A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS

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ABSTRACT. We propose compactifications of the moduli space of Bridgeland stability conditions of a triangulated category. Our construction arises from a viewing a stability condition as a metric on the underlying category and is inspired by the Thurston compactification of the Teichmüller space of hyperbolic metrics on a surface. The key ingredient in the construction are maps from the stability manifold to an infinite projective space. We prove that, under suitable hypotheses, these maps are injective and their image has a compact closure. We identify a family of points in the boundary that are categorical analogous to the intersection functionals in Teichmüller theory.

We study in detail the geometry of the resulting compactification for the 2-Calabi–Yau categories of quivers, and fully work out the cases of the A_2 and $\widehat{A}_1$ quivers. To do so, we carefully examine the dynamics of Harder–Narasimhan multiplicities under auto-equivalences of the category. We introduce finite automata to study this dynamics and employ it in our analysis of the A_2 and $\widehat{A}_1$ categories.

1. INTRODUCTION

The space of Bridgeland stability conditions has emerged as an important invariant of triangulated categories, evidenced by its prominent role in homological algebra, algebraic geometry, mirror symmetry, representation theory, and mathematical physics. This paper takes the point of view of studying stability conditions through the metrics they define on the underlying triangulated category [23]. This perspective infuses the study of stability conditions with ideas from metric geometry, opening up new possibilities inspired by the study of moduli spaces of metrics elsewhere in mathematics.

In recent years, a significant body of work has established connections between metric geometry and homological algebra, particularly by relating Teichmüller theory and Bridgeland stability [10, 11, 12, 14, 17, 23, 31, 33]. A particularly promising prospect of this connection is a theory of dynamics in triangulated categories parallel to the theory of dynamics in surface geometry. To further develop this subject, we must consider not only spaces of Bridgeland stability conditions but also their modular compactifications.

In this paper, we propose a family of such compactifications, inspired by Thurston's compactification of Teichmüller space. The first part of the paper establishes general results about this construction by studying a stability condition via its associated family of metrics. The highlights of this part of the paper are Theorems 3.9 and 4.3. Theorem 3.9 shows that the stable objects of a stability condition (though not necessarily the central charge) are determined by any one of its associated metrics. Theorem 4.3 shows that in fact the entire stability condition (including the central charge) is determined by any pair of its associated metrics.

The second part of the paper focuses on a family of well-studied categories, namely the 2-Calabi–Yau categories associated to quivers. In this case, we prove that a stability condition is completely determined by any one of its associated metrics. In Sections 7

and 8 respectively, we work out our proposed compactification explicitly for two small but highly non-trivial cases, namely the A_2 and $\widehat{A}_1$ quivers. To do so, we introduce a number of general ideas, including the use of automata to describe the evolution of Harder–Narasimhan filtrations (Section 1.4 and Section 5) and the notion of rectifiable filtrations (Appendix A).

1.1. **The construction.** Thurston's compactification of Teichmüller space proceeds by embedding it in an infinite projective space, and then taking its closure. To do this, one first fixes a suitable collection of curves $\mathbf{S}$. The embedding sends a point of Teichmüller space, represented by a metric μ, to the real-valued function on $\mathbf{S}$ defined by the length with respect to μ.

In our categorical analogue of Thurston's construction, Bridgeland stability conditions provide the metrics. The analogue of the set of curves tends to be flexible—the best choice may vary for different categories. In the examples studied in the second part of the paper, we take $\mathbf{S}$ to be the set of spherical objects. Indeed, as described by Khovanov and Seidel [26], in a 2-Calabi–Yau category of type A there is a precise correspondence between spherical objects and curves on the punctured disc. In greater generality, a reasonable choice for "curves" may be the objects that are stable for some stability condition. The categorical analogue of the length of a curve with respect to a metric is the mass of such an object with respect to a stability condition, whose definition we now recall.

Let $\mathcal{C}$ be a triangulated category. Given a stability condition τ on $\mathcal{C}$ with central charge Z, the *mass* of an object x is defined as

$$(1) \qquad m_\tau(x) = \sum_i |Z(a_i)|,$$

where the a_i are the semistable Harder–Narasimhan factors of x. The notion of mass admits a q-analogue for any positive real number $q > 0$. The *q-mass* of an object x is defined as

$$m_{q,\tau}(x) = \sum_i q^{\phi(a_i)} |Z(a_i)|,$$

were $\phi(a_i)$ is the phase of the semistable object a_i.

The q-mass satisfies the triangle inequality [23]. That is, in a distinguished triangle $x \to y \to z \xrightarrow{+1}$, we have $m_{q,\tau}(y) \le m_{q,\tau}(x) + m_{q,\tau}(z)$. It easily follows that the function that sends a morphism f to the non-negative real number $m_{q,\tau}(\mathrm{Cone}(f))$ defines a translation invariant metric on the category in the sense of [28]. In our paper, all the metrics on categories are of this kind.

Let $\mathbf{S}$ be a subset of the set of objects of $\mathcal{C}$. Denote by $\mathbf{R}^{\mathbf{S}}$ the space of functions from $\mathbf{S}$ to $\mathbf{R}$, endowed with the product topology. Let $\mathbf{P}^{\mathbf{S}}$ denote the quotient $(\mathbf{R}^{\mathbf{S}} \setminus \{0\})/\mathbf{R}^{\times}$. Let $\mathrm{Stab}(\mathcal{C})$ be the space of stability conditions on $\mathcal{C}$, and recall that it admits a standard action of $\mathbf{C}$. For each q, the association $\tau \mapsto m_{q,\tau}$ defines a continuous map

$$m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}}.$$

Our proposed "Thurston compactification" of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ is the closure of the image.

1.2. **The results.** Thurston's compactification of Teichmüller space has the following salient features.

(1) The length function up to scaling determines the metric up to isotopy. That is, the map taking a metric to its length function is injective. In fact, it is a homeomorphism onto its image.

(2) The closure of the image is a compact real manifold with boundary.

(3) The boundary contains a distinguished collection of functionals, namely the intersection functionals. This collection is in fact dense in the boundary. Finally, all points of the boundary have a modular interpretation as projective measured foliations.

We now address each of these points in our categorical context. In the remarks below, we have chosen $\mathbf{S}$ to be the set of objects x such that there is a stability condition in which x is semistable. We remind the reader that the best choice for $\mathbf{S}$ may vary in applications.

1.2.1. *Injectivity and homeomorphic embedding.* The family of functions $m_{q,\tau}$ determines the stability condition $\tau \in \mathrm{Stab}(\mathcal{C})/\mathbf{C}$. In fact, we prove in Theorem 4.3 that for any two distinct positive real numbers q_1, q_2, the product map $m_{q_1} \times m_{q_2}$ is injective. We conjecture that a product of mass maps gives a homeomorphic embedding of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$.

We prove in Theorem 3.9 that for any single q, the function $m_{q,\tau}$ determines the stable objects of τ. On the other hand, we note that a single $m_{q,\tau}$ may fail to determine the central charge of τ. As a result, we cannot expect in general that a single m_q gives an embedding of the stability manifold (a phenomenon also observed in [27]).

Nevertheless, for 2-CY categories of quivers, we prove in Proposition 6.14 that for any q, the map m_q is injective. Moreover, in types A_2 and $\widehat{A_1}$, we prove in Propositions 7.8 and 8.9 that $q = 1$ already gives a homeomorphic embedding. (The case of $q \neq 1$ is handled in the follow-up work [1] using similar techniques). Our key technical tool is *Harder–Narasimhan automata*, which we briefly discuss in Section 1.4 and develop in Section 5.

1.2.2. *Properties of the closure.* We prove that the closure of the image of m_q is compact under mild hypotheses. Precisely, as long as the set $\mathbf{S}$ contains a classical generator of the category, the closure of the image of m_q is compact (Proposition 4.1). Categories that have a classical generator include derived categories of coherent sheaves on a quasi-projective variety, and the category of perfect complexes over a dg algebra.

For applications to dynamics, it is particularly important to understand when the compactification is homeomorphic to a closed Euclidean ball, since in that case every autoequivalence will act with a fixed point on the compactification. For the A_2 2-Calabi–Yau category, we prove that the closure of m_1 is homeomorphic to a closed Euclidean ball (Proposition 7.23).

It is an important question to understand analogous results for other categories, for example, 2-CY quiver categories and derived categories of K3 surfaces.

1.2.3. *Functionals on the boundary.* Assume that $\mathcal{C}$ is $\mathbf{k}$-linear and of finite type. Let $a \in \mathcal{C}$ be a spherical object. Associated to a, we have a function on $\mathbf{S}$ defined by

$$x \mapsto \dim \mathrm{Hom}^{\bullet}(a, x) = \sum_i \dim \mathrm{Hom}^i(a, x).$$

In Definition 4.8, we define a close analogue of this function which also incorporates the q, denoted by $\overline{\mathrm{hom}}_q(a)$. This is a categorical analogue of the intersection functional of a curve. We prove in Corollary 4.13 that, in direct analogy with geometry, the function $\overline{\mathrm{hom}}_q(a)$ represents a boundary point in our compactification.

Whether or not the $\overline{\mathrm{hom}}$ functionals are dense in the boundary is a subtle question, whose answer depends on q. In the case of the 2-CY category of type A_2 and $q = 1$, the answer is "yes"—they are essentially the rational numbers $\mathbf{Q} \cup \{\infty\}$ on the boundary circle $\mathbf{R} \cup \{\infty\}$. For the same category and $q \neq 1$, however, they are not dense in the boundary, but a fractal subset, which may be identified with the left q-rational numbers of [1].
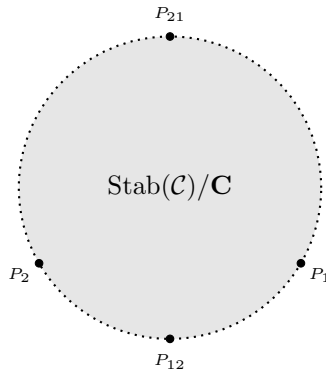
FIGURE 1. The compactification of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$. The spherical objects appear as a dense subset of the boundary.

Giving a moduli interpretation to the boundary points, as a kind of categorical analogue of measured foliations, remains an excellent outstanding problem for future work.

1.3. **Departures from Teichmüller theory.** The categorical picture, although motivated by Thurston's geometric picture, also differs from it interesting ways.

While a stability condition defines a family of metrics on the underlying triangulated category, those metrics are not "hyperbolic" in a meaningful sense. They behave more like flat metrics, which is consonant with [11], where the authors establish a precise relationship between quadratic differentials and stability condition in certain 3-CY settings.

In the 2-CY category of type A_n, the stability manifold and the Teichmüller space of a disk with $(n+1)$ punctures are both universal covers of the complement of the hyperplane arrangement of type A_n. For $n > 2$, however, we believe that no homeomorphism between them extends to a homeomorphism between the Thurston compactification and any of our compactifications. In Thurston's construction, the Teichmüller space maps homeomorphically onto the interior of the Thurston compactification. In our case, on the other hand, we see that for $n > 2$, the interior of our compactification contains additional points which are not themselves stability conditions. (Some of them can be identified with degenerate stability conditions constructed by Bolognese [4].) It seems likely that a closer Teichmüller-theoretic analogue of our construction is a compactification of the moduli space of flat metrics on a surface, constructed, for example, by Duchin, Leininger, and Rafi [13]. It would be interesting to relate our construction and theirs.

Our theory has a natural q-analogue, so we obtain a family of compactifications of the space of stability conditions, depending on a positive real parameter q. This should not be confused with existing literature on quantizations of Teichmüller space (e.g. [16]), which pertain to quantizing algebras of functions.

1.4. **Harder–Narasimhan automata.** After establishing the general results, we focus on the 2-CY categories $\mathcal{C}_\Gamma$ associated to finite connected quivers Γ. The Artin–Tits braid group B_Γ acts (conjecturally faithfully) on the 2-CY category $\mathcal{C}_\Gamma$. To understand the compactification of the stability manifold, we heavily use this action. Along the way, we develop techniques that may be broadly applicable to the study of stability conditions in the presence of a group action.

The key question we address is how the HN filtration of an object transforms under an auto-equivalence. To make this precise, fix a stability condition and let Σ be the set of indecomposable semistable objects up to shift. For an object x, let $\mathrm{HN}(x) \in \mathbf{Z}^\Sigma$ be its HN multiplicity vector—the function that assigns to $s \in \Sigma$ the number of times it appears (up to shift) among the HN factors of x. Suppose a group G acts on our category by auto-equivalences. This action may not induce a compatible linear action on $\mathbf{Z}^\Sigma$. Nevertheless, in many situations—including the A_2 and $\widehat{A}_1$ cases studied in the latter part of the paper—it induces a "piecewise linear" action. Roughly speaking, this means the following. We can partition the objects of the category into subsets, called *states*. For each ordered pair of states, there are certain allowable elements of G that send all objects of the first state into objects of the second state and induce linear transformations on their HN multiplicities. We formalise this structure as a decorated finite automaton called an *HN automaton* (see Section 5).

An HN automaton allows us to express the evolution of HN multiplicities as a linear map. This can lead to a direct method to compute the categorical entropy of an autoequivalence (see [18]).

We construct HN automata in type A_2 and $\widehat{A}_1$. These automata recognize all braids, meaning that every braid can be represented by a sequence of allowable arrows in the decorated graph. We use these automata as a tool to prove all of the basic properties of our compactifications. As an additional application, we recover a theorem of Rouquier–Zimmermann [29, Proposition 4.8] in the type A_2 case and prove a new analogue of their theorem in the type $\widehat{A}_1$ case. We anticipate automata to play an important role in the study of group actions on triangulated categories.

1.5. **Organisation.** In Section 2, we recall some background on Bridgeland stability conditions, spherical objects and spherical twists. In Section 3, we prove foundational results about the mass functions associated to stability conditions, including the fact that the mass function determines the stable objects (Theorem 3.9). In Section 4, we use the mass to define a map from the stability manifold to a projective space, and establish its basic properties. These include the compactness of the closure of the image (Proposition 4.1), injectivity for a pair of masses (Theorem 4.3), and the presence of $\overline{\mathrm{hom}}$ functionals in the closure (Corollary 4.13). In Section 5, we introduce the notion of Harder–Narasimhan automata.

So far, all the discussion applies generally. From Section 6 onwards, we focus on specific categories, namely the 2-Calabi–Yau categories of quivers. Section 6 begins with the definition and basic properties of these categories, and proceeds to prove the injectivity of a single mass map (Proposition 6.14). Section 7 contains the A_2 case in detail. Using a suitable HN automaton, we prove that the mass map is a homeomorphic embedding (Proposition 7.8; the boundary is a circle (Proposition 7.16 and Proposition 7.21); and construct an explicit homeomorphism from the closure to a 2-disk (Proposition 7.23). Along the way, we reprove a result of Rouquier–Zimmermann [29, Proposition 4.8]. Section 8 contains analogous results for the $\widehat{A}_1$ case, including a new analogue of Rouquier–Zimmermann's result.

The paper has two appendices. They contain technical results, which may have broader use. Appendix A addresses the question of when a filtration in a triangulated category can be re-arranged to obtain the Harder–Narasimhan filtration. Appendix B studies homological properties of self-extensions of a spherical object.

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2. Background

2.1. **Stability conditions.** We assume familiarity with the theory of Bridgeland stability conditions (see [8]), but we recall the basic definitions. Let $\mathcal{C}$ be a triangulated category. A stability condition τ on $\mathcal{C}$ is specified by two compatible structures $(\mathcal{P}, Z)$:

(1) the *slicing* $\mathcal{P}$ consists of additive subcategories $\mathcal{P}(\phi)$ for each real number ϕ;
(2) the *central charge* Z is a homomorphism of additive groups from the Grothendieck group $K(\mathcal{C})$ to the complex numbers $\mathbf{C}$.

The slicing and the central charge satisfy the following conditions:

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
(2) if $a \in \mathcal{P}(\phi)$ and $b \in \mathcal{P}(\psi)$ with $\phi > \psi$, then $\mathrm{Hom}(a, b) = 0$;
(3) for every nonzero object $x \in \mathcal{C}$, there exists a sequence of morphisms

$$0 = x_0 \to x_1 \to \cdots \to x_n = x$$

such that if we define a_i so that each triangle in

$$0 = x_0 \longrightarrow x_1 \longrightarrow x_2 \quad \cdots \quad x_{n-1} \longrightarrow x_n = x,$$

is exact, then $a_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \cdots > \phi_n$;
(4) for every nonzero $a \in \mathcal{P}(\phi)$, there is some positive real number r such that

$$Z([a]) = r \cdot e^{i\pi\phi}.$$

The additive category $\mathcal{P}(\phi)$ turns out to be abelian. Its objects are called *semistable of phase ϕ*.

We call a sequence of maps $x_0 \to \cdots \to x_n = x$ a *filtration* of x. For each i, let a_i be such that $x_{i-1} \to x_i \to a_i \xrightarrow{+1}$ is a distinguished triangle. We then say that the objects a_i are *factors* of the filtration.

For a morphism $f \colon x_{i-1} \to x_i$, we use $\mathrm{Cone}(f)$ to denote any object that fits into a distinguished triangle $x_{i-1} \xrightarrow{f} x_i \to \mathrm{Cone}(f) \xrightarrow{+1}$. Such an object is unique up to a possibly non-unique isomorphism. We only use this notation in contexts where the ambiguity of the isomorphism is irrelevant.

The specific filtration described in (3) is called the *Harder–Narasimhan (HN) filtration* of x. Its factors are called *HN factors* and their phases are called *HN phases*. The HN filtration is unique up to isomorphisms (see, e.g. [8, §3]).

Fix a stability condition $\tau = (\mathcal{P}, Z)$ on $\mathcal{C}$. We recall from [8, §3] the induced truncation structure on $\mathcal{C}$. This structure depends on the slicing, but the notation hides this dependence. Let $I \subset \mathbf{R}$ be an interval. Denote by $\mathcal{C}_I \subset \mathcal{C}$ the full subcategory consisting of objects whose HN phases lie in I. The notation $\mathcal{C}_{\geq \alpha}$ means $\mathcal{C}_{[\alpha, \infty)}$, and likewise for $\mathcal{C}_{<\alpha}$

and $\mathcal{C}_{\geq\alpha}$ and $\mathcal{C}_{>\alpha}$. The pair of categories $\mathcal{C}_{>\alpha}$ and $\mathcal{C}_{\leq\alpha+1}$ define a t-structure on $\mathcal{C}$ with the heart $\mathcal{C}_{(\alpha,\alpha+1]}$. As a result, we have a truncation functor

$$\mathsf{tr}_{>\alpha}\colon \mathcal{C} \to \mathcal{C}_{>\alpha},$$

right adjoint to the inclusion $\mathcal{C}_{\leq\alpha} \to \mathcal{C}$ [3, 1.3]. We abbreviate $\mathsf{tr}_{>\alpha}\, x$ to $x_{>\alpha}$. We have similar functors $\mathsf{tr}_{<\alpha}$ and $\mathsf{tr}_{\geq\alpha}$, and $\mathsf{tr}_{\leq\alpha}$, and the corresponding notation $x_{<\alpha}$ and $x_{\geq\alpha}$ and $x_{\leq\alpha}$. For an interval $I = [\alpha, \beta]$, we define tr_I as

$$\mathsf{tr}_I = \mathsf{tr}_{\geq\alpha} \circ \mathsf{tr}_{\leq\beta} = \mathsf{tr}_{\leq\beta} \circ \mathsf{tr}_{\geq\alpha},$$

and set $x_I = \mathsf{tr}_I\, x$. We have analogous functors and notation for other kinds of intervals, namely (α, β), $(\alpha, \beta]$, and $[\alpha, \beta)$.

Let $\mathrm{Stab}(\mathcal{C})$ be the set of all stability conditions satisfying an additional condition called local finiteness [8, Definition 5.7]. The main result of [8] states that each connected component Σ of $\mathrm{Stab}(\mathcal{C})$ is a complex manifold, and the forgetful map

$$\mathrm{Stab}(\mathcal{C}) \ni (\mathcal{P}, Z) \mapsto Z \in \mathrm{Hom}(K(\mathcal{C}), \mathbf{C})$$

is a local homeomorphism to a linear subspace $V(\Sigma)$.

The complex numbers $\mathbf{C}$ act on $\mathrm{Stab}(\mathcal{C})$ as follows. Say $\tau = (\mathcal{P}, Z)$ and $\omega = a + i\pi b$, then

$$(\omega \cdot \mathcal{P})(\phi) = \mathcal{P}(\phi - b) \text{ and } \omega \cdot Z = e^{\omega} Z.$$

The action of $\mathbf{C}$ on $\mathrm{Stab}(\mathcal{C})$ lifts the action of $\mathbf{C}^*$ on $\mathrm{Hom}(K(\mathcal{C}), \mathbf{C})$ by scaling. It follows that the induced forgetful map

$$\mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathrm{Hom}(K(\mathcal{C}), \mathbf{C})/\mathbf{C}^*$$

maps each connected component $\Sigma/\mathbf{C}$ locally homeomorphically to $V(\Sigma)/\mathbf{C}^*$.

Observe that the purely imaginary numbers $i\mathbf{R} \subset \mathbf{C}$ simply shift the slicing and rotate the central charge. We call such transformations *translations*.

2.2. Spherical objects and twists.

Let $\mathbf{k}$ be a field and $\mathcal{C}$ a $\mathbf{k}$-linear triangulated category. Assume that $\mathcal{C}$ is of finite type, that is, for all objects x, y of $\mathcal{C}$, the vector space

$$\mathrm{Hom}^{\bullet}(x, y) = \bigoplus_n \mathrm{Hom}(x, y[n])$$

is a finite-dimensional $\mathbf{k}$-vector space.

We recall the notion of a spherical object introduced in [30, Definition 2.9] (see also [20, Chapter 8]). An object x of $\mathcal{C}$ is said to have a Serre dual if the functor $\mathrm{Hom}^{\bullet}(x, -)$ is representable. If x has a Serre dual, the representing object Sx is unique up to a unique isomorphism, and there are functorial isomorphisms

$$\mathrm{Hom}^{\bullet}(x, y) \cong \mathrm{Hom}^{\bullet}(y, Sx)^{\vee}.$$

A d-*Calabi–Yau* object is an object x such that its shift $x[d]$ is its Serre dual. A d-*spherical object* is a d-Calabi–Yau object x whose endomorphism algebra is isomorphic to the cohomology algebra of the d-sphere:

$$\mathrm{Hom}^{\bullet}(x, x) \cong H^{\bullet}(S^d, \mathbf{k}).$$

Note that this algebra is simply $k[t]/t^2$, where t has degree d. We denote by loop_x an unspecified homogeneous generator of degree d of $\mathrm{Hom}^{\bullet}(x, x)$.

If x is a spherical object and y is any object, then there is a perfect pairing

$$(2) \qquad\qquad \mathrm{Hom}^i(x, y) \times \mathrm{Hom}^{d-i}(y, x) \to \mathrm{Hom}^d(x, x) \cong \mathbf{k},$$

induced by composition [25, Proposition 2.2].

If every object in the category $\mathcal{C}$ is d-Calabi–Yau (for the same d), then the category $\mathcal{C}$ is said to be a d-Calabi–Yau category. Thus, if $\mathcal{C}$ is a d-Calabi–Yau category, then for any $x, y \in \mathcal{C}$, there exist functorial isomorphisms

$$\operatorname{Hom}^i(x, y) \cong \operatorname{Hom}^{d-i}(y, x)^\vee.$$

Moreover, it follows that if x is an indecomposable object in a d-Calabi–Yau category such that the $\mathbf{k}$-vector space $\operatorname{Hom}^\bullet(x, x)$ is two-dimensional, then x is automatically a d-spherical object of $\mathcal{C}$.

There is also a notion of a *strongly d-Calabi–Yau* category, in which the pairing corresponding to the functorial isomorphism (2) is required to be anti-symmetric (see [25]). The categories we study in the later part of the paper are strongly d-Calabi–Yau.

Assume that $\mathcal{C}$ admits a dg-enhancement. This is true, for instance, if $\mathcal{C}$ is algebraic in the sense of [24] or enhanced in the sense of [6], and will hold in all the examples we consider. Fix a dg-enhancement on $\mathcal{C}$, which guarantees that $\mathcal{C}$ has functorial cones. Then any spherical object x gives rise to an autoequivalence $\sigma_x \colon \mathcal{C} \to \mathcal{C}$ called the *spherical twist* in x (see [30, §2.2]). For $y \in \mathcal{C}$, the object $\sigma_x(y)$ is defined to be the cone of the evaluation map

$$x \otimes \operatorname{Hom}^\bullet(x, y) \xrightarrow{\mathrm{ev}} y.$$

The twist σ_x and its inverse σ_x^{-1} give rise to distinguished triangles

$$\operatorname{Hom}^\bullet(x, y) \otimes x \xrightarrow{\mathrm{ev}} y \to \sigma_x(y) \xrightarrow{+1} \text{ and } \sigma_x^{-1}y \to y \xrightarrow{\mathrm{coev}} \operatorname{Hom}^\bullet(y, x)^\vee \otimes x \xrightarrow{+1}.$$

The evaluation map ev is self-explanatory. The co-evaluation map coev is the adjoint to the evaluation map

$$\operatorname{Hom}^\bullet(y, x) \otimes y \to x.$$

3. The mass associated to a stability condition

Let $\mathcal{C}$ be a triangulated category and let τ be a stability condition on $\mathcal{C}$. Fix a positive real number q; for us the most important case is $q = 1$.

Let x be an object of $\mathcal{C}$. Let

$$0 = x_0 \longrightarrow x_1 \longrightarrow x_2 \quad \cdots \quad x_{n-1} \longrightarrow x_n = x,$$

be the HN filtration of x, where a_i is semistable of phase ϕ_i. We define the (q, τ)-*mass* of x by the formula

$$m_{q,\tau}(x) = \sum_i q^{\phi_i} |Z(a_i)|.$$

We take the mass of the zero object to be zero.

Let $H^i(-)$ be the ith cohomology with respect to any fixed heart associated to τ. Then it is easy to verify that

$$(3) \qquad\qquad m_{q,\tau}(x) = \sum_i q^{-i} m_{q,\tau}(H^i(x)).$$

A key property of the mass is that it satisfies a triangle inequality.

Proposition 3.1 (Triangle inequality). *Let $x \to y \to z \xrightarrow{+1}$ be an exact triangle. Then*

$$m_{q,\tau}(y) \le m_{q,\tau}(x) + m_{q,\tau}(z).$$

Proof. See [23, Theorem 3.8]. □

We devote Appendix A to understanding more about when equality holds in Proposition 3.1. The following is a particularly simple instance.

Proposition 3.2. *Let $x \to y \to z \xrightarrow{+1}$ be an exact triangle. Assume that the lowest HN phase of x is greater than or equal to the highest HN phase of z. Then*

$$m_{q,\tau}(y) = m_{q,\tau}(x) + m_{q,\tau}(z).$$

Proof. This proposition is immediate from Example A.3 (4) and Proposition A.9. However, we also give a direct argument here.

Using the exact triangle, we can splice together filtrations of x and z to get a filtration of y, as follows. Suppose we are given $0 = x_0 \to \cdots \to x_n = x$ with factors $a_1, \ldots, a_n$ and $0 = z_0 \to \cdots \to z_m = z$ with factors $b_1, \ldots, b_m$.

Set $y_{m+n} = y$, and recall that $z_m = z$. For each $i = m, \ldots, 1$, define y_{n+i-1} by descending induction so that it fits into a map of distinguished triangles as follows:

$$
\begin{array}{ccccccc}
y_{n+i-1} & \longrightarrow & z_{i-1} & \longrightarrow & x[1] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
y_{n+i} & \longrightarrow & z_i & \longrightarrow & x[1] & \xrightarrow{+1} & .
\end{array}
$$

By the octahedral axiom, we see that for each $i = m, \ldots, 1$, there is a distinguished triangle

$$y_{n+i-1} \to y_{n+i} \to b_i \xrightarrow{+1} .$$

Note that $y_n = x$. For each $i = n, \ldots, 1$, set $y_{i-1} = x_{i-1}$. Then for $i = n, \ldots, 1$, there is a distinguished triangle

$$y_{i-1} \to y_i \to a_i \xrightarrow{+1} .$$

Therefore we have constructed a filtration

(4) $$0 = y_0 \to y_1 \to \cdots \to y_{m+n} = y$$

with factors $a_1, \ldots, a_n, b_1, \ldots, b_m$.

Suppose the filtrations (x_i) and (y_j) are the HN filtrations of x and y. By hypothesis, we have $\phi(a_n) \geq \phi(b_1)$. If $\phi(a_n) > \phi(b_1)$, then (4) is the HN filtration of y, and hence

(5) $$m_{q,\tau}(y) = \sum m_{q,\tau}(a_i) + \sum m_{q,\tau}(b_j) = m_{q,\tau}(x) + m_{q,\tau}(z).$$

If $\phi(a_n) = \phi(b_1)$, then shorten the filtration (4) by dropping the y_n term so that in the middle it looks like

$$\cdots \to y_{n-1} \to y_{n+1} \to \cdots .$$

Let c complete the triangle $y_{n-1} \to y_{n+1} \to c \xrightarrow{+1}$. By the octahedral axiom, we have an exact triangle $a_n \to c \to b_1 \xrightarrow{+1}$. Since a_n and b_1 are semistable of the same phase, so is c. As a result, the shortened (4) is the HN filtration of y. Furthermore, we have

$$m_{q,\tau}(c) = m_{q,\tau}(a_n) + m_{q,\tau}(b_1),$$

and hence (5) holds. □

The following key proposition shows that there are no unexpected mass-preserving decompositions of semistable objects.

Proposition 3.3. *Let y be a τ-semistable object of phase ϕ. Let*

$$x \to y \to z \xrightarrow{+1}$$

be an exact triangle such that

$$m_{q,\tau}(y) = m_{q,\tau}(x) + m_{q,\tau}(z).$$

Then x and z are also semistable of the same phase ϕ.

The proof follows the same ideas as the proof of the triangle inequality in [23]. It uses the following q-analogue of the triangle inequality for vectors, which is [23, Lemma 3.6].

Lemma 3.4. *Let a, b, c be complex numbers of the form*

$$a = r_a e^{i\pi\phi_a}, \quad b = r_b e^{i\pi\phi_b}, \quad c = r_c e^{i\pi\phi_c},$$

where r_a, r_b, r_c are positive real numbers and ϕ_a, ϕ_b, ϕ_c are real numbers lying in an interval of length 1. Suppose $b = a + c$. Then

$$r_a q^{\phi_a} + r_c q^{\phi_c} \geq r_b q^{\phi_b}$$

with equality if and only if $\phi_a = \phi_b = \phi_c$.

Lemma 3.5. *Let I be a half-open interval of length 1, and let x be an object in the I-heart associated to τ. Write $Z(x) = re^{i\pi\phi}$ for $\phi \in I$. Then*

$$m_{q,\tau}(x) \geq rq^{\phi},$$

with equality if and only if x is semistable of phase ϕ.

Proof. Apply Lemma 3.4 inductively to the central charges of the HN factors of x. $\square$

Proof of Proposition 3.3. Abbreviate $m_{q,\tau}$ by m and assume that both x and z are nonzero. The statement is unaffected if we translate the slicing of τ. Translate so that y has phase 0.

We first show that z lies in the $[0, 1)$-heart associated to τ. Taking cohomology with respect to the corresponding t-structure gives

$$0 \to H^{-1}(z) \to H^0(x) \to H^0(y) = y \to H^0(z) \to H^1(x) \to 0.$$

Let $s = \operatorname{coker}(H^{-1}(z) \to H^0(x))$ and $t = \ker(H^0(z) \to H^1(x))$. Then we have the short exact sequence

$$0 \to s \to y \to t \to 0.$$

By the triangle inequality (Proposition 3.1), we have

$$(6) \qquad m(y) \leq m(s) + m(t).$$

The triangle inequality applied to (suitable rotations of) the defining triangles of s and t gives

$$(7) \qquad m(s) \leq m(H^0(x)) + qm(H^{-1}(z))$$

and

$$(8) \qquad m(t) \leq m(H^0(z)) + q^{-1}m(H^1(x)).$$

By adding (7) and (8), we get

$$m(s) + m(t) \leq m(H^0(x)) + q^{-1}m(H^1(x)) + qm(H^{-1}(z)) + m(H^0(z)).$$

By (6), the left hand side dominates $m(y)$. For the right hand side, we use

$$(9) \qquad qm(H^{-1}(z)) + m(H^0(z)) \leq \sum_i q^{-i}m(H^i(z)) = m(z),$$

where the last step uses (3). A similar inequality holds for x. Thus, the right hand side is dominated by $m(x) + m(z)$, which by the hypothesis is also $m(y)$. Hence, equalities must hold in (6), (7), (8), and (9). From (9), we conclude that $H^i(z) = 0$ for all $i \notin \{-1, 0\}$. In the exact sequence

$$0 \to s \to y \to t \to 0,$$

the object y is semistable of the lowest possible phase in the heart, namely 0. It follows that s is also semistable of phase 0.

Now consider the exact sequence

$$0 \to H^{-1}(z) \to H^0(x) \to s \to 0.$$

Since s is semistable of the lowest phase in the heart, Proposition 3.2 implies that

$$m(H^0(x)) = m(s) + m(H^{-1}(z)).$$

On the other hand, we know that (7) is an equality, so

$$m(s) = m(H^0(x)) + qm(H^{-1}(z)).$$

The last two equations imply that $H^{-1}(z) = 0$. That is, z lies in the $[0, 1)$-heart.

A parallel argument shows that x lies in the $(-1, 0]$-heart.

Write

$$Z(x) = r_x e^{i\pi\phi_x}, \quad Z(y) = r_y, \quad Z(z) = r_z e^{i\pi\phi_z},$$

where r_x, r_y, r_z are positive real numbers, $\phi_x \in (-1, 0]$ and $\phi_z \in [0, 1)$. Lemma 3.5 says

$$(10) \qquad\qquad m(x) \geq r_x q^{\phi_x} \text{ and } m(z) \geq r_z q^{\phi_z}$$

with equalities if and only if x and z are semistable. Since $Z(y) = Z(x) + Z(z)$ lies on the non-negative real axis, we must have

$$\phi_z \leq \phi_x + 1.$$

Therefore, Lemma 3.4 gives

$$(11) \qquad\qquad r_x q^{\phi_x} + r_z q^{\phi_z} \geq r_y$$

with equality if and only if $\phi_x = \phi_z = 0$. Inequalities (10) and (11), together with the hypothesis $r_y = m(y) = m(x) + m(z)$ imply that equalities must hold in (10) and (11) Thus, x and z are semistable of phase 0. $\qquad\square$

Corollary 3.6. *Let y be a semistable object of phase ϕ. Let*

$$0 = y_0 \to y_1 \to \cdots \to y_n = y$$

be a filtration of y with factors $a_i = \mathrm{Cone}(y_{i-1} \to y_i)$. Suppose that

$$m_{q,\tau}(y) = \sum_{i=1}^{n} m_{q,\tau}(a_i).$$

Then for all i, the object a_i is semistable of phase ϕ.

Proof. Follows from Proposition 3.3 and induction on n. $\qquad\square$

Corollary 3.7. *Let y be a τ-stable object of phase ϕ. Let*

$$0 = y_0 \to y_1 \to \cdots \to y_n = y$$

be a filtration of y with factors $a_i = \mathrm{Cone}(y_{i-1} \to y_i)$. Suppose that

$$m_{q,\tau}(y) = \sum_{i=1}^{n} m_{q,\tau}(a_i).$$

Then for all i except possibly one, we have $a_i = 0$.

Proof. By Corollary 3.6, we conclude that all a_i are τ semistable of the same phase ϕ. Since y is stable, it is a simple object of abelian category of semistable objects of phase ϕ. Thus, its Jordan–Holder filtration is of length 1.   □

Remark 3.8. The Harder–Narasimhan factors of an object x are semistable, but not necessarily stable. However, the abelian category of semistable objects of a given phase is of finite length [8, Definition 5.7]. As a result, every semistable object has a finite Jordan–Hölder filtration whose factors are stable of the same phase. By splicing together these Jordan–Hölder filtrations, we obtain a filtration of x with stable factors. Note that the mass of x is the sum of the masses of the (stable) factors in this filtration.

Theorem 3.9. *Let τ and τ' be two stability conditions such that for each object x of $\mathcal{C}$, we have $m_{q,\tau}(x) = m_{q,\tau'}(x)$. Then an object of $\mathcal{C}$ is τ-stable if and only if it is τ'-stable.*

Proof. Let x be a τ-stable object. As described in Remark 3.8, consider a filtration of x with τ'-stable factors $y_1, \ldots, y_k$ such that

$$m_{q,\tau'}(x) = \sum_{i=1}^{k} m_{q,\tau'}(y_i).$$

Since $m_{q,\tau}(x) = m_{q,\tau'}(x)$ and $m_{q,\tau}(y_i) = m_{q,\tau'}(y_i)$ for every i, we have:

$$m_{q,\tau}(x) = \sum_{i=1}^{k} m_{q,\tau}(y_i).$$

Since x is τ-stable, we conclude by Corollary 3.7 that all but one of the objects y_i are zero. It follows that x is τ'-stable.   □

It is useful to record a generalisation of Theorem 3.9 that allows us to restrict the objects x. Let $\mathbf{S}$ be a set of objects of $\mathcal{C}$ satisfying the following conditions.

(1) If $x \in \mathbf{S}$ and $x \cong y$, then $y \in \mathbf{S}$.
(2) If $x \in \mathbf{S}$ and τ is a stability condition, then the τ-stable HN factors of x also lie in $\mathbf{S}$.

Examples of such $\mathbf{S}$ include:

(a) the union over τ of τ-stable objects,
(b) if $\mathcal{C}$ is 2-Calabi–Yau, the set of spherical objects.

In the second example, the condition (2) holds thanks to the Mukai lemma (see, e.g., [9, Lemma 12.2] or [21, Corollary 2.3]).

Corollary 3.10. *Let $\mathbf{S}$ be a set of objects of $\mathcal{C}$ satisfying the conditions (1) and (2) as above. Let τ and τ' be two stability conditions such that for all $x \in \mathbf{S}$, we have $m_{q,\tau}(x) = m_{q,\tau'}(x)$. Then an object $x \in \mathbf{S}$ is τ-stable if and only if it is τ'-stable.*

Proof. The proof of Theorem 3.9 goes through verbatim.   □

Proposition 3.11. *Let τ and τ' be stability conditions. Suppose that q_1 and q_2 are two distinct positive real numbers such that for each object x we have*

$$m_{q_1,\tau}(x) = m_{q_1,\tau'}(x) \text{ and}$$
$$m_{q_2,\tau}(x) = m_{q_2,\tau'}(x).$$

Then $\tau = \tau'$.

Proof. Let x be any stable object of τ. By Theorem 3.9, x is also a stable object of τ'. Let ϕ be the τ-phase of x and ϕ' be the τ'-phase of x. We have

$$q_1^{\phi}|Z_\tau(x)| = q_1^{\phi'}|Z_{\tau'}(x)| \text{ and}$$

$$q_2^{\phi}|Z_\tau(x)| = q_2^{\phi'}|Z_{\tau'}(x)|.$$

Noting that both $|Z_\tau(x)|$ and $|Z_{\tau'}(x)|$ are nonzero, we have $(q_1/q_2)^{\phi} = (q_1/q_2)^{\phi'}$, which gives $\phi = \phi'$ and hence $|Z_\tau(x)| = |Z_{\tau'}(x)|$. We conclude that the phases and central charges of x in both τ and τ' are equal. Since this holds for all stable objects x, we must have $\tau = \tau'$. $\square$

4. The projective embedding and its boundary

Let $\mathcal{C}$ be a triangulated category and $\mathrm{Stab}(\mathcal{C})$ the space of stability conditions on $\mathcal{C}$. Let $\mathbf{S}$ be a non-empty set of nonzero objects of $\mathcal{C}$. Let $\mathbf{R^S}$ be the space of functions from $\mathbf{S} \to \mathbf{R}$ with the product topology, and set

$$\mathbf{P^S} = \left(\mathbf{R^S} \setminus \{0\}\right)/\mathbf{R}^{\times},$$

where $\mathbf{R}^{\times}$ acts by simultaneous scaling. Fix a positive real number q.

Every stability condition τ gives a nonzero function

$$m_{q,\tau} \colon \mathbf{S} \to \mathbf{R}$$

that sends $x \in \mathbf{S}$ to its τ-mass $m_{q,\tau}(x)$ defined in Section 3. Changing τ to a translate by the action of $\mathbf{C}$ only rescales the function above. As a result, we have a well-defined map

$$(12) \qquad\qquad m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P^S},$$

which we call the *q-mass map*. If $q = 1$, we drop it from the notation. The choice of $\mathbf{S}$ is flexible. The aim of this section is to prove some general properties of the mass map.

4.1. **Pre-compactness.** Recall that an object $p \in \mathcal{C}$ is called a (classical) generator if every object $y \in \mathcal{C}$ is a direct summand of some object $x \in \mathcal{C}$ that admits a filtration

$$0 \to x_0 \to \cdots \to x_m = x,$$

whose factors are shifts of p.

Proposition 4.1 (Pre-compactness)**.** *Suppose the set $\mathbf{S}$ contains objects $p_1, \ldots, p_n$ such that $p = \oplus p_i$ is a generator of $\mathcal{C}$. Then the closure of the image of m_q in $\mathbf{P^S}$ is compact.*

Proof. We first show that for every $y \in \mathcal{C}$, there is some positive real number $N(y)$ such that for every stability condition τ, we have

$$(13) \qquad\qquad m_{q,\tau}(y) \leq N(y)m_{q,\tau}(p).$$

To see this, take $y \in \mathcal{C}$. Consider an object $x \in \mathcal{C}$ such that y is a direct summand of x, and x has a filtration

$$0 = x_0 \to x_1 \to \cdots \to x_m = x$$

whose factors are $p[a_0], \ldots, p[a_m]$. By the triangle inequality (Proposition 3.1), we have for every stability condition τ that

$$m_{q,\tau}(x) \leq (q^{a_0} + \cdots + q^{a_m})m_{q,\tau}(p).$$

Since y is a summand of x, we also have $m_{q,\tau}(y) \leq m_{q,\tau}(x)$. Taking, $N(y) = (q^{a_0} + \cdots + q^{a_m})$, we have the desired inequality (13).

Note in particular by (13) that for every $\tau \in \mathrm{Stab}(\mathcal{C})/\mathbf{C}$, we have

$$m_{q,\tau}(p) = \sum_i m_{q,\tau}(p_i) > 0.$$

Let $\widetilde{m}_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{R}^{\mathbf{S}}$ be the lift of m_q characterised by

$$\sum_i \widetilde{m}_{q,\tau}(p_i) = 1.$$

Let $\widetilde{B}$ be the closure in $\mathbf{R}^{\mathbf{S}}$ of the image of $\widetilde{m}_q$. By (13), we see that the image of $\widetilde{m}_q$ lies in the compact set

$$\prod_{y \in \mathbf{S}} [0, N(y)].$$

Thus, $\widetilde{B}$ is a closed subset of a compact set, and hence compact. Note that by continuity, we have for all $\mu \in \widetilde{B}$ that $\sum \mu(p_i) = 1$, and therefore $\widetilde{B} \subset \mathbf{R}^{\mathbf{S}} - 0$.

Let $\pi \colon (\mathbf{R}^{\mathbf{S}} - 0) \to \mathbf{P}^{\mathbf{S}}$ be the projection. Since $\widetilde{B}$ is compact, so is $\pi(\widetilde{B})$. By construction, $m_q(\mathrm{Stab}(\mathcal{C})/\mathbf{C})$ is contained in the compact set $\pi(\widetilde{B}) \subset \mathbf{P}^{\mathbf{S}}$. Therefore, its closure is a closed subset of a compact set, and hence compact. □

Corollary 4.2. *Under the same hypotheses as Proposition 4.1, fix positive real numbers $q_1, \ldots, q_n$. Consider the map*

$$m_{q_1} \times \cdots \times m_{q_n} \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}} \times \cdots \times \mathbf{P}^{\mathbf{S}}.$$

Then the closure of its image is compact.

Proof. Let $m = m_{q_1} \times \cdots \times m_{q_n}$. Then the image of m is contained in the product of the images of m_{q_i}. Hence, the closure of the image of m is contained in the product of their closures, which is compact by Proposition 4.1. The result follows. □

4.2. **Injectivity.** The following proposition says that if $\mathbf{S}$ is sufficiently large, then two mass maps together determine the stability condition up to translation.

Theorem 4.3. *Assume that $\mathbf{S}$ contains all the stable objects in every stability condition on $\mathcal{C}$. Then for $n \geq 2$, the map*

$$m = m_{q_1} \times \cdots \times m_{q_n} \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}} \times \cdots \times \mathbf{P}^{\mathbf{S}}$$

is injective.

Proof. It is sufficient to show that m is injective for $n = 2$; it then follows for any $n \geq 2$.

Let $\tau, \tau' \in \mathrm{Stab}(\mathcal{C})$, and suppose $m(\tau) = m(\tau')$. By using the action of $\mathbf{C}$ on $\mathrm{Stab}(\mathcal{C})$, we may replace τ' by a stability condition τ'' so that for all $i = 1, 2$, we have

$$m_{q_i}(\tau) = m_{q_i}(\tau'') \in \mathbf{R}^{\mathbf{S}}.$$

Let $x \in \mathbf{S}$ be stable for τ. By Corollary 3.10, x is also stable for τ''. Now, by the argument in the proof of Proposition 3.11, it follows that $\tau = \tau''$. Thus τ and τ' agree up to the action of $\mathbf{C}$. □

Remark 4.4. Suppose $\mathcal{C}$ is 2-Calabi–Yau and the spherical objects in $\mathcal{C}$ span its Grothendieck group. Then Theorem 4.3 holds even if we replace $\mathbf{S}$ by its subset consisting only of *spherical* objects. See Corollary 3.10.

Remark 4.5. In general, the mass map for a single q, namely

$$m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P^S},$$

need not be injective. See [27, § 6.1] for an example, or consider the following example. For simplicity, take $q = 1$. Consider the quiver $1 \to 2$ of type A_2, and let $\mathcal{C}$ be the bounded derived category of finite-dimensional representations of this quiver. Let S_1 and S_2 be the irreducible representations $k \to 0$ and $0 \to k$ respectively. Let E be the representation $k \xrightarrow{1} k$, which fits into the exact sequence

$$0 \to S_2 \to E \to S_1 \to 0.$$

Recall that any object of $\mathcal{C}$ is a direct sum of triangulated shifts of copies of S_1, S_2, and E. Consider a stability condition in which S_1 and S_2 are stable objects of the standard heart, such that $\phi(S_1) < \phi(S_2)$. Due to this inequality, E is not semistable, and its HN factors are S_2 and S_1. So we have

$$m(E) = m(S_1) + m(S_2).$$

Consider two stability conditions τ and τ' in which S_1 and S_2 are stable objects of the $[0,1)$ heart, satisfying the following property:

$$Z_\tau(S_1) = Z_{\tau'}(S_1) = 1, \text{ and}$$

both $Z_\tau(S_2)$ and $Z_{\tau'}(S_2)$ are complex numbers in the upper half plane such that

$$Z_{\tau'}(S_2) = e^{ic} Z_\tau(S_2)$$

for some small $c > 0$. That is, $Z_{\tau'}(S_2)$ is a slight rotation of $Z_\tau(S_1)$. Then it is clear that τ and τ' are not equal points of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$, whereas $m_\tau = m_{\tau'}$.

Remark 4.6. For specific categories $\mathcal{C}$, we can prove stronger injectivity results. For example, for the 2-CY categories $\mathcal{C}_\Gamma$ associated to a connected graph Γ, we prove in Proposition 6.14 that the mass map for a single q is injective.

Remark 4.7. It is a natural question to ask whether the map m from Theorem 4.3 is a homeomorphism onto its image. Since m is continuous and injective, the question is whether it is an open map. In the case of 2-CY quiver categories, we prove a sufficient criterion for m to be an open map (Proposition 6.18). We check this criterion for the A_2 and $\widehat{A_1}$ cases in Proposition 7.8 and Proposition 8.9 respectively.

4.3. **Points in the boundary of the mass map.** In this section, we obtain a family of elements in $\mathbf{P^S}$ that lie in the closure of the image of the stability manifold. These are analogous to the intersection functionals in Teichmüller theory, and arise as limits under successive applications of a spherical twist.

We assume that the category $\mathcal{C}$ is $\mathbf{k}$-linear of finite type with a fixed dg-enhancement.

Definition 4.8. Let a be a d-spherical object in $\mathcal{C}$ and let $q > 0$ be a fixed real number. Let x be any object of $\mathcal{C}$ such that

$$\mathrm{Hom}^i(x, x) = 0 \text{ if } i < 0.$$

(1) If x is indecomposable, set

$$\overline{\mathrm{hom}}_q(a, x) = \begin{cases} \sum_i q^{-i} \dim(\mathrm{Hom}^i(a, x)) & x \neq a[j] \text{ for any } j \in \mathbf{Z}, \\ q^j \cdot |q^{-d} - q^{-1}| & x \cong a[j]. \end{cases}$$

(2) If $x = \oplus_i y_i$ for indecomposable objects y_i, set

$$\overline{\hom}_q(a, x) = \sum_i \overline{\hom}_q(a, y_i).$$

We should regard $\overline{\hom}_q(x, y)$ as a categorical version of the minimal intersection number between two curves. Then, in analogy with [15, §3.3]), we can regard the function $\overline{\hom}_q(a, -) \colon \mathbf{S} \to \mathbf{R}$ as the intersection functional.

The following theorem shows that the quantity $\overline{\hom}_q(a, x)$ governs the growth of the τ-mass of an object under repeated applications of σ_a.

Theorem 4.9. *Let τ be a stability condition on $\mathcal{C}$. Let a be a τ-semistable d-spherical object, and let x be an object that has no endomorphisms of negative degree. Let $q > 0$ and set $e = d - 1$.*

(1) If $0 < q \leq 1$, then

$$\lim_{n \to \infty} \frac{m_{q,\tau}(\sigma_a^n x)}{q + q^{1-e} + \cdots + q^{1-(n-1)e}} = m_{q,\tau}(a) \cdot \overline{\hom}_q(a, x).$$

(2) If $q \geq 1$, then

$$\lim_{n \to \infty} \frac{m_{q,\tau}(\sigma_a^{-n} x)}{q^{1+e} + q^{1+2e} + \cdots + q^{1+ne}} = m_{q,\tau}(a) \cdot \overline{\hom}_q(a, x).$$

In particular, if $q = 1$, then

$$\tag{14} \lim_{n \to \pm\infty} \frac{m_\tau(\sigma_a^n x)}{n} = m_\tau(a) \cdot \overline{\hom}(a, x).$$

The rest of Section 4.3 is devoted to the proof of Theorem 4.9.

Recall that for a d-spherical object a, we have a perfect pairing

$$\mathrm{Hom}^i(x, y) \otimes \mathrm{Hom}^{d-i}(y, x) \to \mathbf{k},$$

which is obtained by the composition followed by a map

$$\mathrm{Tr} \colon \mathrm{Hom}^d(x, x) \to \mathbf{k}.$$

The graded vector space $\mathrm{Hom}^\bullet(a, a)$ is two-dimensional, generated by the identity map id_a of degree zero, and a map loop_a of degree d, which we choose so that $\mathrm{Tr}(\mathrm{loop}_a) = 1$.

Let x be any object. Then we have the distinguished triangles

$$\tag{15} \mathrm{Hom}^\bullet(a, x) \otimes a \xrightarrow{\mathrm{ev}} x \to \sigma_a(x) \xrightarrow{+1}$$

and

$$\sigma_a^{-1} x \to x \xrightarrow{\mathrm{coev}} \mathrm{Hom}^\bullet(x, a)^\vee \otimes a \xrightarrow{+1}.$$

The two triangles together give the filtration

$$\tag{16} \sigma_a^{-1} x \to x \to \sigma_a x$$

with factors $\mathrm{Hom}^\bullet(x, a)^\vee \otimes a$ and $\mathrm{Hom}^\bullet(a, x) \otimes a[1]$. Using that a is d-spherical, we identify

$$\tag{17} \mathrm{Hom}^\bullet(x, a)^\vee = \mathrm{Hom}^\bullet(a, x)[d].$$

Lemma 4.10. *In the setup above, suppose x does not contain any shift of a as a direct summand. Consider the filtration*

$$\sigma_a^{-1} x \to x \to \sigma_a x.$$

Then, with the identification as in (17), *the connecting map*

(18) $$\mathrm{Hom}^\bullet(a, x) \otimes a[1] \to \mathrm{Hom}^\bullet(a, x) \otimes a[d+1]$$

is a shift of $\mathrm{id} \otimes \mathrm{loop}_a$.

Proof. Shifted by -1, the connecting map is the composite

(19) $$\mathrm{Hom}^\bullet(a, x) \otimes a \xrightarrow{\mathrm{ev}} x \xrightarrow{\mathrm{coev}} \mathrm{Hom}^\bullet(x, a)^\vee \otimes a \xrightarrow{\gamma} \mathrm{Hom}^\bullet(a, x)[d] \otimes a,$$

where γ is the duality isomorphism. By the definition of evaluation and co-evaluation, $\mathrm{coev} \circ \mathrm{ev}$ is adjoint to the composite

$$\mathrm{Hom}^\bullet(a, x) \otimes \mathrm{Hom}^\bullet(x, a) \otimes a \to \mathrm{Hom}^\bullet(a, a) \otimes a \xrightarrow{\mathrm{ev}} a,$$

where the first map is composition and the second is evaluation. Since x does not have any shift of a as a direct summand, no composition $a[j] \to x \to a[j]$ can be the identity. Therefore, the image of the composition map

(20) $$\mathrm{Hom}^\bullet(a, x) \otimes \mathrm{Hom}^\bullet(x, a) \to \mathrm{Hom}^\bullet(a, a)$$

is the one-dimensional subspace spanned by loop_a.

Recall that we have the map

$$\mathrm{Tr} \colon \mathrm{Hom}^d(a, a) \to \mathbf{k}.$$

For convenience, set $\mathrm{Tr} \colon \mathrm{Hom}^i(a, a) \to \mathbf{k}$ to be the zero map for each $i \neq d$. Then we can write (20) as

$$f \otimes g \mapsto \mathrm{Tr}(g \circ f) \cdot \mathrm{loop}_a,$$

The pairing that induces the duality isomorphism

$$\gamma \colon \mathrm{Hom}^\bullet(x, a)^\vee \cong \mathrm{Hom}^\bullet(a, x)[d]$$

is also given by

$$(f, g) \mapsto \mathrm{Tr}(g \circ f).$$

It follows that the map $(\gamma \circ \mathrm{coev} \circ \mathrm{ev})$ is $\mathrm{id} \otimes \mathrm{loop}_a$. $\qquad\square$

We recall the objects a_n studied in Appendix B. The object a_n is characterised uniquely by the existence of a filtration with factors $a[0], a[-(d-1)], \dots, a[-n(d-1)]$ such that the connecting map

$$a[-i(d-1)] \to a[-(i-1)(d-1)]$$

is a shift of loop_a (Proposition B.2). By Proposition B.3, we have nonzero maps

(21) $$\begin{aligned} i_n &\colon a \to a_n, & l_n &\colon a[-n(d-1)-d] \to a_n \\ t_n &\colon a_n \to a[d], & q_n &\colon a_n \to a[-n(d-1)] \end{aligned}$$

forming distinguished triangles

(22) $$\begin{aligned} a_{n-1}[-d] &\xrightarrow{t_{n-1}[-d]} a \xrightarrow{i_n} a_n \xrightarrow{+1} \text{ and} \\ a_n &\xrightarrow{q_n} a[-n(d-1)] \xrightarrow{l_{n-1}[1]} a_{n-1}[1] \xrightarrow{+1} . \end{aligned}$$

We have

(23) $$t_n \circ i_n = \mathrm{loop}_a \text{ and } q_n \circ l_n = \mathrm{loop}_a[-n(d-1)-d].$$

The other composites are zero. Up to scaling and shifts, the maps in (21) are the only nonzero maps between a and a_n.

Recall from (15) that we have a natural map $x \to \sigma_a x$. We can repeatedly apply σ_a to this map and compose the resulting maps to obtain a natural map $x \to \sigma_a^{n+1} x$ for all $n \geq 0$.

Lemma 4.11. *Let $x \in \mathcal{C}$ be any object that does not contain a shift of a as a direct summand and which does not have endomorphisms of negative degree. For every $n \geq 0$, the natural map $x \to \sigma_a^{n+1} x$ completes to a distinguished triangle*

$$\mathrm{Hom}^\bullet(a, x) \otimes a_n \to x \to \sigma_a^{n+1} x.$$

Proof. Recall that we have the triangle

(24) $$\mathrm{Hom}^\bullet(a, x) \otimes a \to x \to \sigma_a x \xrightarrow{+1} .$$

Set $e = d - 1$. By repeatedly applying σ_a to the triangle above, we get the triangles

(25) $$\mathrm{Hom}^\bullet(a, x) \otimes a[-ie] \to \sigma_a^i x \to \sigma_a^{i+1} x \xrightarrow{+1},$$

which assemble into the diagram

(26)

$$x \longrightarrow \sigma_a x \longrightarrow \cdots \longrightarrow \sigma_a^n x \longrightarrow \sigma_a^{n+1} x$$

$$\mathrm{Hom}^\bullet(a, x) \otimes a[1] \qquad \cdots \qquad \cdots \qquad \mathrm{Hom}^\bullet(a, x) \otimes a[1 - ne].$$

By Lemma 4.10, the connecting map between any two successive factors in (26) is a shift of $\mathrm{id} \otimes \mathrm{loop}_a$. Suppose y_n completes $x \to \sigma_a^{n+1} x$ to a distinguished triangle

$$x \to \sigma_a^{n+1} x \to y_n \xrightarrow{+1} .$$

We prove the following two statements by induction on n.

   (1) We have an isomorphism $y_n \cong \mathrm{Hom}^\bullet(a, x) \otimes a_n[1]$.
   (2) Via the above isomorphism, the map

(27) $$y_n \to \mathrm{Hom}^\bullet(a, x) \otimes a[1 - ne]$$

      induced by the diagram (26) is a shift of $\mathrm{id} \otimes q_n$.

The base case $n = 0$ follows from the distinguished triangle (24). Assume the result for $n - 1$. Then (26) collapses to

$$x \longrightarrow \sigma_a^n x \longrightarrow \sigma_a^{n+1} x$$

$$\mathrm{Hom}^\bullet(a, x) \otimes a_{n-1}[1] \qquad \mathrm{Hom}^\bullet(a, x) \otimes a[1 - ne].$$

We now prove that the connecting map

(28) $$\mathrm{Hom}^\bullet(a, x) \otimes a[1 - ne] \to \mathrm{Hom}^\bullet(a, x) \otimes a_{n-1}[2]$$

is $\mathrm{id} \otimes l_{n-1}[2]$.

Choose a homogeneous basis of $\mathrm{Hom}^\bullet(a, x)$ so that the map (28) may be written as a matrix of maps from shifts of a to shifts of a_{n-1}. We know that, up to scaling and shifts, the only two maps from a to a_{n-1} are

   (1) $i_{n-1} \colon a \to a_{n-1}$, and
   (2) $l_{n-1} \colon a \to a_{n-1}[ne + 1]$.

So each entry of the matrix of the map (28) is a multiple of a shift of either i_{n-1} or l_{n-1}. The composite

$$\mathrm{Hom}^\bullet(a,x) \otimes a[1-ne] \xrightarrow{(28)} \mathrm{Hom}^\bullet(a,x) \otimes a_{n-1}[2] \xrightarrow{(27)} \mathrm{Hom}^\bullet(a,x) \otimes a[2-(n-1)e].$$

is the connecting map in the last two steps of (26). By Lemma 4.10, we know that it is a shift of $\mathrm{id} \otimes \mathrm{loop}_a$. By the second statement in the induction hypothesis, the map (27) is $\mathrm{id} \otimes q_{n-1}$.

Recall from the discussion around (23) that $q_{n-1} \circ i_{n-1} = 0$ and $q_{n-1} \circ l_{n-1} = \mathrm{loop}_a$. Since the composite of (28) with (27) is a shift of $\mathrm{id} \otimes \mathrm{loop}_a$, we conclude that the diagonal entries of (28) are shifts of l_{n-1} and the off-diagonal entries are multiples of shifts of i_{n-1}, possibly zero. In fact, we need to prove that the off-diagonal entries are zero. To do so, we need another fact about a_{n-1} from the appendix: that the map i_{n-1} does not factor through $\sigma_a^n x$ (Lemma B.6). We conclude that a nonzero multiple of a shift of i_{n-1} cannot be a matrix entry of (28), and so the off-diagonal entries are indeed zero.

Having proved that (28) is a shift of $\mathrm{id} \otimes l_{n-1}$, it follows from Proposition B.3 that y_{n+1} is isomorphic to $\mathrm{Hom}^\bullet(a,x) \otimes a_n$ and the map $y_{n+1} \to \mathrm{Hom}^\bullet(a,x) \otimes a[1-ne]$ is isomorphic to a shift of $\mathrm{id} \otimes q_{n+1}$. $\qquad \square$

We now employ the machinery of *rectifiable filtrations*, as discussed in Appendix A, specifically Definition A.10.

Lemma 4.12. *Fix a stability condition τ. Let a be a τ-semistable d-spherical object of $\mathcal{C}$. Let $x \in \mathcal{C}$ be any object that does not have endomorphisms of negative degree and does not have any shift of a as a direct summand. There exists N (depending on x and a) such that for every $n \geq N$, the filtration*

$$0 \to \sigma_a^n x \to \sigma_a^{n+1} x$$

is rectifiable (with respect to τ).

Proof. As before, set $e = d - 1$.

The factors of the filtration are $\sigma_a^n x$ and $\mathrm{Hom}^\bullet(a,x) \otimes a[1-ne]$. We need to show that the connecting map

$$\mathrm{Hom}^\bullet(a,x) \otimes a[1-ne] \to \sigma_a^n x[1]$$

is rectifiable. Let us show, equivalently, that its shift

$$\chi \colon \mathrm{Hom}^\bullet(a,x) \otimes a[-ne] \to \sigma_a^n x$$

is rectifiable.

Let α be a real number. We need to show that the map

$$(29) \qquad\qquad (\mathrm{Hom}^\bullet(a,x) \otimes a[-ne])_{\geq \alpha} \to (\sigma_a^n x)_{<\alpha+1}$$

induced by χ is zero. Without loss of generality, assume that $x_{\leq 0} = 0$, which can be achieved by replacing x by a sufficiently positive shift.

First assume that $\alpha \geq -1$. Then, if n is large enough, we have

$$(\mathrm{Hom}^\bullet(a,x) \otimes a[-ne])_{\geq \alpha} = 0.$$

So the map (29) is also zero.

Now assume $\alpha < -1$. By Lemma 4.11, we have the exact triangle

$$(30) \qquad\qquad \mathrm{Hom}^\bullet(a,x) \otimes a_{n-1} \to x \to \sigma_a^n x \xrightarrow{+1} .$$

It is easy to see that, $\alpha < -1$ and $x_{\leq 0} = 0$ imply that the $+1$ map in (30) induces an isomorphism
$$(\sigma_a^n x)_{<\alpha+1} \to (\mathrm{Hom}^\bullet(a, x) \otimes a_{n-1}[1])_{<\alpha+1}.$$
So the map in (29) is equal to the map

(31) $$(\mathrm{Hom}^\bullet(a, x) \otimes a[-ne])_{\geq \alpha} \to (\mathrm{Hom}^\bullet(a, x) \otimes a_{n-1}[1])_{<\alpha+1}.$$

The map above is induced by the connecting map in the filtration
$$x \to \sigma_a^n x \to \sigma_a^{n+1} x.$$

We have seen in the proof of Lemma 4.11 that this map a shift of $\mathrm{id} \otimes l_{n-1}$ (see (28)). Note that l_{n-1} is rectifiable by Example A.3 (4) and so is $\mathrm{id} \otimes l_{n-1}$ by Example A.3 (3). Therefore, the map (31) is zero. $\qquad\square$

Proof of Theorem 4.9. We prove the result for $0 < q \leq 1$; the case of $q \geq 1$ is analogous. We abbreviate $m_{q,\tau}$ by m and $\overline{\mathrm{hom}}_q$ by $\overline{\mathrm{hom}}$.

First, let $x \cong a[j]$. Then
$$\sigma_a^n x = a[j - ne],$$
and hence
$$m(\sigma_a^n x) = m(a) \cdot q^{j-en}.$$
Therefore,
$$\lim_{n \to \infty} \frac{m(\sigma_a^n x)}{q + q^{1-e} + \cdots + q^{1-(n-1)e}} = \lim_{n \to \infty} \frac{m(a) \cdot q^{j-en}}{q + q^{1-e} + \cdots + q^{1-(n-1)e}}$$
$$= m(a) \cdot q^j \left(q^{-d} - q^{-1}\right)$$
$$= m(a) \cdot \overline{\mathrm{hom}}(a, x).$$

Next, suppose x is indecomposable and not isomorphic to a shift of a. Let N be as guaranteed by Lemma 4.12. Then, for every $i \geq N$, we have
$$m(\sigma_a^{i+1} x) = m(\sigma_a^i x) + m\left(\mathrm{Hom}^\bullet(a, x) \otimes a[1 - ie]\right)$$
$$= m(\sigma_a^i x) + q^{1-ie} \cdot m(a) \cdot \overline{\mathrm{hom}}(a, x).$$

By taking the sum as i ranges from N to $n-1$, we get
$$m(\sigma_a^n x) = m(\sigma_a^N x) + \left(q^{1-(n-1)e} + \cdots + q^{1-Ne}\right) \cdot m(a) \cdot \overline{\mathrm{hom}}(a, x).$$

Dividing by $q^{1-(n-1)e} + \cdots + q^1$ and letting $n \to \infty$ yields
$$\lim_{n \to \infty} \frac{m(\sigma_a^n x)}{q + q^{1-e} + \cdots + q^{1-(n-1)e}} = m(a) \cdot \overline{\mathrm{hom}}(a, x),$$
as desired.

Finally, if x is decomposable, then the desired equality follows by adding up the equalities for each indecomposable summand. $\qquad\square$

Let $\mathbf{S}$ be a set of objects of $\mathcal{C}$ such that no object of $\mathbf{S}$ has endomorphisms of negative degree. Recall from (12) the mass map
$$m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}}.$$
We also have a map $h_q \colon \mathbf{S} \to \mathbf{P}^{\mathbf{S}}$ defined as
$$h_q \colon x \mapsto [\overline{\mathrm{hom}}_q(x, -)]$$
For $q = 1$, we write m and h to mean m_q and h_q respectively.

Corollary 4.13. *Let a be a spherical object that is semistable in a stability condition τ. Then, in $\mathbf{P^S}$ we have the following.*

(1) If $0 < q \leq 1$, then

$$\lim_{n\to\infty} m_q(\sigma_a^n x) = h_q(a).$$

(2) If $q \geq 1$, then

$$\lim_{n\to\infty} m_q(\sigma_a^{-n} x) = h_q(a).$$

In particular, if $q = 1$, then

$$\lim_{n\to\pm\infty} m(\sigma_a^n x) = h(a).$$

In all cases, $h_q(a)$ is in the closure of the image of the mass map.

Proof. Follows immediately from Theorem 4.9.   □

Remark 4.14 (Degenerate stability conditions). Theorem 4.9 describes one way to approach the boundary of the stability manifold in $\mathbf{P^S}$, namely by applying powers of a spherical twist. There are more direct ways to approach the boundary. Start with a stability condition τ with heart $\heartsuit$ and central charge $Z \colon K(\mathcal{C}) \to \mathbf{C}$. Recall that Z must map nonzero objects of $\heartsuit$ to nonzero complex numbers. Take a deformation Z_t of Z such that Z_0 sends some objects of $\heartsuit$ to 0. Then the limit of the stability conditions described by $(\heartsuit, Z_t)$ is not a stability condition, but it is in the closure. Such limits correspond to the degenerate stability conditions in the sense of [4]. Note that these two kinds of limits can yield distinct boundary points in general (see, e.g., the results of [1]).

5. HARDER–NARASIMHAN (HN) AUTOMATA

The aim of this section is to introduce some general machinery for a piecewise generalisation of a G-set or a G-representation. In Sections 7.3 and 8.4, we apply this to the action of the auto-equivalence group on the Harder–Narasimhan filtration of an object. The reader may choose to defer reading the remainder of this section until required for the applications.

Let Θ be a quiver with vertex set Θ_0 and edge set Θ_1. For an edge $\alpha \in \Theta_1$, let $s(\alpha)$ and $t(\alpha)$ denote its source and target, respectively. A *path* in Θ is a finite sequence of edges $(\alpha_1, \ldots, \alpha_n)$, such that for each i, we have

$$s(\alpha_{i+1}) = t(\alpha_i).$$

A Θ-representation X with values in a fixed category consists of objects $\{X_v \mid v \in \Theta_0\}$ indexed by vertices, and morphisms $\{\phi(X)_\alpha \colon X_{s(\alpha)} \to X_{t(\alpha)} \mid \alpha \in \Theta_1\}$ indexed by edges. A morphism of Θ-representations from X to Y consists of morphisms $\{f_v \colon X_v \to Y_v \mid v \in \Theta_0\}$ such that all squares of the form

$$
\begin{array}{ccc}
X_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & Y_{s(\alpha)} \\
\downarrow{\scriptstyle \phi(X)_\alpha} & & \downarrow{\scriptstyle \phi(Y)_\alpha} \\
X_{t(\alpha)} & \xrightarrow{f_{t(\alpha)}} & Y_{t(\alpha)}
\end{array}
$$

commute. A Θ-set is a representation of Θ in the category of sets. Similarly, if A is any ring, then a Θ-representation of A-modules is a representation of Θ in the category of A-modules. Clearly, any Θ-representation of A-modules is also a Θ-set. A Θ-subset $T \subset S$ is a morphism of Θ-sets in which for each $v \in \Theta_0$, the corresponding map $T_v \to S_v$ is an inclusion.

Fix a group G. We say that a function $\ell \colon \Theta_1 \to G$ is a G-*labelling* of Θ. We can extend this function to paths in Θ in a natural way, as follows. If $(\alpha_1, \ldots, \alpha_n)$ is a path, then

$$\ell((\alpha_1, \ldots, \alpha_n)) = \ell(\alpha_n) \cdots \ell(\alpha_1).$$

Consider a quiver Θ with a G-labelling $\ell \colon \Theta_1 \to G$. Let S be a set with a G-action. These data naturally give rise to a Θ-set S, in which the object associated to each vertex $v \in \Theta_0$ is simply $S_v = S$, and the map associated to any edge α is the action of $\ell(\alpha)$ on S. Similarly, a G-representation M naturally gives rise to a Θ-representation. In particular, if $\mathcal{C}$ is a (triangulated) category with a G-action, then $\mathrm{ob}\,\mathcal{C}$ is a Θ-set.

We now take up the question of evolutions of HN filtrations under group actions. Let $\mathcal{C}$ be a triangulated category and τ a stability condition on $\mathcal{C}$. Let Σ be the set of all indecomposable τ-semistable objects in the $[0, 1)$-heart.

Definition 5.1. Let x be any object of $\mathcal{C}$. The τ-*Harder–Narasimhan multiplicity vector* $\mathrm{HN}_\tau(x) \in \mathbf{Z}^\Sigma$ is the function that sends $s \in \Sigma$ to

$$\mathrm{HN}_\tau(x)(s) = \sum_{n \in \mathbf{Z}} (\text{Number of occurrences of } s[n] \text{ among the } \tau\text{-HN factors of } x).$$

Remark 5.2. The definition has a natural q-analog $\mathrm{HN}_{q,\tau}(x) \in \mathbf{Z}[q^\pm]^\Sigma$ defined by

$$\mathrm{HN}_{q,\tau}(x)(s) = \sum_{n \in \mathbf{Z}} (\text{Number of occurrences of } s[n] \text{ among the } \tau\text{-HN factors of } x) \cdot q^n.$$

We now come to the key definition that will capture the piecewise linear nature of the evolution of HN multiplicities. Recall that τ is a stability condition on a triangulated category $\mathcal{C}$. The set of indecomposable τ semistable objects in the $[0, 1)$-heart is denoted Σ, and G is a group acting on $\mathcal{C}$.

Definition 5.3. A τ-HN automaton consists of the following data.

(1) A quiver Θ together with a G-labelling $\ell \colon \Theta_1 \to G$.
(2) A Θ-subset $S \subset \mathrm{ob}\,\mathcal{C}$.
(3) A Θ-representation M of $\mathbf{Z}$-modules that assigns the module $\mathbf{Z}^\Sigma$ to every vertex such that the HN-multiplicity vector

$$S \xrightarrow{\mathrm{HN}_\tau} M$$

defines a map of Θ-sets.

Let us make condition (3) explicit. Let $\alpha \colon v \to w$ be an edge of Θ. Then condition (3) means that the following diagram commutes:

$$
\begin{array}{ccc}
S_v & \xrightarrow{\mathrm{HN}_\tau} & \mathbf{Z}^\Sigma \\
\downarrow{\scriptstyle \ell(\alpha)} & & \downarrow{\scriptstyle M(\alpha)} \\
S_w & \xrightarrow{\mathrm{HN}_\tau} & \mathbf{Z}^\Sigma.
\end{array}
$$

Thus, for the objects in S_v, the matrix $M(\alpha)$ captures the evolution of HN multiplicities under the application of $\ell(\alpha)$.

We obtain the mass of an object by applying a linear function to the HN-multiplicity vector. Hence, an HN automaton also allows us to understand the evolution of the mass.

Remark 5.4. We establish some terminology for HN automata. Consider a τ-HN automaton as explained in Definition 5.3. Following standard conventions in the theory of finite automata, we call the vertices of Θ the *states* of the automaton. Given a state v, we say that S_v is the set of objects *supported at the state* v. Note that an object may be supported at multiple states. We say that an object is *recognised by* Θ if it is supported at some state.

We say that an expression $g = g_n \cdots g_1$ in G is *recognised by* Θ if there is a path $(\alpha_1, \ldots, \alpha_n)$ in Θ such that $g_i = \ell(\alpha_i)$, and hence

$$g = \ell((\alpha_1, \ldots, \alpha_n)).$$

Finally, let x be an object of $\mathcal{C}$ supported at a state v. Consider an expression $g_n \cdots g_1 \cdot x$ for some group elements $g_n, \ldots, g_1$. Suppose that $g_n \cdots g_1$ is a recognised expression via a path starting at the state v. Then we say that the expression $g_n \cdots g_1 \cdot x$ is *recognised by* Θ.

6. The category $\mathcal{C}_\Gamma$

In this section we recall the definition and properties of 2-Calabi–Yau categories associated to quivers. Fix Γ, a connected undirected graph without loops or multiple edges.

Definition 6.1. The *Artin–Tits braid group associated to* Γ, denoted by B_Γ is the group generated by σ_i, where i ranges over the set of vertices of Γ, modulo the following relations:

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if there is an edge between } i \text{ and } j,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{otherwise.}$$

Definition 6.2. The *Coxeter group associated to* Γ, denoted by W_Γ, is the quotient of B_Γ by the following additional relations. For each vertex i of Γ, set

$$\sigma_i^2 = 1.$$

Denote the image of σ_i in W_Γ by s_i.

Let V_Γ be the **Z**-module with basis vectors v_i indexed by the vertices of Γ. Define a bilinear form on V_Γ by the following formula:

$$\langle v_i, v_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are connected by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.3. The *standard representation* of W_Γ is the action on V_Γ defined by the formula

$$s_i(v_j) = v_j - \langle v_i, v_j \rangle v_i.$$

We describe a category $\mathcal{C}_\Gamma$ with a (weak) action of B_Γ that categorifies the standard representation of W_Γ. The construction is via the zig-zag algebra, which was introduced in [19]. A construction of $\mathcal{C}_\Gamma$ for type A_n graphs is in [30,33].

We first recall the zig-zag algebra. A version of this construction is also in [2, §2.3], but in that version, we have not been careful with signs; see Remarks 6.6, 6.10 and 6.11.

Let Γ^{dbl} denote the doubled quiver of Γ. This is the directed graph obtain by replacing each edge of Γ by a pair of oppositely oriented arrows between its endpoints. For each original edge e in Γ between i and j, we let e_{ij} and e_{ji} be the corresponding arrows in Γ^{dbl} from i to j and j to i respectively. Let $\mathrm{Path}(\Gamma^{\mathrm{dbl}})$ denote the path algebra of Γ^{dbl}. As a vector space over **k**, the algebra $\mathrm{Path}(\Gamma^{\mathrm{dbl}})$ is spanned by all oriented paths in Γ^{dbl}, and the

multiplication is given by concatenation. The length function on paths that declares arrows to have length one induces a non-negative grading on $\mathrm{Path}(\Gamma^{\mathrm{dbl}})$.

We define the zigzag algebra of Γ, as follows.

Definition 6.4. Suppose Γ has at least two vertices. Fix a sign $s_{ij} \in \{+1, -1\}$ for each arrow e_{ij} in Γ^{dbl} such that $s_{ij} = -s_{ji}$. Let $A(\Gamma) = A_s(\Gamma)$ be the quotient of the path algebra $\mathrm{Path}(\Gamma^{\mathrm{dbl}})$ by the two-sided ideal generated by the following elements:

(1) all length three paths,
(2) length two paths whose source and target differ,
(3) for each triple of vertices i, j, k of Γ such that there are arrows e_{ij} and e_{ik}, the element

$$s_{ij}e_{ij}e_{ji} - s_{ik}e_{ik}e_{ki}.$$

If Γ has one vertex, (type A_1), set $A(\Gamma) = \mathbf{C}[x]/x^2$. We call $A(\Gamma)$ the *zigzag algebra* of Γ.

Note that the third relations in the definition above imply that all degree-two paths starting and ending at the same vertex are equal up to sign. We call one such nonzero path a "loop" at a vertex. When Γ has at least two vertices, the relations of $A(\Gamma)$ are all homogeneous, and so the natural grading on $\mathrm{Path}(\Gamma^{\mathrm{dbl}})$ descends to a grading on $A(\Gamma)$. When Γ has a single vertex, grade $A(\Gamma) = \mathbf{C}[x]/x^2$ by setting $\deg(x) = 2$.

The choice of signs does not matter.

Proposition 6.5. *Consider another choice of signs $s'_{ij} \in \{+1, -1\}$ such that $s'_{ij} = -s'_{ji}$. Then the map*

$$e_{ij} \mapsto s_{ij}s'_{ij}e_{ij}$$

induces an isomorphism of graded algebras $A_s(\Gamma) \cong A_{s'}(\Gamma)$.

Proof. We leave the proof to the reader. ☐

Remark 6.6. In [2], we gave a definition of the zig-zag algebra that ignored the choice of signs in Definition 6.4, in effect taking $s_{ij} = s_{ji} = 1$. This unsigned version also appears in [19] and [33] for type A. Let $B(\Gamma)$ denote this unsigned version of the zig-zag algebra, namely the algebra with the same generators and relations as $A(\Gamma)$ from Definition 6.4, but in which we set $s_{ij} = 1$.

We prefer to work with the algebra $A(\Gamma)$ in this paper, as it gives rise to a strongly 2-Calabi–Yau category $\mathcal{C}_\Gamma$; see Remarks 6.10 and 6.11. However, if the unoriented graph Γ is bipartite, the two definitions are equivalent, as stated in Proposition 6.7. In particular, the definitions are equivalent for finite type ADE.

The following is easy to check; we omit the proof.

Proposition 6.7. *Let $\Gamma = \{\Gamma_1, \Gamma_2\}$ be an unoriented bipartite graph without self-loops or multiple edges, in which each edge connects a vertex of Γ_1 with a vertex of Γ_2. Let $A(\Gamma)$ be the zig-zag algebra as in Definition 6.4 and $B(\Gamma)$ the unsigned zig-zag algebra. Consider the map from $A(\Gamma)$ to $B(\Gamma)$ in which*

$$e_{ij} \mapsto \begin{cases} s_{ij}e_{ij} & \text{if } i \in \Gamma_1 \text{ and } j \in \Gamma_2, \\ e_{ij} & \text{if } i \in \Gamma_2 \text{ and } j \in \Gamma_1. \end{cases}$$

This map is an algebra isomorphism.

The minimal idempotents of $A(\Gamma)$ are the length zero paths (i) for $i \in \Gamma$. The modules $P_i = A(\Gamma)(i)$ are indecomposable projective left $A(\Gamma)$ modules. Any finitely-generated graded projective left $A(\Gamma)$ module is isomorphic to a direct sum of grading shifts of the modules P_i.

The category of complexes of graded projective left $A(\Gamma)$ modules admits two "shift" functors, namely the homological shift and the grading shift. We work in a simpler setting where the two shifts are identified. We construct the desired category as follows.

Regard $A(\Gamma)$ as a differential graded algebra (dga) where all differentials are zero. Let $K(\mathrm{dgmod}\text{-}A)$ be the category of finite-dimensional differential graded modules (dgms) over the dga $A(\Gamma)$. Morphisms in $K(\mathrm{dgmod}\text{-}A)$ are homotopy classes of chain maps that are compatible with the action of $A(\Gamma)$. Then $K(\mathrm{dgmod}\text{-}A)$ is a triangulated category (see, e.g. [32, §22.8]).

Definition 6.8. Set $\mathcal{C}_\Gamma$ to be the smallest full and strict triangulated subcategory generated by the objects P_i as i ranges over the vertices of Γ.

Remark 6.9. The categories considered in [30] and [33] are described in a slightly different way. However, [2, Proposition 2.1] shows that they coincide with our category $\mathcal{C}_\Gamma$. The category considered in [19] is the category of graded projective $A(\Gamma)$-modules; see [2, § 2.3.3] for a discussion on how it is related to $\mathcal{C}_\Gamma$.

We state the salient properties of $\mathcal{C}_\Gamma$ from [2, § 2.3].

(1) $\mathcal{C}_\Gamma$ is a $\mathbf{k}$-linear triangulated category.
(2) $\mathcal{C}_\Gamma$ is (strongly) 2-Calabi–Yau. That is, for every pair of objects $x, y \in \mathcal{C}$ and $i \in \mathbf{Z}$, we have a natural non-degenerate graded symmetric pairing

$$\mathrm{Hom}^i(x, y) \otimes \mathrm{Hom}^{2-i}(y, x) \to \mathbf{k}.$$

That is, the pairing is symmetric for even i and skew-symmetric for odd i.
(3) $\mathcal{C}_\Gamma$ is classically generated by the objects P_i. These objects satisfy

$$\mathrm{Hom}(P_i, P_i[n]) = \begin{cases} \mathbf{k} & \text{if } n = 0 \text{ or } n = 2, \\ 0 & \text{otherwise}; \end{cases}$$

$$\mathrm{Hom}(P_i, P_j[n]) = \begin{cases} \mathbf{k} & \text{if } n = 1 \text{ and } i \text{ and } j \text{ are neighbours}, \\ 0 & \text{otherwise, for } i \neq j. \end{cases}$$

(4) The extension closure of the objects P_i is an abelian category, which is the heart of a bounded t-structure on $\mathcal{C}_\Gamma$. We call this t-structure the *standard t-structure* on $\mathcal{C}_\Gamma$, and refer to its heart as the *standard heart* $\heartsuit_{\mathrm{std}}$.

The Grothendieck group K_Γ of $\mathcal{C}_\Gamma$ is a $\mathbf{Z}$-module with a pairing given by

$$\langle [x], [y] \rangle = \bigoplus_i (-1)^i \dim \mathrm{Hom}^i(x, y).$$

It is easy to check that we have an isomorphism $V_\Gamma \to K_\Gamma$ compatible with the pairing, defined by $v_i \mapsto [P_i]$.

Remark 6.10. The pairing

$$\mathrm{Hom}^i(x, y) \otimes \mathrm{Hom}^{2-i}(y, x) \to \mathbf{k}$$

is defined exactly as in [2, § 2.3]. The choice of signs in Definition 6.4 gives the graded symmetry, and hence a strongly 2-Calabi–Yau category (see [25]).

Remark 6.11. The zig-zag algebra $A(\Gamma)$ arises naturally as follows. Let $\{P_i \mid i \in \Gamma\}$ be a collection of spherical objects in a strongly 2-CY triangulated category. Assume that for $i \neq j$, we have

$$\operatorname{Hom}(P_i, P_j[1]) \cong \begin{cases} \mathbf{k} & \text{if } i\text{---}j \text{ is an edge of } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Set $P = \bigoplus_{i \in \Gamma} P_i$. Then the endomorphism algebra of P is isomorphic to the zig-zag algebra $A(\Gamma)$ as defined in Definition 6.4. We describe an explicit isomorphism

$$\psi \colon A(\Gamma) \to \operatorname{End}(P).$$

Choose a sign $s_{ij} \in \{\pm 1\}$ for each edge ij of Γ^{dbl} such that $s_{ij} = -s_{ji}$. For edge ij with $s_{ij} = 1$, let $\psi(e_{ij}) \in \operatorname{Hom}^1(P_i, P_j)$ be any nonzero element and $\psi(e_{ji})$ be the unique element such that

$$\langle \psi(e_{ij}), \psi(e_{ji}) \rangle = 1.$$

Then it is easy to check that $\psi \colon A(\Gamma) \to \operatorname{End}(P)$ an isomorphism of graded algebras.

Recall that the indecomposable projective modules P_i in $\mathcal{C}_\Gamma$ are spherical. Furthermore, the spherical twists in the P_i satisfy the defining relations of the Artin–Tits braid group B_Γ associated to Γ (see, e.g., [19]). That is,

$$\sigma_{P_i} \sigma_{P_j} \sigma_{P_i} \cong \sigma_{P_j} \sigma_{P_i} \sigma_{P_j} \text{ when } i \text{ and } j \text{ are connected by an edge of } \Gamma,$$

$$\sigma_{P_i} \sigma_{P_j} \cong \sigma_{P_j} \sigma_{P_i} \text{ when } i \text{ and } j \text{ are not connected by an edge of } \Gamma.$$

Via the homomorphism $B_\Gamma \to \operatorname{Aut}(\mathcal{C}_\Gamma)$ defined by $\sigma_i \mapsto \sigma_{P_i}$, we have a (weak) action of B_Γ on $\mathcal{C}_\Gamma$. This action is known to be faithful in some cases—for example when Γ is an ADE Dynkin diagram [7, 26]—and is conjecturally faithful for all Γ. The induced action on the Grothendieck group K_Γ is the standard (geometric) representation of the Coxeter group associated to Γ.

6.1. **Injectivity of the mass map.** We now fix the graph Γ and let $\mathcal{C} = \mathcal{C}_\Gamma$ be the associated strongly 2-CY category. We say that a stability condition τ is *standard* if its $[0, 1)$ heart is the standard heart $\heartsuit_{\mathrm{std}}$. It is easy to see that the set of standard stability conditions is connected. Let $\operatorname{Stab}^\circ \mathcal{C}_\Gamma \subset \operatorname{Stab} \mathcal{C}_\Gamma$ be the connected component containing the standard stability conditions.

Let $\mathbf{S}$ be the set of spherical objects x of $\mathcal{C}_\Gamma$ such that x is stable with respect to some stability condition in $\operatorname{Stab}^\circ \mathcal{C}_\Gamma$. The goal of Section 6.1 is to show that the map $m_q \colon \operatorname{Stab}^\circ(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}}$ is injective. We need two lemmas.

Lemma 6.12. *Let τ be a standard stability condition on $\mathcal{C}$. Let τ' be any stability condition such that for all $x \in \mathbf{S}$, we have*

$$m_{q,\tau}(x) = m_{q,\tau'}(x).$$

Suppose that $\phi_\tau(P_1) < \phi_\tau(P_2)$ and $\phi_{\tau'}(P_1) = \phi_\tau(P_1)$. Then $\phi_{\tau'}(P_2) = \phi_\tau(P_2)$.

Proof. We introduce some notation. Given an edge ij in Γ, let P_{ij} denote the cone of a nonzero morphism $P_i[-1] \to P_j$. Note that P_{ij} is spherical and fits in an exact sequence

$$0 \to P_i \to P_{ij} \to P_j \to 0$$

in the standard heart.

Since we have $\phi_\tau(P_1) < \phi_\tau(P_2)$, the argument of the central charge of P_{21} lies strictly between these two; that is

$$\phi_\tau(P_1) < \arg(Z_\tau(P_{21})) < \phi_\tau(P_2).$$

Since the only proper sub-object of P_{21} in the standard heart is P_1, we observe that P_{21} is τ-stable. The previous inequality gives us

$$(32) \qquad \phi_\tau(P_1) < \phi_\tau(P_{21}) < \phi_\tau(P_1) < \phi_\tau(P_1) + 1.$$

Recall from Corollary 3.10 that τ and τ' have the same stable spherical objects. In particular, P_{21} is also τ'-stable. Since we know that

$$\mathrm{Hom}(P_1, P_{21}) \neq 0, \quad \mathrm{Hom}(P_{21}, P_2) \neq 0, \quad \text{and } \mathrm{Hom}(P_2, P_1[1]) \neq 0,$$

the analogous inequalities hold for $\phi_{\tau'}$:

$$(33) \qquad \phi'_\tau(P_i) < \phi'_\tau(P_{ji}) < \phi'_\tau(P_j) < \phi'_\tau(P_i) + 1.$$

We also know that

$$m_{q,\tau'}(P_1) = m_{q,\tau}(P_1), \quad m_{q,\tau'}(P_{21}) = m_{q,\tau}(P_{21}), \quad m_{q,\tau'}(P_2) = m_{q,\tau}(P_2).$$

By [1, Lemma 5.2], it follows that the pair of complex numbers $(Z_\tau(P_1), Z_\tau(P_2))$ is equal to the pair $(Z_{\tau'}(P_1), Z_{\tau'}(P_2))$ possibly after a rotation and a reflection. The inequalities (32) and (33) imply that no reflection is necessary, and the equality $\phi_\tau(P_1) = \phi_{\tau'}(P_1)$ implies that no rotation is necessary. We conclude that $Z_\tau(P_2) = Z_{\tau'}(P_2)$; call this number z. Since both $\phi = \phi_\tau(P_2)$ and $\phi = \phi_{\tau'}(P_2)$ have the property that $e^{i\pi\phi}$ lies on the same real ray as z, and both lie in the interval $[\phi_\tau(P_1), \phi_\tau(P_1) + 1)$, we conclude that they are equal. $\qquad \square$

Lemma 6.13. *Let τ be a standard stability condition on $\mathcal{C}$. Let τ' be any stability condition such that for all $x \in \mathbf{S}$, we have*

$$m_{q,\tau}(x) = m_{q,\tau'}(x).$$

Moreover, suppose $\phi_\tau(P_1) = \phi_\tau(P_2)$. Then $\phi_{\tau'}(P_1) = \phi_{\tau'}(P_2)$. In particular, if $\phi_{\tau'}(P_1) = \phi_\tau(P_1)$, then we also have $\phi_{\tau'}(P_2) = \phi_\tau(P_2)$.

Proof. Recall from Corollary 3.10 that τ and τ' have the same stable spherical objects. In particular P_1 and P_2 are also τ'-stable.

Let $x = P_{12}$. We claim that the τ'-stable factors in the τ'-HN filtration of x are precisely P_1 and P_2. To see this, consider a filtration

$$(34) \qquad 0 = x_0 \to x_1 \to \cdots \to x_n = x$$

whose factors a_i are τ'-stable and appear in the order of non-increasing phase. By the Mukai lemma [9, Lemma 12.2] and [21, Corollary 2.3], the a_i are spherical. By Corollary 3.10, they are also τ-stable. We have

$$m_{q,\tau'}(x) = \sum_i m_{q,\tau'}(a_i),$$

and hence

$$m_{q,\tau}(x) = \sum_i m_{q,\tau}(a_i).$$

But x is τ-semistable. Therefore by Proposition 3.3, we conclude that the τ-stable objects a_i all have τ-phase $\phi = \phi_\tau(x)$. Since the a_i are τ-stable, they are simple objects of the category $\mathcal{C}_{\tau,\phi}$. Thus, (34) is a Jordan–Hölder filtration of x in $\mathcal{C}_{\tau,\phi}$. But we know that x has a unique Jordan–Hölder filtration in $\mathcal{C}_{\tau,\phi}$, namely

$$(35)$$

Therefore, the filtration (34) must coincide with (35). That is, P_1 and P_2 are the τ'-stable HN factors of x, with

$$\phi_{\tau'}(P_2) \geq \phi_{\tau'}(P_1).$$

By the same argument, the τ'-stable factors of $y = P_{21}$ are also P_1 and P_2, with

$$\phi_{\tau'}(P_1) \geq \phi_{\tau'}(P_2).$$

It follows that $\phi_{\tau'}(P_1) = \phi_{\tau'}(P_2)$.   □

Proposition 6.14 (Injectivity)**.** *Let $\mathcal{C}$ be the 2-CY category associated to a finite connected graph Γ. Let $q > 0$ be a real number. Then the mass map $m_q \colon \mathrm{Stab}^\circ(\mathcal{C})/\mathbf{C} \to \mathbf{P^S}$ is injective.*

Proof. Let τ be a stability condition in $\mathrm{Stab}^\circ(\mathcal{C})$. By [22, Proposition 4.13], τ is in the braid group orbit of a standard stability condition up to the action of $\mathbf{C}$. (Although [22] treats the case of preprojective algebras, the same proof works in our setting.) By applying a braid and an element of $\mathbf{C}$, suppose that τ is standard. Since the objects P_i are simple objects of the standard heart, they are τ-stable.

Let τ' be another stability condition such that $m_q(\tau) = m_q(\tau') \in \mathbf{P^S}$. By scaling the central charge of τ', we may assume that $m_{q,\tau} = m_{q,\tau'}$ on $\mathbf{S}$. By Corollary 3.10, τ and τ' have the same spherical stable objects.

Label the vertices of Γ by $\{1, \ldots, n\}$ so that for each $i > 1$ there is some $j < i$ such that the subgraph induced by $\{i, j\}$ is of type A_2. By translating the slicing of τ' if necessary, assume that $\phi_{\tau'}(P_1) = \phi_\tau(P_1)$. Rescale the central charge again to ensure that $m_{q,\tau} = m_{q,\tau'}$ continues to hold on $\mathbf{S}$. Then $Z_\tau(P_1) = Z_{\tau'}(P_1)$.

We now prove by induction on i that $\phi_\tau(P_i) = \phi_{\tau'}(P_i)$ and $Z_\tau(P_i) = Z_{\tau'}(P_i)$. The base case $i = 1$ holds by construction. For the induction step, let $i > 1$. Consider $j < i$ such that the subgraph induced by $\{i, j\}$ is of type A_2. By the induction hypothesis, we have $\phi_{\tau'}(P_j) = \phi_\tau(P_j)$ and $Z_{\tau'}(P_j) = Z_\tau(P_j)$. Depending on whether or not the quantities $\phi_\tau(P_i)$ and $\phi_\tau(P_j)$ are equal, we are now in the setting of one of Lemmas 6.12 and 6.13. In each case, we conclude that $\phi_{\tau'}(P_i) = \phi_\tau(P_i)$. Since we already have $m_{q,\tau}(P_i) = m_{q,\tau'}(P_i)$, we conclude that $Z_\tau(P_i) = Z_{\tau'}(P_i)$.

Since $\phi_\tau(P_i) = \phi_{\tau'}(P_i)$ and $0 \leq \phi_\tau(P_i) < 1$, the same holds for $\phi_{\tau'}(P_i)$. In particular, P_i lies in the $[0, 1)$ heart of τ'. As a result, $\heartsuit_{\mathrm{std}}$ is contained in the $[0, 1)$-heart of τ'. Since both are hearts of bounded t-structures, they must be equal. Furthermore, since the P_i span the Grothendieck group and $Z_\tau(P_i) = Z_{\tau'}(P_i)$, we have $Z_\tau = Z_{\tau'}$. So τ' is a stability condition with the same heart and central charge as τ. We conclude that $\tau' = \tau$.   □

6.2. **Homeomorphism onto the image.** Having proved injectivity, we take up the question of m_q being a homeomorphism onto its image.

As before, let $\mathcal{C}$ be the 2-CY category associated to a finite connected quiver. We begin by explicitly identifying convenient open neighbourhoods of points in $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ whose closure is compact.

Let $W \subset \mathrm{Stab}(\mathcal{C})$ be the set of standard stability conditions. It is easy to see that $W \subset \mathrm{Stab}(\mathcal{C})$ is locally closed. Let $H \subset \mathbf{C}$ be the semi-closed upper half plane

$$H = \{z \in \mathbf{C} \mid z = re^{i\pi\phi} \text{ for } r \in \mathbf{R}_{>0} \text{ and } \phi \in [0, 1)\}.$$

We have a homeomorphism

(36) $$H^n \to W$$

that sends $(z_1, \ldots, z_n)$ to the stability condition whose $[0, 1)$-heart is the standard heart and whose central charge is defined by $[P_i] \mapsto z_i$.

Fix a small $\epsilon > 0$. Let $W_\epsilon \subset W$ be the open subset of stability conditions τ that satisfy

$$\epsilon < m_{q,\tau}(P_i) < \epsilon^{-1}.$$

Likewise, let $H_\epsilon \subset H$ be the open subset containing $z = re^{i\pi\phi}$ such that

$$\epsilon < rq^\phi < \epsilon^{-1}.$$

Then (36) restricts to a homeomorphism

$$H_\epsilon^n \to W_\epsilon.$$

Let $\overline{H}_\epsilon \subset \mathbf{C}$ be the closure of H_ϵ. Then $\overline{H}_\epsilon$ is closed and bounded, and hence compact. Let $\overline{W}_\epsilon \subset \mathrm{Stab}(\mathcal{C})$ be the closure of W_ϵ.

Proposition 6.15. *The map $H_\epsilon^n \to W_\epsilon$ extends to a homeomorphism $\overline{H}_\epsilon^n \to \overline{W}_\epsilon$. As a result, $\overline{W}_\epsilon$ is compact.*

Proof. We first extend the map $H_\epsilon^n \to \mathrm{Stab}(\mathcal{C})$ to $\overline{H}_\epsilon^n$. The only points of $\overline{H}_\epsilon$ not in H_ϵ are the complex numbers of argument 1. Given $(z_1, \ldots, z_n) \in \overline{H}_\epsilon^n$, let $S \subset \{1, \ldots, n\}$ be the indices such that the argument of z_i is 1. Let $\mathcal{A} \subset \mathcal{C}$ be the extension closure of P_i for $i \notin S$ and $P_i[-1]$ for $i \in S$. This is a tilt of the standard heart, and hence also the heart of a bounded t-structure. We send $(z_1, \ldots, z_n)$ to the stability condition whose $[0, 1)$-heart is $\mathcal{A}$ and whose central charge is defined by $[P_i] \mapsto z_i$. It is easy to check that the resulting map is continuous, and defines the required homeomorphism. $\square$

Let $U \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$ be the image of W under the quotient map $\mathrm{Stab}(\mathcal{C}) \to \mathrm{Stab}(\mathcal{C})/\mathbf{C}$. This is the set of stability conditions (up to the $\mathbf{C}$-action) in which the objects P_i are stable and their phases lie in an open interval of length 1. It is easy to check that $U \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$ is an open subset. Let $U_\epsilon \subset U$ be the image of W_ϵ.

Proposition 6.16. *The open set $U_\epsilon \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$ has a compact closure.*

Proof. The closure $\overline{U}_\epsilon$ is the image of $\overline{W}_\epsilon$, which is compact by Proposition 6.15. $\square$

The following proposition will be used when we show that the mass map is a homeomorphism onto its image in type A_2 and $\widehat{A}_1$. First let us establish some notation.

Definition 6.17. Fix some $q > 0$. Let τ be a stability condition. We say that τ *collapses* a distinguished triangle $x \to y \to z \xrightarrow{+1}$ if

$$m_{q,\tau}(y) = m_{q,\tau}(x) + m_{q,\tau}(z).$$

If τ does not collapse $x \to y \to z \xrightarrow{+1}$, then we have a strict triangle inequality

$$m_{q,\tau}(y) < m_{q,\tau}(x) + m_{q,\tau}(z).$$

Proposition 6.18. *Let τ be a standard stability condition. Suppose there exists a finite set T of distinguished triangles of objects in $\mathbf{S}$ with the following properties:*

(1) no triangle in T is collapsed by τ;
(2) if τ' is any stability condition such that no triangle in T is collapsed by τ', then τ' is standard up to the action of $\mathbf{C}$.

Then

$$m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P^S}$$

is a local homeomorphism onto its image at τ.

Proof. We abbreviate m_q by m.

Since τ is standard, it represents a point of $U \subset \text{Stab}(\mathcal{C})/\mathbf{C}$, which we also denote by τ. For a small enough ϵ, we have $\tau \in U_\epsilon$. The set $\overline{U}_\epsilon$ is compact by Proposition 6.16 and m is injective by Proposition 6.14. Therefore the map

$$(37) \qquad\qquad\qquad m \colon \overline{U}_\epsilon \to m(\overline{U}_\epsilon)$$

is a continuous bijection between two compact sets and hence a homeomorphism.

Let V be the subset of the image of m consisting of points $[f]$ that satisfy the inequalities

$$f(y) < f(x) + f(z)$$

for each non-collapsed triangle $x \to y \to z \xrightarrow{+1}$ asserted by the hypotheses. Then V is an open subset of the image of m containing $m(\tau)$. Furthermore, by the hypotheses, we have

$$m^{-1}(V) \subset U.$$

Let $V_\epsilon \subset V$ be the open subset defined by the conditions

(1) $f(P_1) \neq 0$, and
(2) for all i, we have

$$\epsilon < f(P_i)/f(P_1) < \epsilon^{-1}.$$

Then it is easy to check that $m^{-1}(V_\epsilon) \subset U_\epsilon$. If ϵ is small enough, then $m(\tau) \in V_\epsilon$ and hence $\tau \in m^{-1}(V_\epsilon)$. The map

$$m \colon m^{-1}(V_\epsilon) \to V_\epsilon$$

is a restriction of the homeomorphism (37), and hence a homeomorphism. Thus, m maps an open neighbourhood of τ homeomorphically to an open neighbourhood of $m(\tau)$, and hence is a local homeomorphism at τ. $\qquad\square$

Remark 6.19 (Existence of triangles). We prove the existence of the triangles T required in Proposition 6.18 in the A_2 and $\widehat{A}_1$ cases. We expect that such sets of triangles exist more generally.

Remark 6.20 (Functionals in the closure). It follows from Corollary 4.13 that the $\overline{\text{hom}}$ functionals are in the closure of the image of the mass map. Whether these functionals are dense in the boundary depends on q. In the next two sections we answer this question for $q = 1$ in the A_2 and $\widehat{A}_1$ cases. [1] answers this question for other values of q in the A_2 case.

7. The A_2 case

The aim of this section is to understand the Thurston compactification at $q = 1$ of the stability space for the 2-Calabi–Yau category $\mathcal{C}_\Gamma$ where Γ is the A_2 graph $\bullet \!\!\!\!\! \xrightarrow{\hspace{1cm}} \!\!\!\!\! \bullet$.

7.1. **Standard stability conditions.** Let $\mathcal{C}$ be the 2-Calabi–Yau category associated to the A_2 graph, as defined in Section 6. It is a graded, $\mathbf{k}$-linear triangulated category classically generated by two spherical objects P_1 and P_2. The standard heart $\heartsuit_{\text{std}}$ is the extension closure of P_1 and P_2. It has two simple objects, P_1 and P_2, and two additional indecomposable objects, denoted by P_{12} and P_{21}. The object P_{ij} is the unique extension of P_i by P_j.

Recall that $\tau \in \text{Stab}(\mathcal{C})$ is standard if its $[0, 1)$ heart is $\heartsuit_{\text{std}}$ and a point in $\text{Stab}(\mathcal{C})/\mathbf{C}$ is standard if it is the image of a standard τ.

We divide the set of standard stability conditions in two subsets as follows. Recall that for a standard τ, both P_1 and P_2 must be τ-stable. We say that τ is of *type I* if $\phi(P_1) \leq \phi(P_2)$ and of *type II* if $\phi(P_2) \leq \phi(P_1)$. If $\phi(P_1) = \phi(P_2)$, then τ is of both types, and we say that

it is *on-the-wall*; otherwise, it is *off-the-wall*. See Figure 2 for a sketch of the central charges of the stability conditions of the two types.

The two types are distinguished by which of P_{12} or P_{21} is semistable. In type I, P_{21} is semistable; in type II, P_{12} is semistable; on the wall, both are semistable.
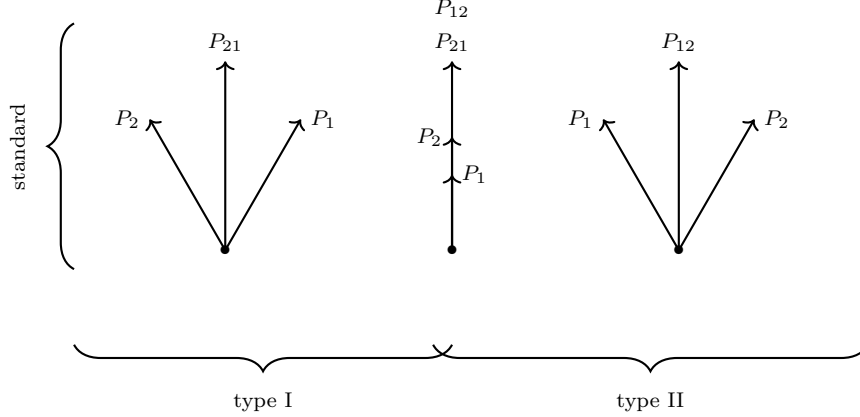

FIGURE 2. Central charges in a standard stability condition of types I and II. The intersection of the two types are on-the-wall stability conditions.

The subset of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ represented by standard stability conditions of type I is locally closed. Let Λ be the closure of this set in $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$. Note that Λ includes some non-standard stability conditions, namely those with $\phi(P_2) = \phi(P_1) + 1$. The set Λ tiles $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ under the action of B_3. That is, Λ satisfies the following properties (see, e.g., [10, Proposition 4.2]).

(1) Each point of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ lies in the B_3-orbit of a point of Λ.
(2) The stabiliser of Λ is the subgroup generated by $\gamma = \sigma_2\sigma_1$ in B_3.
(3) For any $g \in B_3$ not in the stabiliser, the interiors of Λ and $g\Lambda$ have empty intersection.

See Figure 6 for a picture of the tiling of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ by the orbit of Λ. The interior of Λ is the set of standard off-the-wall stability conditions of type I.

7.2. **The spherical objects.** Let $\mathbf{S}$ be the set of spherical objects of $\mathcal{C}$, up to isomorphism and shift. It turns out that elements of $\mathbf{S}$ can be thought of as rational points on the circle $\mathbf{P}^1(\mathbf{R})$. We now explain this correspondence.

Recall that the Artin–Tits braid group of the A_2 quiver is

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

It acts on $\mathcal{C}$ via the homomorphism

$$\sigma_i \mapsto \sigma_{P_i}.$$

It turns out to act transitively on the set of all the spherical objects of $\mathcal{C}$, and hence on $\mathbf{S}$ [2]. The element $(\sigma_2\sigma_1)^3$ generates the center of B_3 and acts on the category by $x \mapsto x[-2]$. As a result, the action of B_3 on $\mathbf{S}$ factors through $B_3/Z(B_3)$. On the other hand, we have an isomorphism $B_3/Z(B_3) \to \mathrm{PSL}_2(\mathbf{Z})$ given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The action of B_3 on $\mathcal{C}$ is faithful [26, 29]. It is easy to check that the stabiliser of $P_1 \in \mathbf{S}$ is generated by σ_1. As a result, we have a $\mathrm{PSL}_2(\mathbf{Z})$-equivariant bijection

$$(38) \qquad\qquad i \colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$$

defined uniquely by the choice

$$P_1 \mapsto [1 : 0].$$

Let us describe the spherical object corresponding to a point $[a : c] \in \mathbf{P}^1(\mathbf{Z})$. Assume $c \neq 0$. Write the rational number a/c as a continued fraction with an odd number of terms:

$$\frac{a}{c} = n_0 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_{2k}}}}.$$

Here, each n_i is an integer, with $n_i > 0$ for $i = 1, \ldots, 2k$. Then $[a : c]$ corresponds to the object of $\mathbf{S}$ given by

$$(39) \qquad\qquad \sigma_1^{n_0} \sigma_2^{-n_1} \cdots \sigma_1^{n_{2k}}(P_2).$$

For example, we get

$$P_1 \mapsto [1 : 0], \quad P_2 \mapsto [0 : 1], \quad P_{21} \mapsto [1 : -1], \text{ and } P_{12} \mapsto [1 : 1].$$

7.3. **The automaton.** Fix an off-the-wall standard stability condition τ of type I. Let $\Sigma = \{P_1, P_2, P_{21}\}$, the set of indecomposable τ-stable objects of the heart. Set $X = P_{21}$. Define the braid $\gamma \in B_3$ by

$$\gamma = \sigma_2 \sigma_1 = \sigma_X \sigma_2 = \sigma_1 \sigma_X.$$

Figure 3 describes a τ-HN automaton Θ.

Formally (following Definition 5.3), the automaton Θ is defined by the B_3-labeled graph with three vertices and three edges from each vertex, as shown in Figure 3. Note that the states are called $[X, P_1]$, $[P_1, P_2]$, and $[P_2, X]$. The Θ-set S is defined by setting $S_{[a,b]}$ to be the set of spherical objects whose τ-stable HN factors are shifts of a and b. For example, the object $\sigma_1 \sigma_1(P_2)$ has stable HN factors $P_1[-1]$, P_1, and P_2, and hence is supported at the state $[P_1, P_2]$. The objects P_1, P_2, and X are supported at two states. Recall that the representation M associates the rank 3 module $\mathbf{Z}^\Sigma$ to each state. However, by construction, at the state $[a, b]$, the image of $\mathrm{HN}_\tau \colon S \to \mathbf{Z}^\Sigma$ lands in the rank 2 sub-module $\mathbf{Z}^{\{a,b\}}$. We replace $\mathbf{Z}^\Sigma$ by this sub-module. Then the maps $M(e)$ are given by 2×2 matrices in the standard basis, as shown in Figure 3. The arrows labelled $\gamma^{\pm 1}$ correspond to the identity matrix, which we omit in the figure.

To check that this setup defines an HN automaton, we need to check that an arrow takes objects supported at its source to objects supported at its target. We also need to check that the HN multiplicities transform according to the matrix associated to the arrow, as in Definition 5.3. We carry out both these checks in the next proposition.

Proposition 7.1. *Let $v = [a, b]$ and $w = [c, d]$ be two states of Θ. Let $e \colon v \to w$ be an arrow labelled by the element $g \in B_3$. Let $x \in \mathcal{C}$ be any object whose stable τ-HN factors are shifts of a and b. The following hold.*

(1) The stable τ-HN factors of $g \cdot x$ are shifts of c and d.
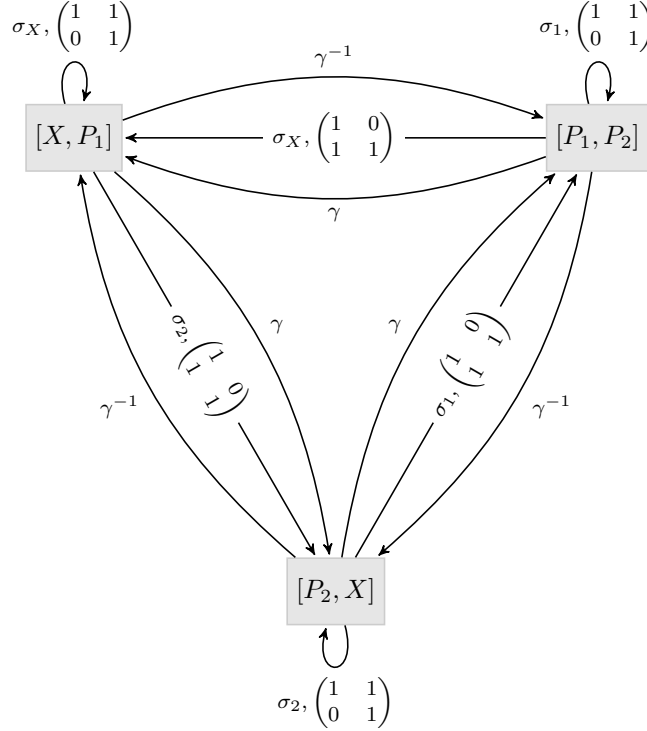
(2) The following diagram commutes.

FIGURE 3. An automaton describing the dynamics of HN filtrations in a stability condition with stable objects P_1, P_2, and $X = P_{21}$.

$$x \xmapsto{\mathrm{HN}_\tau} \mathrm{HN}_\tau(x) \in \mathbf{Z}^{\{a,b\}}$$
$$\downarrow{\scriptstyle g} \qquad\qquad \downarrow{\scriptstyle M(e)}$$
$$gx \xmapsto{\mathrm{HN}_\tau} \mathrm{HN}_\tau(gx) \in \mathbf{Z}^{\{c,d\}}$$

In other words,

$$\mathrm{HN}_\tau(gx) = M(e) \cdot \mathrm{HN}_\tau(x).$$

Proof. Let

$$0 \to x_0 \to x_1 \to \cdots \to x_n = x$$

be a filtration of x with whose factors $z_i = \mathrm{Cone}(x_{i-1} \to x_i)$ are stable and appear in non-increasing order of phase. Then, by hypothesis each z_i is a shift of a or b. By applying g, we get a filtration

(40) $$0 \to gx_0 \to gx_1 \to \cdots \to gx_n = gx$$

with factors gz_i.

We check that the filtration (40) is rectifiable in the sense of Definition A.10. We use Theorem A.18.

We outline the case $v = [P_1, P_2]$, leaving the others to the reader. Denote by $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) the HN factor of x of highest (resp. lowest) phase. To apply Theorem A.18 to the filtration (40), we must check the following: for $i < j$, either

(1) $\mathrm{Hom}(gz_j, gz_i[1]) = \mathrm{Hom}(gz_i, gz_j[1]) = 0$, or
(2) $\lfloor gz_i \rfloor \geq \lceil gz_j \rceil$.

If $z_i \cong z_j$, then (1) holds. Let us assume otherwise, so that $\phi(z_i) > \phi(z_j)$. Since we are in a 2-CY category, $\mathrm{Hom}(gz_j, gz_i[1])$ vanishes if and only if $\mathrm{Hom}(gz_i, gz_j[1])$ does. We enumerate the pairs (z_i, z_j) such that

(1) z_i and z_j are shifts of P_1 or P_2,
(2) $\phi(z_i) > \phi(z_j)$, and
(3) $\mathrm{Hom}(z_j, z_i[1]) \neq 0$.

Up to a simultaneous shift, the only such pairs are $(P_1[1], P_1)$, $(P_2[1], P_2)$, and (P_2, P_1). The outgoing edges e from v are labeled $\sigma_1, \sigma_2, \gamma$, or γ^{-1}. It is a simple check that for each of these edges and each of the enumerated pair, condition (2) holds.

Since the filtration (40) is rectifiable, it follows from Proposition A.9 that

$$\mathrm{HN}_\tau(gx) = \sum \mathrm{HN}_\tau(gz_i).$$

Therefore, it suffices to prove the proposition for $x = z_i$, and hence, for $x = a$ and $x = b$. This is another straightforward calculation. $\qquad\square$

The automaton recognises every spherical object in the sense of Remark 5.4.

Proposition 7.2. *The following hold.*

(1) Every $\beta \in B_3$ has an expression of the form

$$\beta = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k},$$

where n is an integer, k is a non-negative integer, the m_i are positive integers, and the sequence $(a_1, a_2 \ldots, a_k)$ is a contiguous subsequence of the sequence

$$(\ldots, X, 1, 2, X, 1, 2, \ldots).$$

(2) Every spherical object $s \in \mathcal{C}$ has an expression

$$s = \beta_1 \cdots \beta_n P_1$$

that is recognised by Θ.

Proof. For (1), we repeatedly use the commutation relations

$$\gamma \sigma_2 \gamma^{-1} = \sigma_1, \quad \gamma \sigma_X \gamma^{-1} = \sigma_2, \quad \gamma \sigma_1 \gamma^{-1} = \sigma_X.$$

Begin by writing β as any product of the generators σ_1 and σ_2, along with their inverses. Eliminate the inverses of the generators by rewriting as follows:

$$\sigma_1^{-1} = \sigma_X \gamma^{-1}, \quad \sigma_2^{-1} = \sigma_1 \gamma^{-1}.$$

Next, use the commutation relations to rewrite

(41) $$\gamma^i \sigma_X = \sigma_2 \gamma^i, \quad \gamma^i \sigma_1 = \sigma_X \gamma^i, \quad \gamma^i \sigma_2 = \sigma_1 \gamma^i,$$

and thus move all powers of to the left. The rest of β is now a product of elements from $\{\sigma_1, \sigma_2, \sigma_X\}$. Replace any occurrences of $\sigma_2 \sigma_1$, $\sigma_1 \sigma_X$, or $\sigma_X \sigma_2$ by γ and again move γ to the left as before. It is easy to see that eventually, β reaches the desired form.

For (2), begin by writing $s = \beta P_1$ for some $\beta \in B_3$. Write

$$\beta = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k}$$

as in (1). Note that P_1 is supported at two states, namely $[P_1, P_2]$ and $[X, P_1]$. Each of the letters $\{\sigma_1, \sigma_2, \sigma_X\}$ is applicable at least at one of the two states above, so we can apply σ_{a_k}.

The condition on our cyclic writing guarantees that we can apply each subsequent letter. Therefore, Θ recognises the expression

$$s = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k} P_1.$$

$\square$

As a consequence of Proposition 7.2, every spherical object is supported on at least one of the three states. In particular, its HN factors involve (up to shift) at most 2 out of the 3 objects P_1, P_2, and X.

Proposition 7.3. *Let s be a spherical object supported at the state $[P_1, P_2]$. Assume that s is not a shift of P_1 or P_2. Then*

(1) we can write $s = \sigma_1^r t$ for some $r > 0$ and an object t supported at $[P_2, X]$;
(2) we can write $s = \sigma_2^{-r} t$ for some $r > 0$ and an object t supported at $[X, P_1]$.

Analogous statements hold for the other two states.

Proof. Write a recognised expression $s = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k} P_1$ as in the proof of Proposition 7.2. Since s is not a shift of P_1, P_2, or X, we have $k \geq 1$. Using the commutation relations (41), move γ to the right to obtain another expression $s = \sigma_{b_1}^{m_1} \sigma_{b_2}^{m_2} \cdots \sigma_{b_k}^{m_k} \gamma^n P_1$, which is also recognised by Θ. Observe that the state at which the recognising path ends is dictated by b_1. For $b_1 = 1$, the end state is $[P_1, P_2]$; for $b_1 = 2$, it is $[P_2, X]$; and for $b_1 = X$, it is $[X, P_1]$. By assumption, we must have $b_1 = 1$.

Taking $r = m_1$ and $t = \sigma_{b_2}^{m_2} \cdots \sigma_{b_k}^{m_k} \gamma^n P_1$ yields the first assertion of the proposition. For the second assertion, consider

$$\sigma_2 s = \sigma_2 \sigma_{b_1}^{m_1} \cdots \sigma_{b_k}^{m_k} \gamma^n P_1.$$

Write $\sigma_2 \sigma_{b_1} = \sigma_2 \sigma_1$ as γ and propagate γ to the right using the commutation relations (41), to obtain a recognised expression for $\sigma_2 s$, as follows:

$$\sigma_2 s = \sigma_{b_1'}^{m_1 - 1} \sigma_{b_2'}^{m_2} \cdots \sigma_{b_k'}^{m_k} \gamma^{n+1} P_1,$$

where $\sigma_{b_i'} = \gamma \sigma_{b_i} \gamma^{-1}$. In particular, $\sigma_{b_1'} = \sigma_X$. Depending on the values of m_1 and k, the leftmost letter of this expression is one of γ, σ_X, or σ_1. In the first two cases, we stop; in the last case we consider

$$\sigma_2^2 s = \sigma_2^2 \sigma_{b_1}^{m_1} \cdots \sigma_{b_k}^{m_k} \gamma^n P_1$$

and continue as above. After finitely many steps say r, we obtain $t = \sigma_2^r s$, which has a recognised expression whose leftmost letter is either γ or σ_X.

If the leftmost letter is γ, then $t = \gamma^{n+r} P_1$, and so t is one of P_1, P_2, or X, up to shift. By assumption, s is not a shift of P_2, so neither is t. In particular, t is supported at $[X, P_1]$ If the leftmost letter is σ_X, then t is supported at $[X, P_1]$ as observed earlier in the proof. $\square$

Remark 7.4. Let τ be an on-the-wall standard stability condition. Figure 4 describes a τ-HN automaton that recognises all spherical objects. Since the automaton in Figure 3 suffices for mass computations, we omit the details.

Remark 7.5. For every off-the-wall stability condition, we have an HN automaton obtained by applying an appropriate braid to the automaton in (3). Likewise, for an on-the-wall stability condition, we have an HN automaton obtained by applying a braid to the automaton in (4). Thus, we have an automaton at every point of the stability space. Observe that these automata exhibit a wall-and-chamber behaviour: they remain constant in a chamber, get more complicated along a wall, and flip to a different automaton as we cross the wall.
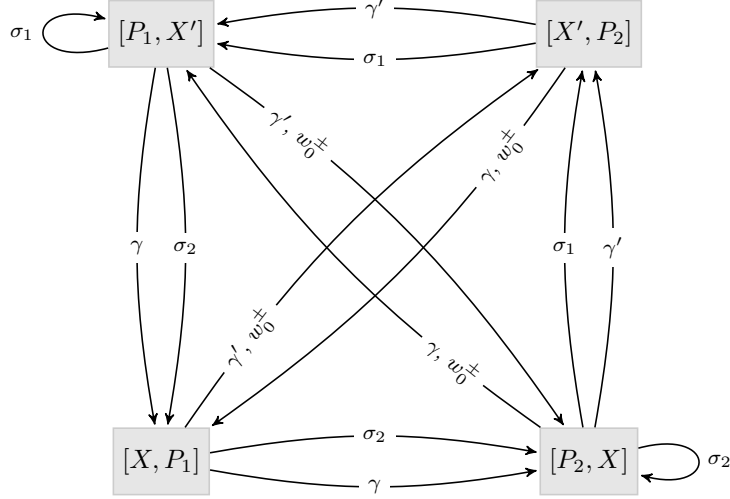
FIGURE 4. An automaton describing the dynamics of HN filtrations in an on-the-wall stability condition with semistable objects P_1, P_2, $X = P_{21}$, and $X' = P_{12}$. Here $\gamma = \sigma_2\sigma_1$; $\gamma' = \sigma_1\sigma_2$; and $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

It would be interesting to understand the wall-crossing formulas for recognized expressions of a fixed braid in different chambers.

7.4. **Consequences of the automaton.** In this section, we use the automaton from Section 7.3 to prove results about $m\colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P^S}$ and $h\colon \mathbf{S} \to \mathbf{P^S}$. We prove that m is a homeomorphism onto its image (Proposition 7.8) and describe the closure of h (Proposition 7.16). In subsequent sections, we establish that the closure of h is indeed the boundary of the image of m.

Recall that $X = P_{21}$. Recall from Definition 6.17 that $x \to y \to z \xrightarrow{+1}$ is a collapsed triangle if $m(y) = m(x) + m(z)$.

Proposition 7.6. *Consider the following collection of distinguished triangles:*

$$T = \{P_1 \to X \to P_2 \xrightarrow{+1}, \quad X \to P_2 \to P_1[1] \xrightarrow{+1}, \quad P_2[-1] \to P_1 \to X \xrightarrow{+1}\}.$$

If τ is an off-the-wall standard stability condition of type I, then no triangle in T is collapsed by τ. Furthermore, if τ' is any stability condition such that no triangle in T is collapsed by τ', then τ' is standard of type I, up to the action of $\mathbf{C}$.

Proof. It is immediate from Proposition 3.3 (or a direct check) that no triangle in T is collapsed by any off-the-wall standard stability condition of type *I*.

Let τ' be any stability condition. Recall that τ' is in the braid group orbit of a standard stability condition τ of type I. Write $\tau' = \beta\tau''$ for some braid β and some standard stability condition τ'' of type I.

If β is a power of γ, then τ' is already standard of type I. Otherwise, we exhibit a triangle of T that is collapsed by τ'. Equivalently, we exhibit a triangle of $\beta^{-1}T$ that is collapsed by τ''.

Suppose that τ'' is off-the-wall. Consider a cyclic writing of β^{-1} as described in Proposition 7.2. Since β is not a power of γ, this writing has rightmost letter either σ_1, σ_2, or

σ_X. Suppose the rightmost letter of β^{-1} is σ_1. Consider the first triangle in T, namely $X \to P_2 \to P_1[1] \xrightarrow{+1}$. Observe from the automaton in Figure 3 that the objects P_1, P_2, and X are all sent to the state $[P_1, P_2]$ after applying σ_1. Furthermore, we have the equation

$$\text{HN}_{\tau''}(\sigma_1(X)) = \text{HN}_{\tau''}(\sigma_1(P_1)) + \text{HN}_{\tau''}(\sigma_1(P_2)).$$

The equation above persists by linearity as we apply each successive letter of the cyclic writing of β^{-1}, resulting in the equation

$$\text{HN}_{\tau''}(\beta^{-1}(X)) = \text{HN}_{\tau''}(\beta^{-1}(P_1)) + \text{HN}_{\tau''}(\beta^{-1}(P_2)),$$

and hence

$$m_{\tau''}(\beta^{-1}(X)) = m_{\tau''}(\beta^{-1}(P_1)) + m_{\tau''}(\beta^{-1}(P_2)).$$

Hence the triangle $\beta^{-1}X \to \beta^{-1}P_2 \to \beta^{-1}P_1[1] \xrightarrow{+1}$ is collapsed by τ''. Similarly, we can check that if the cyclic writing of β^{-1} ends in σ_2 (respectively σ_X), then the second (respectively third) triangle of T is collapsed by τ'.

If τ'' is on-the-wall, then it is a limit of an on-the-wall stability condition. By continuity of the masses, we still conclude that $\tau' = \beta\tau''$ collapses one of the triangles in T. $\qquad\square$

Recall that $X' = P_{21}$.

Proposition 7.7. *Consider the following collection of distinguished triangles:*

$$T = \{X \to P_2 \to P_1[1] \xrightarrow{+1}, \quad P_2[-1] \to P_1 \to X \xrightarrow{+1}, \quad X' \to P_1 \to P_2[1] \xrightarrow{+1}\}.$$

If τ is an on-the-wall standard stability condition, then no triangle in T is collapsed by τ. Furthermore, if τ' is any stability condition such that no triangle in T is collapsed by τ', then τ' is standard of type I, up to the action of $\mathbf{C}$.

Proof. Analogous to the proof of Proposition 7.6. $\qquad\square$

Proposition 7.8. *The map $m \colon \text{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}}$ is a homeomorphism onto its image.*

Proof. Proposition 6.14 shows that m is injective. Proposition 6.18 together with Proposition 7.6 and Proposition 7.7 shows that m is a local homeomorphism at a standard stability condition of type I. Since every $\tau \in \text{Stab}(\mathcal{C})/\mathbf{C}$ is in the braid group orbit of a standard stability condition of type I, it follows that m is a local homeomorphism at every τ. As a result, m is a homeomorphism onto its image. $\qquad\square$

Recall that we have a surjective map $B_3 \to \text{PSL}_2(\mathbf{Z})$. We use the notation $/\pm$ to denote the quotient that identifies every element and its negative. The standard action of $\text{PSL}_2(\mathbf{Z})$ on $\mathbf{Z}^2/\pm$ gives an action of B_3 on $\mathbf{Z}^2/\pm$. As a result, $\mathbf{Z}^2/\pm$ becomes a Θ-set. Recall that we also have the Θ-representation M shown in Figure 3, and hence the Θ-set $M/\pm$.

For each of the three states v of Θ, we define a linear map $\phi_v \colon M_v \to \mathbf{Z}^2$ as follows:

$$\phi_{[P_1, P_2]} \colon (1, 0) \mapsto (1, 0) \text{ and } (0, 1) \mapsto (0, 1),$$
$$\phi_{[P_2, X]} \colon (1, 0) \mapsto (0, 1) \text{ and } (0, 1) \mapsto (1, -1),$$
$$\phi_{[X, P_1]} \colon (1, 0) \mapsto (1, -1) \text{ and } (0, 1) \mapsto (1, 0).$$

Proposition 7.9. *The maps ϕ_v give an isomorphism of Θ-sets $M/\pm \to \mathbf{Z}^2/\pm$.*
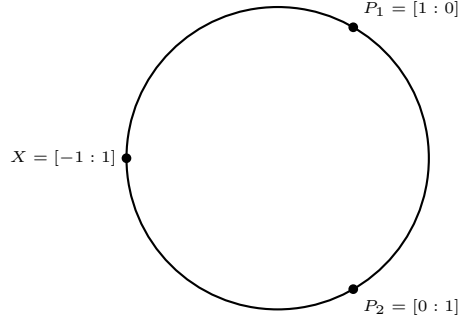
FIGURE 5. The points P_1, P_2, and $X = P_{21}$ divide $\mathbf{P}^1(\mathbf{R})$ into three arcs. The HN factors of an object (up to shift) only include the two endpoints of the arc on which the object lies.

Proof. All three maps ϕ_v are clearly bijective. It remains to check that they are Θ-equivariant. That is, up to sign, for every edge $e\colon v \to w$, we have

$$e \circ \phi_v = \phi_w \circ M(e).$$

We omit this straightforward verification. $\square$

Recall that we have a B_3-equivariant map $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$ defined in (38). The following proposition identifies $i(s)$ explicitly in terms of the HN multiplicities.

Proposition 7.10. *For a spherical object s supported at the state v, we have the equality*

$$i(s) = \phi_v(\mathrm{HN}_\tau(s)) \in \mathbf{P}^1(\mathbf{Z}).$$

Moreover $\phi_v(\mathrm{HN}_\tau(s)) \in \mathbf{Z}^2/\pm$ is the unique representative of $i(s)$ whose coordinates are relatively prime.

Proof. The equation evidently holds for $s = P_1$. If it holds for s supported at v, and $e\colon v \to w$ is an edge of Θ with label $\beta \in B_3$, then using the Θ-equivariance of i, ϕ_v, and HN_τ, we see that it also holds for βs. By Proposition 7.2, it holds for all spherical objects.

By Proposition 7.2, the vector $\mathrm{HN}_\tau(s)$ is obtained by applying a sequence of matrices in Figure 3 to the vector $(1,0)$ or $(0,1)$. Since all these matrices have determinant 1, we see that the coordinates of $\mathrm{HN}_\tau(s)$ are relatively prime. Since the linear maps defining ϕ are also of determinant ± 1, the same is true for $\phi_v(\mathrm{HN}_\tau(s))$. $\square$

By Proposition 7.2, the set $\mathbf{S}$ is the union of the three sets supported at the three vertices of Θ. Under the map $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$, this division corresponds to a geometric division of the circle $\mathbf{P}^1(\mathbf{R})$, which we now describe. The three points $i(P_1) = [1:0]$ and $i(P_2) = [0:1]$, and $i(X) = [-1:1]$ divide $\mathbf{P}^1(\mathbf{R})$ into three closed arcs (see Figure 5). We denote these arcs by $[P_1, P_2]$, $[P_2, X]$, and $[X, P_1]$. While this notation is identical to the state labels of our HN automaton, the following proposition justifies this choice.

Proposition 7.11. *The map i sends the objects of $\mathbf{S}$ supported at the state $[P_1, P_2]$ to the arc $[P_1, P_2] \subset \mathbf{P}^1(\mathbf{R})$, and likewise for the other two states.*

Proof. By Proposition 7.10, the map $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{R})$ is equal to $s \mapsto \phi(\mathrm{HN}_\tau(s))$. For s supported at $[P_1, P_2]$, the vector $\mathrm{HN}_\tau(s)$ lies in the non-negative cone in $\mathbf{Z}^{\{P_1, P_2\}}$, which is mapped by $\phi_{[P_1,P_2]}$ to the arc $[P_1, P_2]$. The argument for the other two states is similar. $\square$

We now use our automaton to give a new and simpler proof of a theorem of Rouquier and Zimmermann [29, Proposition 4.8]. This will be the content of Proposition 7.13. Rouquier and Zimmermann work in the homotopy category of complexes of projective modules over the zig-zag algebra, which is formally a different category from the 2-CY category we study in this section. As explained, e.g., in [2, §2.3.3], the effect of passing from their category to ours is to collapse one grading. Since [29, Proposition 4.8] involves forgetting the grading, Proposition 7.13 implies [29, Proposition 4.8].

To explain the argument in more detail, let us first introduce some basic notation.

Definition 7.12. Let x be an object of $\mathcal{C}$ belonging to the standard heart. The *Jordan–Hölder (JH) multiplicity* of P_1 (resp. P_2) in x is the multiplicity of P_1 (resp. P_2) in any Jordan–Hölder filtration of x. More generally, the JH multiplicity of P_i in an arbitrary x is the sum of the JH multiplicities of P_i in each cohomology of x, where cohomology is taken with respect to the t-structure whose heart is the standard heart.

Proposition 7.13. *Let s be a spherical object with $i(s) = [a : c] \in \mathbf{P}^1(\mathbf{Z})$, where a, c are relatively prime integers. Then the JH multiplicities of P_1 and P_2 in s are $|a|$ and $|c|$ respectively.*

Proof. Since the τ-HN filtration refines the cohomology filtration with respect to the standard heart, the JH multiplicities of P_1 and P_2 are linear functions of the τ-HN multiplicities. Up to sign, these linear functions are precisely the maps ϕ in Proposition 7.9. The result now follows by Proposition 7.10. $\square$

The next proposition relates the JH multiplicities and the $\overline{\mathrm{hom}}$ functionals. It is an important tool for understanding the closure of the image of the mass map.

Proposition 7.14. *Let x be a spherical object with $i(x) = [a : c] \in \mathbf{P}^1(\mathbf{Z})$, where a, c are relatively prime integers. Then $\overline{\mathrm{hom}}(P_2, x) = |a|$ and $\overline{\mathrm{hom}}(P_1, x) = |c|$.*

Proof. We prove the assertion for $\overline{\mathrm{hom}}(P_1, x)$. The other case follows by applying γ. It is easy to check the result for $x = P_1, P_2$, and X by hand.

Fix a standard off-the-wall stability condition. By Proposition 7.13, $|c|$ gives the JH-multiplicity of P_2, which is a linear function of the HN-multiplicities. The key idea is to prove that $\overline{\mathrm{hom}}(P_1, -)$ is also a linear function of the HN-multiplicities. Then the cases $x = P_1, P_2$, and X imply the result for all x.

We now prove the linearity. Begin with an object x supported at $[P_2, X]$. Let

$$0 = x_0 \to \cdots \to x_n = x.$$

be the HN filtration with factors $z_i = \mathrm{Cone}(x_{i-1} \to x_i)$. We must prove that

$$\overline{\mathrm{hom}}(P_1, x) = \sum \overline{\mathrm{hom}}(P_1, z_i).$$

Note that since neither x nor z_i is a shift of P_1, we have $\overline{\mathrm{hom}} = \dim \mathrm{Hom}^\bullet$.

Apply $\mathrm{Hom}^\bullet(P_1, -)$ to the HN filtration of x to get the following filtration in the bounded derived category of graded vector spaces:

(42) $$0 = \mathrm{Hom}^\bullet(P_1, x_0) \to \cdots \to \mathrm{Hom}^\bullet(P_1, x_n) = \mathrm{Hom}^\bullet(P_1, x).$$

The factors in (42) are $\mathrm{Hom}^\bullet(P_1, z_i)$. We argue that the connecting map

$$\mathrm{Hom}^\bullet(P_1, z_i) \to \mathrm{Hom}^\bullet(P_1, x_i[1])$$

vanishes. To see this, consider the map $z_i \to x_{i-1}[1]$. It suffices to prove that for every HN factor y of $x_{i-1}[1]$ and every map $z_i \to y$, the induced map

$$\mathrm{Hom}^\bullet(P_1, z_i) \to \mathrm{Hom}^\bullet(P_1, y)$$

vanishes. But the HN factors of $x_{i-1}[1]$ are simply $z_j[1]$ for $j < i$. Using that both z_i and z_j are shifts of P_2 or X and that $\phi(z_j) \geq \phi(z_i)$, we see that (up to isomorphisms and shifts) the only nonzero possibilities for $z_i \to z_j[1]$ are

$$P_2 \to P_2[2], \quad X \to X[2], \text{ and } P_2 \to X[2].$$

All three are killed by the functor $\mathrm{Hom}^\bullet(P_1, -)$.

Since the connecting maps in (42) vanish, we get

$$\mathrm{Hom}^\bullet(P_1, x) \cong \bigoplus \mathrm{Hom}^\bullet(P_1, z_i),$$

and by taking dimensions

$$\dim \mathrm{Hom}^\bullet(P_1, x) = \sum \dim \mathrm{Hom}^\bullet(P_1, z_i).$$

We now treat the case of x supported at the other two states. Instead of directly proving linearity, we argue using the group action. By Proposition 7.13, the JH multiplicities of P_2 in x and $\sigma_1^r x$ are equal. We also have

$$\overline{\mathrm{hom}}(P_1, x) = \overline{\mathrm{hom}}(P_1, \sigma_1^r x).$$

Hence, the proposition for x implies the proposition for $\sigma_1^r x$. By Proposition 7.3, any x supported at $[X, P_1]$ or $[P_1, P_2]$ can be written as $x = \sigma_1^r y$ for some $r \in \mathbf{Z}$ and y supported at $[P_2, X]$. Hence the proposition holds for x. $\qquad\square$

Consider the action of B_3 on $\mathbf{R}^2/\pm$ via the homomorphism $B_3 \to \mathrm{PSL}_2\,\mathbf{Z}$. Let $\pi\colon \mathbf{R}^2 \to \mathbf{R}$ be the second projection.

Proposition 7.15. *Let $\beta \in B_3$. Set $s = \beta P_1$ and let x be any spherical object with $i(x) = [a : c]$, where $a, c \in \mathbf{Z}$ are relatively prime. Then*

$$\overline{\mathrm{hom}}(s, x) = |\pi(\beta^{-1}(a, c))|.$$

In particular, we have

$$\overline{\mathrm{hom}}(P_1, x) = |c|, \quad \overline{\mathrm{hom}}(P_2, x) = |a|, \quad \overline{\mathrm{hom}}(P_{21}, x) = |a + c|, \text{ and } \overline{\mathrm{hom}}(P_{12}, x) = |a - c|.$$

Proof. We have

$$\overline{\mathrm{hom}}(s, x) = \overline{\mathrm{hom}}(\beta P_1, x) = \overline{\mathrm{hom}}(P_1, \beta^{-1}x).$$

By Proposition 7.14, the last quantity is $|\pi(\beta^{-1}(a, c))|$, as required. Taking $\beta = 1, \gamma^{-1}, \sigma_2$, and σ_2^{-1} yield the four equations. $\qquad\square$

Recall the map $h\colon \mathbf{S} \to \mathbf{P^S}$ defined by

$$h\colon x \mapsto [\overline{\mathrm{hom}}(x, -)].$$

Consider $\mathbf{S}$ as a subset of $\mathbf{P}^1(\mathbf{R})$ via the bijection $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$ as defined in (38).

Proposition 7.16. *The map $h\colon \mathbf{S} \to \mathbf{P^S}$ extends to a continuous map $\mathbf{P}^1(\mathbf{R}) \to \mathbf{P^S}$, which is a homeomorphism onto its image.*

Proof. For an $s \in \mathbf{S}$, chose a $\beta_s \in B_3$ such that $s = \beta_s P_1$. Define $\widetilde{h} \colon \mathbf{P}^1(\mathbf{R}) \to \mathbf{P}^{\mathbf{S}}$ by

$$\widetilde{h} \colon [a : c] \mapsto \left[\left| \pi(\beta_s^{-1}(a, c)) \right| \right].$$

Then $\widetilde{h}$ is evidently continuous and by Proposition 7.15, it extends h.

We check that $\widetilde{h}$ is a homeomorphism onto its image. Since the domain is compact and the target is Hausdorff, it suffices to check that $\widetilde{h}$ is injective. Consider the composition of $\widetilde{h}$ with the projection onto the homogenous coordinates corresponding to the objects P_1, P_2, and P_{21}. By Proposition 7.15, the composition is given by

$$[a : c] \mapsto [|a| : |c| : |a + c|].$$

This map is injective, and hence so is $\widetilde{h}$. □

7.5. **Gromov coordinates of a stability condition.** Let τ be a stability condition of type I. Since the three positive real numbers $m_\tau(P_1)$, $m_\tau(P_2)$, and $m_\tau(X)$ satisfy the triangle inequalities, there exist non-negative real numbers x, y, z such that

$$m_\tau(P_1) = y + z, \quad m_\tau(P_2) = z + x, \quad m_\tau(X) = x + y.$$

We call the x, y, z, the *Gromov coordinates* of τ. Note that if τ is off-the-wall, then the Gromov coordinates are all positive. If τ is on-the-wall (and type I), then $z = 0$, while x and y are positive.

Recall that $m_\tau \colon \mathbf{S} \to \mathbf{R}$ is the mass function associated to τ. Let $\overline{\mathrm{hom}}(s) \colon \mathbf{S} \to \mathbf{R}$ be the function $\overline{\mathrm{hom}}(s, -)$ as defined in Definition 4.8.

Proposition 7.17 (Linearity)**.** *We have*

$$m_\tau = x\,\overline{\mathrm{hom}}(P_1) + y\,\overline{\mathrm{hom}}(P_2) + z\,\overline{\mathrm{hom}}(X).$$

Proof. Let $s \in \mathbf{S}$ be a spherical object. Suppose τ is off-the-wall. Observe that the HN filtration of s is the same for all off-the-wall type I stability conditions, and its stable factors are P_1, P_2, and X, up to shift. Denoting by $a(s)$, $b(s)$, and $c(s)$ the multiplicities of these three, we have

$$m_\tau(s) = (y + z)a(s) + (x + z)b(s) + (x + y)c(s).$$

In particular, $m_\tau(s)$ is linear in the Gromov coordinates. Using Proposition 7.13 and Proposition 7.14, we have

$$b(s) + c(s) = \text{JH multiplicity of } P_2 \text{ in } s$$
$$= \overline{\mathrm{hom}}(P_1, s);$$
$$a(s) + c(s) = \text{JH multiplicity of } P_1 \text{ in } s$$
$$= \overline{\mathrm{hom}}(P_2, s).$$

By applying γ to the first equality, we get $a(s) + b(s) = \overline{\mathrm{hom}}(X, s)$. It follows that

$$m_\tau(s) = x\,\overline{\mathrm{hom}}(P_1, s) + y\,\overline{\mathrm{hom}}(P_2, s) + z\,\overline{\mathrm{hom}}(X, s).$$

The case of an on-the-wall τ follows by continuity. □

Let τ be an arbitrary stability condition. Then there are three τ semistable objects, say A, B, C, whose classes in the Grothendieck group are (up to sign) the classes of P_1, P_2, and X. These are obtained simply by applying an appropriate braid to P_1, P_2, and X. Define the Gromov coordinates x, y, z for τ by the condition

$$m_\tau(A) = y + z, \quad m_\tau(B) = x + z, \quad m_\tau(C) = x + y.$$

Then Proposition 7.17 implies that we have

$$m_\tau = x \, \overline{\hom}(A) + y \, \overline{\hom}(B) + z \, \overline{\hom}(C).$$

Note that the Gromov coordinates are all non-negative, and at most one may be zero.

The Gromov coordinates give a nice geometric picture of $\mathrm{Stab}(\mathcal{C})/\mathbf{C} \subset \mathbf{P^S}$. Let Δ denote the following clipped triangle:

$$\Delta = \{(x, y, z) \in \mathbf{R}^3_{\geq 0} \mid \text{at least two coordinates nonzero}\} / \, \mathbf{R}_{>0}$$
$$\cong \text{a closed planar triangle minus the three vertices.}$$

Recall from Section 7.1 that $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$ is the closure of the set of type I stability conditions. We have a homeomorphism $\Delta \to m(\Lambda) \subset \mathbf{P^S}$ given by the Gromov coordinates:

$$(43) \qquad\qquad (x, y, z) \mapsto x \cdot \overline{\hom}(P_1) + y \cdot \overline{\hom}(P_2) + z \cdot \overline{\hom}(X).$$

Note that the homeomorphism extends to the closed (unclipped) triangle $\overline{\Delta}$, under which the three vertices are mapped to the points $h(P_1)$, $h(P_2)$, and $h(X)$. By applying the B_3 action, we obtain the picture as shown in Figure 6.

Let us describe Figure 6 in more detail. Recall that the element $\gamma = \sigma_2\sigma_1$ generates the stabiliser of Λ in $\mathrm{PSL}_2(\mathbf{Z})$. So, we label the various (distinct) translates of Λ by $\langle\gamma\rangle$-cosets. Two translates of Λ are either disjoint or intersect along an edge, which is a copy of the open interval. The three translates that intersect Λ along its three edges are $\sigma_1\Lambda$, $\sigma_X\Lambda$, and $\sigma_2\Lambda$. Consequently, the three translates that intersect $\beta\Lambda$ are $\beta\sigma_1\Lambda$, $\beta\sigma_X\Lambda$, and $\beta\sigma_2\Lambda$. We can encode the translates and their intersections in a graph (called the *exchange graph* following [10]). Its vertices are left γ-cosets in $\mathrm{PSL}_2(\mathbf{Z})$, and a coset $\beta\langle\gamma\rangle$ is connected by an edge with $\beta\sigma_1\langle\gamma\rangle$, $\beta\sigma_X\langle\gamma\rangle$, and $\beta\sigma_2\langle\gamma\rangle$.

Remark 7.18. The complement of Λ in $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ has three connected components. We can describe these three components in two ways, as follows.

(1) Write τ' in the complement of Λ as $\tau' = \beta\tau$, where $\beta \in B_3$ and $\tau \in \Lambda$. Write β in a form recognised by the automaton in Figure 3. Then the three components correspond to the three possible end states of the automaton when it reads the writing of β.

(2) Recall from the proof of Proposition 7.6 that the three numbers

$$m_{\tau'}(P_1), m_{\tau'}(P_2), \text{ and } m_{\tau'}(X)$$

satisfy a collapsed triangle inequality—one is the sum of the other two. The three connected components correspond to the ways in which the inequality collapses.

7.6. **The closure of the image of the mass map.** We know by Proposition 4.1 that the closure of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ in $\mathbf{P^S}$ is compact. Recall that the map $h \colon \mathbf{S} \to \mathbf{P^S}$ is defined as

$$h \colon x \mapsto [\overline{\hom}(x, -)].$$

Set

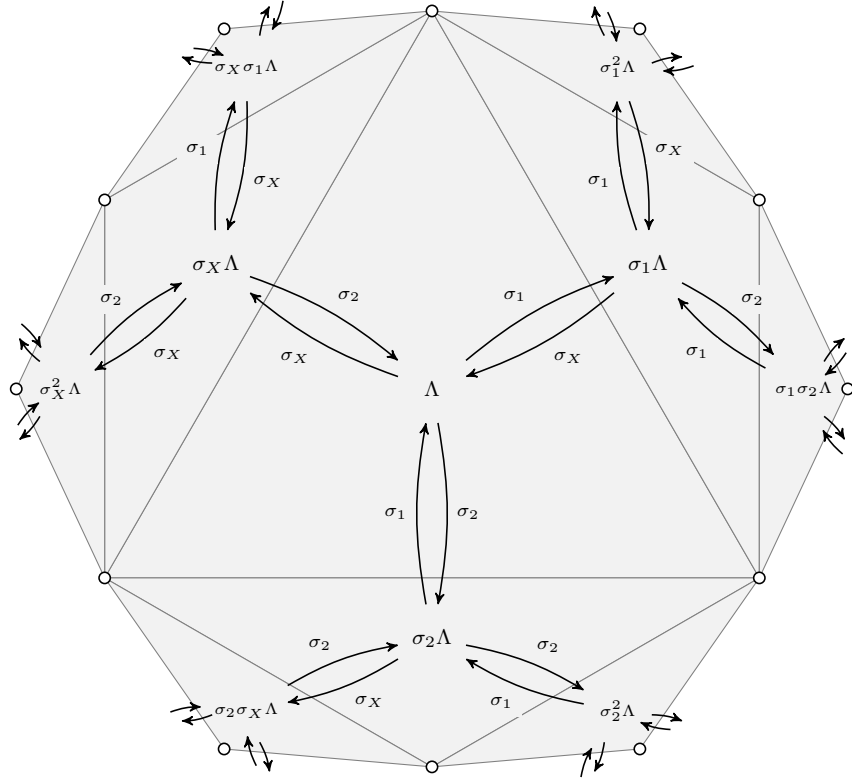$$P = \overline{h(\mathbf{S})} = h(\mathbf{P}^1(\mathbf{R})) \subset \mathbf{P^S}, \text{and}$$
$$M = m\left(\mathrm{Stab}(\mathcal{C})/\mathbf{C}\right) \subset \mathbf{P^S}.$$

The goal of Section 7.6 is to prove that $\overline{M} = M \cup P$.

Let

$$\overline{\Delta} = \left(\mathbf{R}^3_{\geq 0} \setminus 0\right) / \, \mathbf{R}_{>0}.$$

Then $\overline{\Delta}$ is homeomorphic to a closed triangle. An immediate consequence of the linearity (from Proposition 7.17) is the following.

FIGURE 6. The tiling of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ by the images Λ under the braid group.

Proposition 7.19 (Closure of Λ). *The closure of Λ in $\mathbf{P^S}$ is*

$$\overline{\Lambda} = \Lambda \cup \left\{ \overline{\mathrm{hom}}(P_1), \overline{\mathrm{hom}}(P_2), \overline{\mathrm{hom}}(X) \right\}.$$

This closure is homeomorphic to $\overline{\Delta}$.

Proof. Identify Λ with Δ using the Gromov coordinates as in (43). From Proposition 7.17, we see that the map $m \colon \Delta \to \mathbf{P^S}$ extends to a continuous map $\overline{\Delta} \to \mathbf{P^S}$. Its image is closed, and equals the union of Λ and the three additional points described above. The map from $\overline{\Delta}$ is injective, and hence an isomorphism onto its image. $\qquad\square$

It is convenient to have a measure of closeness for two elements of a projective space. Given two nonzero vectors in $\mathbf{R}^n$ for some positive integer n, denote by $\sphericalangle(v, w)$ the (acute) angle between them. By a slight abuse of notation, we use $\sphericalangle(v, w)$ also for two points $v, w \in \mathbf{P}^{n-1}$; this is just the angle between any two representatives in $\mathbf{R}^n$.

Given a finite subset $T \subset \mathbf{S}$, let $h_T \colon \mathbf{S} \to \mathbf{P}^T$ be the composition of $h \colon \mathbf{S} \to \mathbf{P^S}$ and the projection onto the T-coordinates.

Proposition 7.20 (Group action contracts). *Fix two elements $x, y \in \mathbf{S}$ and a finite subset $T \subset \mathbf{S}$. Given an $\epsilon > 0$, for all but finitely many elements g of $\mathrm{PSL}_2(\mathbf{Z})$, we have*

$$\sphericalangle(h_T(gx), h_T(gy)) < \epsilon.$$

Proof. Recall the bijection $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$. Set $a = i(x)$ and $b = i(y)$. The map $h_T\colon \mathbf{P}^1(\mathbf{R}) \to \mathbf{P}^T$ is continuous, and hence uniformly continuous. Therefore, it suffices to check that for all but finitely many g, the angle between ga and gb in $\mathbf{R}^2$ is small. Let X be the matrix with columns a and b. Then,

$$|\sin\left(\sphericalangle(ga, gb)\right)| = \frac{|\det g| \cdot |\det(X)|}{|ga| \cdot |gb|} = \frac{|\det(X)|}{|ga| \cdot |gb|}.$$

Since $g \in \mathbf{PSL}_2(\mathbf{Z})$ has integer entries, the quantity $|ga| \cdot |gb|$ is greater than $\epsilon^{-1}|\det(X)|$ for all but finitely many g. The claim follows. $\square$

Proposition 7.21 (Closure of M). *The sets M and P are disjoint and their union is the closure of M.*

Proof. The mass of every object in a stability condition is positive. On the other hand, $\overline{\mathrm{hom}}(x, x) = 0$, by definition. Therefore, M and P are disjoint.

By Proposition 7.19, we know that P is contained in the closure of M. We now prove that $M \cup P$ is closed. Let τ_n be a sequence in $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ whose images $m(\tau_n)$ in M approach a limit $t \in \mathbf{P^S}$. We must show that t lies in $M \cup P$.

Write $\tau_n = \beta_n \tau_n'$, where β_n is a braid and $\tau_n' \in \Lambda$ is a stability condition of type I. Let $\overline{\beta_n}$ be the image of β_n in $\mathrm{PSL}_2(\mathbf{Z})$. We have two possibilities: the set $\{\overline{\beta_n}\}$ is finite or infinite.

If $\{\overline{\beta_n}\}$ is finite, then the sequence $\{\tau_n\}$ is contained in the union of finitely many translates of the set of standard stability conditions Λ. By Proposition 7.19, the closure of Λ is contained in $M \cup P$. Hence, the closure of the union of finitely many translates of Λ is also contained in $M \cup P$. So the limit t lies in $M \cup P$.

Suppose $\{\overline{\beta_n}\}$ is infinite. Set $a_n = \beta_n(P_1)$, $b_n = \beta_n(P_2)$, and $c_n = \beta_n(X)$. Since P is compact, we may assume, after passing to a subsequence if necessary, that $h(a_n)$, $h(b_n)$, and $h(c_n)$ have limits in P, say a, b, and c.

We first prove that $a = b = c$. To see this, let $T \subset \mathbf{S}$ be an arbitrary finite set, and use the subscript T to denote projections to $\mathbf{P}^T$. It suffices to show that $a_T = b_T = c_T$. Since $\{\overline{\beta_n}\}$ is infinite, by Proposition 7.20, we note that the angles between $h_T(a_n)$, $h_T(b_n)$, and $h_T(c_n)$ approach 0 as n approaches ∞. Therefore, the three sequences have the same limits. But these limits are a_T, b_T, and c_T.

Next, we claim that $t = h(a) = h(b) = h(c)$. Indeed, by linearity (Proposition 7.17), we know that there exist non-negative real numbers x_n, y_n, z_n such that

$$m_T(\tau_n) = x_n h_T(A_n) + y_n h_T(B_n) + z_n h_T(C_n).$$

Taking the limit as $n \to \infty$ in projective space yields

$$t = h(a) = h(b) = h(c).$$

In particular, t lies in $M \cup P$. The proof is thus complete. $\square$

7.7. **Homeomorphism to the closed disk.** We take up the final part of the main theorem, namely that $(\overline{M}, P)$ is a manifold with boundary homeomorphic to the unit disk. We explicitly construct a homeomorphism from $\overline{M}$ to the disk, using the unprojectivised, but suitably normalised, mass and $\overline{\mathrm{hom}}$ functions.

Letting $X = P_{21}$, set

$$T = \{P_1, P_2, X\}.$$

Define a map

$$\mu\colon \mathrm{Stab}(\mathcal{C})/i\,\mathbf{R} \to \mathbf{R}^T = \mathbf{R}^3$$

by
$$\mu \colon \tau \mapsto (m_\tau(P_1), m_\tau(P_2), m_\tau(X)).$$
Similarly, define
$$\eta \colon \mathbf{S} \to \mathbf{R}^T = \mathbf{R}^3$$
by
$$\eta(s) = (\overline{\hom}(s, P_1), \overline{\hom}(s, P_2), \overline{\hom}(s, X)).$$
Thinking of $\mathbf{S}$ as $\mathbf{P}^1(\mathbf{Z}) \subset \mathbf{P}^1(\mathbf{R})$, it is easy to see that η extends to a continuous map
$$\eta \colon \mathbf{P}^1(\mathbf{R}) \to \mathbf{R}^3.$$
Indeed, using Proposition 7.15, we get
$$\eta \colon [a : c] \mapsto (|c|, |a|, |a + c|).$$

Let τ be a stability condition with (semi)-stable objects A, B, and C of class $[P_1]$, $[P_2]$, and $[X]$ in the Grothendieck group. Let x, y, z be the Gromov coordinates of τ, namely the non-negative real numbers such that
$$m_\tau(A) = y + z, \quad m_\tau(B) = z + x, \quad m_\tau(C) = x + y.$$
Denote by $|-|$ the standard Euclidean norm in $\mathbf{R}^3$. We say that τ is *normalised* if
$$(44) \qquad x|\eta(A)| + y|\eta(B)| + z|\eta(C)| = 1.$$

Remark 7.22. We point out one subtle aspect of the definition. If τ is on-the-wall, then one of the A, B, or C is not uniquely determined (even up to shift), since on-the-wall stability conditions have non-isomorphic semistable objects with the same class in the Grothendieck group. In this case, however, the corresponding Gromov coordinate is 0, and hence (44) holds for every possible A, B, C.

The normalisation yields a continuous section of
$$\mathrm{Stab}(\mathcal{C})/i\,\mathbf{R} \to \mathrm{Stab}(\mathcal{C})/\mathbf{C}.$$
That is, for every stability condition τ up to rotation and scaling, there is a unique normalised stability condition τ^ν up to rotation, and furthermore the map $\tau \to \tau^\nu$ is continuous.

We now identify $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ with its image M in $\mathbf{P^S}$ and $\mathbf{P}^1(\mathbf{R})$ with its image P in $\mathbf{P^S}$. Define
$$(45) \qquad \pi \colon \overline{M} = M \cup P \to \mathbf{R}^3$$
as follows. For $\tau \in M$, set
$$(46) \qquad \pi(\tau) = \mu(\tau^\nu),$$
and for $s \in P$, set
$$\pi(s) = \eta(s)/|\eta(s)|.$$

By construction, $\pi([a : c])$ is the unit vector in the direction of
$$\eta([a : c]) = (|a|, |c|, |a + c|).$$
That is, π maps P to the unit sphere in $\mathbf{R}^3$. It is easy to see that $\pi|_P$ is injective, and hence a homeomorphism onto its image. The image consists of three circular arcs, one for each pair of end-points from the three points
$$\frac{1}{\sqrt{2}}(0, 1, 1), \quad \frac{1}{\sqrt{2}}(1, 0, 1), \text{ and } \frac{1}{\sqrt{2}}(1, 1, 0).$$
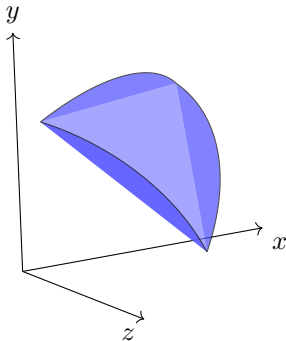
FIGURE 7. The homeomorphic image $\Phi \subset \mathbf{R}^3$ of the compactified stability manifold.

The arc joining each pair is a geodesic arc on the unit sphere; that is, the plane it spans passes through the origin.

Equation (44) and the triangle inequality imply that for every stability condition τ, we have $|\pi(\tau)| \leq 1$. In fact, for every stability condition, at least two of the Gromov coordinates are nonzero. Therefore, we actually have a strict inequality $|\pi(\tau)| < 1$.

Recall from Section 7.1 that Λ is the closure of the type I stability conditions in $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$. Recall also that the B_3-translates of Λ tile $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$. To get a better understanding of π, let us study it on the translates of Λ. Let β be a braid. Consider the translate $\beta\overline{\Lambda}$. Set

$$A = \beta(P_1), \quad B = \beta(P_2), \quad C = \beta(X).$$

These are the (semi)-stable objects for the stability conditions in $\beta\Lambda$. Recall that $\beta\overline{\Lambda}$ is homeomorphic to the projectivised octant $\overline{\Delta} = \left(\mathbf{R}^3_{\geq 0} \setminus 0\right)/\mathbf{R}_{>0}$; the homeomorphism is given by the Gromov coordinates. The normalisation condition (44) is a linear condition on the Gromov coordinates. It cuts out an affine hyperplane slice of the octant, and gives a section of the projectivisation map. Since π is linear in the Gromov coordinates for normalised stability conditions, it maps $\beta\overline{\Lambda}$ linearly, and homeomorphically, onto the triangle in $\mathbf{R}^3$ with vertices $\pi(A)$, $\pi(B)$, and $\pi(C)$.

Let $\Phi \subset \mathbf{R}^3$ be the union of the triangle

$$\{(x,y,z) \mid x+y+z = \sqrt{2},\, x+y-z \geq 0,\, y+z-x \geq 0,\, z+x-y \geq 0\},$$

and the three circular segments, each bounded by an edge of the triangle above and the arc in the image of $\pi(P)$ with the same end-points as the edge (see Figure 7).

Observe that π maps $\overline{\Lambda}$ to the central triangle of Φ. On the other hand, the translates of $\overline{\Lambda}$ on the three sides of the identity on the exchange graph (Figure 6) are mapped to the three circular segments. Indeed, the segment to which π sends a non-standard stability condition τ depends on how the triangle inequality collapses among the τ-masses of P_1, P_2, and X, and this in turn, is determined by where τ lies on the exchange graph.

Proposition 7.23. *The map $\pi \colon \overline{M} \to \Phi$ is a homeomorphism, where $\overline{M}$ is given the subspace topology from $\mathbf{P}^\mathbf{S}$.*

Proof. Let us first show that π is continuous. Since π is linear on the translates of $\overline{\Lambda}$, and these translates cover M, it follows that π is continuous on M. The restriction of π to P is

given by

$$\pi \colon [a : c] \mapsto \frac{1}{\sqrt{a^2 + c^2 + (a + c)^2}} \left(|a|, |c|, |a + c|\right),$$

which is continuous. We must show that π is continuous on $\overline{M}$ at a point $p \in P$.

Let τ_n be a sequence of normalised stability conditions converging to $p \in P$. We already know that $m_T(\tau_n)$ converges to $h_T(p)$ in the projective space $\mathbf{P}^T$. Therefore, it suffices to show that $\pi(\tau_n)$ approaches a vector of norm 1 in $\mathbf{R}^3$.

Let $\epsilon > 0$ be given. Write $\tau_n = \beta_n \tau_n'$ for some braid β_n and standard stability condition τ_n. Denote by $\overline{\beta}_n$ the image of β_n in $\mathrm{PSL}_2(\mathbf{Z})$. If the set $\{\overline{\beta}_n\}$ is finite, then the sequence τ_n lies in finitely many translates of Λ. By passing to a subsequence, we may assume that it lies in one translate, say $\beta\Lambda$. Recall that $\pi \colon \overline{\beta}\Lambda \to \mathbf{R}^3$ is linear in the Gromov coordinates and hence continuous. Therefore, $\pi(\tau_n) \to \pi(p)$ as $n \to \infty$.

The harder case is when the set $\{\overline{\beta}_n\}$ is infinite. Set $A_n = \beta_n(P_1)$, $B_n = \beta_n(P_2)$, $C_n = \beta_n(X)$, and let x_n, y_n, z_n be the Gromov coordinates of τ_n. In this case, we know by Proposition 7.20 that the angle between $\eta(A_n)$, $\eta(B_n)$, and $\eta(C_n)$ approaches 0 as n approaches ∞. Therefore, the difference between

$$|x_n \eta(A_n) + y_n \eta(B_n) + z_n \eta(C_n)| \text{ and } x_n |\eta(A_n)| + y_n |\eta(B_n)| + z_n |\eta(C_n)|$$

approaches 0 as n approaches ∞. Since τ_n is normalised, the right-hand quantity is 1, and the left-hand quantity is $|\pi(\tau_n)|$.

We have now proved that $\pi \colon \overline{M} \to \Phi$ is continuous. Since $\overline{M}$ is compact, π is a homeomorphism once we know that it is a bijection. We know that the map

$$\pi \colon P \to \partial\Phi = \Phi \cap \{v \mid |v| = 1\}$$

is a bijection and π maps M to

$$\Phi^\circ = \Phi \cap \{v \mid |v| < 1\}.$$

Recall that M is the union of the translates of $\beta\Lambda$, and each translate is homeomorphic to a clipped triangle (a planar triangle minus the vertices). The map π maps each translate bijectively to a (clipped) triangle in Φ°. To check that π is an injection, we must check that the clipped triangles $\pi(\beta\Lambda)$ have disjoint interiors, and two of them, say $\pi(\beta\Lambda)$ and $\pi(\beta'\Lambda)$, intersect along an edge if and only if β and β' are adjacent in the exchange graph. To check that π is a surjection, we must check that the union $\bigcup_\beta \pi(\beta\Lambda)$ is Φ°.

Take $\beta = \mathrm{id}$. Then $\pi(\Lambda)$ is the central triangle of Φ°. The only other triangles that intersect this triangle are $\pi(\sigma_1\Lambda)$, $\pi(\sigma_2\Lambda)$, and $\pi(\sigma_X\Lambda)$, and the intersections are along the three edges, as required.

The exchange graph divides the non-trivial translates of Λ into three connected components, and the translates in each of the three components map to the three distinct circular segments in Φ°. So it suffices to restrict our attention to one component and the corresponding circular segment. Since γ permutes the components, it suffices to look at only one of them.

Let us consider the component containing $\sigma_1\Lambda$. The translates in this component are $A\Lambda$ where $A \in \mathrm{PSL}_2(\mathbf{Z})$ is a matrix that has a cyclic writing (as in Proposition 7.2) that starts with σ_1. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\pi(A\Lambda)$ is the clipped triangle with vertices

$$(47) \qquad \pi([a : c]), \quad \pi([b : d]), \quad \pi([a - b : c - d]).$$

Since we are considering the translates in the $\sigma_1\Lambda$ component, the points $[a : c]$, $[b : d]$, and $[a - b : c - d]$ lie in the arc $[P_1, P_2]$ of $\mathbf{P}^1(\mathbf{Z}) \subset \mathbf{P}^1(\mathbf{R})$. Under the bijection $\mathbf{P}^1(\mathbf{Z}) = \mathbf{Q} \cup \{\infty\}$

given by $[a : c] \to a/c$, the set of all such points corresponds to $\mathbf{Q}_{\geq 0} \cup \{\infty\}$. By construction, the points $\pi([a : c])$ and $\pi([b : d])$ form an edge of a clipped triangle if and only if $|ad - bc| = 1$. Thus, the triangles (47) form the Farey tiling [5, Chapter 8] of the circular segment, which has the intersection properties as dictated by the exchange graph. $\qquad\square$

8. The $\widehat{A}_1$ case

The aim of this section is to construct the Thurston compactification at $q = 1$ of stability space for the 2-Calabi–Yau category $\mathcal{C}_\Gamma$ where Γ is the $\widehat{A}_1$ graph.

8.1. **The 2-Calabi–Yau category of type $\widehat{A}_1$.** Since the $\widehat{A}_1$ graph is not simply-laced, some mild modifications of the definitions from Section 6 are required in order to define the associated zigzag algebra and 2-Calabi–Yau category. We give the precise definitions here.

8.1.1. *The Coxeter system of type $\widehat{A}_1$.* Let Γ be the Dynkin diagram of type $\widehat{A}_1$, which has two vertices, called 0 and 1, and two (unoriented) edges connecting them.

$$0 \xaligned= 1$$

Definition 8.1. The *Artin–Tits braid group associated to* Γ is the free group on two letters F_2:
$$B_\Gamma \cong F_2 = \langle \sigma_0, \sigma_1 \rangle.$$

Definition 8.2. The *Coxeter group associated to* Γ is isomorphic to the quotient of B_Γ by the relations
$$\sigma_0^2 = \sigma_1^2 = 1.$$
We denote this group by W_Γ, and denote the images of σ_i in W_Γ by s_i.

Let V_Γ be the rank two **Z**-module with basis vectors v_i indexed by the vertices of Γ. Define a bilinear form on V_Γ by the following formula:
$$\langle v_i, v_j \rangle = \begin{cases} 2 & i = j, \\ -2 & i \neq j. \end{cases}$$

Definition 8.3. The *standard representation* of W_Γ is the action on V_Γ defined by the formula
$$s_i(v_j) = v_j - \langle v_i, v_j \rangle v_i.$$

8.1.2. *The zigzag algebra of type $\widehat{A}_1$.* Let Γ^{dbl} denote the doubled quiver of Γ. If the two edges of Γ are denoted e and f, then there are four oriented edges denoted $e_{01}, e_{10}, f_{01}, f_{10}$ in Γ^{dbl}, where the subscript ij means that the arrow points from i to j. Fix a sign $s_{ij} \in \{\pm 1\}$ for each arrow e_{ij}, and a sign $t_{ij} \in \{\pm 1\}$ for each arrow f_{ij}, such that $s_{ij} = -s_{ji}$ and $t_{ij} = -t_{ji}$.
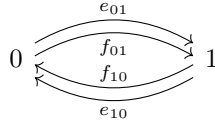
The zigzag algebra is the quotient of the path algebra $\mathrm{Path}(\Gamma^{dbl})$ by the two-sided ideal generated by

(1) the elements $e_{ij} f_{ji}$ and $f_{ij} e_{ji}$, for $i \neq j$, and

(2) the elements

$$s_{ij}e_{ij}e_{ji} - t_{ij}f_{ij}f_{ji},$$

for $i \neq j$.

The algebras corresponding to the two kinds of sign choices $s_{ij} = t_{ij}$ and $s_{ij} = -t_{ij}$ are isomorphic, as was the case in simply laced type.

8.1.3. *The 2-Calabi–Yau category of type* $\widehat{A}_1$. The strongly 2-Calabi–Yau category associated to the $\widehat{A}_1$ quiver, which we denote as $\mathcal{C}$ in what follows, is now defined exactly as in simply-laced type in Section 6, using the above $\widehat{A}_1$ zigzag algebra. The category $\mathcal{C}$ is a graded, $\mathbf{k}$-linear triangulated category classically generated by two spherical objects, which we denote P_0 and P_1. The hom spaces between these generators are given as follows:

$$\mathrm{Hom}^\bullet(P_i, P_i[n]) = \begin{cases} \mathbf{k} & \text{if } n = 0 \text{ or } n = 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{Hom}^\bullet(P_i, P_j[n]) = \begin{cases} \mathbf{k}^2 & \text{if } n = 1 \text{ and } i \neq j, \\ 0 & \text{otherwise, for } i \neq j. \end{cases}$$

The extension closure of P_0 and P_1 in $\mathcal{C}$ is an abelian category; it is the heart of a (bounded) t-structure. We refer to it as the *standard heart* $\heartsuit_{\mathrm{std}}$.

8.2. **The spherical objects and the boundary circle.** Denoting the spherical twist in P_i as σ_{P_i}, we have a weak action of $B_\Gamma \cong F_2$ on $\mathcal{C}$ given by

$$\sigma_i \to \sigma_{P_i}.$$

The image of B_Γ in $\mathrm{Aut}(\mathcal{C})$ is precisely the subgroup generated by the spherical twists σ_{P_0} and σ_{P_1}. It is not difficult to check directly, via e.g. a ping-pong argument, that the action of B_Γ on $\mathcal{C}$ is faithful, so we can regard B_Γ as a subgroup of $\mathrm{Aut}(\mathcal{C})$.

Set $\gamma = \sigma_1\sigma_0$. One can check that for any integer n, the object $\sigma_1\gamma^n(P_0)$ is the cone of a morphism from a direct sum of $|2n + 2|$ copies of $P_1[-1]$ to a direct sum of $|2n + 1|$ copies of P_0. Similarly, the object $\gamma^n P_1$ is the cone of a morphism from a direct sum of $|2n + 1|$ copies of $P_1[-1]$ to a direct sum of $|2n|$ copies of P_0. In other words, we may write

$$\sigma_1\gamma^n(P_0) = \mathrm{Cone}(P_1^{\oplus|2n+2|}[-1] \to P_0^{\oplus|2n+1|}), \text{ and}$$

$$\gamma^n P_1 = \mathrm{Cone}(P_1^{\oplus|2n+1|}[-1] \to P_0^{\oplus|2n|}).$$

The objects obtained as above are all spherical, and their union consists of each of the objects $\mathrm{Cone}(P_1^{\oplus n} \to P_0^{\oplus(n\pm1)})$ exactly once, as n ranges over the integers. For convenience, we set

$$P_n = \mathrm{Cone}(P_1^{\oplus|n|} \to P_0^{\oplus|n-1|}) \text{ for } n \in \mathbf{Z}.$$

The objects P_n fit into distinguished triangles as follows:

(48)
$$P_{2i+1}[-1] \to P_{2i} \oplus P_{2i} \to P_{2i-1} \xrightarrow{+1},$$

$$P_{2i+2} \to P_{2i+1} \oplus P_{2i+1} \to P_{2i}[1] \xrightarrow{+} .$$

Regard $\mathbf{Z}$ as the subset of $\mathbf{P}^1(\mathbf{Z})$ consisting of points of the form $[k : 1]$. We can write a homomorphism $B_\Gamma \to \mathrm{PSL}_2(\mathbf{Z})$, as follows:

(49)
$$\sigma_0 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \sigma_1 \mapsto \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Let $\mathbf{S}$ be the set of all spherical objects, up to shift, that are stable with respect to some stability condition in $\mathrm{Stab}^\circ(\mathcal{C})$. It can be checked, for example using the explicit construction of spherical stable objects in [2, § 4], that $\mathbf{S}$ is the B_Γ orbit of $\{P_0, P_1\}$. We have a B_Γ-equivariant map $i\colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$ characterised by $P_n \mapsto [n:1]$. It is easy to check that i is injective. For convenience, let P_∞ be any one of the indecomposable extensions of P_0 by P_1. We extend the embedding above to $\mathbf{S} \cup \{P_\infty\}$ by sending P_∞ to $[1:0]$, which is the point at infinity.

The element $\gamma = \sigma_1 \sigma_0$ acts on the objects P_n by an infinite-order cyclic rotation by two steps, lowering the phases. More precisely,

$$\gamma\colon P_n \mapsto P_{n+2} \text{ for } n \neq 0, -1, \infty,$$
$$\gamma\colon P_n \mapsto P_{n+2}[-1] \text{ for } n = 0, -1,$$
$$\gamma\colon P_\infty \mapsto P_\infty.$$

For each $k \in \mathbf{Z}$, set

$$\sigma_{2k} = \gamma^k \sigma_0 \gamma^{-k} \text{ and } \sigma_{2k+1} = \gamma^k \sigma_1 \gamma^{-k}.$$

It is easy to see that the image of σ_k in $\mathrm{Aut}(\mathcal{C})$ is the spherical twist in the object P_k. The image of σ_k in $\mathrm{PSL}_2(\mathbf{Z})$ is given by

$$(50) \qquad \sigma_k \mapsto \begin{pmatrix} 2k+1 & -2k^2 \\ 2 & -2k+1 \end{pmatrix}.$$

8.3. **Standard stability conditions.** We say that a stability condition τ is *standard* if its heart $\heartsuit([0,1))$ is the standard heart $\heartsuit_{\mathrm{std}}$. Let ϕ denote the phase function for τ. Then we either have $\phi(P_0) \leq \phi(P_1)$ or $\phi(P_1) \leq \phi(P_0)$. In the first case, we say τ is standard of *type I*, and in the latter case, of *type II*. Note that, in type I, all objects P_n are τ-semistable.

Let $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$ be the closure of the set of standard stability conditions of type I. Then Λ includes non-standard stability conditions where $\phi(P_1) = \phi(P_0)+1$. We continue to call these stability conditions as of *type I*, reserving the adjective standard for those where $\phi(P_1) < \phi(P_0) + 1$. Figure 8 shows the central charges of the objects P_n in a standard off-the-wall stability condition of type I. The dotted line is the direction of the sum of the central charges of P_0 and P_1, which is also the central charge of P_∞.

8.4. **The automaton.** Fix an off-the-wall stability condition τ of type I. Recall that Σ is the set of indecomposable τ-semistable objects in the standard heart, which contains P_n for $n \in \mathbf{Z}$ as well as the objects P_∞. Figure 9 describes a τ-HN-automaton Θ by illustrating the incoming and outgoing edges at one of the vertices.

Here is a more formal description of the automaton, following Definition 5.3.

(1) We have a B_Γ-labelled graph with vertices indexed by $\mathbf{Z}$ and edges as indicated in Figure 9: an edge from state k to state j labelled $\sigma_j = \sigma_{P_j}$ (unless $j = k+1$); an edge from state k to state $(k+2)$ labelled γ; and an edge from state k to state $(k-2)$ labelled γ^{-1}.

(2) The kth state is labelled $[P_k, P_{k+1}]$. The set $S_{[P_k, P_{k+1}]}$ consists of the spherical objects of $\mathcal{C}$ whose HN factors are shifts of P_k and P_{k+1}.

(3) The representation M associates the free abelian group $\mathbf{Z}^\Sigma$ to each state. However, as we show in Proposition 8.4, the image of the HN multiplicity map from S to M will only be supported on the sub-representation that assigns $\mathbf{Z}^{\{P_k, P_{k+1}\}}$ to the state labelled as $[P_k, P_{k+1}]$. So we simplify notation by setting M to be this sub-representation. The map associated to the edge $\sigma_j\colon k \to j$ is represented by the
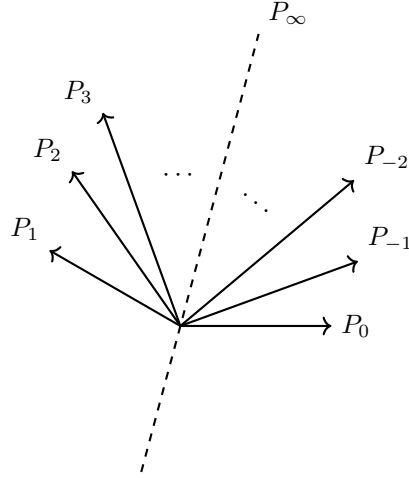
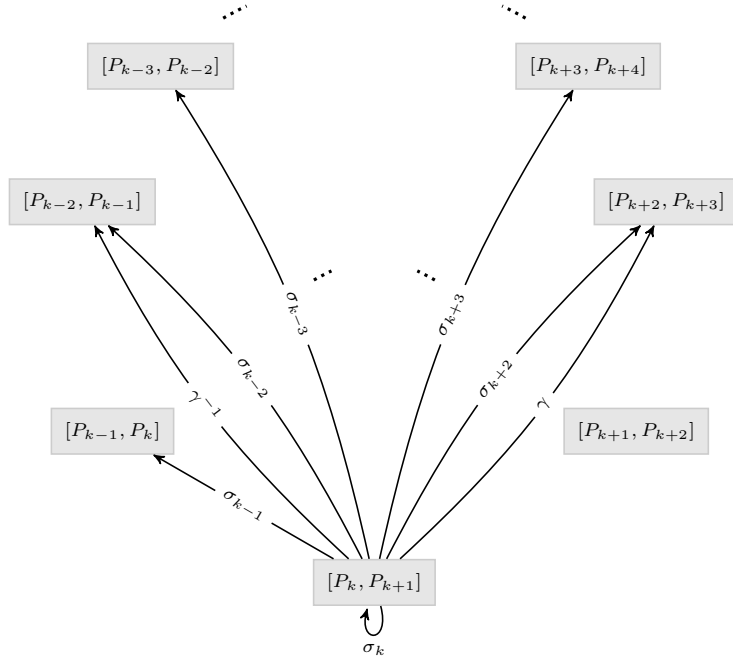FIGURE 8. The central charges of the objects P_i in a standard off-the-wall stability condition of type I.

FIGURE 9. An HN automaton for an off-the-wall type I stability condition for $\widehat{A}_1$, showing the outgoing edges from $[P_k, P_{k+1}]$.

2×2 matrix

(51)
$$\begin{pmatrix} |j-k-1| & |j-k-2| \\ |j-k| & |j-k-1| \end{pmatrix}$$

in the standard bases. The maps associated to the edges labelled $\gamma^{\pm}$ are represented by the identity matrix.

Analogously to Proposition 7.1, the next proposition shows that the description above fits into the framework of Definition 5.3.

Proposition 8.4. *Let* $e\colon [P_k, P_{k+1}] \to [P_j, P_{j+1}]$ *be an edge of the automaton labelled by the group element* $g \in B_\Gamma$. *Let* $x \in \mathcal{C}$ *be an object whose HN filtration consists of shifts of* P_k *and* P_{k+1}.

(1) *The HN filtration of* gx *consists of shifts of* P_j *and* P_{j+1}.

(2) *The following diagram commutes.*

$$
\begin{array}{ccc}
x & \xmapsto{\ \mathrm{HN}_\tau\ } & \mathrm{HN}_\tau(x) \in \mathbf{Z}^{\{P_k, P_{k+1}\}} \\
\Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle M(e)} \\
gx & \xmapsto{\ \mathrm{HN}_\tau\ } & \mathrm{HN}_\tau(gx) \in \mathbf{Z}^{\{P_j, P_{j+1}\}}
\end{array}
$$

In other words,

$$\mathrm{HN}_\tau(gx) = M(e) \cdot \mathrm{HN}_\tau(x).$$

Proof. By an argument similar to Proposition 7.1, the image under e of the HN filtration of x is rectifiable. So it suffices to prove the proposition for $x = P_k$ and $x = P_{k+1}$.

Suppose $g = \gamma^{\pm}$ so that $j = k \pm 2$. Since $\gamma^{\pm}$ sends P_i to $P_{i\pm 2}$ (up to shift), preserving the ordering of the phases, it is clear that $e \cdot x$ is supported at $[P_j, P_{j+1}]$, and that the HN multiplicities transform by the identity matrix.

Otherwise, $g = \sigma_j$. Suppose that $k = 0$. We can check explicitly that if $j \neq 1$, we have

$$
\begin{aligned}
\sigma_j P_0 &= \mathrm{Cone}(P_j^{\oplus |j-1|}[-1] \to P_{j+1}^{\oplus |j|}), \text{ and} \\
\sigma_j P_1 &= \mathrm{Cone}(P_j^{\oplus |j-2|}[-1] \to P_{j+1}^{\oplus |j-1|}).
\end{aligned}
\tag{52}
$$

As a result, both $\sigma_j(P_0)$ and $\sigma_j(P_1)$ are both supported at state j and their HN multiplicities transform according to the matrix $M(e)$.

Let k be arbitrary. Observe that we have

$$
\sigma_j(P_k) = \begin{cases} \gamma^m \sigma_{j-2m} P_0, & \text{if } k = 2m \text{ is even,} \\ \gamma^m \sigma_{j-2m} P_1, & \text{if } k = 2m+1 \text{ is odd.} \end{cases}
$$

Using the above, we see that for any k except $k = j - 1$, we have

$$\sigma_j P_k = \mathrm{Cone}(P_j^{\oplus |j-k-1|}[-1] \to P_{j+1}^{\oplus |j-k|}).\tag{53}$$

As a result, σP_k and σP_{k+1} are both supported at state j and their HN multiplicities transform according to the matrix $M(e)$. $\qquad\square$

The next proposition shows that the automaton recognises every braid and every object of $\mathbf{S}$.

Proposition 8.5. (1) *Every* $\beta \in B_\Gamma$ *has an expression of the form*

$$\beta = \gamma^n \sigma_{a_1} \sigma_{a_2} \ldots \sigma_{a_k},$$

such that $a_i - a_{i+1} \neq 1$ *for each* i.

(2) *Every object* $s \in \mathbf{S}$ *has an expression*

$$s = \beta_1 \cdots \beta_n P_i$$

where $i \in \{0, 1\}$, *which is recognised by the automaton in Figure 9.*

Proof. For (1), first write β as any expression in the generators $\{\sigma_0^{\pm 1}, \sigma_1^{\pm 1}\}$. For any $i \in \mathbf{Z}$, we have

$$\sigma_i^{-1} = \sigma_{i-1}\gamma^{-1}, \quad \gamma^{-1}\sigma_i = \sigma_{i-2}\gamma^{-1}, \quad \gamma = \sigma_i\sigma_{i-1}.$$

Using these relations, we first rewrite β solely in terms of the elements σ_i for $i \in \mathbf{Z}$ and $\gamma^{\pm}$. Then we successively replace any instances of $\sigma_i\sigma_{i-1}$ by γ, and commute all powers of γ to the left. This process terminates, resulting in an expression of the desired form.

For (2), write $s = \beta P_i$ where i is either 0 or 1. Assume that $i = 0$; the other case is similar. Rewrite

$$\beta = \gamma^n \sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_k}$$

as in (1). Note that the object P_0 is supported at two states, namely $[P_0, P_1]$ and $[P_{-1}, P_0]$. We can apply σ_{a_k} at least at one of these two states. From the condition on our expression for β, it is clear that all subsequent letters are applicable. To finish, any power of γ is applicable at any vertex. Therefore s is recognised by the automaton. $\qquad\square$

As a consequence of Proposition 8.5, every spherical object is supported on at least one of the states of the automaton in Figure 9. In particular, its HN factors involve (up to shift) at most the objects P_i and P_{i+1} for some integer i.

Proposition 8.6. *Let x be a spherical object of $\mathcal{C}$ supported at the state $[P_k, P_{k+1}]$. Assume that x is not a shift of P_k or P_{k+1}. Then*

(1) *we can write $s = \sigma_k^r t$ for some $r > 0$ and an object t supported at $[P_{k+1}, P_{k+2}]$;*
(2) *we can write $s = \sigma_{k+1}^{-r} t$ for some $r > 0$ and an object t supported at $[P_{k-1}, P_k]$.*

Proof. Analogous to the proof of Proposition 7.3. $\qquad\square$

Remark 8.7. Let τ be an on-the-wall standard stability condition. Set P_i' to be the object $\sigma_0 P_{i-1}$. The objects P_i' are also semistable in τ, in addition to the objects P_i. Note that up to shift, we have $P_1' = P_0$ and $P_0' = P_1$. Recall that $\gamma = \sigma_1\sigma_0$. Let $\gamma' = \sigma_0\sigma_1$.

Figure 10 shows an automaton that computes τ-HN multiplicities. Only the outgoing arrows labelled σ_0 and σ_1 are depicted. The remaining possible labels for outgoing arrows are $\gamma^{\pm}$ and $\gamma'^{\pm}$, which act as follows:

$$\gamma \colon [P_i, P_{i+1}] \to [P_{i+2}, P_{i+3}] \text{ unless } i = -2, \quad \gamma \colon [P_i', P_{i+1}'] \to [P_1, P_2];$$
$$\gamma^{-1} \colon [P_i, P_{i+1}] \to [P_{i-2}, P_{i-3}], \text{ unless } i = 2, \quad \gamma^{-1} \colon [P_i', P_{i+1}'] \to [P_{-2}, P_{-1}];$$
$$\gamma' \colon [P_i', P_{i+1}'] \to [P_{i+2}', P_{i+3}'] \text{ unless } i = -2, \quad \gamma' \colon [P_i, P_{i+1}] \to [P_1', P_2'];$$
$$\gamma'^{-1} \colon [P_i', P_{i+1}'] \to [P_{i-2}', P_{i-3}'] \text{ unless } i = 2, \quad \gamma'^{-1} \colon [P_i, P_{i+1}] \to [P_{-2}', P_{-1}'].$$

Since Θ suffices for mass computations, we omit further details.

8.5. **Consequences of the automaton.** In this section, we use the automaton from Section 8.4 to prove results about our projective embedding of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$. We have the following analogue of Proposition 7.6.

Proposition 8.8. *Consider the following distinguished triangles:*

$$P_1[-1] \to P_0 \oplus P_0 \to P_{-1} \xrightarrow{+1} \text{ and } P_3[-1] \to P_2 \oplus P_2 \to P_1 \xrightarrow{+1} .$$

If τ is a standard stability condition of type I, then neither of the above triangles is collapsed by τ in the sense of Definition 6.17. Furthermore, if τ' is any stability condition such that neither of the above triangles is collapsed by τ', then τ' is standard of type I up to the action of $\mathbf{C}$.
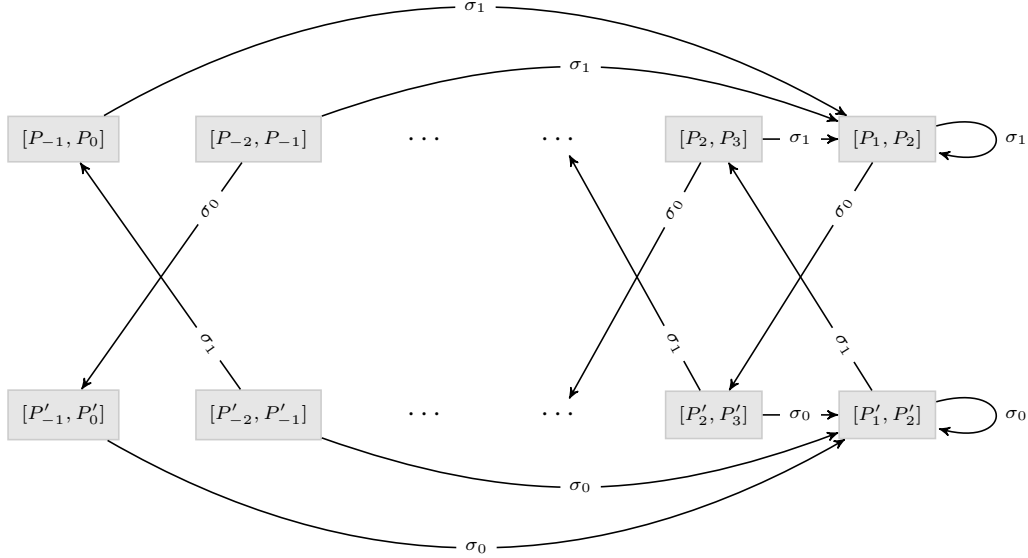
FIGURE 10. An HN automaton for an on-the-wall standard stability condition, depicting outgoing arrows labelled σ_0 and σ_1.

Proof. It is immediate from Proposition 3.3 that neither of the above triangles is collapsed by any standard stability condition of type I. Now let τ' be any stability condition. Recall that τ' is in the braid group orbit of a standard stability condition of type I. Write $\tau' = \beta\tau''$ for some braid β and some standard stability condition τ''. As in the proof of Proposition 7.6, we may assume that τ'' is off-the-wall; otherwise the result follows by continuity.

If β is a power of γ, then τ' is already standard of type I up to the action of $\mathbf{C}$. Otherwise, we show that one of the two triangles above is collapsed by τ'. Equivalently, we show that β^{-1} applied to one of the two triangles above is collapsed by τ''.

Consider a recognised expression of β^{-1} as described in Proposition 8.5. Since β is not a power of γ, this expression ends in σ_i for some $i \in \mathbf{Z}$. We observe using the automaton in Figure 9, analogous to the argument in the proof of Proposition 7.6 that the first triangle is collapsed by τ' if $i \neq 0, 1$. Otherwise, the second triangle is collapsed by τ'. We omit the details. $\qquad\square$

Proposition 8.9. *The map $m\colon \mathrm{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^{\mathbf{S}}$ is a homeomorphism onto its image.*

Proof. The proof is analogous to the proof of Proposition 7.8, using the result of Proposition 8.8. $\qquad\square$

Recall that we set M to be the Θ-representation that associates the free abelian group $\mathbf{Z}^{\{P_k, P_{k+1}\}}$ to the state $[P_k, P_{k+1}]$. As in the A_2-case, the Θ-set $M/\pm$ isomorphic to $\mathbf{Z}^2/\pm$ with the standard action of $\mathrm{PSL}_2(\mathbf{Z})$, as follows. Corresponding to each state $[k, k+1]$ of Θ, we define a linear map $\phi_k\colon M_{[k,k+1]} \to \mathbf{Z}^2$ by

$$\phi_k\colon (1,0) \mapsto (k,1), \quad (0,1) \mapsto (k+1,1).$$

Proposition 8.10. *The maps ϕ_k give an isomorphism of Θ-sets $M/\pm \to \mathbf{Z}^2/\pm$.*

Proof. Let us check that ϕ is Θ-equivariant. It is evident that ϕ_k intertwines the edges labelled $\gamma^\pm$. To check that it also intertwines $\sigma_j \colon k \to j$, we must check that

$$\sigma_j \circ \phi_k = \phi_j \circ M(e).$$

Recalling the matrix of σ_{P_j} from (50) and $M(e)$ from (51), we must check

$$\begin{pmatrix} 2j+1 & -2j^2 \\ 2 & -2j+1 \end{pmatrix} \begin{pmatrix} k & k+1 \\ 1 & 1 \end{pmatrix} = \pm \begin{pmatrix} j-k-1 & j-k-2 \\ j-k & j-k-1 \end{pmatrix} \begin{pmatrix} j & j+1 \\ 1 & 1 \end{pmatrix}.$$

This is straightforward. $\qquad\square$

Recall that we have a B_Γ-equivariant map $i \colon \mathbf{S} \to \mathbf{P}^1(\mathbf{Z})$. The following proposition identifies $i(s)$ explicitly in terms of the HN multiplicities.

Proposition 8.11. *For a spherical object s supported at the kth state, we have the equality*

$$i(s) = \phi_k(\mathrm{HN}_\tau(s)) \in \mathbf{P}^1(\mathbf{Z}).$$

Moreover, $\phi_k(\mathrm{HN}_\tau(s)) \in \mathbf{Z}^2/\pm$ is the unique representative of $i(s)$ whose coordinates are relatively prime.

Proof. Analogous to the proof of Proposition 7.10 $\qquad\square$

The division of $\mathbf{S}$ according to the HN support corresponds nicely to a geometric division of the circle $\mathbf{P}^1(\mathbf{R})$, which we now describe. The objects P_i divide $\mathbf{P}^1(\mathbf{R})$ into a collection of closed arcs, namely $\{[P_i, P_{i+1}] \mid i \in \mathbf{Z}\}$. Together with P_∞, these arcs cover all of $\mathbf{P}^1(\mathbf{R})$. We show that this decomposition of the circle is compatible with the automaton.

Proposition 8.12. *For each $i \in \mathbf{Z}$, the objects supported at state i are mapped to the arc $[P_i, P_{i+1}]$.*

Proof. Analogous to the proof of Proposition 7.11. $\qquad\square$

As a consequence of the automaton, we also obtain a Rouquier–Zimmermann–type theorem for $\widehat{A}_1$. Just as in type A_2, this theorem relates the parametrization of rational points $[a : c] \in \mathbf{P}^1(\mathbf{Z})$ with Jordan–Hölder multiplicities of the corresponding spherical object.

Recall the notion of Jordan–Hölder (or JH) multiplicity from Definition 7.12. We use the same definition for the $\widehat{A}_1$ category, with respect to the standard heart.

Proposition 8.13. *Let $x \in \mathbf{S}$ such that $i(x) = [a : c] \in \mathbf{P}^1(\mathbf{Z})$, with a and c relatively prime integers. Then the JH multiplicities of P_0 and P_1 in x are exactly $|a - c|$ and $|a|$ respectively.*

Proof. Analogous to the proof of Proposition 7.13, using the result of Proposition 8.10 in lieu of Proposition 7.9. $\qquad\square$

The following statement computes values of the $\overline{\mathrm{hom}}$ function of a spherical object from the JH multiplicities of P_0 and P_1 in that object.

Proposition 8.14. *Let $x \in \mathbf{S}$ such that the JH multiplicities of P_0 and P_1 in x are n_0 and n_1 respectively. Then $\overline{\mathrm{hom}}(x, P_1) = 2n_0$ and $\overline{\mathrm{hom}}(x, P_0) = 2n_1$.*

Proof. We prove the assertion for $\overline{\mathrm{hom}}(x, P_1)$. The proof for $\overline{\mathrm{hom}}(x, P_0)$ is similar. We can compute directly from the definitions that $\overline{\mathrm{hom}}(P_1, P_k) = 2|k - 1|$ for each $k \in \mathbf{Z}$.

Fix a standard off-the-wall stability condition of type I. The key idea, as in the proof of Proposition 7.14, is to prove that $\overline{\mathrm{hom}}(-, P_1)$ is a linear function of the HN multiplicities. Then the cases $x = P_i$ imply the result for all x.

Suppose that x is supported at $[P_k, P_{k+1}]$ for $k \neq 0, 1$. Consider the Harder–Narasimhan filtration of x:

$$0 = x_0 \to x_1 \to \cdots \to x_n = x,$$

with factors $z_i = \mathrm{Cone}(x_i \to x_{i+1})$. Each z_i is either P_k or P_{k+1}, up to shift.

By applying $\mathrm{Hom}^\bullet(P_1, -)$, we obtain the following filtration in the bounded derived category of graded vector spaces:

$$(54) \qquad 0 = \mathrm{Hom}^\bullet(P_1, x_0) \to \cdots \to \mathrm{Hom}^\bullet(P_1, x_n) = \mathrm{Hom}^\bullet(P_1, x).$$

As in the first case in the proof of Proposition 7.14, we obtain

$$\mathrm{Hom}^\bullet(P_1, x) = \bigoplus \mathrm{Hom}^\bullet(P_1, z_i).$$

We now treat the cases where x is supported either at $[P_0, P_1]$ or at $[P_1, P_2]$. By Proposition 8.13, the JH multiplicities of P_0 in the objects x and $\sigma_1^r x$ are equal, and both also have the same value for $\overline{\mathrm{hom}}(-, P_1)$. Hence the proposition for x implies the proposition for $\sigma_1^r x$. By Proposition 8.6, any x supported either at $[P_0, P_1]$ or $[P_1, P_2]$ can be rewritten as $x = \sigma_1^r y$ for some $r \in \mathbf{Z}$ and y supported at $[P_k, P_{k+1}]$ for $k \neq 0, 1$. Hence the proposition holds for x. $\qquad\square$

8.6. Gromov coordinates.

Definition 8.15 (Gromov coordinates). Let τ be a stability condition of type I. For each integer $i \in \mathbf{Z}$, let $x_i(\tau)$ be the rational number defined as follows:

$$x_i(\tau) = \frac{m_\tau(P_{i-1}) + m_\tau(P_{i+1}) - 2m_\tau(P_i)}{2}.$$

We call the numbers $(x_i(\tau))$ the *Gromov coordinates* of τ.

Using the triangle inequality (Proposition 3.1) on the triangles (48), we deduce that all Gromov coordinates of τ are non-negative. If τ is off-the-wall, they are all positive.

Proposition 8.16 (Linearity). *Let τ be a stability condition of type I with Gromov coordinates (x_i). For any object $X \in \mathbf{S}$, we have*

$$(55) \qquad\qquad m_\tau(X) = \frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \, \overline{\mathrm{hom}}(P_i, X).$$

Proof. Recall that for any $i \in \mathbf{Z}$, we have

$$P_{2i} = \gamma^i P_0, \quad P_{2i+1} = \gamma^i P_1.$$

Therefore, we get

$$\overline{\mathrm{hom}}(P_{2i}, X) = \overline{\mathrm{hom}}(P_0, \gamma^{-i} X), \quad \overline{\mathrm{hom}}(P_{2i+1}, X) = \overline{\mathrm{hom}}(P_1, \gamma^{-i} X).$$

Proposition 8.14 allows us to compute $\overline{\mathrm{hom}}(P_0, \gamma^{-i} X)$ and $\overline{\mathrm{hom}}(P_1, \gamma^{-i} X)$ from the HN multiplicities of $\gamma^{-i} X$. In turn, the automaton in Figure 9 allows us to compute the HN multiplicities of $\gamma^{-i} X$ from the HN multiplicities of X. Suppose X is supported at the kth state of the automaton. Suppose the HN filtration of X consists of α copies of P_k and β copies of P_{k+1}, up to shift. Then $\gamma^{-i} X$ is supported on the $(k - 2i)$th state, and its HN

filtration contains α copies of P_{k-2i} and β copies of P_{k+1-2i}. In particular, for any $i \in \mathbf{Z}$, the JH multiplicities of P_1 and P_0 in $\gamma^{-i}X$ are exactly

$$(\alpha|k - 2i| + \beta|k - 2i + 1|) \text{ and } (\alpha|k - 2i - 1| + \beta|k - 2i|)$$

respectively. Using Proposition 8.14, we deduce that for any $j \in \mathbf{Z}$, we have

$$(56) \qquad \overline{\hom}(P_j, X) = 2\left(\alpha|k - j| + \beta|k - j + 1|\right).$$

Substitute

$$x_i = \frac{m_\tau(P_{i-1}) + m_\tau(P_{i+1}) - 2m_\tau(P_i)}{2}$$

in

$$\frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \overline{\hom}(P_i, X).$$

Then the coefficient of $m_\tau(P_j)$ is exactly

$$\frac{\overline{\hom}(P_{j-1}, X) + \overline{\hom}(P_{j+1}, X) - 2\overline{\hom}(P_j, X)}{4}.$$

Using (56), we see that the expression above equals α if $j = k$, is β if $j = k + 1$, and is 0 otherwise. Therefore we see that

$$\frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \overline{\hom}(P_i, X) = \alpha \cdot m_\tau(P_k) + \beta \cdot m_\tau(P_{k+1}),$$

which is precisely the τ-mass of X. $\qquad \square$

Recall that Λ is the closure in $\operatorname{Stab}(\mathcal{C})/\mathbf{C}$ of the image of the standard off-the-wall type I stability conditions. We now study the topology of Λ, and compute the closure of $m(\Lambda)$ in $\mathbf{P^S}$.

Proposition 8.17. *Let $\overline{\mathbf{H}}$ be the closed upper half plane. Consider the subset*

$$L = \{-1\} \cup \left\{ \frac{1-n}{n} \mid n \in \mathbf{Z} \setminus 0 \right\}.$$

Then the map

$$\tau \mapsto \frac{Z_\tau(P_1)}{Z_\tau(P_0)}$$

is a homeomorphism of Λ onto $\overline{\mathbf{H}} \setminus L$.

Proof. The map is clearly continuous. We first show that its image lies in $\overline{\mathbf{H}} \setminus L$. Recall that Λ is the closure of off-the-wall stability conditions of type I (up to the $\mathbf{C}$-action). These are stability conditions in which P_0 and P_1 are stable and satisfy the phase inequalities

$$\phi(P_0) < \phi(P_1) < \phi(P_0) + 1.$$

It follows that for any point $\tau \in \Lambda$ the quantity $\omega_\tau = Z_\tau(P_1)/Z_\tau(P_0)$ lies in the closed upper half plane. To see why ω_τ avoids L, note that the objects P_n are τ-semistable for each $n \in \mathbf{Z}$ as well as $n = \infty$. Therefore, we must have $Z_\tau(P_n) \neq 0$ for $n \in \mathbf{Z} \cup \{\infty\}$. Since

$$[P_n] = |n|[P_1] + |n - 1|[P_0] \text{ and } [P_\infty] = [P_0] + [P_1],$$

it is clear that $\omega_\tau \notin L$.

Now fix some $\omega \in \overline{\mathbf{H}} \setminus L$. We now prove that there is a unique $\tau \in \Lambda$ with $\omega = Z_\tau(P_1)/Z_\tau(P_0)$. Consider the central charge $Z \colon K(\mathcal{C}) \to \mathbf{C}$ defined as

$$Z(P_0) = 1, \quad Z(P_1) = \omega.$$

Recall from [8, Proposition 5.3] that a stability condition is uniquely specified by choosing:

(a) the heart of a bounded t-structure, and
(b) a central charge that sends the heart to the semi-closed upper half plane $\mathbf{H} \cup \mathbf{R}_{>0}$ with the Harder–Narasimhan property.

The standard heart on $\mathcal{C}$ is the extension closure of P_0 and P_1. Being a finite-length category, any central charge as above with respect to the standard heart automatically enjoys the Harder–Narasimhan property. Thus if ω lies in the semi-closed upper half plane, the central charge above uniquely specifies a type I standard stability condition.

We now treat the remaining cases; namely those where ω lies on the negative real axis. In this case we can no longer use the standard heart for our reconstruction, because the central charge function does not land in the semi-closed upper half plane. We will show nevertheless that we can uniquely reconstruct points of Λ in these cases using tilts of the standard heart. These tilts will also be finite-length categories for which the Harder–Narasimhan property is automatic.

The set $\mathbf{R} \setminus L$ is a disjoint union of open intervals of two kinds:

$$\mathbf{R} \setminus L = \bigcup_{n \geq 0} \left(-\frac{n}{n+1}, -\frac{n-1}{n} \right) \cup \bigcup_{n \geq 0} \left(-\frac{n+1}{n}, -\frac{n+2}{n+1} \right).$$

The tilt we choose will depend on the interval that contains ω. Regard $(0, \infty)$ as an interval of the first kind at the value $n = 0$; we have already treated this case.

Suppose ω lies in $(-1/2, 0)$, which is the interval of the first kind at the value $n = 1$. Consider the central charge $Z' = Z \circ \sigma_1$. Recall that σ_1^{-1} takes stability conditions of type I to those (possibly in the closure) of type II. Also note that $\omega' = Z'(P_0)/Z'(P_1)$ lies in the interval $(0, +\infty)$. The previous argument applied to type II conditions shows that there is a unique τ' of type II with central charge Z'. Then $\tau = \sigma_1 \tau'$ gives the unique point of Λ with central charge Z.

Now suppose $\omega = Z(P_1)/Z(P_0)$ lies in $\left(-\frac{n}{n+1}, -\frac{n-1}{n} \right)$ for some $n \geq 2$. We construct the corresponding stability condition using the action of the element $\gamma = \sigma_1 \sigma_0$, which tilts the standard heart. Let $m = \lfloor n/2 \rfloor$ and set $Z' = Z \circ \gamma^m$. Then $\omega' = Z'(P_1)/Z'(P_0)$ lies in one of the two intervals $(-1/2, 0)$ or $(0, +\infty)$, depending on the parity of n. In either case, the previous argument shows that there is a unique stability condition τ' of type I with central charge Z'. Since γ preserves Λ, the stability condition $\tau = \gamma^m \tau'$ gives the unique point of Λ with central charge Z.

For $\omega = Z(P_1)/Z(P_0)$ in the intervals of the second kind, we use a similar argument with $Z' = Z \circ \gamma^{-m}$. We omit the details.

We have shown that the map $\tau \mapsto \omega_\tau$ is a bijection from Λ to $\overline{\mathbf{H}} \setminus L$. It is evident from the proof that the inverse map is continuous; therefore the map $\tau \mapsto \omega_\tau$ is a homeomorphism from Λ to $\overline{\mathbf{H}} \setminus L$. $\qquad \square$

It will be convenient in the remainder of this section to consider a change of coordinates from the previous proposition. Let $\overline{\Delta}$ be a closed disk, considered as the one-point compactification of the closed upper half plane. Consider the map

$$\delta \colon \Lambda \to \overline{\Delta}$$

defined as

(57)
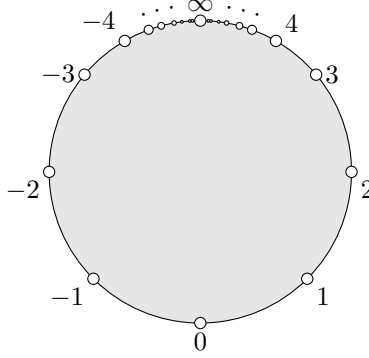$$\delta_\tau = \frac{Z_\tau(P_0)}{Z_\tau(P_0) + Z_\tau(P_1)}.$$

FIGURE 11. The set Λ of stability conditions of type I for the CY2 category associated to $\widehat{A}_1$. The stability conditions are labelled by the ratio $\frac{Z(P_0)}{Z(P_0)+Z(P_1)}$. The interior of the disk represents the upper half plane.

By changing coordinates from Proposition 8.17, we see that δ is injective and a homeomorphism onto its image, and also that the image of δ is precisely

$$\overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\}).$$

Figure 11 shows a sketch of Λ as a subset of the closed disk $\overline{\Delta}$.

8.7. **The closure.** We use the embedding $\Lambda \overset{\delta}{\hookrightarrow} \overline{\Delta}$ to prove the following proposition.

Proposition 8.18 (Closure of Λ). *The closure of $m(\Lambda)$ in $\mathbf{P}(\mathbf{R^S})$ is precisely*

$$\overline{m(\Lambda)} = m(\Lambda) \cup \{\overline{\hom}(P_i) \mid i \in \mathbf{Z}\} \cup \{\overline{\hom}(P_1) - \overline{\hom}(P_0)\}.$$

This closure is homeomorphic to $\overline{\Delta}$.

Proof. Via the identification $\Lambda \cong \overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\})$ via the map δ from (57), the set $\overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\})$ maps to $\mathbf{P}(\mathbf{R^S})$ by $m \circ \delta^{-1}$. We now show that this map extends continuously to $\overline{\Delta}$. By Proposition 8.16, we can express the mass functional m_τ of $\tau \in \Lambda$ as an infinite linear combination of the Gromov coordinates of τ. We now compute the limit of each Gromov coordinate of τ as δ_τ approaches one of the points in $\mathbf{Z} \cup \{\infty\}$, and thereby check that the map m_τ extends continuously.

Consider some $\tau \in \Lambda$ with central charge Z_τ; let us compute the limit of the Gromov coordinates as δ_τ approaches some point of $\mathbf{Z} \cup \{\infty\}$. We know that up to a complex scalar, we have

$$Z_\tau(P_0) = 1, \quad Z_\tau(P_1) = \frac{1 - \delta_\tau}{\delta_\tau}.$$

We see that

$$Z_\tau(P_i) = \frac{|i|(1 - \delta_\tau) + |i - 1|\delta_\tau}{\delta_\tau}.$$

First suppose that δ_τ approaches some $n \in \mathbf{Z}$. In this case,

$$Z_\tau(P_i) \to \frac{|i|(1 - n) + |i - 1|n}{n}.$$

We can compute from Definition 8.15 that the limit of the jth Gromov coordinate is

$$x_j(\tau) \to \begin{cases} 0, & j \neq n, \\ 1, & j = n. \end{cases}$$

The sum in (55) obviously converges in the limit. Up to a simultaneous scalar, the limit of the mass functional is

$$m_\tau(-) \to \overline{\hom}(P_n, -).$$

Now suppose that δ_τ approaches ∞. In this case,

$$Z_\tau(P_j) \to -|i| + |i - 1| = \begin{cases} -1, & i > 0, \\ 1, & i \leq 0. \end{cases}$$

We compute that

$$x_j(\tau) \to \begin{cases} 0, & i < 0 \text{ or } i > 1, \\ -1, & i = 0, \\ 1, & i = 1. \end{cases}$$

Again, the sum in (55) obviously converges in the limit. Up to a simultaneous scalar, the limit of the mass functional is

$$m_\tau(-) \to \overline{\hom}(P_1, -) - \overline{\hom}(P_0, -).$$

We have exhibited a factoring

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\quad m \quad} & \mathbf{P}(\mathbf{R}^{\mathbf{S}}) \\
& \searrow{\scriptstyle \delta} \quad \nearrow{\scriptstyle d} & \\
& \overline{\Delta} &
\end{array}
$$

and identified the additional points in the image of $\overline{\Delta}$. Since $\overline{\Delta}$ is compact, no other points are in the closure of $m(\Lambda)$. $\qquad \square$

Remark 8.19. The point ∞ is already a limit point of the subset $\mathbf{Z} \cup \{\infty\} \subset \overline{\Delta}$. Therefore the additional point in the closure, namely the functional $\overline{\hom}(P_1) - \overline{\hom}(P_0)$, is already a limit point of the set $\{\overline{\hom}(P_i) \mid i \in \mathbf{Z}\}$.

It is now easy to identify the closure of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$, using the following contraction property, analogous to Proposition 7.20. Fix a metric on $\mathbf{P}^1(\mathbf{R}) = S^1$.

Proposition 8.20. *Let $\epsilon > 0$ and let $I \subset \mathbf{P}^1(\mathbf{R})$ be a compact subset. Then for all but finitely many elements $g \in B_\Gamma$, the image $g(I)$ has diameter less than ϵ.*

Proof. The action of B_Γ on $\mathbf{P}^1(\mathbf{R})$ is via the map $B_\Gamma \to \mathrm{PSL}_2(\mathbf{Z})$ of (49). The claim is true for $\mathrm{PSL}_2(\mathbf{Z})$, as shown in the proof of Proposition 7.20, and hence also for B_Γ. $\qquad \square$

Let M and P be the images of $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ and $\mathbf{S}$ in $\mathbf{P}^{\mathbf{S}}$, respectively.

Proposition 8.21 (Closure of M)**.** *The sets M and P are disjoint, and their union is the closure of M.*

Proof. The proof is analogous to the proof of Proposition 7.21. $\qquad \square$

We conjecture that the union $M \cup P$ is homeomorphic to a closed disk, as in the A_2 case.

Appendix A. Rectifiable filtrations

In this section, we formulate sufficient conditions on a filtration to ensure that it can be re-arranged to the Harder–Narasimhan (HN) filtration.

Fix a triangulated category $\mathcal{C}$ and stability condition on $\mathcal{C}$. For a semistable object x, let $\phi(x)$ denote its phase. For any object x, denote by $\lfloor x \rfloor$ (resp $\lceil x \rceil$) the semistable factor of lowest (resp. highest) phase in its HN filtration. We retain the subscript notation for the various truncations induced by the stability condition from Section 2.1.

The first basic question is as follows. Consider a distinguished triangle

$$x \to y \to z \to x[1].$$

When can we easily obtain the HN filtration of y from that of x and z? The following property of the map $z \to x[1]$ leads to the answer.

Definition A.1 (Rectifiable map). We say that a map $z \to x[1]$ is *rectifiable* if for every $\alpha \in \mathbf{R}$, the induced map $z_{>\alpha} \to x_{\leq\alpha}[1]$ obtained as the composition

$$
\begin{array}{ccc}
z_{>\alpha} & \longrightarrow & z \\
& & \downarrow \\
x[1] & \longrightarrow & x_{\leq\alpha}[1]
\end{array}
$$

vanishes. Furthermore, we say that a triangle $x \to y \to z \xrightarrow{+1}$ is rectifiable if the connecting map $z \to x[1]$ is rectifiable.

Remark A.2. A given object has only finitely many HN pieces, and hence only finitely many distinct truncations. Therefore, checking the rectifiability of a given map involves only finitely many vanishing conditions. In particular, $z \to x[1]$ is rectifiable if and only if for all α, the induced map

$$z_{\geq\alpha} \to x_{<\alpha}[1]$$

vanishes.

Example A.3.   (1) The zero map $z \to x[1]$ is rectifiable.
   (2) If both $z \to x_1[1]$ and $z \to x_2[1]$ are rectifiable, then so is the direct sum $z \to (x_1 \oplus x_2)[1]$.
   (3) Analogously, if $z_1 \to x[1]$ and $z_2 \to x[1]$ are rectifiable, then so is the direct sum $(z_1 \oplus z_2) \to x[1]$.
   (4) Suppose $\phi(\lfloor x \rfloor) \geq \phi(\lceil z \rceil)$. Then any map $z \to x[1]$ is rectifiable. Indeed, either $z_{>\alpha}$ or $x_{\leq\alpha}$ is zero.

Proposition A.4. *Suppose $z \to x[1]$ is rectifiable. Then for any maps $y \to z$ and $x \to w$, the induced map $y \to w[1]$ is rectifiable. That is, rectifiable maps form a two-sided ideal.*

Proof. The map $y_{>\alpha} \to w_{\leq\alpha}[1]$ factors as

$$y_{>\alpha} \to z_{>\alpha} \to x_{\leq\alpha}[1] \to w_{\leq\alpha}[1].$$

Since $z \to x[1]$ is rectifiable, the map $z_{>\alpha} \to x_{\leq\alpha}[1]$ vanishes. Hence, the map $y_{>\alpha} \to w_{\leq\alpha}[1]$ vanishes too. $\square$

We will need the following simple observations.

Lemma A.5. *Let $x \to y \to z \xrightarrow{+1}$ be a distinguished triangle.*

(1) Let $f\colon a \to y$ be a map such that the composite $a \xrightarrow{f} y \to z$ is zero. Further, suppose $\mathrm{Hom}(a, z[-1]) = 0$. Then there is a unique map $\widehat{f}\colon a \to x$ such that the composite $a \xrightarrow{\widehat{f}} x \to y$ is f.

(2) Dually, let $f\colon y \to a$ be a map such that the composite $x \to y \xrightarrow{f} a$ is zero. Further, suppose $\mathrm{Hom}(x[1], a) = 0$. Then there is a unique map $\widehat{f}\colon z \to a$ such that the composite $y \to z \xrightarrow{\widehat{f}} a$ is f.

Proof. The existence of $\widehat{f}$ follows from the axioms of triangulated categories. The difference of two such maps factors through a map $x \to z[-1]$, which must be zero by assumption.

The proof of the dual statement is similar. $\square$

Lemma A.6. *Consider a filtration*

$$x_1 \longrightarrow x_2 \longrightarrow x_3.$$

Suppose that the connecting map $y_3 \to y_2[1]$ vanishes. Then we can flip y_2 and y_3. More precisely, for some object x_2', we have a filtration

$$x_1 \longrightarrow x_2' \longrightarrow x_3,$$

such that the connecting map $y_2 \to y_3[1]$ also vanishes.

Proof. Let z be such that $x_1 \to x_3 \to z \xrightarrow{+1}$ is a distinguished triangle, where the first map is the composition $x_1 \to x_2 \to x_3$ from the given diagram. We see by the octahedral axiom that $y_2 \to z \to y_3 \xrightarrow{+1}$ is also a distinguished triangle. The connecting map $y_3 \to y_2[1]$ in this distinguished triangle is the same as the one in the given filtration, and is hence zero. Therefore, the distinguished triangle is split and we have $z \cong y_2 \oplus y_3$.

Now consider the composition of $x_3 \to z$ with the projection $z \to y_2$. Let x_2' be an object such that the triangle

$$x_2' \to x_3 \to y_2 \xrightarrow{+1}$$

is distinguished. Again, the octahedral axiom shows that

$$x_1 \to x_2' \to y_3 \xrightarrow{+1}$$

is a distinguished triangle, and we obtain the desired filtration. It is easy to see that the connecting map $y_2 \to y_3[1]$ is also zero, as it is part of the split distinguished triangle $y_3 \to z \to y_2 \xrightarrow{+1}$. $\square$

Proposition A.7. *Suppose $e\colon z \to x[1]$ is rectifiable.*

(1) *There exist unique maps* $z_{\leq\alpha} \xrightarrow{e_{\alpha}^{\leq}} x_{\leq\alpha}[1]$ *and* $z_{>\alpha} \xrightarrow{e_{\alpha}^{>}} x_{>\alpha}[1]$ *fitting in a morphism of distinguished triangles as follows.*

$$
\begin{array}{ccccc}
z_{>\alpha} & \longrightarrow & z & \longrightarrow & z_{\leq\alpha} \xrightarrow{+1} \\
\downarrow{\scriptstyle e_{\alpha}^{>}} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_{\alpha}^{\leq}} \\
x_{>\alpha}[1] & \longrightarrow & x[1] & \longrightarrow & x_{\leq\alpha}[1] \xrightarrow{+1}
\end{array}
$$

(2) *The maps* $e_{\alpha}^{\leq}$ *and* $e_{\alpha}^{>}$ *are rectifiable.*

(3) *Complete* e *to a distinguished triangle*

$$
y \to z \xrightarrow{e} x[1] \xrightarrow{+1} .
$$

Then we have distinguished triangles

$$
y_{\leq\alpha} \to z_{\leq\alpha} \xrightarrow{e_{\alpha}^{\leq}} x_{\leq\alpha}[1] \xrightarrow{+1} \quad and
$$

$$
y_{>\alpha} \to z_{>\alpha} \xrightarrow{e_{\alpha}^{>}} x_{>\alpha}[1] \xrightarrow{+1} .
$$

Proof. Let us prove each part in sequence.

(1) Recall that we have the following distinguished triangles:

(58)
$$
z_{>\alpha} \to z \to z_{\leq\alpha} \xrightarrow{+1} \quad and
$$
$$
x_{>\alpha}[1] \to x[1] \to x_{\leq\alpha}[1] \xrightarrow{+1} .
$$

The composition $z_{>\alpha} \to z \xrightarrow{e} x[1] \to x_{\leq\alpha}[1]$ vanishes because e is rectifiable. Furthermore, there are no nonzero maps from $z_{>\alpha}$ to $x_{\leq\alpha}$. Therefore, by Lemma A.5, we obtain a unique map

$$
e_{\alpha}^{\leq} : z_{>\alpha} \to x_{>a}[1]
$$

that makes the following square commute.

$$
\begin{array}{ccc}
z_{>\alpha} & \longrightarrow & z \\
\downarrow{\scriptstyle e_{\alpha}^{>}} & & \downarrow{\scriptstyle e} \\
x_{>\alpha}[1] & \longrightarrow & x[1]
\end{array}
$$

The argument for $e_{\alpha}^{\leq}$ is similar.

(2) Let us check that $e_{\alpha}^{>}$ is rectifiable. We must check that for every β, the map

$$
z_{>\max(\alpha,\beta)} \to x_{(\alpha,\beta]}[1]
$$

is zero. This is clear if $\beta \leq \alpha$, so assume $\beta > \alpha$. Then the map above becomes

$$
z_{>\beta} \to x_{(\alpha,\beta]}[1].
$$

Consider the distinguished triangle

$$
x_{\leq\alpha} \to x_{(\alpha,\beta]}[1] \to x_{\leq\beta}[1] \xrightarrow{+1} .
$$

The composite

$$
z_{>\beta} \to x_{(\alpha,\beta]}[1] \to x_{\leq\beta}[1]
$$

is zero because e is rectifiable. Thus there is an induced map $z_{>\beta} \to x_{\leq\alpha}$ making the following diagram commute.

$$
\begin{array}{ccc}
 & & z_{>\beta} \\
 & \swarrow & \downarrow \\
x_{\leq\alpha} & \longrightarrow & x_{(\alpha,\beta]}[1]
\end{array}
$$

However, since $\beta > \alpha$, any such induced map must be zero. Therefore the original map $z_{>\beta} \to x_{(\alpha,\beta]}[1]$ is zero.

By a similar argument, we see that $e_{\alpha}^{\leq}$ is also rectifiable.

(3) The distinguished triangle

$$x \to y \to z \xrightarrow{+1}$$

combined with (appropriate shifts of) the distinguished triangles (58) gives us a filtration as follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & y_1 & \longrightarrow & y_2 & \longrightarrow & y_3 & \longrightarrow & y_4 = y. \\
 & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & \\
{}^{+1} & & x_{>\alpha} & & x_{\leq\alpha} & & z_{>\alpha} & & z_{\leq\alpha}
\end{array}
$$

Since $z \to x[1]$ is rectifiable, the connecting map $z_{>\alpha} \to x_{\leq\alpha}[1]$ vanishes. Using Lemma A.6, we obtain a new filtration

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & y_1 & \longrightarrow & y_2' & \longrightarrow & y_3 & \longrightarrow & y_4 = y. \\
 & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & & \nwarrow\!\swarrow & \\
{}^{+1} & & x_{>\alpha} & & z_{>\alpha} & & x_{\leq\alpha} & & z_{\leq\alpha}
\end{array}
$$

It is clear from the proof of Lemma A.6 and the uniqueness statement of Proposition A.7 that the connecting maps $z_{\leq\alpha} \to x_{\leq\alpha}[1]$ and $z_{>\alpha} \to x_{>\alpha}[1]$ in the diagram above are $e_{\alpha}^{\leq}$ and $e_{\alpha}^{>}$ respectively.

Let y_2'' be an object such that $y_2' \to y \to y_2'' \xrightarrow{+1}$ is a distinguished triangle. By the octahedral axiom, we have distinguished triangles

$$x_{>\alpha} \to y_2' \to z_{>\alpha} \xrightarrow{e_{\alpha}^{>}} x_{>\alpha}[1]$$

$$x_{\leq\alpha} \to y_2'' \to z_{\leq\alpha} \xrightarrow{e_{\alpha}^{\leq}} x_{\leq\alpha}[1] \xrightarrow{+1}.$$

We note because of the distinguished triangles above that

$$(y_2')_{\leq\alpha} = 0 \text{ and } (y_2')_{>\alpha} = y_2',$$

and so $y_{>\alpha} = y_2'$. Similarly, $y_{\leq\alpha} = y_2''$. The proof is complete.

$\square$

Warning A.8. The maps $e_{\alpha}^{\leq}$ and $e_{\alpha}^{>}$ constructed in the previous proof are not the same as the maps induced by the truncation functors $(-)_{\leq\alpha}$ and $(-)_{>\alpha}$. Indeed, the object $x_{\leq\alpha}[1]$ is not the same as the object $(x[1])_{\leq\alpha}$, and similarly $x_{>\alpha}[1]$ is not the same as $(x[1])_{>\alpha}$.

For every $\alpha \in \mathbf{R}$, recall that $\mathcal{C}_\alpha$ is the abelian category of semistable objects of phase α.

Proposition A.9. *Let $x \to y \to z \xrightarrow{e} x[1]$ be a rectifiable triangle. For every $\alpha \in \mathbf{R}$ we have an exact sequence*

$$0 \to x_\alpha \to y_\alpha \to z_\alpha \to 0$$

in $\mathcal{C}_\alpha$. Consequently, we have

$$m_q(y) = m_q(x) + m_q(z).$$

Proof. Since e is rectifiable, we can apply Proposition A.7 twice to obtain a distinguished triangle

$$x_\alpha \to y_\alpha \to z_\alpha \xrightarrow{+1} .$$

All three objects lie in the abelian category $\mathcal{C}_\alpha$, which is an abelian subcategory of the $\mathcal{C}_{[\alpha,\alpha+1)}$. Since the latter is the heart of a t-structure, the distinguished triangle becomes a short exact sequence.

Since Z is additive for short exact sequences, we get

$$Z(y_\alpha) = Z(x_\alpha) + Z(z_\alpha).$$

But the three numbers all lie on the ray of argument $\pi\alpha$, so we get

$$|Z(y_\alpha)| = |Z(x_\alpha)| + |Z(z_\alpha)|.$$

Multiplying throughout by q^α gives

$$m_q(y_\alpha) = m_q(x_\alpha) + m_q(z_\alpha).$$

There are only finitely many numbers α such that at least one of the objects x_α, z_α, and y_α is nonzero. Summing over all such α yields $m_q(y) = m_q(x) + m_q(z)$. $\qquad\square$

We now take up filtrations.

Definition A.10 (Rectifiable filtration). Consider a filtration

$$x_0 \to x_1 \to \cdots \to x_n.$$

We inductively define what it means for such a filtration to be rectifiable. All filtrations for $n = 0$ and 1 are rectifiable. For $n \geq 2$, the filtration is rectifiable if the following hold.

(1) The filtration $x_1 \to \cdots \to x_n$ is rectifiable.
(2) Suppose $x(i, j)$ completes $x_i \to x_j$ to a distinguished triangle

$$x_i \to x_j \to x(i, j) \xrightarrow{+1} .$$

Then the connecting map $x(1, n) \to x(0, 1)[1]$ arising from

is rectifiable.

Remark A.11. In the previous definition, the objects $x(i, j)$ are unique up to (possibly non-unique) isomorphism, so the rectifiability of the connecting map is independent of the particular choices.

Henceforth in this section, if $f \colon x \to z$ is a morphism, we will denote by $\mathrm{Cone}(f)$ any object that fits into a distinguished triangle $x \xrightarrow{f} z \to \mathrm{Cone}(f) \xrightarrow{+1}$. Although such an object is not canonically defined in general, it will be sufficient for the arguments that it is unique up to isomorphism.

Remark A.12. The definition of a rectifiable filtration is "translation invariant" in the following sense. Given a filtration $x_0 \to x_1 \to \cdots \to x_n$, set $x_i' = \mathrm{Cone}(x_0 \to x_i)$ and consider the filtration $0 \to x_1' \to \cdots \to x_n'$. Then the first filtration is rectifiable if and only if the second one is. Indeed, the cones $x(i, j)$ and the connecting maps between them in the first filtration are isomorphic to the corresponding objects in the second filtration.

Proposition A.13. *Consider a filtration*

$$x_0 \to \cdots \to x_n,$$

and set $a_i = \mathrm{Cone}(x_{i-1} \to x_i)$. If for all i, the lowest HN phase of a_i is greater than or equal to the highest HN phase of a_{i+1}, then the filtration is rectifiable. In particular, the HN filtration is rectifiable.

Proof. We induct on n. The base cases $n = 0, 1$ are evident. Let us use the notation of Definition A.10. To go from $n - 1$ to n, consider the map $x(1, n) \to x(0, 1) = a_1$. The hypothesis implies that the lowest HN phase of $x(0, 1)$ is greater than or equal to the highest HN phase of $x(1, n)$, so the map is rectifiable by Example A.3. $\square$

Proposition A.14. *Suppose $x_0 \to \cdots \to x_n$ is rectifiable. Then for every i, j with $i < j$, the filtration $x_i \to \cdots \to x_j$ is rectifiable.*

Proof. This is an easy consequence of Proposition A.4. $\square$

Recall that $x(i, j) = \mathrm{Cone}(x_i \to x_j)$.

Proposition A.15. *Consider a filtration $x_0 \to x_1 \to \cdots \to x_n$. The following are equivalent:*

    (1) $x_0 \to x_1 \to \cdots \to x_n$ is rectifiable.
    (2) for all i, both $x_0 \to \cdots \to x_i$ and $x_i \to \cdots \to x_n$ are rectifiable, and the triangle

$$x(0, i) \to x(0, n) \to x(i, n) \xrightarrow{+1}$$

    is rectifiable.
    (3) for some i, both $x_0 \to \cdots \to x_i$ and $x_i \to \cdots \to x_n$ are rectifiable, and the triangle

$$x(0, i) \to x(0, n) \to x(i, n) \xrightarrow{+1}$$

    is rectifiable.

Proof. Let us show that (1) implies (2). We induct on i. The case $i = 1$ follows from the definition of a rectifiable filtration. Assume $i \geq 2$. Proposition A.14 implies that the smaller filtrations $x_0 \to \cdots \to x_i$ and $x_i \to \cdots \to x_n$ are both rectifiable. Recall that $x(i, j) = \mathrm{Cone}(x_i \to x_j)$. We now prove that the map $x(i, n) \to x(0, i)[1]$ is rectifiable. Given $\alpha \in \mathbf{R}$, we must prove that

(59) $$x(i, n)_{>\alpha} \to x(0, i)_{\leq \alpha}[1]$$

vanishes. By Proposition A.7, we have the triangle

$$x(0, 1)_{\leq \alpha} \to x(0, i)_{\leq \alpha} \to x(1, i)_{\leq \alpha} \xrightarrow{+1}$$

By the inductive hypothesis applied to the rectifiable filtration $x_1 \to \cdots \to x_n$ for $(i - 1)$, we get that the map

(60) $$x(i, n) \to x(1, i)[1]$$

is rectifiable. Therefore, the composite

$$x(i,n)_{>\alpha} \to x(0,i)_{\leq\alpha}[1] \to x(1,i)_{\leq\alpha}[1]$$

already vanishes. By Lemma A.5, (59) factors uniquely through a map

(61) $$x(i,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1].$$

By Proposition A.7 applied to the rectifiable map in (60), we have the triangle

$$x(1,i)_{>\alpha} \to x(1,n)_{>\alpha} \to x(i,n)_{>\alpha} \xrightarrow{+1} .$$

Consider the diagram

$$x(1,n)_{>\alpha} \longrightarrow x(i,n)_{>\alpha} \longrightarrow x(1,i)_{>\alpha}[1] \xrightarrow{+1}$$
$$\downarrow{\scriptstyle(61)}$$
$$x(0,1)_{\leq\alpha}[1].$$

The composite

$$x(1,n)_{>\alpha} \to x(i,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1]$$

vanishes since $x_0 \to \cdots \to x_n$ is rectifiable. Therefore, the map (61) factors through a map $x(1,i)_{>\alpha}[1] \to x(0,1)_{\leq\alpha}[1]$. But the last map must vanish for phase reasons. As a result, (61) vanishes and so does (59). We have thus proved that $x(i,n) \to x(0,i)[1]$ is rectifiable.

That (2) implies (3) is a tautology.

Let us show that (3) implies (1). We again induct on i. The case $i = 1$ is the definition. Assume $i \geq 2$. By using Proposition A.14 and the inductive hypothesis, we conclude that

$$x_1 \to \cdots \to x_n$$

is rectifiable. It remains to show that the map $x(1,n) \to x(0,1)[1]$ is rectifiable. Given $\alpha \in \mathbf{R}$, consider the map

(62) $$x(1,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1].$$

We have the rectifiable triangle

$$x(1,i) \to x(1,n) \to x(i,n) \xrightarrow{+1},$$

which by Proposition A.7 gives a distinguished triangle

$$x(1,i)_{>\alpha} \to x(1,n)_{>\alpha} \to x(i,n)_{>\alpha} \xrightarrow{+1} .$$

Since $x_0 \to \cdots \to x_i$ is rectifiable, the composite

$$x(1,i)_{>\alpha} \to x(1,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1]$$

vanishes. By Lemma A.5, (62) factors uniquely through a map

(63) $$x(i,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1].$$

Since $x_0 \to \cdots \to x_i$ is rectifiable, we have the triangle

$$x(0,1)_{\leq\alpha} \to x(0,i)_{\leq\alpha} \to x(1,i)_{\leq\alpha} \xrightarrow{+1} .$$

Consider the diagram

$$x(i,n)_{>\alpha}$$
$$\downarrow{\scriptstyle(63)}$$
$$x(1,i)_{\leq\alpha} \longrightarrow x(0,1)_{\leq\alpha}[1] \longrightarrow x(0,i)_{\leq\alpha}[1] \xrightarrow{+1} .$$

The composite
$$x(i,n)_{>\alpha} \to x(0,1)_{\leq\alpha}[1] \to x(0,i)_{\leq\alpha}[1]$$
vanishes since $x(i,n) \to x(0,i)[1]$ is rectifiable. As a result, (63) factors through a map $x(i,n)_{>\alpha} \to x(1,i)_{\leq\alpha}$. But the last map must vanishes for phase reasons. As a result, (63) vanishes and so does (62). The proof is now complete. □

Theorem A.16. *Suppose that a filtration $0 = x_0 \to x_1 \to \cdots \to x_n = x$ is rectifiable. Set $a_i = \mathrm{Cone}(x_{i-1} \to x_i)$. Then for any $\alpha \in \mathbf{R}$, we have an induced filtration*
$$0 \subset (x_1)_\alpha \subset \cdots \subset (x_n)_\alpha = x_\alpha$$
in the abelian category $\mathcal{C}_\alpha$, with factors
$$(x_i)_\alpha/(x_{i-1})_\alpha = (a_i)_\alpha.$$
Consequently, we have
$$m_q(x) = \sum m_q(a_i).$$

Proof. We induct on n. The base cases $n = 0, 1$ are immediate. Suppose the statement holds for $n - 1$. Since $0 = x_0 \to x_1 \to \cdots \to x_n$ is rectifiable, the triangle
$$x(0, n-1) \to x(0,n) \to x(n, n-1) \xrightarrow{+1}$$
is rectifiable by Proposition A.15. Note that this triangle is equal to
$$x_{n-1} \to x_n \to a_n \xrightarrow{+1}.$$
By Proposition A.9, in $\mathcal{C}_\alpha$ we have the short exact sequence
(64) $$0 \to (x_{n-1})_\alpha \to (x_n)_\alpha \to (a_n)_\alpha \to 0.$$
That is, we have an inclusion $(x_{n-1})_\alpha \subset (x_n)_\alpha$ whose quotient is $(a_n)_\alpha$. We now apply the inductive hypothesis to the rectifiable filtration
$$0 = x_0 \to \cdots \to x_{n-1}$$
to obtain the filtration
$$0 \subset (x_1)_\alpha \subset \cdots \subset (x_n)_\alpha = x_\alpha$$
with the desired factors. Summing over the α as in Proposition A.9, we see that the q-mass of x is the sum of the q-masses of the a_i. □

Proposition A.17. *Suppose*
$$x_0 \to \cdots \to x_{i-1} \to x_i \to x_{i+1} \to \cdots \to x_n$$
is rectifiable, and the map $x(i, i+1) \to x(i-1, i)[1]$ is zero. Consider a filtration
$$x_0 \to \cdots \to x_{i-1} \to x_i' \to x_{i+1} \to \cdots \to x_n = x,$$
obtained by flipping $x(i-1, i)$ and $x(i, i+1)$ as in Lemma A.6. Then this new filtration is also rectifiable.

Proof. By Proposition A.15, it suffices to show that
  (1) the filtration $x_0 \to \cdots \to x_{i-1}$ is rectifiable;
  (2) the filtration $x_{i-1} \to x_i' \to \cdots \to x_n$ is rectifiable; and
  (3) the triangle $x(0, i-1) \to x(0, n) \to x(i-1, n) \xrightarrow{+1}$ is rectifiable.
The first and third conditions follow directly from the rectifiability of the original filtration, so it remains to check the second. Once again, we apply Proposition A.15. It suffices to show that

(1) the filtration $x_{i-1} \to x_i' \to x_{i+1}$ is rectifiable;

(2) the filtration $x_{i+1} \to \cdots \to x_n$ is rectifiable; and

(3) the triangle $x(i-1, i+1) \to x(i-1, n) \to x(i+1, n) \xrightarrow{+1}$ is rectifiable.

The second and third conditions follow directly from the rectifiability of the original filtration, so it remains to check the first. However, the connecting map in the first filtration is zero by Lemma A.6, and so it is rectifiable. The proof is complete. $\qquad\square$

Theorem A.18. *Consider a filtration*

$$x_0 \to \cdots \to x_n = x.$$

Set $a_i = \mathrm{Cone}(x_{i-1} \to x_i)$. *Suppose for every* i, j *with* $i < j$, *we have*

$$\mathrm{Hom}(a_j, a_i[1]) = \mathrm{Hom}(a_i, a_j[1]) = 0 \ \text{or} \ \phi(\lfloor a_i \rfloor) \geq \phi(\lceil a_j \rceil).$$

Then the filtration is rectifiable.

Proof. We induct on n. The base cases $n = 0, 1$ are trivial. Assume $n \geq 2$.

Suppose i is such that

$$\phi(\lceil a_i \rceil) < \phi(\lceil a_{i+1} \rceil).$$

Then we also have $\phi(\lfloor a_i \rfloor) < \phi(\lceil a_{i+1} \rceil)$. By the hypothesis, we must have $\mathrm{Hom}(a_{i+1}, a_i[1]) = 0$. In particular, the connecting map $a_{i+1} \to a_i[1]$ vanishes. Consider a filtration obtained by flipping a_i and a_{i+1} as in Lemma A.6. The sequence of factors of the new filtration is

$$a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots a_n.$$

Observe that the new filtration also satisfies the hypotheses of the theorem. By Proposition A.17, it suffices to check that the new filtration is rectifiable.

By repeatedly swapping adjacent terms as above, we may assume that

$$\phi(\lceil a_1 \rceil) \geq \cdots \geq \phi(\lceil a_n \rceil).$$

By the inductive hypothesis, the filtration

$$x_1 \to \cdots \to x_n$$

is rectifiable. It remains to check that the triangle

(65) $$\qquad\qquad\qquad x(0,1) \to x(0,n) \to x(1,n) \xrightarrow{+1}$$

is rectifiable.

Note that $x(0,1) = a_1$. Let $b = \phi(\lfloor a_1 \rfloor)$. Partition $\{2, \ldots, n\}$ into two subsets $\{2, \ldots, i\}$ and $\{i+1, \ldots, n\}$ such that:

(1) for all $j \in \{2, \ldots, i\}$ we have $b < \phi(\lceil a_j \rceil)$, and

(2) for all $j \in \{i+1, \ldots, n\}$ we have $b \geq \phi(\lceil a_j \rceil)$.

If the first set is empty, take $i = 1$; in this case, $x(1, i) = 0$. If the second set is empty, take $i = n+1$, and set $x(i, n) = 0$. Consider the diagram

$$x(1,i) \longrightarrow x(1,n) \longrightarrow x(i,n) \xrightarrow{+1}$$
$$\downarrow$$
$$a_1[1]$$

in which the vertical map is the connecting map of (65). The object $x(1, i)$ has the filtration

$$0 = x(1,1) \to x(1,2) \to \cdots \to x(1,i)$$

with factors $a_2, \ldots, a_i$. By construction, for $j \in \{2, \ldots, i\}$, we have

$$b = \phi(\lfloor a_1 \rfloor) < \phi(\lceil a_j \rceil)$$

and hence, by hypothesis, we must have

$$\mathrm{Hom}(a_j, a_1[1]) = 0.$$

As a result, we get

$$\mathrm{Hom}(x(1, i), a_1[1]) = 0.$$

Therefore, the vertical map of the diagram above factors as a composition

(66) $$x(1, n) \to x(i, n) \to a_1[1].$$

Now, $x(i, n)$ has the filtration

$$0 = x(i, i) \to x(i, i+1) \to \cdots \to x(i, n)$$

with factors $a_{i+1}, \ldots, a_n$. For $j \in \{i+1, \ldots, n\}$, we have

$$b \geq \phi(\lceil a_j \rceil),$$

and therefore

$$b = \phi(\lfloor a_1 \rfloor) \geq \phi(\lceil x(i, n) \rceil).$$

By Example A.3, any map $x(i, n) \to a_1[1]$ is rectifiable. In particular, the second map in (66) is rectifiable. Proposition A.4 implies that $x(1, n) \to a_1[1]$ is rectifiable; that is, the triangle (65) is rectifiable. $\qquad\square$

Appendix B. Self-extensions of a spherical object

The aim of this section is to record some properties of self-extensions of a spherical object. The results of this section plays a significant role in the proof of Theorem 4.9, and may also be of independent interest.

Fix $d \geq 2$, a field $\mathbf{k}$, a $\mathbf{k}$-linear triangulated category $\mathcal{C}$, and a d-spherical object a of $\mathcal{C}$. Fix a nonzero element $\mathrm{loop}_a \in \mathrm{Hom}^d(a, a)$. It will be convenient to set $e = d - 1$.

Proposition B.1. *Let a_n be an object of $\mathcal{C}$ that admits a filtration*

(67)
$$0 = a_{-1} \longrightarrow a_0 \longrightarrow \cdots \longrightarrow a_n$$
$$a \qquad\qquad a[-e] \quad \cdots \quad a[-ne],$$

where the connecting maps

$$a[-ie] \to a[-(i-1)e + 1]$$

are shifts of loop_a. Then we have the following.

(1) *The space $\mathrm{Hom}^\bullet(a, a_n)$ is two dimensional, with generators*

$$i_n \colon a \to a_n \ \text{ and } \ l_n \colon a[-ne - d] \to a_n$$

in degrees 0 and $ne + d$ respectively. The map i_n is a nonzero multiple of the map $a = a_0 \to a_n$ in (67).

(2) *The space $\mathrm{Hom}^\bullet(a_n, a)$ is two dimensional, with generators*

$$t_n \colon a_n \to a[d] \ \text{ and } \ q_n \colon a_n \to a[-ne]$$

in degrees d and $-ne$ respectively. The map q_n is a nonzero multiple of $a_n \to a[-ne]$ in (67).

(3) The compositions

$$t_n \circ i_n \colon a \to a[d],$$
$$q_n \circ l_n \colon a[-ne - d] \to a[-ne]$$

are nonzero multiples of shifts of loop_a.

Proof. We prove the first two statements by induction on n.

For $n = 0$, the proposition follows from the the d-CY property of a. Let us assume it for $(n - 1)$, and prove it for n.

Consider the distinguished triangle

(68) $$a_{n-1} \to a_n \to a[-ne] \xrightarrow{+1} .$$

The connecting map $a[-ne] \to a_{n-1}[1]$ composed with the map $a_{n-1}[1] \to a[-(n-1)e+1]$ is a shift of loop_a. In particular, the connecting map is nonzero.

Apply $\mathrm{Hom}(a, -)$ to (68) to get the long exact sequence

$$\cdots \to \mathrm{Hom}^i(a, a_{n-1}) \to \mathrm{Hom}^i(a, a_n) \to \mathrm{Hom}^i(a, a[-ne]) \to \cdots .$$

Both $\mathrm{Hom}^i(a, a_{n-1})$ and $\mathrm{Hom}^i(a, a[-ne])$ are zero unless $i \in \{0, ne, ne + 1, ne + d\}$. So $\mathrm{Hom}^i(a, a_n) = 0$ for all i unless $i \in \{0, ne, ne+1, ne+d\}$. For $i = 0$, the long exact sequence gives

$$0 \to \mathrm{Hom}^0(a, a_{n-1}) \to \mathrm{Hom}^0(a, a_n) \to 0.$$

From the inductive hypothesis, we conclude that $\mathrm{Hom}^0(a, a_n)$ is one-dimensional and is spanned by a copy of the map $i_n \colon a_0 \to a_n$ from (67). For $i = ne$ and $i = ne + 1$, the hom spaces fit in the sequence

$$0 \to \mathrm{Hom}^{ne}(a, a_n) \to \mathrm{Hom}^{ne}(a, a[-ne]) \xrightarrow{\delta} \mathrm{Hom}^{ne+1}(a, a_{n-1}) \to \mathrm{Hom}^{ne+1}(a, a_n) \to 0.$$

The source of δ isomorphic to $\mathbf{k}$, spanned by the identity. The target of δ isomorphic to $\mathbf{k}$ by the inductive hypothesis. The map δ is induced by the connecting map $a[-ne] \to a_{n-1}[1]$ in (68), which is nonzero. It follows that δ is an isomorphism, and hence $\mathrm{Hom}^i(a, a_n) = 0$ for $i = ne$ and $ne + 1$.

Apply $\mathrm{Hom}(-, a)$ to (68) to get the long exact sequence

$$\cdots \to \mathrm{Hom}^i(a[-ne], a) \to \mathrm{Hom}^i(a_n, a) \to \mathrm{Hom}^i(a_{n-1}, a) \to \cdots .$$

The d-CY property of a and the vanishing of $\mathrm{Hom}^i(a, a_n)$ for all $i \notin \{0, -ne\}$ imply that $\mathrm{Hom}^i(a_n, a) = 0$ for all $i \notin \{-ne + d, d\}$. For $i = -ne$, the long exact sequence gives

$$0 \to \mathrm{Hom}^{-ne}(a[-ne], a) \to \mathrm{Hom}^{-ne}(a_n, a) \to 0.$$

We conclude that $\mathrm{Hom}^{-ne}(a_n, a)$ is one-dimensional and is spanned by a copy of the map $q_n \colon a_n \to a[-ne]$ from (67).

We have proved that $\mathrm{Hom}^0(a, a_n)$ and $\mathrm{Hom}^{-ne}(a_n, a)$ are one-dimensional and spanned by the maps i_n and q_n in (67). The d-CY property of a implies that $\mathrm{Hom}^d(a_n, a)$ and $\mathrm{Hom}^{ne+d}(a, a_n)$ are one-dimensional and spanned by the maps that compose with i_n and q_n to give loop_a. We have also proved that all other hom spaces vanish. The induction step is complete. $\qquad\square$

Proposition B.2. *For each $n \geq 0$, there exists an object a_n, unique up to isomorphism, which admits a filtration whose factors are $a, a[-e], \cdots, a[-ne]$ such that for all i, the connecting maps $a[-ie] \to a[-ie + d]$ are shifts of* loop_a.

Proof. We induct on n. For the base case $n = 0$, we have $a_0 \cong a$.

We assume the result for n, and prove it for $n + 1$. Observe that Proposition B.1 applies to a_n. Let a_{n+1} be the object that fits into a distinguished triangle

(69)
$$a[-ne - d] \xrightarrow{l_n} a_n \to a_{n+1} \xrightarrow{+1} .$$

Using the rotated triangle

$$a_n \to a_{n+1} \to a[-(n+1)e] \xrightarrow{+1}$$

and the assumed filtration of a_n, we obtain the desired filtration for a_{n+1}.

It remains to prove that the object a_{n+1} is unique up to isomorphism. Consider another object a'_{n+1} that also admits a filtration

(70)
$$0 \to a'_0 \to \cdots \to a'_n \to a'_{n+1}$$

satisfying the hypotheses. By the inductive hypothesis, we have an isomorphism $a'_n \cong a_n$. The last map in (70) then fits into a distinguished triangle

$$a[-ne - d] \xrightarrow{\iota'} a_n \to a'_{n+1} \xrightarrow{+1} .$$

By assumption, the composite of ι' with the next map $a'_n \to a[-(n-1)e]$ in (70) is the loop map; so $\iota' \neq 0$. By Proposition B.1, $\mathrm{Hom}^{ne+d}(a, a_n)$ is one-dimensional, so ι' is a nonzero scalar multiple of l_n. By comparing (69) and (70), we obtain an isomorphism $a_{n+1} \cong a'_{n+1}$. □

Proposition B.3. *For each $n \geq 0$, consider nonzero maps*

$$i_n \colon a \to a_n, \quad l_n \colon a[-ne - d] \to a_n$$
$$t_n \colon a_n \to a[d], \quad q_n \colon a_n \to a[-ne]$$

as in Proposition B.1. Then we have distinguished triangles

$$a_{n-1}[-d] \xrightarrow{t_{n-1}[-d]} a \xrightarrow{i_n} a_n \xrightarrow{+1} \quad and$$
$$a_n \xrightarrow{q_n} a[-ne] \xrightarrow{l_{n-1}[1]} a_{n-1}[1] \xrightarrow{+1} .$$

Proof. Using Proposition B.2, the objects a_i fit into a filtration

(71)

where the connecting maps are loop maps. By the uniqueness statement in Proposition B.2, the cone of the map $i_n \colon a = a_0 \to a_n$ is $a_{n-1}[-e]$. It gives rise to the first triangle. Likewise, the cone of the map $q_n \colon a_n \to a[-ne]$ is $a_{n-1}[1]$. It gives rise to the second triangle. □

Proposition B.4. *For each $n \geq 0$, the object a_n is indecomposable in $\mathcal{C}$.*

Proof. The statement is clear for $n = 0$: spherical objects are indecomposable because they have a one-dimensional space of endomorphisms of degree zero. It can be checked, e.g., via induction on $0 \leq k \leq n$ and using the triangles

$$a_{k-1} \to a_k \to a[-ke] \xrightarrow{+1}$$

that $\mathrm{Hom}^0(a_k, a_n)$ is one-dimensional for each $k \leq n$. In particular, $\mathrm{Hom}^0(a_n, a_n)$ is one-dimensional for each n, and so the objects a_n are indecomposable. □

In the remainder of the section, we prove some homological properties of the objects a_n. These results are important ingredients in the proof of Theorem 4.9. Suppose we have a stability condition τ on $\mathcal{C}$ such that a is a τ-semistable object in the $[0,1)$-heart, with phase strictly between 0 and 1. Then the defining filtration of a_n from Proposition B.2 is also its τ-Harder–Narasimhan filtration, because the factors are semistable and of decreasing phase.

Let I be the interval $[0, e)$. For $m \in \mathbf{Z}$, let by $I - me$ be the translate $[-me, -(m-1)e)$. For an object $x \in \mathcal{C}$, we study the truncations x_{I-me}. For the object a_n, observe that

$$(a_n)_{I-me} = \begin{cases} a[-m] & \text{if } m \in \{0, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

The objects a_n satisfy the following "lifting property".

Lemma B.5. *Let $f \colon x \to a_n$ be a map such that the induced map*

$$x_{I-ne} \to (a_n)_{I-ne}$$

vanishes. Then the map $x_{\geq -(n-1)e} \to a_{n-1}$, obtained by truncating f, extends to a map $x \to a_{n-1}$.

Proof. We have the distinguished triangle

$$x_{\geq -(n-1)e} \to x \to x_{< -(n-1)e} \xrightarrow{+1} .$$

It suffices to prove that the composition $x_{< -(n-1)e}[-1] \to x_{\geq -(n-1)e} \to a_{n-1}$ vanishes. We have the commutative diagram

$$
\begin{array}{ccccc}
x_{< -(n-1)e}[-1] & \longrightarrow & x_{\geq -(n-1)e} & \longrightarrow & x \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow & & \downarrow \\
a[-ne-1] & \xrightarrow{\;b\;} & a_{n-1} & \longrightarrow & a_n.
\end{array}
$$

We prove that the composite $b \circ \overline{f}$ vanishes. The map

$$x_{I-ne}[-1] \to a[-ne-1]$$

induced by f vanishes by assumption. Therefore, $\overline{f}$ factors through a map g as follows

$$
\begin{array}{ccccc}
x_{I-ne}[-1] & \longrightarrow & x_{< -(n-1)e}[-1] & \longrightarrow & x_{< -ne}[-1] \\
 & & \downarrow{\scriptstyle \overline{f}} & \swarrow{\scriptstyle g} & \\
 & & a[-ne-1]. & &
\end{array}
$$

Now it suffices to prove that the composite

$$b \circ g \colon x_{< -ne}[-1] \to a_{n-1}$$

vanishes. The phases of the HN factors of the domain lie in the interval $(-\infty, -ne-1)$. The HN factors of the codomain are d-CY and their phases lie in the interval $[-(n-1)e, +\infty)$. Let s be a semistable HN factor of the domain and t a semistable HN factor of the codomain. Then $\phi(t) > \phi(s) + d$. By the d-CY property of t, we have

$$\dim \operatorname{Hom}(s, t) = \dim \operatorname{Hom}(t, s[d]),$$

and the right hand side must vanish since $\phi(t) > \phi(s[d])$. We conclude that

$$\operatorname{Hom}(x_{< -ne}[-1], a_{n-1}) = 0,$$

and hence the composite $b \circ g$ vanishes. $\qquad\square$

We can use the previous lemma to show that the map $i_n \colon a \to a_n$ of degree zero is irreducible in the following way.

Lemma B.6. *Let x be an object that does not contain a as a direct summand and has no nonzero endomorphisms of negative degree. Then, for any $n \geq 0$, the map $i_n \colon a \to a_n$ does not factor as $a \to x \to a_n$.*

Proof. We prove the contrapositive. We continue to use truncations to the translates of the interval I.

Suppose we have maps $s \colon a \to x$ and $q \colon x \to a_n$ that compose to $i_n \colon a \to a_n$. Let n be the smallest with this property. If $n = 0$, then we obtain a as a direct summand. If $n > 0$, we claim that the composite map

$$\tag{72} x \xrightarrow{q} a_n \to a[-ne] \xrightarrow{s[-ne]} x[-ne]$$

is nonzero.

To see this, we first consider the map

$$\tag{73} x_{I-ne} \to (a_n)_{I-ne}.$$

If this map is zero, Lemma B.5 yields a map $r \colon x \to a_{n-1}$ which also has the property that the composite of $s \colon a \to x$ and $r \colon x \to a_{n-1}$ is $i_{n-1} \colon a \to a_{n-1}$. Since we chose n to be the smallest with this property, we conclude that the map (73) is nonzero.

Second, we see that the composite

$$a_I \to x_I \to (a_n)_I$$

is the identity. Therefore, the map $a_I \to x_I$ has a left inverse.

Now, consider the maps on the truncations induced by (72)

$$x_{I-ne} \xrightarrow{q} (a_n)_{I-ne} \xrightarrow{=} a[-ne] \xrightarrow{s[-ne]} (x[-ne])_{I-ne}.$$

The first map is nonzero, the middle map is an equality, and the last map has a left inverse. It follows that the composite is nonzero, and hence the composite in (72) is also nonzero. $\square$

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