatorname{ch}_2(E)$$

PROOF. By definition, we have

$$[\mu(E)] = -c_1 R p_*(\operatorname{End} E \otimes \mathcal{O}(-1))$$

By Grothendieck–Riemann–Roch, we get

$$\operatorname{ch} R p_*(\operatorname{End} E \otimes \mathcal{O}(-1)) = p_* \left(\operatorname{ch}(E) \otimes \operatorname{ch}(E^\vee) \operatorname{ch} \mathcal{O}(-1) \operatorname{td}(P/B) \right)$$
$$= 2r \operatorname{ch}_2(E) - c_1^2(E).$$

$\square$

Let us call the expression $c_1^2(E) - 2r \operatorname{ch}_2(E)$ the *Bogomolov expression* and denote it by $\operatorname{Bog}(N_i)$. Note that $\operatorname{Bog}(N_i) = \operatorname{Bog}(N_i \otimes L)$ for any line bundle L, which should be expected from the geometric interpretation.

4.2. The Koszul resolution. By Proposition 4.1, the problem of finding the divisor class of μ_i is reduced to finding $c_1(N_i)$ and $\operatorname{ch}_2(N_i)$. To calculate the Chern classes of the bundles N_i, we express them as cohomology bundles of a resolution involving more familiar bundles. This is the Koszul resolution, which we now recall.

Let R be a (Noetherian) ring and E a locally free R-module of rank r. Let $S = \operatorname{Sym}^*(E)$ be the symmetric algebra on E and let M be a graded S-module. Suppose we have a graded resolution

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where

$$F_i = \bigoplus_{j \geq 0} N_{ij} \otimes_R S(-i-j)$$

and the N_{ij} are locally free R-modules. Suppose the resolution is minimal in the sense that all the maps $F_{i+1} \to F_i$ have graded components in positive degree. Then we have the identification

(4.1) $$N_{ij} = \mathrm{Tor}_S^i(M, R)_{i+j},$$

where the subscript denotes the graded component. The right hand side can be computed in another way. Instead of using an S-resolution of M, we use the S-resolution of R given by the Koszul complex

$$0 \to \wedge^r E \otimes_R S(-r) \to \cdots \to \wedge^p E \otimes_R S(-p) \to \cdots \to E \otimes_R S(-1) \to S \to R \to 0.$$

Tensoring by M and taking the $(i+j)$th graded component yields the complex

$$K_{i+j} : \wedge^r E \otimes_R M_{i+j-r} \to \cdots \xrightarrow{d_{p-1}} \wedge^p E \otimes_R M_{i+j-p} \xrightarrow{d_p} \cdots \to E \otimes_R M_{i+j-1} \to M_{i+j}.$$

Let $H^p(K_{i+j}) = \ker d_p / \operatorname{im} d_{p-1}$ be the cohomology. Then we get the identification

$$\mathrm{Tor}_S^j(M, R)_{i+j} = H^i(K_{i+j}).$$

Combining with (4.1), we get

$$N_{ij} = H^i(K_{i+j}).$$

Let us now turn to the Casnati–Ekedahl resolution of the universal finite cover $\alpha \colon \mathcal{C} \to \mathcal{P}$. Let $E = \ker(\omega_\phi \to \mathcal{O}_\mathcal{P})$ be the Tschirnhausen bundle and $\iota \colon \mathcal{C} \to \mathbf{P}E$ the relative canonical embedding. Let $I \subset S = \mathrm{Sym}^* E$ be the homogeneous ideal of $\mathcal{C}$. The Koszul complex K_{i+1} for the S-module S/I is the following

$$K_{i+1} : \wedge^{i+1} E \to \wedge^i E \otimes E \to \wedge^{i-1} E \otimes \alpha_*(\omega_\alpha^2) \to \cdots \to \alpha_*(\omega_\alpha^{i+1}).$$

Denote by $K_{i+1}(j)$ the jth term in the above complex, starting from $j = 0$ and counting from the right to the left.

PROPOSITION 4.2. *Let $1 \le i \le d-3$ and let N_i be the ith syzygy bundle of α. Then we have*

$$\mathrm{ch}\, N_i = \sum_{j=0}^{i+1} (-1)^{j-i-1} \mathrm{ch}\left(K_{i+1}(j)\right).$$

PROOF. From the Casnati–Ekedahl resolution of α and the identification of the syzygy bundles with the cohomology of the Koszul complex, we know that

$$H^p(K_{i+1}) = \begin{cases} N_i & \text{if } p = i \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, we have the equality

$$N_i = \sum_{j=0}^{i+1} (-1)^{j-i-1} K_{i+1}(j)$$

in the K-ring, from which the formula for the Chern character follows.   □

4.3. The computation. We now compute $\operatorname{ch} N_i$ using the expression in Proposition 4.2. Since we are ultimately only interested in c_1 and ch_2, we ignore all terms of degree higher than 2. We may assume, for example, that the computation is happening over a general curve $B \to \widetilde{\mathcal{H}}_{d,g}$. Denote by $\pi\colon C \to B$ and $p\colon P \to B$ the two projections.

From Proposition 4.2, we have

$$(4.2) \quad \begin{aligned} \operatorname{ch} N_i &= \sum_{j=0}^{i+1} (-1)^{j-i-1} \operatorname{ch}\left(K_{i+1}(j)\right) \\ &= \left(\sum_{j=0}^{i+1} (-1)^{j-1} \operatorname{ch}(\wedge^{i+1-j}E)\operatorname{ch}(\alpha_*\omega_\alpha^j)\right) - \operatorname{ch}(\wedge^i E) + \operatorname{ch}(\wedge^{i+1}E)\operatorname{ch}(E^\vee). \end{aligned}$$

The two correction terms at the end are needed because the $j = 0$ and $j = 1$ terms in the summation are differ from the corresponding terms of the Koszul resolution in the following way (the computation is in the K-ring):

$$\begin{aligned} [\wedge^{i+1}E] \otimes [\alpha_*\omega_\alpha^0] &= [\wedge^{i+1}E] \otimes [\mathcal{O} + E^\vee] \\ &= [K_{i+1}(i+1)] + [\wedge^{i+1}E] \otimes [E^\vee], \text{ and} \\ [\wedge^i E] \otimes [\alpha_*\omega_\alpha] &= [\wedge^i E] \otimes [\mathcal{O} + E] \\ &= K_{i+1}(i) + [\wedge^i E]. \end{aligned}$$

Next, by Grothendieck–Riemann–Roch applied to α we get

$$(4.3) \quad \operatorname{ch}\alpha_*\omega_\alpha^\ell = \alpha_*\left(1 + \ell \cdot c_1(\omega_\alpha) + \frac{\ell^2 c_1(\omega_\alpha)^2}{2}\right)\left(1 - \frac{c_1(\omega_\alpha)^2}{2} + \frac{c_1(\omega_\alpha)^2 + c_2(\Omega_{C/P})}{12}\right).$$

Note that $c_1(\omega_\alpha)$ is the class of the ramification divisor of $\alpha\colon C \to P$. In particular, $\alpha_* c_1(\omega_\alpha)$ is the class of the branch divisor, which is cut out by a section of $(\det E)^{\otimes 2}$. Therefore, we get

$$(4.4) \quad \alpha_* c_1(\omega_\alpha) = 2c_1 E.$$

Specializing (4.3) to the case $\ell = 0$ and comparing the degree two terms yields

$$(4.5) \quad \operatorname{ch}_2 E = \alpha_*\left(\frac{c_1(\omega_\alpha)^2 + c_2(\Omega_{C/P})}{12}\right).$$

After using (4.4) and (4.5) to simplify (4.3), we get

$$(4.6) \quad \operatorname{ch}\alpha_*\omega_\alpha^\ell = d + (2\ell - 1)c_1(E) + \left(\operatorname{ch}_2(E) + \frac{\ell^2 + \ell}{2}\pi_* c_1(\omega_\alpha)^2\right).$$

For a vector bundle E of rank $d - 1$, we have

$$\begin{aligned} \operatorname{ch}_0 \wedge^\ell E &= \binom{d-1}{\ell}, \\ c_1(\wedge^\ell E) &= \binom{d-2}{\ell-1}c_1(E), \text{ and} \\ \operatorname{ch}_2(\wedge^\ell E) &= \binom{d-2}{\ell-1}\operatorname{ch}_2(E) + \frac{1}{2}\binom{d-3}{l-2}(c_1(E)^2 - 2\operatorname{ch}_2(E)). \end{aligned}$$

Using these identities and using (4.6), we expand the terms $\mathrm{ch}(\wedge^{i+1-j}E)\,\mathrm{ch}(\alpha_*\omega_\alpha^j)$ and carry out the summation. To evaluate the summation in a closed form, we use the following combinatorial identities[1]:

$$\sum_{l=0}^{p}(-1)^l\binom{a}{p-l}=\binom{a-1}{p}$$

$$\sum_{l=0}^{p}(-1)^l\binom{a}{p-l}l=-\binom{a-2}{p-1}$$

$$\sum_{l=0}^{p}(-1)^l\binom{a}{p-l}l(l-1)=2\binom{a-3}{p-2}.$$

The result is the following:

$$\mathrm{ch}_0(N_i)=\frac{i(d-2-i)}{d-1}\binom{d}{i+1}$$

(4.7) $$\mathrm{c}_1(N_i)=(d-2-i)\binom{d-2}{i-1}\mathrm{c}_1(E)$$

$$\mathrm{ch}_2(N_i)=\binom{d-4}{i-1}\left(d\,\mathrm{ch}_2(E)+\frac{(d-4)i+2}{2(d-i-1)}\mathrm{c}_1^2(E)-\mathrm{c}_1(\omega_\alpha)^2\right)$$

We use this computation to finish a postponed proof from Proposition 2.4.

COROLLARY 4.3. $\deg N_i=(d-2-i)(g+d-1)\binom{d-2}{i-1}$.

PROOF. Follows from (4.7) and that $\deg\mathrm{c}_1(E)=(g+d-1)$. $\qquad\square$

THEOREM 4.4. *The push-forward to $\widetilde{H}_{d,g}$ of the Bogomolov expression for N_i is the following linear combination the standard divisor classes:*

$$p_*\mathrm{Bog}(N_i)=A_i\left(6(gd-6g+d+6)\cdot\zeta-d(d-12)\cdot\kappa-d^2\cdot\delta\right),$$

where the coefficient A_i is given by

$$A_i=\binom{d-4}{i-1}^2\frac{(d-2)(d-3)}{6(i+1)(d-i-1)}.$$

PROOF. This is a direct consequence of the results of the Chern class computation collected in (4.7) and the relations in Proposition 2.8. $\qquad\square$

Note that $\mathrm{Bog}(N_i)$ is symmetric with respect to the change $i\longleftrightarrow d-2-i$, consistent with the fact that N_i and N_{d-2-i} are isomorphic up to twisting and taking duals.

The main theorem (Theorem 1.1) follows from Proposition 2.5, the interpretation of the Bogomolov expression (§ 4.1), and Theorem 4.4.

4.4. The supports of μ_i. Given that the divisor classes $[\mu_i]$ are proportional, it is natural to wonder if the divisors μ_i are supported on the same set. It would be surprising if it were true, but we cannot yet preclude this.

Some evidence towards this is provided by the work of Christian Bopp. Using his Macaulay2 package [**2**], he has found many examples where the jumping loci of syzygy bundles are not supported on the same set. Although his examples are for higher codimension loci, we expect there to be examples also in the divisorial case.

---

[1]along with ample help from the computer algebra system `Maple` with its `sumtools` package

References

[1] D. Abramovich, A. Corti, and A. Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003.

[2] C. Bopp and M. Hoff. `RelativeCanonicalResolution.m2`: Construction of relative canonical resolutions and eagon–northcott type complexes. `Macaulay2` package.

[3] G. Bujokas and A. Patel. Invariants of a general branched cover of $\mathbf{P}^1$. *arXiv:1504.03756 [math.AG]*, Apr. 2015.

[4] G. Casnati and T. Ekedahl. Covers of algebraic varieties. I. A general structure theorem, covers of degree 3, 4 and Enriques' surfaces. *J. Algebraic Geom.*, 5:439–460, 1996.

[5] A. Clebsch. Zur theorie der riemann'schen fläche. *Mathematische Annalen*, 6(2):216–230, 1873.

[6] A. Deopurkar and A. Patel. Sharp slope bounds for sweeping families of trigonal curves. *Math. Res. Lett.*, 20(3):869–884, 2013.

[7] A. Deopurkar and A. Patel. The Picard rank conjecture for the Hurwitz spaces of degree up to five. *Algebra Number Theory*, 9(2):459–492, 2015.

[8] S. Diaz and D. Edidin. Towards the homology of Hurwitz spaces. *J. Differential Geom.*, 43(1):66–98, 1996.

[9] M. Fried. Fields of definition of function fields and Hurwitz families—groups as Galois groups. *Comm. Algebra*, 5(1):17–82, 1977.

[10] M. D. Fried and H. Völklein. The inverse Galois problem and rational points on moduli spaces. *Math. Ann.*, 290(4):771–800, 1991.

[11] W. Fulton. Hurwitz schemes and irreducibility of moduli of algebraic curves. *The Annals of Mathematics*, 90(3):pp. 542–575, 1969.

[12] F. Geiss. The unirationality of Hurwitz spaces of 6-gonal curves of small genus. *Doc. Math.*, 17:627–640, 2012.

[13] J. Harris and D. Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982.

[14] A. Hurwitz. Ueber die anzahl der riemann'schen flächen mit gegebenen verzweigungspunkten. *Mathematische Annalen*, 55(1):53–66, 1901.

[15] R. Lazarsfeld. Brill-Noether-Petri without degenerations. *J. Differential Geom.*, 23(3):299–307, 1986.

[16] S. Mochizuki. The geometry of the compactification of the Hurwitz scheme. *Publ. Res. Inst. Math. Sci.*, 31(3):355–441, 1995.

[17] A. Patel. Special codimension one loci in Hurwitz spaces. *arXiv:1508.06016 [math.AG]*, Aug. 2015.

[18] Z. Ran. Deformations of maps. In *Algebraic curves and projective geometry (Trento, 1988)*, volume 1389 of *Lecture Notes in Math.*, pages 246–253. Springer, Berlin, 1989.

[19] B. Riemann. Theorie der Abel'schen Functionen. *J. Reine Angew. Math.*, 54:101–155, 1857.

[20] M. Romagny and S. Wewers. Hurwitz spaces. In *Groupes de Galois arithmétiques et différentiels*, volume 13 of *Sémin. Congr.*, pages 313–341. Soc. Math. France, Paris, 2006.

[21] F. Severi. *Vorlesungen über Algebraische Geometrie*. Teubner Verlag, 1921.

[22] J. H. Silverman. The space of rational maps on $\mathbf{P}^1$. *Duke Math. J.*, 94(1):41–77, 1998.

[23] G. van der Geer and A. Kouvidakis. The cycle classes of divisorial Maroni loci. *arXiv:1509.08598 [math.AG]*, Sept. 2015.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027
E-mail address: `anandrd@math.columbia.edu`

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CARNEY HALL, CHESTNUT HILL, MA 02467
E-mail address: `anand.patel@bc.edu`