AN INTRODUCTION TO INTERSECTION HOMOLOGY

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1. Introduction

There is an extremely fruitful homology and cohomology theory for smooth manifolds. It comes in several different guises — de Rham cohomology, simplicial (co)homology, singular (co)homology, sheaf cohomology, etc — which all lead to the same answer. There is a functorial product (cup product) having a beautiful geometric interpretation as intersection of cycles. The product gives a duality between the cohomology groups of complimentary dimensions. Furthermore, there is additional structure in special cases like Hodge decomposition for Kähler manifolds and Lefschetz hyperplane theorems for complex projective varieties.

The theory loses a lot its features in the case of singular spaces. Although the cup product survives, it does not lead to a duality. Moreover, the product cannot be interpreted as intersections of chains, partly because of the loss of duality between homology and cohomology.

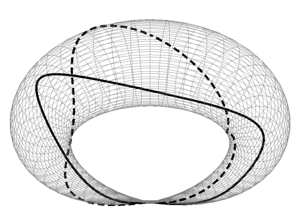


Figure 1. A pinched torus with cycles that cannot be made transverse

As an example, consider the 'pinched torus,' which is constructed, say, by shrinking $S^1 \times \{1\}$ on $S^1 \times S^1$ (Figure 1). Consider the two curves C and D shown in the figure. One can try to define the intersection number $C \cdot D$ by trying to move them to a transverse position. However, the idea of transversality breaks down at the singular point, and one cannot move C or D away from the singularity. Thus, the inability to define an intersection pairing for singular spaces can be attributed to the inability to move cycles away from singularities.

Mark Goresky and Robert MacPherson realized that to have a well defined intersection products in homology, one must restrict the cycles to certain 'intersectable' ones by controlling how they were allowed to pass through the singularities. They found that the class of spaces on which this would make sense is that of 'stratified pseudomanifolds.' These are stratified spaces

$$X = X_n \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0,$$

where $X_{i-1} \subset X_i$ is closed, $X_i \backslash X_{i-1}$ an i-dimensional manifold, and the stratification satisfies certain local niceness conditions (Definition 2.1). One must then control how the chains are allowed to intersect the various strata. On one extreme, all the chains are required to be transverse to the strata and on the other extreme there are no such restrictions. If X is normal, the former gives cohomology groups $H^{\cdot}(X)$ (Proposition 6.7), the later homology groups $H_{\cdot}(X)$; one has the usual cap product $H^i(X) \otimes H_j(X) \to H_{j-i}(X)$. Goresky and MacPherson's insight was to have chains lying between these two extremes, their deviation from transversality to the strata measured by sequences of integers called 'perversities.' Given a perversity p, one obtains a chain complex $IC_p^{\cdot}(X)$ consisting of simplicial chains whose intersection with the strata is controlled by p (Definition 3.2). The cohomology groups of this complex are the *intersection homology groups* $IH_{\cdot}^p(X)$.

In [GM80], Goresky and MacPherson define these intersection homology groups. They construct non-degenerate products in complimentary perversities generalizing the cap product between cohomology and homology. In the special case where X has only even dimensional strata (e.g. complex varieties), they obtain a duality for the self-complimentary 'middle' perversity, generalizing Poincaré duality for compact manifolds. They also prove that the intersection homology groups retain other desirable properties, namely the Mayer-Vietoris sequence, the Künneth formula, Poincaré duality, and for complex projective varieties, the Lefschetz hyperplane theorem. They work in the piecewise linear setting, using simplicial methods. Their proofs are explicit and geometrical.

While they were developing the theory, Goresky and MacPherson communicated their ideas to Deligne. He suggested that using sheaves may prove technically advantageous. Inspired by his ideas, Goresky and MacPherson worked out the sheaf theoretic formulation and published entirely different proofs of their earlier results in *Inventiones Mathematicae* in 1983 ([GM83]). The new approach, albeit much more technical and less geometric, proved to be technically superior. Not only did they obtain their previous results, but they also proved that the intersection homology groups were homeomorphism invariants! See Kleiman's fascinating article [Kle07] for more on the history of the development of the subject.

The point of departure of [GM83] is to give a formulation of intersection homology groups that makes them amenable to sheaf theoretic techniques, and then use these techniques to prove various properties. For example, one obtains Poincaré duality as a result of Verdier duality. Before we dive into the theory, let us think about why using sheaves might be beneficial. Despite being mathematically vague (or probably even nonsensical), the following will hopefully help the reader see the point of the endeavor.

The first step is to construct a complex of sheaves chains $\mathcal{IC}^{\bullet}_p$, whose global sections give the complex $IC^{\bullet}_p$. This, in itself, is not much better since the global section functor of sheaves behaves in a fairly complicated way. However, we observe (Proposition 4.1) that the individual sheaves $\mathcal{IC}^{i}_p$ are soft and hence acyclic with respect to the global section functor. Hence the intersection homology groups $H^i(\Gamma\mathcal{IC}^{\bullet}_p)$ are isomorphic to the hypercohomology groups $\mathbb{H}(\mathcal{IC}^{\bullet}_p)$, which depend only on the quasi isomorphism class of the complex $\mathcal{IC}^{\bullet}_p$. In effect, the study of the complex $IC^{\bullet}_p$ is reduced to the study of the complex $\mathcal{IC}^{\bullet}_p$.

On the face of it, this does not seem like progress. We have replaced the complex of *groups* $IC^{\bullet}_p$ by the complex of *sheaves* $\mathcal{IC}^{\bullet}_p$, and sheaves seem more complicated than groups. However, the complex $IC^{\bullet}_p$ depends highly on the global geometry, whereas the complex $\mathcal{IC}^{\bullet}_p$, being a complex of sheaves, can be described locally. In particular, if a local study of our spaces gives us a usable characterization of the complex $\mathcal{IC}^{\bullet}_p$ then we would be set. This is exactly what happens! The local computation of cohomology in Section 5 gives a list of properties of $\mathcal{IC}^{\bullet}_p$ that characterize the complex up to quasi isomorphism (Theorem 6.2). We obtain Poincaré duality by showing that the Verdier dual complex of $\mathcal{IC}^{\bullet}_p$ satisfies the axioms for the complimentary perversity q, and hence must be quasi isomorphic to $\mathcal{IC}^{\bullet}_q$. A more careful analysis, as done in [GM83, §4] gives a set of characterizing properties that only depends on the topological properties of X. This implies that the groups $IH^p_{\bullet}$ depend only on the homeomorphism class of X. We, however, do not go into the details of the second characterization.

The paper is organized as follows. In Section 2, we review the basic notions of piecewise linear topology. In Section 3, we define the intersection homology groups using simplicial chains and do a simple example computation. In Section 4, we take up the sheaf theoretic approach, constructing the complex $\mathcal{IC}^{\bullet}_p$ and proving some basic properties. In Section 5, we compute the local cohomology of $\mathcal{IC}^{\bullet}_p$. In Section 6, we use the local computation to extract a set of characterizing properties for the quasi isomorphism class of $\mathcal{IC}^{\bullet}_p$. We also describe Deligne's complex $\mathcal{P}$, a particularly simple object quasi isomorphic to $\mathcal{IC}^{\bullet}_p$. In Section 7, we use the apparatus of Verdier duality to outline a proof of Poincaré duality for intersection homology. In Section 8, we use Deligne's complex $\mathcal{P}$ to compute $IH^{\bullet}_p$ for some Schubert varieties. Appendix A and Appendix B give a summary of results from homological algebra and sheaf theory required for some of the later sections. Our discussion of intersection homology is heavily based on [Bor84a].

2. PL Spaces and stratified pseudomanifolds

The intersection homology groups are defined for stratified spaces endowed with a piecewise linear structure. In spite of the heavily simplicial nature of the basic definitions, our discussion of piecewise linear topology will be fairly brief. The reader should consult [Hud69] for a more rigorous treatment of the subject. Lurie's notes [Lur09] give a quick overview.

Roughly, a *piecewise linear space* or *pl space* is a topological space obtained by gluing together polyhedra in a piecewise linear fashion. Pl maps are the maps that are piecewise linear when restricted to these polyhedral pieces. What follows is a precise formulation of this idea.

A *simplex* in $\mathbb{R}^n$ is the convex hull of finitely many points that are linearly independent in the affine sense. A *polyhedron* is a finite union of simplices. Finite unions, finite intersections and finite products of polyhedra are polyhedra. Let $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ be polyhedra and $f \colon P \to Q$ a map between them. Then f is *piecewise linear* (or simply *pl*) if we can write P as a union of simplices Δ_i such that $f|_{\Delta_i} \colon \Delta_i \to \mathbb{R}^m$ is the restriction of an affine linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Composition of two pl maps is pl; pl maps are continuous; and if a pl map is a homeomorphism, then the inverse is also pl.

Having defined polyhedra, which are some distinguished subsets of $\mathbb{R}^n$, and pl maps, which are a distinguished class of morphisms between them, we can define an abstract pl space by the familiar recipe of patching. Thus, a *pl space* is a second countable, Hausdorff topological space with a family $\mathscr{F}$ of *coordinate charts* $f \colon P \to X$, where P is a polyhedron, such that

1. if $f \colon P \to X$ is in $\mathscr{F}$, then f is a homeomorphism onto its image;
2. every $x \in X$ lies in the interior of $f(P)$ for some $f \colon P \to X$ in $\mathscr{F}$;
3. if $f \colon P \to X$ and $g \colon Q \to X$ are in $\mathscr{F}$ and $f(P) \cap f(Q) \neq \varnothing$ then there exists $h \colon R \to X$ in $\mathscr{F}$ with $h(R) = f(P) \cap f(Q)$ and $f^{-1}h \colon R \to P$ and $g^{-1}h \colon R \to Q$ are pl;
4. $\mathscr{F}$ is maximal satisfying the above properties.

The notion of a pl map between two pl space is as expected. We say that a map $\phi \colon (X, \mathscr{F}) \to (Y, \mathscr{G})$ is pl if for every chart $f \colon P \to X$ in $\mathscr{F}$ and every chart $g \colon Q \to Y$ in $\mathscr{G}$ with $g(Q) \subset \phi \circ f(P)$, the map $g^{-1} \circ \phi \circ f \colon P \to Q$ is pl. Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be pl spaces such that $X \subset Y$. Then X is a *pl subspace* if the inclusion $X \hookrightarrow Y$ is pl. An open subspace U of a pl space $(X, \mathscr{F})$ is naturally a pl subspace with charts given by $\{f \in \mathscr{F} \mid \operatorname{im} f \subset U\}$. The Euclidean space $\mathbb{R}^n$ is a pl space with charts given by inclusions of polyhedra. A product of two pl spaces is a pl space.

The topological realization $|K|$ of a locally finite simplicial complex K is naturally a pl space. A *triangulation* of a pl space X is a pl isomorphism $t \colon |K| \to X$. Every pl space admits a triangulation; every compact pl space admits a finite triangulation. Moreover, if $f \colon X \to Y$ is a pl map between pl spaces then there exists triangulations $t \colon |K| \to X$ and $s \colon |L| \to Y$ such that the map $s^{-1} \circ f \circ t \colon |K| \to |L|$ sends simplices linearly to simplices. The most fruitful way to think about pl spaces for our purposes is to imagine them as topological spaces equipped with a class of triangulations such that any two triangulations have a common refinement and a linear subdivision of a triangulation is a triangulation.

Out of the various operations one can perform on pl spaces (joins, suspensions, products, etc), one will be of considerable importance: forming the cone. Let L be a compact pl space. The closed cone $\overline{\mathrm{c}^{\circ}}L$ is the topological space $L \times [0,1]/L \times \{0\}$. The point corresponding to $[L \times \{0\}]$ is called the *vertex* of the cone, often denoted by v. The pl structure on $\overline{\mathrm{c}^{\circ}}L$ is best described by specifying a triangulation. A triangulation of $\overline{\mathrm{c}^{\circ}}L$ is obtained by simply taking the closed cones of the simplices in a (finite) triangulation of L. The open cone $\mathrm{c}^{\circ}L$ is just the image of $L \times [0,1)$ in $\overline{\mathrm{c}^{\circ}}L$ with the induced pl structure. For an $\epsilon > 0$, we call the image of $L \times [0, \epsilon)$ in $\mathrm{c}^{\circ}L$ a *conical neighborhood* of the vertex. The open (and closed) cone on an empty set is defined to be a point.

An n dimensional *pl manifold* is a pl space X such that every point in X has an open neighborhood that is pl isomorphic to an open subset of $\mathbb{R}^n$.

We now come to the objects of prime interest. As we have seen, the aim of intersection homology is to have a good homology theory for singular spaces. However, one cannot expect it to work for spaces with arbitrary bad behavior (whatever that means). A sufficiently general, but workable notion is the following.

Definition 2.1. *(*[Hae84]*) A* stratified pl pseudomanifold *of dimension n is defined inductively as follows. For $n = 0$, it is simply a countable discrete set. In general, it is a pl space X with a stratification*

$$X = X_n \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0$$

satisfying the following properties:

(1) $X \backslash X_{n-2}$ is dense in X;
(2) X_{k-1} is a closed pl subspace of X_k;
(3) $X_k \backslash X_{k-1}$ is empty or a k dimensional pl manifold;
(4) for every x in $X_k \backslash X_{k-1}$, there exists a stratified pl pseudomanifold

$$L_x = L_{n-k-1} \supset L_{n-k-3} \supset \cdots \supset L_0,$$

and a neighborhood U of x in X with a pl isomorphism $U \xrightarrow{\sim} \mathbb{R}^k \times \mathrm{c}^{\circ} L_x$ which maps $U \cap X_j$ isomorphically to $\mathbb{R}^k \times \mathrm{c}^{\circ} L_{j-k-1}$ for $j > k$ and maps $U \cap X_k$ isomorphically to $\mathbb{R}^k \times \{v\}$, where v is the vertex of $\mathrm{c}^{\circ} L_x$.

The set $X_k \backslash X_{k-1}$ is called the codimension $n-k$ stratum*, and L_x is called the* link *of x.*

The last condition is called 'local normal triviality.' It roughly says that small neighborhoods of nearby points in the stratum $X_k \backslash X_{k-1}$ look 'the same.' In other words, the stratification of X does not degenerate as we move in a particular stratum $X_k \backslash X_{k-1}$. The second condition guarantees that X has 'pure dimension' n. If $X_k \backslash X_{k-1}$ is empty, we do not mention X_k in the stratification.

For example, the pinched torus T (Figure 1) is a stratified pl pseudomanifold with the stratification $T_2 = T$ and $T_0 = \{p\}$, where p is the singularity. All complex quasiprojective varieties have a stratified pl pseudomanifold structure. In fact, one can take all nonempty strata to be even dimensional.

Given a complex irreducible quasiprojective variety X of complex dimension n, a naïve attempt at a stratification would be the following. We set X_{2n} to be X, set X_{2n-2} to be a (complex) codimension 1 subvariety of X_{2n} containing the singular locus of X_{2n}, and likewise, in general, set X_{2k-2} to be a codimension 1 subvariety X_{2k} that contains the singular locus of X_{2k}. Although this process produces a stratification in which the open strata $X_{2k} \backslash X_{2k-2}$ are manifolds, it does not guarantee local normal triviality. For example ([Hae84, 1.5]), consider the surface X in $\mathbb{C}^3$ defined by $y^2 = tz^2$ (it is a family of a pair of lines degenerating to a double line). The singular locus is the line L given by $y = z = 0$. The stratification $X_4 = X$ and $X_2 = L$ fails local normal triviality at $(0, 0, 0)$. Adding another stratum $X_0 = \{(0, 0, 0)\}$ rectifies the situation, however, and gives us a stratified pl pseudomanifold.

Thus, although the fact that complex quasiprojective varieties admit a stratification as in Definition 2.1 is not completely trivial, we will rest assured that it can always be done. For more details, see the references listed in [Hae84, 1.5].

3. Intersection homology groups

The idea behind intersection homology is to restrict how simplicial chains are allowed to intersect various strata. As an indexing device for this purpose, we define *perversities.*

Definition 3.1. *A* perversity *on X is a sequence of integers $(p_2, p_3, \dots, p_n, \dots)$ satisfying $p_2 = 0$, and $p_k \leq p_{k+1} \leq p_k + 1$.*

The following are some examples of perversities:

$$\begin{aligned} \overline{0} &= (0, \dots, 0, \dots), \\ \overline{t} &= (0, 1, \dots, n-2, \dots), \\ \overline{m}_1 &= (0, 0, 1, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor - 1, \dots), \\ \overline{m}_2 &= (0, 1, 1, 2, \dots, \left\lceil \frac{k}{2} \right\rceil - 1, \dots). \end{aligned}$$

For a perversity p, the sequence $\overline{t} - p$ is also a perversity, called the *complimentary* or *dual* perversity of p. Thus, $\overline{0}$ and $\overline{t}$ are complimentary and so are $\overline{m}_1$ and $\overline{m}_2$.

Before we introduce the perverse chain complex, we recall the usual simplicial chain complex on X. Fix a (commutative) ring R and let X be a pl space. For a triangulation T of X we set

$$C_T^{-i}(X, R)_c = \{\text{Finite } R\text{-linear combinations of } i\text{-simplices in } T\}. \tag{3.1}$$

We have simplicial boundary maps $\partial^{-i} \colon C_T^{-i}(X, R)_c \to C_T^{-i+1}(X, R)_c$ that make $C_T^{\bullet}(X, R)_c$ a chain complex. A linear subdivision S of T induces chain maps $C_T^{\bullet}(X, R)_c \to C_S^{\bullet}(X, R)_c$. We set

$$C^{\bullet}(X, R)_c = \varinjlim_T C_T^{\bullet}(X, R)_c.$$

The elements of $C^{-i}(X, R)_c$ are called i-chains on X. For an i-chain ξ, we denote by $|\xi|$ the support of ξ. This is simply the union of the i-simplices of T that have nonzero coefficient in ξ. Clearly, the support of a chain does not change under a subdivision, letting us talk about supports of chains in $C^{-i}(X, R)_c$. Note that the supports are *compact*, which is the reason for the subscript c. The cohomology groups of $C^{\bullet}(X, R)_c$ are simply the homology groups $H_c^{\bullet}(X, R)$ [1].

For our purposes, a more useful idea will be that of homology with *closed supports*. This is obtained by dropping the finiteness restriction in (3.1). Explicitly, we set

$$C_T^{-i}(U, R) = \{\text{(Possibly infinite) } R\text{-linear combinations of } i\text{-simplices in } T\}.$$

$$C^{-i}(U, R) = \varinjlim_T C_T^{-i}(U)$$

Note that although the chains ξ in $C_T^{-i}(X)$ are infinite, any given point of X is contained in only finitely many simplices of ξ as the triangulation T is locally finite. The usual formula for boundaries makes $C^{\bullet}(X)$ a chain complex. The notion of the support of a chain ξ still makes sense. Observe that the support is a *closed* subset of X. The cohomology groups of $C^{\bullet}(X)$ are called *homology groups with closed supports*, denoted by $H^{\bullet}(X, R)$. Visibly, if X is compact then $H_c^{\bullet}(X, R) = H^{\bullet}(X, R)$.

[1] usually denoted without the subscript c

A few remarks are in order. First of all, the strange sign convention for the indices i is to make all complexes go towards the right (i.e. the differentials *raise* the degree.) This makes the homological algebra more uniform. Secondly, at this point, the coefficient ring can be arbitrary. However, in later sections, we do not hesitate to take coefficients in a field of characteristic zero. For simplicity, we often drop R from the notation.

Having defined simplicial chains, we turn to perverse chains. We let X be a stratified pl pseudomanifold, T a triangulation on X, and p a perversity.

Definition 3.2. *The group* $IC_p^{-i}(X)$ *of* perverse i-chains on X *is a subgroup of* $C^{-i}(X)$ *consisting of chains* $\xi \in C^{-i}(X)$ *satisfying*

(1) $\dim\ |\xi| \cap X_{n-k} \leq i - k + p(k)$*,*

(2) $\dim\ |\partial\xi| \cap X_{n-k} \leq i - 1 - k + p(k)$*.*

The subcomplex $IC_p^{\bullet}(X)$ *of* $C^{\bullet}(X)$ *is called the* perverse chain complex *(or more respectably the* intersection chain complex*) for the perversity* p*. The* intersection homology groups *are defined by*

$$IH_i^p(X) = H^{-i} IC_p^{\bullet}(X).$$

Since we are dealing with simplicial chains, the notion of dimension is straightforward. We take a triangulation of X with respect to which $|\xi|$ and X_{n-k} are subcomplexes. Then $\dim\ |\xi| \cap X_{n-k}$ is the largest j such that both $|\xi|$ and X_{n-k} have a common j-simplex.

The second condition ensures that $\partial^{-i} \colon C^{-i}(X) \to C^{-i+1}(X)$ restricts to a differential $\partial^{-i} \colon IC_p^{-i}(X) \to IC_p^{-i+1}(X)$. We sometimes use $IC_{p,T}^{-i}(X)$ to denote $IC_p^{-i}(X) \cap C_T^{-i}(X)$ — these are the perverse i chains that can be defined using the triangulation T.

To decode the restrictions in Definition 3.2 (referred to as *perversity restrictions*), note that if ξ intersects $X_{n-k} \backslash X_{n-k-1}$ 'transversely', then the intersection has dimension $i - k$. Thus, $p(k)$ can be thought of as the 'excess intersection' allowed for codimension k. Also, see that if $X_{n-k} \backslash X_{n-k-1} = \varnothing$, then the value of $p(k)$ is irrelevant. In particular, for a complex quasi projective variety, only the even perversities $p(2i)$ are relevant. Thus, in this context, the perversities m_1 and m_2 are both denoted by m, given by $m(2i) = i - 1$. Note that m is complimentary to itself.

One can restrict to finite linear combinations and obtain a subcomplex $IC_p^{\bullet}(X)_c$ of $C^{\bullet}(X)_c$. The resulting homology groups are denoted by $IH_p^{\bullet}(X)_c$. As before, if X is compact, then they coincide with $IH_p^{\bullet}(X)$.

Perhaps the best way to get acquainted with the definitions is to look at some simple examples. Consider a manifold X stratified as $X_n = X$, and $X_{n-1} = \cdots = X_0 = \varnothing$. We clearly have $IH_i^p(X) = H_i(X)$ for any perversity p.

The next computationally simplest case is the case of an X with an isolated singularity x. We take the stratification $X_n = X$ and $X_{n-1} = \cdots = X_0 = \{x\}$. The only relevant value in the perversity is $p(n)$. In this case, Definition 3.2 says

$$IC_p^{-i}(X) = \begin{cases} \{\xi \in C^{-i}(X) \mid x \notin |\xi| \text{ and } x \notin |\partial\xi|\} & \text{if } i - n + p(n) < 0, \\ \{\xi \in C^{-i}(X) \mid x \notin |\partial\xi|\} & \text{if } i - n + p(n) = 0, \\ C^{-i}(X) & \text{if } i - n + p(n) > 0. \end{cases}$$

This immediately gives

$$IH_i^p(X) = \begin{cases} H_i(X \backslash \{x\}) & \text{if } i < n - p(n) - 1, \\ \operatorname{im}\left(H_i(X\backslash\{x\}) \to H_i(X)\right) & \text{if } i = n - p(n) - 1, \\ H_i(X) & \text{if } i > n - p(n) - 1. \end{cases}$$

Of course, we have analogous results for $IH_i^p(X)_c$. In particular, the groups $IH_\bullet^p(X)_c$ are not homotopy invariant unlike the groups $H^\bullet(X)_c$. One can have a contractable X with an isolated singularity such that $IH_i^p(X)_c$ is nonzero for small i.

4. The complex $\mathcal{IC}_p^\bullet$

Let X be a stratified pl pseudomanifold and p a perversity. In this section, we construct a complex of sheaves $\mathcal{IC}_p^\bullet(X)$ whose global sections form the complex of perverse chains $IC_p^\bullet(X)$.

Observe that an open subset U of X is naturally a stratified pl pseudomanifold with the stratification obtained by intersecting the strata of X with U. Thus, we have a complex of perverse chains $IC_p^\bullet(U)$ on every open subset of X. An inclusion of open sets $U \hookrightarrow V$ gives a map of chain complexes $IC_p^\bullet(V) \to IC_p^\bullet(U)$, which we now describe. Consider a chain $\xi \in C^{-i}(V)$ defined using a triangulation T of V. Say $\xi = \sum a_\tau \tau$, where τ ranges over the i-simplices of T. We take a triangulation S of U such that every simplex σ of S is contained in a simplex $t(\sigma)$ of T. We set $i^*(\xi) = \sum a_{t(\sigma)}\sigma$, where the sum is taken over i-simplices of S. It may happen that $t(\sigma)$ is not an i-simplex of T, but a j-simplex for some $j > i$. In that case, $a_{t(\sigma)}$ is understood to be zero.

It can be checked that this gives a well defined map $i^* \colon C^{-i}(V) \to C^{-i}(U)$. Observe that it is essential that we allow infinite chains for this to work.. We often denote restriction of an element ξ to U by $\xi|_U$. See that $|i^*\xi| = |\xi| \cap U$.

The restriction maps give us a presheaf $\mathcal{C}^{-i}$ on X of modules over the coefficient ring. It is not hard to see that it is actually a sheaf. The boundary maps $\partial^{-i} \colon C^{-i}(U) \to C^{-i+1}(U)$ commute with the restrictions and give us a complex of sheaves $\mathcal{C}^\bullet(X)$.

We define a subcomplex $\mathcal{IC}_p^\bullet(X)$ of $\mathcal{C}^\bullet(X)$ by the same recipe as that in Definition 3.2. Recall that $IC_p^\bullet(U) \subset C^\bullet(U)$ consists of the chains ξ that obey the perversity restrictions:

$$\dim\ |\xi| \cap X_{n-k} \leq i - k + p(k), \tag{4.1}$$

$$\dim\ |\partial\xi| \cap X_{n-k} \leq i - 1 - k + p(k). \tag{4.2}$$

Consider an inclusion of open sets $i \colon U \hookrightarrow V$. Since $|i^*\xi| = |\xi| \cap U$, the map $i^* \colon C^{-i}(U) \to C^{-i}(V)$ sends $IC_p^{-i}(U)$ to $IC_p^{-i}(V)$ and gives us a subsheaf $\mathcal{IC}_p^{-i}(X)$ of $\mathcal{C}^{-i}(X)$. The condition (4.2) implies that the boundary $\partial^{-i} \colon \mathcal{C}^{-i} \to \mathcal{C}^{-i+1}$ restricts to a boundary $\partial^{-i} \colon \mathcal{IC}_p^{-i} \to IC^{-i+1}$, making $\mathcal{IC}_p^\bullet(X)$ a chain complex. We recover the complex $IC_p^\bullet(X)$ by taking global sections:

$$IC_p^\bullet(X) = \Gamma\mathcal{IC}_p^\bullet(X).$$

We abbreviate $\mathcal{IC}_p^\bullet(X)$ by $\mathcal{IC}_p^\bullet$.

The following proposition is the key that lets us reduce the study of $IC_p^\bullet(X)$ to that of $\mathcal{IC}_p^\bullet$.

Proposition 4.1. *For all $i \geq 0$, the sheaf $\mathcal{IC}_p^{-i}$ is soft.*

Proof. [Hab84, §5] Let Z be a closed subset of X and $\xi \in \Gamma(\mathcal{IC}_p^{-i}, Z)$. We want to show that ξ is the restriction of an i-chain $\tilde{\xi} \in \Gamma(\mathcal{IC}_p^{-i}, X)$. Suppose ξ is in $IC_p^{-i}(U)$ for some open set U containing Z. More precisely, say $\xi \in IC_{p,T}^{-i}(U)$ for some triangulation T of U in which the strata $X_{n-k} \cap U$ are subcomplexes. We want to produce a global chain that restricts to $\xi|_Z$ on Z.

The idea of the proof is as follows. We would like to extend ξ by zero. However, the support $|\xi|$, although closed in U, may not be closed in X. The idea is to 'clip off' parts of ξ that are 'away from Z' to obtain a chain that restricts to $\xi|_Z$ with support that is closed in X. We then extend this new chain by zero. The 'clipping off' is achieved by barycentric subdivision. The details follow.

Assume that the union of all simplices in T that intersect Z forms a closed set in X. This can be achieved by replacing T by a finer triangulation, if necessary. Let T' be the first barycentric subdivision of T. For a point $v \in X$, denote by T'_v the *star of v in T'* — the union of all simplices of T' containing v. Clearly, T'_v is a closed subset of X.

Write $\xi = \sum a_{\tau'} \tau'$, where τ' ranges over the i-simplices of T'. Denote by ξ_v the chain $\sum_{v \in \tau'} a_{\tau'} \tau'$. It has support $T'_v \cap |\xi|$.

Claim 1. *The chain ξ_v belongs to $IC_p^{-i}(U)$.*

Proof. We need to check the perversity restrictions. Since $|\xi_v| \subset |\xi|$, the perversity condition (4.1) is automatically satisfied. It remains to check (4.2).

The idea is to decompose the boundary $\partial \xi_v$ in two parts: one that is contained in $\partial \xi$ and a residual one. We then analyze the parts separately. It is most helpful to have a picture of the barycentric subdivision in mind.

Write $\partial \xi_v = \sum b_{\sigma'} \sigma' + \sum b_{\pi'} \pi'$, where $\{\sigma'\}$ are the $(i-1)$ faces of T' that contain v and $\{\pi'\}$ are those which do not. Here $b_{\sigma'}$ and $b_{\tau'}$ are nonzero. See that $|\sigma'|$ is contained in $|\partial \xi|$. On the other hand, each π' has the property that no j-face of it is contained in a j-simplex in T. These two crucial observations yield the result. In detail, we have

$$\dim\ |\sigma'| \cap X_{n-k} \leq \dim\ |\partial \xi| \cap X_{n-k} \leq i - 1 - k + p(k). \tag{4.3}$$

In the other case, recall that

$$\dim\ |\pi'| \cap X_{n-k} = \max\{j \text{ such that } |\pi'| \text{ and } X_{n-k} \text{ share a } j\text{-simplex}\}.$$

However, no j-face of π' is contained in a j-simplex in T. Since $X_{n-k} \cap U$ and $|\xi|$ are complexes in the triangulation T and $|\pi'| \subset |\xi|$, we conclude that if $|\pi'|$ and $X_{n-k} \cap U$ share a j-simplex, then $|\xi|$ and $X_{n-k} \cap U$ must share a $(j+1)$-simplex. This gives

$$\dim\ \left|\sum b_{\pi'} \pi'\right| \cap X_{n-k} \leq (\dim\ |\xi| \cap X_{n-k}) - 1 \leq i - 1 - k + p(k). \tag{4.4}$$

The assertions (4.3) and (4.4) show that ξ_v obeys the perversity condition (4.2), finishing the proof of the claim. □

Continuing the proof of the proposition, consider the simplices in T that intersect Z. Let Σ denote union of the vertices of such simplices. Set $T'_Z = \bigcup_{v \in \Sigma} T'_v$. See that T'_Z is a subcomplex of the union of all simplices in T that intersect Z. Since the latter set is closed in X, we conclude that T'_Z is closed in X. Thus we can extend the triangulation of T'_Z (given by T') to a triangulation S of X. Denote by T'°_v the open star of T' at v — the union of the interiors of simplices in T' containing v. The chain $\sum_{v\in\Sigma}\xi_v$ is an element of $IC^{-i}_{p,S}(X)$ that agrees with ξ on the open set $\bigcup T'^{\circ}_{v\in\Sigma}$ containing Z. Thus, the proof of the proposition is complete. ☐

Kirwan gives another proof, using a generalization of partitions of unity [Kir88, §5.2].

As a corollary, we obtain the following result.

Corollary 4.2. *The global sections map induces an isomorphism*

$$\mathbb{H}^{-i}\mathcal{IC}^{\bullet}_p \xrightarrow{\sim} IH^p_i(X).$$

Proof. Since $\Gamma\mathcal{IC}^{\bullet}_p = IC^{\bullet}_p(X)$, and the sheaves $\mathcal{IC}^{\bullet}_p$ are soft, this is a standard result in homological algebra. ☐

5. Computation of local cohomology

Thanks to Corollary 4.2, we focus on the complex of sheaves $\mathcal{IC}^{\bullet}_p$ for the rest of the paper. Since the hypercohomology depends on the complex only up to quasi isomorphism[2], our aim will be to characterize $\mathcal{IC}^{\bullet}_p$ up to quasi isomorphism. The natural step in this direction is to compute the cohomology sheaves $H^{-i}\mathcal{IC}^{\bullet}_p$. By definition, $H^{-i}\mathcal{IC}^{\bullet}_p$ is the sheaf associated to the presheaf

$$U \mapsto IH^p_i(U).$$

Hence, we must compute the homology groups $IH^p_i(U)$ for small open subsets U of X. Thankfully, since X is a stratified pl pseudomanifold, we have good control over its local geometry — sufficiently small neighborhoods of points of X look like $\mathbb{R}^{n-k} \times c^{\circ}L$. This suggests that we investigate how the operations of coning and taking the product with $\mathbb{R}$ alter the intersection homology groups.

We first treat the case of taking the product with $\mathbb{R}$. Let $X = X_n \supset \cdots \supset X_0$ be a stratified pl pseudomanifold. The product $X \times \mathbb{R}$ is naturally a stratified pl pseudomanifold with the stratification

$$X \times \mathbb{R} = X_n \times \mathbb{R} \supset \cdots \supset X_0 \times \mathbb{R} \supset \varnothing.$$

A chain $\xi \in IC^{-i}_p(X)$ gives a chain $\xi \times \mathbb{R}$ in $IC^{-i-1}_p(X \times \mathbb{R})$. In fact, $\xi \mapsto \xi \times \mathbb{R}$ gives a chain map $IC^{\bullet}_p(X) \to IC^{\bullet-1}_p(X \times \mathbb{R})$, called the *suspension*.

Proposition 5.1. *The suspension $IC^{\bullet}_p(X) \to IC^{\bullet-1}_p(X\times\mathbb{R})$ induces isomorphisms $IH^p_i(X) \xrightarrow{\sim} IH^p_{i+1}(X \times \mathbb{R})$ for all i.*

First we prove a small lemma.

Lemma 5.2. *Let $i \geq 0$ and $\xi \in IC^{-i}_p(X \times \mathbb{R})$ be a cycle supported on $X \times [0,\infty)$. Then ξ is a boundary.*

Proof. The idea is to observe that ξ is the boundary of the $(i+1)$-chain obtained by 'translating ξ infinitely towards the right.' To make this precise, see that we have

[2] an isomorphism in the derived category of complexes

a proper pl map $\phi\colon X \times [0,\infty)^2 \to X \times [0,\infty)$ that sends (x, r_1, r_2) to $(x, r_1 + r_2)$. Consider the chain $\xi \times [0,\infty)$ on $X \times [0,\infty)^2$. We have,

$$\begin{aligned}\partial\phi_*(\xi \times [0,\infty)) &= \phi_*\partial(\xi \times [0,\infty)) \\ &= \phi_*(\xi \times \{0\}) + \phi_*(\partial\xi \times [0,\infty)) \\ &= \xi.\end{aligned}$$

This exhibits ξ as a boundary. □

Proof of the proposition. Without loss of generality, X is connected. Otherwise, we work on individual connected components.

We first prove surjectivity. Let T be a triangulation of $X \times \mathbb{R}$ and ξ a cycle in $IC^{-i}_{p,T}(X \times \mathbb{R})$. By second countability, T has countably many vertices. Therefore, there is a $t \in \mathbb{R}$ such that $X \times \{t\}$ contains no vertices of T. Let ξ_t be the chain $\xi \cap X \times \{t\}$. The condition on t guarantees that $X \times \{t\}$ intersects every j-simplex of T in a $(j-1)$-simplex, and therefore $\dim \xi_t = i - 1$. Set $\xi^+ = \xi \cap (X \times [t, +\infty))$ and $\xi^- = \xi \cap (X \times (-\infty, t])$. Then ξ^+ and ξ^- are chains in $IC^{-i}_p(X)$ and $\xi = \xi^+ + \xi^-$. Also, we have $\partial\xi^+ = -\partial\xi^- = \xi_t$.

Now, $\xi^+ - \xi_t \times [t, +\infty)$ is a cycle in $IC^{-i}_p(X)$ supported on $X \times [t, +\infty)$. By Lemma 5.2, we get that $\xi^+ - \xi_t \times [t, +\infty)$ is homologous to zero. Similarly, the cycle $\xi^- - \xi_t \times (-\infty, t]$ is homologous to zero. In other words, ξ is homologous to $\xi_t \times \mathbb{R}$.

For injectivity, let η be a cycle in $IC^{-(i-1)}_p(X)$ such that $\eta \times \mathbb{R} = \partial\gamma$ for some $\gamma \in IC^{-(i+1)}_p(X \times \mathbb{R})$. We must show that η is the boundary of a chain in $IC^{-i}_p(X)$. Let $\eta \times \mathbb{R}$ and γ be defined in a triangulation T of $X \times \mathbb{R}$. As before, let t be such that $X \times \{t\}$ does not contain any vertex of T. Then we have $\gamma_t \in IC^{-i}_p(X)$. Since $\eta \times \mathbb{R} = \partial\gamma$, we obtain $\eta = \partial\gamma_t$ by taking intersections with $X \times \{t\}$. □

Having taken care of products with $\mathbb{R}$, we turn to coning. Consider a compact $(k-1)$-dimensional stratified pl pseudomanifold $L = L_{k-1} \supset \cdots \supset L_0$. Take the open cone $\mathrm{c}^{\circ}L$ and denote its vertex by v. The cone $\mathrm{c}^{\circ}L$ is a stratified pl pseudomanifold with the stratification

$$\mathrm{c}^{\circ}L = \mathrm{c}^{\circ}L_{k-1} \supset \cdots \supset \mathrm{c}^{\circ}L_0 \supset \{v\}.$$

A chain ξ in $IC^{-(i-1)}_p(L)$ gives a chain $\mathrm{c}^{\circ}\xi$ in $C^{-i}(\mathrm{c}^{\circ}L)$. Although $\mathrm{c}^{\circ}\xi$ obeys all perversity restrictions on $\mathrm{c}^{\circ}L\backslash\{v\}$, it may not do so at v. We find out when it does. See that $\mathrm{c}^{\circ}\xi$ always contains v, and so does $\partial(\mathrm{c}^{\circ}\xi)$ unless ξ is a cycle. Hence, we have

$$\mathrm{c}^{\circ}\xi \in IC^{-i}_p(\mathrm{c}^{\circ}L) \quad \text{if} \quad \begin{cases} i - k + p(k) > 0 \text{ or} \\ \partial\xi = 0 \text{ and } i - k + p(k) \geq 0. \end{cases}$$

In other words, the map $\xi \mapsto \mathrm{c}^{\circ}\xi$ gives a map of complexes

$$\mathrm{tr}_{\bullet \leq p(k)-k} IC^{\bullet+1}_p(L) \xrightarrow{\mathrm{c}^{\circ}} IC^{\bullet}_p(\mathrm{c}^{\circ}L). \tag{5.1}$$

Proposition 5.3. *The map in* (5.1) *induces isomorphisms on cohomology for all* i

$$H^{-i}(\mathrm{tr}_{\bullet \leq p(k)-k} IC^{\bullet}_p(L)) \xrightarrow{\sim} H^{-(i+1)}(IC^{\bullet}_p(\mathrm{c}^{\circ}L)).$$

Proof. The proof has the same flavor as the proof of Proposition 5.1. Recall the definition $\mathrm{c}^{\mathrm{o}}L = L \times [0,\infty)/L \times \{0\}$.

First, we make an observation similar to Lemma 5.2. If $\xi \in IC_p^{-i}(\mathrm{c}^{\mathrm{o}}L)$ is a cycle supported 'away from v' (i.e. $v \notin |\xi|$), then it is a boundary. Indeed, if $v \notin |\xi|$ then $|\xi|$ is contained in the image of $L \times [\epsilon,\infty)$ for some $\epsilon > 0$. Now, ξ is the boundary of the chain obtained by translating ξ 'infinitely to the right' along the cone coordinate. This is made rigorous exactly as in the proof of Proposition 5.1.

We first prove surjectivity. Let T be a triangulation of $\mathrm{c}^{\mathrm{o}}L$ and ξ a cycle in $IC_{p,T}^{-i}(\mathrm{c}^{\mathrm{o}}L)$. For $\epsilon > 0$, call the image of $L \times [0,\epsilon) \to \mathrm{c}^{\mathrm{o}}L$ the *conical neighborhood* N_ϵ of v. Take ϵ be so small that N_ϵ contains only the vertex v of T. Let ξ_ϵ be the cycle on L given by $\xi_\epsilon = \xi \cap (L \times \{\epsilon\})$. Then ξ_ϵ lies in $IC_p^{-(i-1)}(L)$ and $\mathrm{c}^{\mathrm{o}}\xi_\epsilon$ in $IC_p^{-i}(\mathrm{c}^{\mathrm{o}}L)$. Moreover, $\mathrm{c}^{\mathrm{o}}\xi_\epsilon - \xi$ is supported away from v, and hence a boundary. Thus, $\xi = \mathrm{c}^{\mathrm{o}}\xi_\epsilon$ in homology. This completes the proof of surjectivity.

The proof of injectivity parallels the one in Proposition 5.1. Consider a chain $\eta \in \mathrm{tr}_{\bullet \le p(k)-k} IC_p^{-(i-1)}(L)$ such that $\mathrm{c}^{\mathrm{o}}\eta = d\gamma$ for some $\gamma \in IC_p^{-i}$. As before, let γ and $\mathrm{c}^{\mathrm{o}}\eta$ be defined in a triangulation T and $\epsilon > 0$ such that N_ϵ only contains the vertex v of T. Setting $\gamma_\epsilon = L \times \{\epsilon\} \cap \gamma$, we see that $\gamma_\epsilon \in IC_p^{-i}$ and $\partial\gamma_\epsilon = \eta$. This completes the proof of injectivity. □

Fix a stratified pl pseudomanifold $X = X_n \supset X_{n-2} \supset \cdots \supset X_0$. We are ready to describe locally the complex $\mathcal{IC}_p^{\cdot}$.

Proposition 5.4. *Let $x \in X_{n-k} \backslash X_{n-k-1}$ be a point of X with a neighborhood U that is pl isomorphic to $\mathbb{R}^{n-k} \times \mathrm{c}^{\mathrm{o}}L$ as in Definition 2.1. We have*

$$H^{-i}(\mathcal{IC}_p^{\bullet})_x = \begin{cases} 0 & \text{if } i > n - p(k) \\ IH^p_{i-1+k-n}(L) & \text{otherwise.} \end{cases}$$

Moreover, for all i and $k \ge 2$, the restriction $H^{-i}(\mathcal{IC}_p^{\bullet})|_{X_{n-k}\backslash X_{n-k-1}}$ is a locally constant sheaf on $X_{n-k}\backslash X_{n-k-1}$.

Proof. We have the chain maps

$$\mathrm{tr}_{\bullet \le p(k)-n} IC_p^{\bullet+1+(n-k)}(L) \to IC_p^{\bullet}(U) \to (\mathcal{IC}_p^{\bullet})_x.$$

The first map is obtained by coning followed by $(n-k)$ suspensions and hence induces isomorphisms on cohomology by Proposition 5.1 and Proposition 5.3.

To see that the second map also induces isomorphism on cohomology, recall that $(\mathcal{IC}_p^{\bullet})_x = \varinjlim_V IC_p^{\bullet}(V)$, where V ranges over all open neighborhoods of x. Recall that N_ϵ denotes a conical neighborhood of the vertex of $\mathrm{c}^{\mathrm{o}}L$. For $\epsilon, \delta > 0$, the inclusion

$$(-\delta,\delta)^{n-k} \times N_\epsilon \hookrightarrow \mathbb{R}^{n-k} \times \mathrm{c}^{\mathrm{o}}L = U$$

induces isomorphisms $IH^p_{\bullet}(U) \xrightarrow{\sim} IH^p_{\bullet}((-\delta,\delta)^{n-k} \times \mathrm{c}^{\mathrm{o}}L)$. Furthermore, open sets of the form $(-\delta,\delta)^{n-k} \times \mathrm{c}^{\mathrm{o}}L$ are cofinal in the direct system of open neighborhoods of x. Therefore $IC_p^{\bullet}(U) \to (\mathcal{IC}_p^{\bullet})_x$ induces isomorphisms in cohomology.

Thus, the composite $\mathrm{tr}_{\bullet \le p(k)-n} IC_p^{\bullet+1+(n-k)}(L) \to (\mathcal{IC}_p^{\bullet})_x$ induces isomorphisms on cohomology. This implies the first part of the proposition. Finally, the map $H^{-i}(IC_p^{\bullet}(U)) \to H^{-i}(\mathcal{IC}_p^{\bullet})_x$ is an isomorphism for all $x \in U \cap (X_{n-k}\backslash X_{n-k-1})$. Hence the cohomology sheaves $H^{-i}(\mathcal{IC}_p^{\bullet})|_{X_{n-k}\backslash X_{n-k-1}}$ are locally constant. □

Definition 5.5. *A complex of sheaves $S^{\bullet}$ on a stratified pseudomanifold X is called* cohomologically constructible *(with respect to the stratification) if the cohomology sheaves $H^i(S^{\bullet})$ are locally constant on the open strata $X_{n-k}\backslash X_{n-k-1}$ and have finitely generated stalks.*

There is a more general notion of cohomological constructibility, independent of the stratification, which we do not consider. See [Bor84b, §3] for more details.

Proposition 5.6. *Let X be a stratified pl pseudomanifold. The complex $\mathcal{IC}^{\bullet}_p$ is cohomologically constructible with respect to the given stratification on X.*

Proof. By Proposition 5.4, we know that the cohomology sheaves of $\mathcal{IC}^{\bullet}_p$ are locally constant on the open strata $X_{n-k}\backslash X_{n-k-1}$. It remains to prove that the stalks are finitely generated.

We proceed by induction on the dimension of X. If the dimension is zero, then the result is trivial. Otherwise, by Proposition 5.4, it suffices to prove that the groups $IH^p_i(L)$ are finitely generated, for various links L. However, the links L are compact and of a smaller dimension than that of X. Hence, the intersection chain complex $\mathcal{IC}^{\bullet}_p(L)$ is cohomologically constructible. Now, the cohomology groups of a compact stratified space with coefficients in a cohomologically constructible sheaf are finitely generated [Bor84b, §3]. This completes the induction step. □

Having computed the stalks of the complex $\mathcal{IC}^{\bullet}_p$, we study how it varies 'stratum by stratum.' Before we begin, we introduce notation that will be used throughout. For $k \geq 2$, set

$$U_k = X\backslash X_{n-k},$$
$$Z_{n-k} = U_{k+1}\backslash U_k = X_{n-k}\backslash X_{n-k-1}.$$

We have an increasing chain of open sets

$$U_2 \subset U_3 \subset \cdots \subset U_{n+1} = X.$$

Let $i_k \colon U_k \hookrightarrow U_{k+1}$ be the inclusion. We look at how the complex $\mathcal{IC}^{\bullet}_p$ changes as we move from U_k to U_{k+1}. More precisely, we study the natural inclusion

$$\mathcal{IC}^{\bullet}_p|_{U_{k+1}} \to i_{k*}(\mathcal{IC}^{\bullet}_p|_{U_k}).$$

The map is an isomorphism over U_k. Consider a point x in Z_{n-k} and let U be its neighborhood such that $U \xrightarrow{\sim} \mathbb{R}^{n-k} \times \mathrm{c}^{\mathrm{o}}L$ as in Definition 2.1. By Proposition 5.4, we have

$$H^{-i}(\mathcal{IC}^{\bullet}_p)_x = IH^p_i(\mathbb{R}^{n-k} \times \mathrm{c}^{\mathrm{o}}L).$$

On the other hand, we have

$$\begin{aligned} H^{-i}(i_{k*}(\mathcal{IC}^{\bullet}_p|_{U_k}))_x &= \varinjlim_V IH^p_i(V \cap U_k) \\ &= \varinjlim_V IH^p_i(V\backslash Z_{n-k}) \\ &= IH^p_i(\mathbb{R}^{n-k} \times (\mathrm{c}^{\mathrm{o}}L\backslash\{v\})). \end{aligned}$$

In the last step, we use that the system of neighborhoods of x contains the system of distinguished neighborhoods (of the form $(-\delta,\delta)^{n-k} \times \mathrm{c}^{\mathrm{o}}L$) as a cofinal system.

Thus, we see that the map on stalk cohomology induced by the natural map $\mathcal{IC}^{\bullet}_p|_{U_{k+1}} \to i_{k*}(\mathcal{IC}^{\bullet}_p|_{U_k})$ fits in the diagram

$$\begin{array}{ccc} H^{-i}(\mathcal{IC}^{\bullet}_{p}|_{U_{k+1}})_x & \xleftrightarrow{\sim} & IH^p_i(\mathbb{R}^{n-k} \times \mathrm{c}^{\mathrm{o}} L) \\ \downarrow & & \downarrow \\ H^{-i}(i_{k*}(\mathcal{IC}^{\bullet}_{p}|_{U_k}))_x & \xleftrightarrow{\sim} & IH^p_i(\mathbb{R}^{n-k} \times (\mathrm{c}^{\mathrm{o}} L\backslash\{v\})) \end{array}$$

Hence, we need to understand the map $IH^p_i(\mathrm{c}^{\mathrm{o}}L) \to IH^p_i(\mathrm{c}^{\mathrm{o}}L\backslash\{v\})$, induced by the inclusion $\mathrm{c}^{\mathrm{o}}L\backslash\{v\} \hookrightarrow \mathrm{c}^{\mathrm{o}}L$. We begin by computing the cohomology of $\mathrm{c}^{\mathrm{o}}L - \{v\}$.

We have a map of complexes

$$IC^{\bullet}_p(L) \xrightarrow{\mathrm{c}^{\mathrm{o}}} C^{\bullet -1}(\mathrm{c}^{\mathrm{o}}L) \to IC^{\bullet -1}_p(\mathrm{c}^{\mathrm{o}}L\backslash\{v\}). \tag{5.2}$$

The first map is obtained by coning and the second by restriction.

Proposition 5.7. *For all i, the map of coning followed by restriction induces isomorphisms*

$$IH^p_i(L) \to IH^p_{i+1}(\mathrm{c}^{\mathrm{o}}L\backslash\{v\}).$$

Proof. Since we have a homeomorphism $\phi\colon \mathrm{c}^{\mathrm{o}}L\backslash\{v\} \to L \times \mathbb{R}$ that commutes with the projection to L and $\mathbb{R}$, it is tempting to use Proposition 5.1. However ϕ is not piecewise linear, and hence Proposition 5.1 does not apply.

A correct proof is not difficult, however. Since it is skipped in [Hab84], we outline it here. The details are almost exactly as in the proof of Proposition 5.1.

Denote by i the inclusion $\mathrm{c}^{\mathrm{o}}L\backslash\{v\} \hookrightarrow \mathrm{c}^{\mathrm{o}}L$. Observe that the projection map $\pi\colon \mathrm{c}^{\mathrm{o}}L\backslash\{v\} \to (0,\infty)$ is pl. By an argument similar to Lemma 5.2, we see that if a cycle in $IC^{-i}_p(\mathrm{c}^{\mathrm{o}}L\backslash\{v\})$ is supported on $\pi^{-1}[t,\infty)$ for some $t > 0$, then it is a boundary. Similarly, a cycle supported in $\pi^{-1}(0,t]$ is a boundary.

To prove surjectivity, consider an arbitrary cycle $\xi \in IC^{-(i+1)}_p(\mathrm{c}^{\mathrm{o}}L\backslash\{v\})$, defined in a triangulation T, say. Let $\xi_t = \xi \cap \pi^{-1}(t)$ be a slice that contains no vertex of T. Then ξ is homologous to $i^*(\mathrm{c}^{\mathrm{o}}\xi_t)$.

To prove injectivity, consider $\eta \in IC^{-i}_p(L)$ and $\gamma \in IC^{-(i+1)}_p(\mathrm{c}^{\mathrm{o}}L\backslash\{v\})$ such that $i^*(\mathrm{c}^{\mathrm{o}}\eta) = \partial\gamma$. Let γ and η be defined in a triangulation T. Taking a t such that $\pi^{-1}(t)$ contains no vertex of T, we get $\eta = \partial(\gamma \cap \pi^{-1}(t))$. □

Combining Proposition 5.1, Proposition 5.3 and Proposition 5.7, we have the following corollary.

Corollary 5.8. *Let L be a compact $k-1$ dimensional stratified pl pseudomanifold. The inclusion $i\colon \mathrm{c}^{\mathrm{o}}L \to \mathrm{c}^{\mathrm{o}}L\backslash\{v\}$ induces isomorphisms*

$$i^*\colon IH^p_j(\mathrm{c}^{\mathrm{o}}L) \to IH^p_j(\mathrm{c}^{\mathrm{o}}L\backslash\{v\}) \quad \textit{for } j \geq k - p(k).$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{tr}_{\bullet \le k-p(k)} IC^{\bullet+1}_p(L) & \longrightarrow & IC^{\bullet+1}_p(L) \\ \downarrow \mathrm{c}^{\mathrm{o}} & & \downarrow i^* \circ \mathrm{c}^{\mathrm{o}} \\ IC^{\bullet}_p(\mathrm{c}^{\mathrm{o}}L) & \xrightarrow{i^*} & IC^{\bullet}_p(\mathrm{c}^{\mathrm{o}}L\backslash\{v\}) \end{array}$$

The vertical arrows are isomorphisms on cohomology by Proposition 5.3 and Proposition 5.7. Hence, the lower arrow is an isomorphism on IH^p_j for $-j \leq p(k) - k$. □

Now we are ready to analyze $\mathcal{IC}_p^\bullet$ stratum by stratum, as promised.

Proposition 5.9. *The natural map*

$$\mathcal{IC}_p^\bullet|_{U_{k+1}} \to i_{k*}(\mathcal{IC}_p^\bullet|_{U_k})$$

induces isomorphisms on cohomology sheaves $H^i(\mathcal{IC}_p^\bullet|_{U_{k+1}}) \xrightarrow{\sim} H^i(i_{k*}(\mathcal{IC}_p^\bullet|_{U_k}))$ *for* $i \le p(k) - n$).

Proof. We already have the essential ingredients of the proof. Since we are testing a map between sheaves to be an isomorphism, it is enough to do so on stalks. Over U_k, the map $\mathcal{IC}_p^\bullet|_{U_{k+1}} \to i_{k*}\mathcal{IC}_p^\bullet|_{U_k}$ is an isomorphism. Therefore, we only need to check the statement for the stalks at $x \in Z_{n-k}$.

Take $x \in Z_{n-k}$ and let a neighborhood of x be pl isomorphic to $\mathrm{c}^{\mathrm{o}}L \times \mathbb{R}^{n-k}$ as in Definition 2.1. We have the following commutative diagram

$$\begin{array}{ccc}
IC_p^{\bullet+n-k}(\mathrm{c}^{\mathrm{o}}L) & \xrightarrow{\text{restrict}} & IC_p^{\bullet+n-k}(\mathrm{c}^{\mathrm{o}}L\backslash\{v\}) \\
\Big\downarrow{\scriptstyle \text{suspend}} & & \Big\downarrow{\scriptstyle \text{suspend}} \\
IC_p^\bullet(\mathrm{c}^{\mathrm{o}}L \times \mathbb{R}^{n-k}) & \xrightarrow{\text{restrict}} & IC_p^\bullet((\mathrm{c}^{\mathrm{o}}L\backslash\{v\}) \times \mathbb{R}^{n-k}) \\
\Big\downarrow & & \Big\downarrow \\
(\mathcal{IC}_p^\bullet|_{U_{k+1}})_x & \longrightarrow & (i_{k*}\mathcal{IC}_p^\bullet|_{U_k})_x
\end{array}$$

The vertical maps are isomorphisms on cohomology. The map at the top induces isomorphisms $H^i(IC_p^{\bullet+n-k}(\mathrm{c}^{\mathrm{o}}L)) \to H^i(IC_p^{\bullet+n-k}(\mathrm{c}^{\mathrm{o}}L\backslash\{v\})$ for $i+n-k \le p(k)-k$ by Corollary 5.8. We conclude that the lower-most arrow induces isomorphisms $H^i(\mathcal{IC}_p^\bullet|_{U_{k+1}})_x \to H^i(i_{k*}\mathcal{IC}_p^\bullet|_{U_k})_x$ for $i \le p(k) - n$. ☐

6. Axiomatic characterization and Deligne's complex

We have enough information about the complex $\mathcal{IC}_p^\bullet$ to characterize it up to quasi isomorphism. We begin by collecting the scattered information about it in a list of axioms. We then prove that any complex of sheaves satisfying those axioms is quasi isomorphic to $\mathcal{IC}_p^\bullet$. Finally, we construct a particularly simple complex that satisfies the axioms by design.

For simplicity, we restrict ourselves to orientable X. That is, we assume that $X\backslash X_{n-2}$ is orientable. This restriction is not essential — one can work with the orientation sheaf, or even a system of local coefficients — but we put it for simplicity. For concreteness, we fix our coefficient field to be $\mathbb{R}$. We denote by $\mathbb{R}_U$ the constant sheaf $\mathbb{R}$ on U.

We say that a complex $S^\bullet$ of sheaves on X satisfies (AX) for a perversity p if

(AX1) $S^\bullet$ is bounded and $H^i(S^\bullet) = 0$ for $i < -n$. (*boundedness*)
(AX2) $S^\bullet|_{U_2} \cong \mathbb{R}_{U_2}[n]$. (*normalization*)
(AX3) For $k \ge 2$, we have $H^i(S^\bullet|_{U_{k+1}}) = 0$ for $i > p(k) - n$. (*vanishing*)
(AX4) For $k \ge 2$ the map $H^i(S^\bullet|_{U_{k+1}}) \to H^i(Ri_{k*}S^\bullet|_{U_k})$ is an isomorphism for $i \le p(k) - n$ (*attaching*).

Theorem 6.1. *The intersection chain complex* $\mathcal{IC}_p^\bullet$ *satisfies (AX).*

Proof. (AX1) is clear from the construction. For (AX2), observe that on U_2 the complex $\mathcal{IC}^{\bullet}_p$ is the complex of ordinary homology with closed supports. Since U_2 is orientable, (AX2) follows. The vanishing condition (AX3) is a consequence of Proposition 5.4. The attaching condition (AX4) follows from Proposition 5.9 once we note that $i_{k*}\mathcal{IC}^{\bullet}_p|_{U_k} = Ri_{k*}\mathcal{IC}^{\bullet}_p|_{U_k}$, because the sheaves $\mathcal{IC}^{\bullet}_p$ are soft. □

Although the construction of $\mathcal{IC}^{\bullet}_p$ is somewhat intricate, the conditions (AX) are enough to determine its quasi isomorphism class. We have the following theorem.

Theorem 6.2. *Let $S^{\bullet}$ and $T^{\bullet}$ be a complexes of sheaves on X that satisfy (AX) and $\phi\colon S^{\bullet}|_{U_2} \to T^{\bullet}|_{U_2}$ a quasi isomorphism. Then ϕ extends to a quasi isomorphism $\tilde{\phi}\colon S^{\bullet} \to T^{\bullet}$.*

The proof will be immediate after we prove a little lemma.

Lemma 6.3. *Let $S^{\bullet}$ be a complex of sheaves on X satisfying (AX). For all $k \geq 2$, the complexes $S^{\bullet}|_{U_{k+1}}$ and $\operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*}(S^{\bullet}|_{U_k})$ are quasi isomorphic.*

Proof. We have the commutative diagram:

$$\begin{array}{ccc} \operatorname{tr}_{\bullet \leq p(k)-n} S^{\bullet}|_{U_{k+1}} & \longrightarrow & S^{\bullet}|_{U_{k+1}} \\ \downarrow & & \downarrow \\ \operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*}(S^{\bullet}|_{U_k}) & \longrightarrow & Ri_{k*}(S^{\bullet}|_{U_k}) \end{array}$$

The top horizontal map is a quasi isomorphism by (AX3). The left vertical map is a quasi isomorphism by (AX4). Hence $S^{\bullet}|_{U_{k+1}}$ and $\operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*}(S^{\bullet}|_{U_k})$ are quasi isomorphic. □

Proof of the theorem. The proof is a straightforward induction. Assume that we have a quasi isomorphism $\phi_k\colon S^{\bullet}|_{U_k} \to T^{\bullet}|_{U_k}$. We have the diagram

$$\begin{array}{ccc} \operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*}(S^{\bullet}|_{U_k}) & \longrightarrow & \operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*}(T^{\bullet}|_{U_k}) \\ \uparrow & & \uparrow \\ S^{\bullet}_{U_{k+1}} & \xrightarrow{\phi_{k+1}} & T^{\bullet}_{U_{k+1}} \end{array}$$

The top map is induced by ϕ_k and hence a quasi isomorphism. The two vertical maps are quasi isomorphisms given by Lemma 6.3. Hence we obtain a quasi isomorphism $\phi_{k+1}\colon S^{\bullet}_{U_{k+1}} \to T^{\bullet}_{U_{k+1}}$. □

As Deligne observed, one can construct much more directly a complex that satisfies (AX). We now describe his construction. We proceed by induction on the codimension. Set

$$\mathcal{P}_2 = \mathbf{D}_{U_2} = \mathbb{R}_{U_2}[n] \text{ on } U_2$$
$$\mathcal{P}_{k+1} = \operatorname{tr}_{\bullet \leq p(k)-n} Ri_{k*} P_k \text{ for } 2 \leq k \leq n.$$

The complex $\mathcal{P} = \mathcal{P}_{n+1}$, defined on X, is called *Deligne's complex*.

Theorem 6.4. *The complex $\mathcal{P}$ satisfies (AX). Consequently, a quasi isomorphism $\mathcal{IC}^{\bullet}_p|_{U_2} \to \mathbb{R}_{U_2}[n]$ gives a quasi isomorphism $\mathcal{IC}^{\bullet}_p \to \mathcal{P}$. In particular, we have*

$$IH^p_i(X) \cong \mathbb{H}^{-i}(\mathcal{P}).$$

Proof. It is easy to see that $\mathcal{P}$ satisfies (AX) — it does so by design. The second statement is a consequence of Theorem 6.1 and Theorem 6.2. Lastly, a quasi isomorphism $\mathcal{IC}^{\bullet}_{p} \to \mathcal{P}$ induces isomorphisms

$$IH_i^p(X) = \mathbb{H}^{-i}(\mathcal{IC}^{\bullet}_{p}) \xrightarrow{\sim} \mathbb{H}^{-i}(\mathcal{P}).$$

□

An immediate corollary is the pl-independence of IH.

Corollary 6.5. *The intersection homology groups $IH^p_{\bullet}(X)$ are independent of the piecewise linear structure of X.*

Proof. We have $IH_i^p = \mathbb{H}^{-i}(\mathcal{P})$ and $\mathcal{P}$ does not depend on the pl structure. □

The simplicity of the construction of $\mathcal{P}$ makes it very useful in proving theorems about intersection homology. More importantly, and rather surprisingly, it gives a *definition* of intersection homology groups in settings where we have a notion of stratification and a category of sheaves, but no rich underlying topological structure, the quintessential example being varieties in positive characteristic. Intersection homology turns out to be a powerful tool even in this setting, providing a proof of the Weil conjectures for singular varieties and leading to the resolution of the so-called Kazhdan-Lusztig conjecture in representation theory. See [Kir88] for more details.

As an application, we prove a result mentioned in the introduction.

Definition 6.6. *A stratified pl pseudomanifold $X = X_n \supset X_{n-2} \supset \cdots \supset X_0$ is called* normal *if every $x \in X$ has a neighborhood U in X such that $U \backslash X_{n-2}$ is connected.*

It can be shown that normal algebraic varieties are normal in this sense.

Proposition 6.7. *Let X be a normal stratified pl pseudomanifold. Then*

$$IH_i^0(X, \mathbb{R}) = H^{n-i}(X, \mathbb{R}).$$

Proof. We use the complex $\mathcal{P}$ to compute the intersection homology. We have

$$\mathcal{P} = tr_{\bullet \le -n} Ri_{n*} \mathrm{tr}_{\bullet \le -n} Ri_{n-1*} \dots \mathrm{tr}_{\bullet \le -n} Ri_{2*} \mathbb{R}_{U_2}[n].$$

Therefore $\mathcal{P} = i_* \mathbb{R}_{U_2}[n]$, where $i\colon U_2 \hookrightarrow X$ is the inclusion. Since X is normal, we have $i_*\mathbb{R} = \mathbb{R}$ and hence $\mathcal{P} = \mathbb{R}_X[n]$. Thus, we conclude that

$$IH_i^0(X, \mathbb{R}) = \mathbb{H}^{-i}(\mathbb{R}_X[n]) = H^{n-i}(X, \mathbb{R}).$$

□

The reader may jump to Section 8 to see the complex $\mathcal{P}$ used to compute some concrete examples.

7. Poincaré-Verdier duality

In this section, we outline the proof of Poincaré duality for intersection homology using the machinery of Verdier duality. The proof is not self-contained — we accept several results about cohomological constructibility. We give references for the unproved assertions.

As in Section 6, fix an n-dimensional orientable stratified pseudomanifold X. Let p and q be complimentary perversities, i.e. $p(k) + q(k) = t(k) = k - 2$. We fix our ring of coefficient to be $\mathbb{R}$.

Denote by $\mathbf{D}_X$ the dualizing sheaf on X. For a bounded complex of sheaves $A^\bullet$, let $\mathcal{D}_X A^\bullet$ be the complex $R\mathscr{H}om(A^\bullet, \mathbf{D}_X)$. We prove that $\mathcal{IC}^\bullet_p$ is isomorphic to $\mathcal{D}_X \mathcal{IC}^\bullet_q[n]$. Taking hypercohomology, this translates into

$$\begin{aligned}\mathbb{H}^{-i}(\mathcal{IC}^\bullet_p) &\cong \mathbb{H}^{-i}(\mathbf{D}_X \mathcal{IC}^\bullet_q[n]) = \mathbb{H}^{n-i}(\mathbf{D}_X \mathcal{IC}^\bullet_q) \\ &\cong \operatorname{Hom}(\mathbb{H}^{i-n}_c(\mathcal{IC}^\bullet_q), \mathbb{R}).\end{aligned}$$

Therefore, for a compact X, we obtain

$$IH^q_{n-i}(X) \cong \operatorname{Hom}(IH^p_i(X), \mathbb{R}).$$

In other words, we have a non-degenerate pairing, called the 'intersection pairing'

$$IH^q_{n-i}(X) \otimes IH^p_i(X) \to \mathbb{R}.$$

In particular, for a complex projective X we get self dual homology groups for the middle perversity $m(2k) = k - 1$.

The idea of the proof is to show that the complex $\mathcal{D}_X \mathcal{IC}^\bullet_q$ satisfies the axioms (AX) characterizing $\mathcal{IC}^\bullet_p$. Our treatment is based loosely on [Ban00, §4.4].

Before we begin the proof, we replace (AX) by an equivalent set of axioms, which is better suited for our purposes. Recall that $U_k = X \backslash X_{n-k}$ and $Z_{n-k} = X_{n-k} \backslash X_{n-k-1}$ for $k \geq 2$. We have the open and closed inclusions

$$U_k \xhookrightarrow{i} U_{k+1} \xhookleftarrow{j} Z_{n-k}.$$

We begin by noting that the vanishing condition (AX3) can be replaced by

(AX3') $\quad H^i(S^\bullet)_x = 0$ for $k \geq 2$, for $i > p(k) - n$ and $x \in Z_{n-k}$.

Proposition 7.1. *The pair {(AX2), (AX3)} is equivalent to {(AX2), (AX3')}.*

Proof. By (AX3), we have $H^i(S^\bullet|_{U_{k+1}}) = 0$ for all $k \geq 2$. Since $Z_{n-k} \subset U_{k+1}$, this implies that the stalk $H^i(S^\bullet|_{U_{k+1}})_x = 0$, for all $x \in Z_{n-k}$. Hence (AX3) implies (AX3').

For the converse, consider a point $x \in U_{k+1}$. If $x \in U_2$, then by (AX2), we have $H^i(S^\bullet)_x = H^i(\mathbb{R}[n])_x = 0$, for all $i > -n$ and hence for all $i > p(k) - n$. Otherwise, $x \in X_{n-j} \backslash X_{n-j-1}$ for some $j \leq k$. By (AX3'), we have $H^i(S^\bullet)_x = 0$ for $i > p(j) - n$ and hence, for $i > p(k) - n$. □

Next, we reformulate (AX4). For a point $x \in Z_{n-k}$, denote by j_x the inclusion $\{x\} \hookrightarrow Z_{n-k}$. Consider the following replacement.

(AX4') $\quad H^i(j_x^! S^\bullet) = 0$ for $k \geq 2$, for $i \leq p(k) - k + 1$, and $x \in Z_{n-k}$.

Proposition 7.2. *For a complex $S^\bullet$ that is cohomologically constructible with respect to the stratification of X, the conditions {(AX1-3),(AX4)} are equivalent to {(AX1-3),(AX4')}.*

We use the following result in the proof. Recall that a complex $A^\bullet$ is *cohomologically locally constant* if the cohomology sheaves $H^i(A^\bullet)$ are locally constant.

Proposition 7.3. *Let M be a k-dimensional manifold, $j_x\colon \{x\} \to M$ be the inclusion of a point, and $A^\bullet$ a cohomologically locally constant complex of sheaves on M. Then, $j_x^! A^\bullet = j_x^* A^\bullet[-k]$.*

This is a consequence of Poincaré-Verdier duality on M. For a proof, see [Bor84b, 3.7(b)].

Proof of the proposition. By (B.5), we have the distinguished triangle:

$$
\begin{array}{ccc}
j_* j^! S^\bullet & \longrightarrow & S^\bullet|_{U_{k+1}} \\
{\scriptstyle [1]}\nwarrow & & \swarrow \\
& Ri_*(S^\bullet|_{U_k}) &
\end{array} \qquad (7.1)
$$

By the attaching condition (AX4), the map $H^i(S^\bullet|_{U_{k+1}}) \to H^i(Ri_*(S^\bullet|_{U_k}))$ is an isomorphism for $i \le p(k) - n$. Therefore, the long exact sequence in cohomology gives $H^i(j^! S^\bullet) = 0$ for $i \le p(k) - n$ and $H^{p(k)-n+1}(j^! S^\bullet) \hookrightarrow H^{p(k)-n+1}(S^\bullet|_{U_{k+1}})$ is an injection. By the vanishing condition (AX3), we have $H^{p(k)-n+1}(S^\bullet|_{U_{k+1}}) = 0$. Thus, $H^i(j^! S^\bullet) = 0$ for $i \le p(k) - n + 1$. Conversely, if we have $H^i(j^! S^\bullet) = 0$ for $i \le p(k) - n + 1$, then the attaching condition (AX4) follows by the long exact sequence in cohomology associated to the triangle (7.1).

Hence, in the presence of (AX1-3), the condition (AX4) is equivalent to

$$H^i(j^! S^\bullet) = 0 \text{ for } k \ge 2, \text{ for and } i < p(k) - n. \qquad (7.2)$$

This condition can be checked by checking it on the stalks of all points of Z_{n-k}. In other words, (7.2) is equivalent to the following: For all $k \ge 2$ and $x \in Z_{n-k}$, we have

$$H^i(j_x^* j^! S^\bullet) = 0 \text{ for } i \le p(k) - n + 1. \qquad (7.3)$$

Since $S^\bullet$ is cohomologically constructible with respect to the stratification of X, the complex $j^! S^\bullet$ is cohomologically locally constant on Z_{n-k} [Bor84b, 3.10(b)]. Therefore, by Proposition 7.3, the equation (7.3) is equivalent to

$$H^i(j_x^! j^! S^\bullet[n-k]) = 0 \text{ for } i \le p(k) - n + 1.$$

In other words,

$$H^i(j_x^! S^\bullet) = 0 \text{ for } i \le p(k) - k + 1.$$

The proof is now complete. ☐

Consolidating Proposition 7.1 and Proposition 7.2, we see that for a cohomologically constructible $S^\bullet$, the set of axioms (AX) is equivalent to the following.

(AX1') $S^\bullet$ is bounded and $H^i(S^\bullet) = 0$ for $i < -n$.
(AX2') $S^\bullet|_{U_2} \cong \mathbb{R}_{U_2}[n]$.
(AX3') $H^i(S^\bullet)_x = 0$ for $k \ge 2$, for $i > p(k) - n$ and $x \in Z_{n-k}$.
(AX4') $H^i(j_x^! S^\bullet) = 0$ for $k \ge 2$, for $i \le p(k) - k + 1$, and $x \in Z_{n-k}$.

The stage is now set for duality. We state the theorem at once.

Theorem 7.4 (Poincaré duality)**.** *Let X be an orientable stratified pl pseudomanifold and p, q complimentary perversities. We have a quasi isomorphism*

$$\mathcal{IC}_p^\bullet \cong \mathcal{D}_X \mathcal{IC}_q^\bullet[n].$$

We need a preparatory lemma, whose proof we skip.

Lemma 7.5. [Ban00, 4.4] *Let $A^\bullet$ be a cohomologically constructible complex of sheaves on X and i an integer. There exist arbitrarily small open sets U around a point $x \in X$, such that we have isomorphisms:*

$$H^i(A^\bullet)_x \cong \mathbb{H}^i(A^\bullet|_U),$$
$$H^i(j_x^! A^\bullet) \cong \mathbb{H}_c^i(A^\bullet|_U).$$

Proof of the theorem. The dual $\mathcal{D}_X \mathcal{IC}_q^\bullet[n]$ is cohomologically constructible with respect to the stratification on X ([Bor84b, 8.6]). Hence, it suffices to check that $\mathcal{D}_X \mathcal{IC}_q^\bullet[n]$ satisfies the axioms (AX').

We begin by checking (AX1'). Boundedness is clear, since both $\mathbf{D}_X$ and $\mathcal{IC}_q^\bullet$ are bounded. For $x \in X$ and a small open set U around it, we have

$$\begin{aligned} H^i(\mathcal{D}_X \mathcal{IC}_q^\bullet[n])_x &= \mathbb{H}^{i+n}(\mathcal{D}_X \mathcal{IC}_q^\bullet|_U) \\ &= \mathrm{Hom}(\mathbb{H}_c^{-i-n}(\mathcal{IC}_q^\bullet|_U), \mathbb{R}) \\ &= \mathrm{Hom}(H^{-i-n}(j_x^! \mathcal{IC}_q^\bullet), \mathbb{R}). \end{aligned}$$

Now, $\mathcal{IC}_q^\bullet$ is a complex of soft sheaves which is zero for $\bullet > 0$. Hence, we get $H^{-i-n}(j_x^! \mathcal{IC}_q^\bullet) = 0$ for $-i-n > 0$, or, equivalently, for $i < -n$.

For the rest of the proof, consider a point $x \in Z_{n-k}$ and an open set U around it for which Lemma 7.5 is true.

For (AX2'), we observe

$$\begin{aligned} (\mathcal{D}_X \mathcal{IC}_q^\bullet)|_{U_2}[n] &= \mathcal{D}_{U_2}(\mathcal{IC}_q^\bullet|_{U_2})[n] \\ &\cong (\mathcal{D}_{U_2}\mathbb{R}[n])[n] = \mathbb{R}[n]. \end{aligned}$$

For (AX3'), we have

$$\begin{aligned} H^i(\mathcal{D}_X \mathcal{IC}_q^\bullet[n])_x &= \mathbb{H}^{i+n}(\mathcal{D}_X \mathcal{IC}_q^\bullet|_U) \\ &= \mathrm{Hom}(\mathbb{H}_c^{-i-n}(\mathcal{IC}_q^\bullet|_U), \mathbb{R}) \\ &= \mathrm{Hom}(H^{-i-n}(j_x^! \mathcal{IC}_q^\bullet), \mathbb{R}). \end{aligned}$$

By (AX4') applied to $\mathcal{IC}_q^\bullet$, we see that $H^{-i-n}(j_x^! \mathcal{IC}_q^\bullet) = 0$ for $-i-n \le q(k) - k + 1$, or, equivalently, for $i > -n + (k-2) - q(k) = p(k) - n$.

Finally, for (AX4'), we compute

$$\begin{aligned} H^i(j_x^! \mathcal{D}_X \mathcal{IC}_q^\bullet[n]) &= \mathbb{H}_c^{i+n}(\mathcal{D}_X \mathcal{IC}_q^\bullet|_U) \\ &= \mathrm{Hom}(\mathbb{H}^{-i-n}(\mathcal{IC}_q^\bullet|_U), \mathbb{R}) \\ &= \mathrm{Hom}(H^{-i-n}(\mathcal{IC}_q^\bullet)_x, \mathbb{R}). \end{aligned}$$

By (AX3') applied to $\mathcal{IC}_q^\bullet$, we have $H^{-i-n}(\mathcal{IC}_q^\bullet)_x = 0$ for $-i-n > q(k) - n$, or, equivalently, for $i \le -1 - q(k) = p(k) - k + 1$. The proof is thus complete. □

8. Computational examples

Let us use Deligne's complex $\mathcal{P}$ to do some calculations. In particular, let us compute the intersection homology groups of some Schubert varieties.

Our first example is the subvariety of the grassmannian of lines in $\mathbb{CP}^3$ consisting of those that intersect a fixed line $l \subset \mathbb{CP}^3$. In symbols,

$$X = \{m \in \mathbb{G}(1,3) \mid m \cap l \neq \varnothing\}.$$

The variety X has complex dimension 3. Set $U = X\backslash[l]$; let $i\colon U \hookrightarrow X$ and $j\colon \{[l]\} \to X$ be the inclusions. The stabilizer of l in PGl(4) acts transitively on U, and hence U is nonsingular. Using local coordinates on the grassmannian, we see that a neighborhood of $[l]$ in X is isomorphic to the cone C in $\mathbb{C}^4$ described by $xy - zw = 0$. We denote by 0 the point $(0,0,0,0)$ on C.

We use the stratification of X given by $X_6 = X$ and $X_0 = \{[l]\}$. Recall that $\mathcal{P}_2 = \mathbf{D}_U = \mathbb{R}_U[6]$ and

$$\mathcal{P} = \mathcal{P}_7 = \mathrm{tr}_{\bullet \le p(6)-6} Ri_* \mathcal{P}_2.$$

The intersection homology of X is simply the hypercohomology of $\mathcal{P}$. To compute the hypercohomology, we use the spectral sequence

$$H^i(X, H^j(\mathcal{P})) \implies \mathbb{H}^{i+j}(\mathcal{P}). \tag{8.1}$$

To calculate the cohomology sheaves $H^j(\mathcal{P})$, we take the de Rham resolution $\mathbb{R}_U[6] \to \Omega^\bullet_U[6]$ on U. Since the sheaves of differential forms $\Omega^\bullet$ are soft, we have $Ri_*\mathcal{P}_2 = Ri_*\mathbb{R}_U[6] = i_*\Omega^\bullet_U[6]$. Let us forget the shift by 6 for a moment and look at the complex $i_*\Omega^\bullet_U$. We have $H^0(i_*\Omega^\bullet_U) = i_*\mathbb{R}_U = \mathbb{R}_U$ (since $C\backslash 0$ is connected). For $j > 0$ the cohomology sheaf $H^j(i_*\Omega^\bullet_U)$ is supported at the point $[l]$ with the stalk $H^j(C\backslash 0)$. In other words, $H^j(i_*\Omega^\bullet_U) = j_*H^j(C\backslash 0)$ for $j > 0$. Using this (and remembering the shift by 6), we can write out the E_2 page of the spectral sequence (8.1) as follows (the group $j_*H^j(C\backslash 0)$ is abbreviated as j_*H^j, and $H^i(X,\mathbb{R})$ as H^iX.).

0	j_*H^6						
	j_*H^5						
-2	j_*H^4						
	j_*H^3						
-4	j_*H^2						
	j_*H^1						
-6	$\mathbb{R}$	H^1X	H^2X	H^3X	H^4X	H^5X	H^6X
	0		2		4		6

As such, this spectral sequence computes the hypercohomology of $Ri_*\mathbb{R}_U[6]$, and not of $\mathcal{P}$. However, $\mathcal{P}$ is simply $Ri_*\mathbb{R}_U[6]$ truncated at $p(6)-6$. Therefore, the spectral sequence for the hypercohomology of $\mathcal{P}$ is obtained by erasing the rows above $p(6)-6$. In particular, the differentials in the spectral sequence for $\mathcal{P}$ are exactly the differentials in the spectral sequence for $Ri_*(\mathbb{R}_U[6])$ that originate in rows with index at most $p(6)-6$.

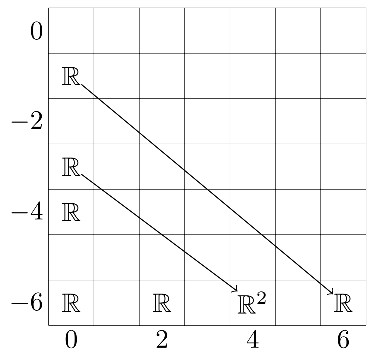
Observe that

$$\begin{aligned}\mathbb{H}^i(X, Ri_*\mathbb{R}_U[6]) &= H^i(R\Gamma \circ Ri_*\mathbb{R}_U[6]) \\ &= H^i(R\Gamma\mathbb{R}_U[6]) = H^{i+6}(U,\mathbb{R})\end{aligned}$$

This can be used to recover the differentials in the spectral sequence if we already know $H^\bullet(X,\mathbb{R})$, $H^\bullet(C\backslash 0,\mathbb{R})$ and $H^\bullet(U,\mathbb{R})$. To focus on intersection homology, we are going to suppress the details about singular cohomology by only saying a few words in the footnotes about their computation. Computing the cohomology of $C\backslash 0$, U and X, we obtain [3]:

$$H^\bullet(C\backslash 0, \mathbb{R}) = (\mathbb{R}, 0, \mathbb{R}, \mathbb{R}, 0, \mathbb{R}),$$
$$H^\bullet(U, \mathbb{R}) = (\mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}),$$
$$H^\bullet(X, \mathbb{R}) = (\mathbb{R}, 0, \mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}).$$

Thus, the E_2 page of the spectral sequence is:



Since the spectral sequence abuts to the cohomology $H^\bullet(U, \mathbb{R})$, we see that the only nonzero differentials are the ones shown in the diagram.

This is all the information we need to compute $IH^p_\bullet(X)$ for any perversity p: we simply forget the rows above $p(6) - 6$ and then read off the diagonals. For example, for the middle perversity ($p(6) = 2$), we only keep the rows indexed -6, -5 and -4 to obtain:

$$IH_0(X) = \mathbb{H}^0(\mathcal{P}) = \mathbb{R},$$
$$IH_2(X) = \mathbb{H}^{-2}(\mathcal{P}) = \mathbb{R}^2,$$
$$IH_4(X) = \mathbb{H}^{-4}(\mathcal{P}) = \mathbb{R}^2,$$
$$IH_6(X) = \mathbb{H}^{-6}(\mathcal{P}) = \mathbb{R}.$$

Observe that $IH_i(X) \cong IH_{6-i}(X)$ as expected by Poincaré duality.

Now we are in a position to do a more complicated example. The basic ideas are the same, but the computation is a bit more involved. We consider the set of lines in $\mathbb{C}\mathrm{P}^4$ meeting a fixed plane Π. In symbols,

$$X = \{m \in \mathbb{G}(1, 4) \mid m \cap \Pi \neq \varnothing\}.$$

See that X has complex dimension 5. We let U be the subset of X consisting of lines meeting Π in a point and Z the subset consisting of lines contained in Π. We have an open inclusion $i \colon U \to X$ and a closed inclusion $j \colon Z \to X$, with $X = U \cup Z$. Both U and Z are nonsingular — the respective stabilizers in PGl(5) act transitively on them. In fact, Z is isomorphic to $\mathbb{C}\mathrm{P}^2$. For a point $x \in Z$, we find that a neighborhood in X is isomorphic to $\mathbb{C}^2 \times C$, where C is the cone in $\mathbb{C}^4$ described by $xy - zw = 0$.

Using the stratification $X_{10} = X$ and $X_4 = Z$, we write Deligne's sheaf on X:

$$\mathcal{P} = \mathrm{tr}_{\bullet \leq p(6)-10} Ri_* \mathbf{D}_U = \mathrm{tr}_{\bullet \leq p(6)-10} Ri_* \mathbb{R}_U[10].$$

[3]The space $C\backslash 0$ is a nontrivial $\mathbb{C}^*$ bundle on $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. The space X has a Schubert cell decomposition. The open set U is a $\mathbb{C}\mathrm{P}^2\backslash\{pt\}$ bundle on $\mathbb{C}\mathrm{P}^1$.

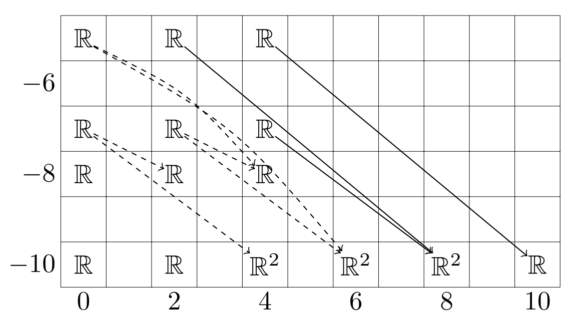
As before, we use the spectral sequence $H^i(X, H^j(\mathcal{P})) \implies \mathbb{H}^{i+j}(\mathcal{P})$ to compute the hypercohomology of $\mathcal{P}$. We forget the shift by 6 and the truncation for a moment and identify the cohomology sheaves $H^j(\mathcal{P})$. If x is a point of Z then we can find its neighborhood U in X such that $U\backslash x$ is connected (take $U \cong \mathbb{C}^2 \times C$). Hence $H^0(Ri_*\mathbb{R}_U) = i_*\mathbb{R}_U$ is the constant sheaf $\mathbb{R}_X$ on X. Next, for $j > 0$, the sheaf $H^j(Ri_*\mathbb{R}_U)$ is supported on Z with stalks $\varinjlim_U H^j(U\backslash x) = H^j(C\backslash 0)$ (one can see this, as we did before, by looking at the de Rham resolution of $\mathbb{R}_U$ on U). Furthermore, one can see that $H^j(Ri_*\mathbb{R}_U)|_Z$ is the constant sheaf $H^1(C\backslash 0)$. This follows by observing that $H^j(Ri_*\mathbb{R}_U)|_Z$ is locally constant, and hence constant, since Z is simply connected. Thus, we have

$$H^j(Ri_*\mathbb{R}_U) = j_* H^j(C\backslash 0), \quad \text{for } j > 0.$$

With this information, we can write out the E_2 page of the spectral sequence $H^i(X, H^j(\mathcal{P})) \implies \mathbb{H}^{i+j}(\mathcal{P})$. To get our hands on the differentials, we observe that $H^i(X, j_* H^j(C\backslash 0)) = H^i(Z, H^j(C\backslash 0))$ and $\mathbb{H}^i(Ri_*\mathbb{R}_U) = H^i(U, \mathbb{R})$. Thus, knowledge of $H^\bullet(Z,\mathbb{R})$, $H^\bullet(X,\mathbb{R})$, $H^\bullet(C\backslash 0,\mathbb{R})$ and $H^\bullet(U,\mathbb{R})$ lets us deduce, up to a large extent, which differentials are nonzero. We compute [4]

$$\begin{aligned} H^\bullet(Z,\mathbb{R}) &= (\mathbb{R}, 0, \mathbb{R}, 0, \mathbb{R}), \\ H^\bullet(X,\mathbb{R}) &= (\mathbb{R}, 0, \mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}), \\ H^\bullet(C\backslash 0,\mathbb{R}) &= (\mathbb{R}, 0, \mathbb{R}, \mathbb{R}, 0, \mathbb{R}), \\ H^\bullet(U,\mathbb{R}) &= (\mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}). \end{aligned}$$

This lets us write the E_2 page of the spectral sequence as



To identify the nonzero differentials from the many possible ones, we use that the sequence (up to a shift by 10 on the Y axis) abuts to $H^\bullet(U,\mathbb{R})$. This lets us deduce that the differentials depicted by solid arrows are nonzero, and out of the two depicted by dashed arrows from a common source, exactly one is nonzero.

This is enough information to be able to compute $IH^p_\bullet(X)$ for any perversity. For example, to compute the middle perversity groups ($p(6) = 2$), we erase the top

[4]The space Z is isomorphic to $\mathbb{CP}^2$. The space X has a Schubert cell decomposition. The open subset U is a $\mathbb{CP}^3\backslash\mathbb{CP}1$ bundle on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

3 rows and obtain:

$$\begin{aligned} IH_0(X) &= \mathbb{H}^0(\mathcal{P}) = \mathbb{R}, \\ IH_2(X) &= \mathbb{H}^{-2}(\mathcal{P}) = \mathbb{R}^2, \\ IH_4(X) &= \mathbb{H}^{-4}(\mathcal{P}) = \mathbb{R}^3, \\ IH_6(X) &= \mathbb{H}^{-6}(\mathcal{P}) = \mathbb{R}^3, \\ IH_8(X) &= \mathbb{H}^{-8}(\mathcal{P}) = \mathbb{R}^2, \\ IH_{10}(X) &= \mathbb{H}^{-10}(\mathcal{P}) = \mathbb{R}. \end{aligned}$$

The symmetry is a manifestation of Poincaré duality.

For the top perversity ($p(6) = 4$), we get:

$$IH^t_\bullet(X) = (\mathbb{R}, 0, \mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}).$$

These groups are dual to the ones for the zero perversity ($p(6) = 0$):

$$IH^0_\bullet(X) = (\mathbb{R}, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}^2, 0, \mathbb{R}, 0, \mathbb{R}).$$

Appendix A. Homological algebra

We give an overview of the necessary background from homological algebra. We summarize the basic definitions and state the main theorems. This is, by no means, a sufficient introduction to the subject. A detailed exposition can be found in [KS90]. Nocolaescu's notes [Noc] are also quite helpful.

Let $\mathbf{A}$ be an additive category. A *complex* in $\mathbf{A}$ is a sequence $\{C^n \mid n \in \mathbb{Z}\}$ of objects of $\mathbf{A}$ along with maps $d^n \colon C^n \to C^{n+1}$ such that $d^{n+1} \circ d^n = 0$. The maps d^n are called *differentials*. We almost always drop the differentials from the notation and denote the complex by $C^\bullet$. A complex $C^\bullet$ is *bounded below* (resp. *bounded above*) if $C^n = 0$ for all $n < N$ (resp. for all $n > N$) for some integer N; it is *bounded* if it is both bounded below and bounded above.

Let $A^\bullet$ and $B^\bullet$ be complexes. A map of complexes $f \colon A^\bullet \to B^\bullet$ is a sequence of maps $f^n \colon A^n \to B^n$ which commute with the differentials:

$$\begin{array}{ccc} A^n & \xrightarrow{d_A^n} & A^{n+1} \\ \downarrow f^n & & \downarrow f^{n+1} \\ B^n & \xrightarrow{d_B^n} & B^{n+1} \end{array}.$$

One can form the category $C(\mathbf{A})$, whose objects are complexes in $\mathbf{A}$ and morphisms are maps of complexes. We denote by $C^+(\mathbf{A})$, $C^-(\mathbf{A})$ and $C^b(\mathbf{A})$ the full subcategories of bounded below, bounded above and bounded complexes, respectively.

Definition A.1. *Let k be an integer. The* kth cohomology functor $H^k \colon C(\mathbf{A}) \to \mathbf{A}$ *is defined by*

$$H^k(C^\bullet) = \frac{\ker\ d^k}{\text{im}\ d^{k+1}}.$$

We define two simple operations on complexes: shift and truncation.

Definition A.2. *Let $C^\bullet$ be a complex and k an integer. The* shift of C by k *is the complex denoted by $C[k]$ and defined by*

$$C[k]^n = C^{k+n},$$
$$d^n_{C[k]} = (-1)^k d^{k+n}_C.$$

We frequently denote the shifted complex $C[k]$ by $C^{\bullet+k}$.

See that $H^n(C[k]) = H^{n+k}(C)$.

Definition A.3. *We define the* truncation of $C^\bullet$ at k, *denoted by* $\mathrm{tr}_{\bullet\leq k}C^\bullet$, *as follows:*

$$(\mathrm{tr}_{\bullet\leq k}C)^n = \begin{cases} C^n & \text{if } n < k, \\ \ker d^n \colon C^n \to C^{n+1} & \text{if } n = k, \\ 0 & \text{if } n > k. \end{cases}$$

The differentials in the truncation are the ones induced from those in the original complex. Clearly, $\mathrm{tr}_{\bullet\leq k}C^\bullet$ is bounded above. Furthermore, we have

$$H^n(\mathrm{tr}_{\bullet\leq k}C^\bullet) = \begin{cases} H^n(C^\bullet) & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

In fact, the natural map $\mathrm{tr}_{\bullet\leq k}C^\bullet \to C^\bullet$ induces isomorphism on the nth cohomology for $n \leq k$.

Let $A^\bullet$ and $B^\bullet$ be two complexes and f and g two morphisms between then. We say that f and g are *homotopic* if there is a sequence of maps $H^n \colon A^n \to B^{n-1}$:

$$\begin{array}{ccccccccc} & & \cdots & \longrightarrow & A^n & \xrightarrow{d^n} & A^{n+1} & \longrightarrow & \cdots \\ & & & \swarrow_{H^n} & \downarrow{\scriptstyle f-g} & \swarrow_{H^{n+1}} & & & \\ \cdots & \longrightarrow & B^{n-1} & \xrightarrow{d^{n-1}} & B^n & \longrightarrow & \cdots & & \end{array},$$

such that $f_n - g_n = H^{n+1} \circ d^n + d^{n-1} \circ H^n$. Homotopic maps f and g induce equal maps on cohomology:

(A.1) $$f_* = g_* \colon H^n(A^\bullet) \to H^n(B^\bullet) \text{ for homotopic } f \text{ and } g.$$

We denote by $\mathrm{Ht}(A^\bullet, B^\bullet)$ the subgroup of $\mathrm{Hom}(A^\bullet, B^\bullet)$ consisting of maps homotopic to zero. It is easy to see that the composition

$$\mathrm{Hom}(A^\bullet, B^\bullet) \times \mathrm{Hom}(B^\bullet, C^\bullet) \to \mathrm{Hom}(A^\bullet, C^\bullet)$$

sends $\mathrm{Ht}(A^\bullet, B^\bullet) \times \mathrm{Hom}(B^\bullet, C^\bullet)$ and $\mathrm{Hom}(A^\bullet, B^\bullet) \times \mathrm{Ht}(B^\bullet, C^\bullet)$ to $\mathrm{Ht}(A^\bullet, C^\bullet)$. This lets us define the *homotopy category* $K(\mathbf{A})$ of complexes as a quotient category of $C(\mathbf{A})$. The objects of $K(\mathbf{A})$ are the same as the objects of $C(\mathbf{A})$, namely the complexes in $\mathbf{A}$. The maps $\mathrm{Hom}_{K(\mathbf{A})}(A^\bullet, B^\bullet)$ are the *homotopy classes* of maps in $\mathrm{Hom}_{C(\mathbf{A})}(A^\bullet, B^\bullet)$. In other words, we set

$$Ob(K(\mathbf{A})) = Ob(C(\mathbf{A}))$$
$$\mathrm{Hom}_{K(\mathbf{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{C(\mathbf{A})}(A^\bullet, B^\bullet)/\mathrm{Ht}(A^\bullet, B^\bullet).$$

The hom group $\mathrm{Hom}_{K(\mathbf{A})}(A^\bullet, B^\bullet)$ is denoted more succinctly by $[A^\bullet, B^\bullet]$. The composition law in $K(\mathbf{A})$ is induced from that in $C(\mathbf{A})$, and thus we have a natural quotient functor from $C(\mathbf{A})$ to $K(\mathbf{A})$. We have full subcategories $K^+(\mathbf{A})$, $K^-(\mathbf{A})$

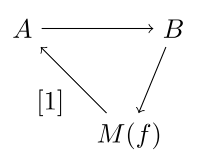
and $K^b(\mathbf{A})$ consisting of bounded below, bounded above and bounded complexes. By observation (A.1), we see that the functors H^k descend to $H^k\colon K(\mathbf{A}) \to \mathbf{A}$.

The category $K(\mathbf{A})$ is an additive category. It is, in general, not an abelian category even if $\mathbf{A}$ is one. However, it is more than just an additive category — it is a *triangulated category.* It has *distinguished triangles*, which are somewhat analogous to exact sequences. We now describe what these are.

We begin with the idea of the mapping cone. Let $A^\bullet$ and $B^\bullet$ be two objects and $f\colon A^\bullet \to B^\bullet$ a morphism in $C(\mathbf{A})$. Construct a complex $M(f)$, called the *mapping cone* of f by

$$M(f)^n = A^{n+1} \oplus B^n,$$
$$d^n_{M(f)}\colon (a,b) \mapsto (-d_A a, d_B b + fa).$$

We have the morphism $\alpha(f)\colon B \to M(f)$ given by the inclusion into the second factor and $\beta(f)\colon M(f) \to A[1]$ given by the projection on the second factor. The sequence of maps $A \to B \to M(f) \to A[1]$ is often pictured as a triangle



A *triangle* in $K(\mathbf{A})$ is a sequence of morphisms $A \to B \to C \to A[1]$. A map between two triangles $A \to B \to C \to A[1]$ and $X \to Y \to Z \to X[1]$ is a triad of maps $A \to X$, $B \to Y$ and $C \to Z$ commuting with the maps in the two triangles.

Definition A.4. *A triangle in $K(\mathbf{A})$ is called* distinguished *if it is isomorphic (in $K(\mathbf{A})$) to a triangle $A \to B \to M(f) \to A[1]$ for some $f\colon A \to B$ in $C(\mathbf{A})$.*

Proposition A.5. *A distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} Z$ induces an exact sequence in cohomology*

$$\cdots \to H^i(X) \xrightarrow{f} H^i(Y) \xrightarrow{g} H^i(Z) \xrightarrow{h} H^{i+1}(X) \to \cdots.$$

The collection of distinguished triangles in $K(\mathbf{A})$ satisfies the following properties ([KS90, 1.4.4]):

(TR1) A triangle isomorphic to a distinguished triangle is distinguished.

(TR2) For any object X, the triangle $X \to\xrightarrow{\text{id}} X \to 0 \to X[1]$ is distinguished.

(TR3) Any morphism $f\colon X \to Y$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR4) A triangle $X \xrightarrow{f} Y \to \mathbb{Z} \to X[1]$ is distinguished if and only if the triangle $Y \to Z \to X[1] \xrightarrow{f[1]} Y[1]$ is.

(TR5) Given two distinguished triangles $A \xrightarrow{\alpha} B \to C \to A[1]$ and $X \xrightarrow{\xi} Y \to Z \to X[1]$, and morphisms $f\colon A \to X$ and $g\colon B \to Y$ such that $g \circ \alpha = \xi \circ f$, there exists a map $h\colon C \to Z$ such that (f, g, h) gives a map of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X & \xrightarrow{\xi} & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

It turns out that the properties (TR1)–(TR5), along with another axiom called the *octahedral axiom* (which is too cumbersome to state), capture the essential properties of the class of distinguished triangles.

Definition A.6. [KS90, 1.5.1] *A triangulated category* $\mathbf{C}$ *is an additive category with an automorphism* $T\colon \mathbf{C} \to \mathbf{C}$ *and a class of distinguished triangles satisfying the axioms (TR1)–(TR5) and the octahedral axiom (with the shift* $[1]$ *replaced by* T*). A functor between two triangulated categories is called a functor of triangulated categories if it sends distinguished triangles to distinguished triangles.*

We are much more interested in the cohomology of complexes than the complexes themselves. Thus, we prefer $K(\mathbf{A})$ to $C(\mathbf{A})$ because we realize that what matters for cohomology is the homotopy classes of maps and not the maps themselves. However, it is advantageous to go one step further. To describe what that step is, we first introduce the notion of quasi isomorphisms.

Definition A.7. *We say that a map* $f\colon A^\bullet \to B^\bullet$ *(in* $C(\mathbf{A})$ *or* $K(\mathbf{A})$*) is a* quasi isomorphism *if the induced maps on cohomology* $f\colon H^i(A^\bullet) \to H^i(B^\bullet)$ *are isomorphisms for all* i*.*

It is easy to check that if $f\colon A \to B$ is a homotopy equivalence (i.e. an isomorphism in $K(\mathbf{A})$), then f is a quasi isomorphism. Not all quasi isomorphisms are homotopy equivalences.

From now on, let $\mathbf{A}$ denote an abelian category. One can construct from $K(\mathbf{A})$ a category $D(\mathbf{A})$ 'by adding the inverses of all quasi isomorphisms.' The construction is an instance of a general categorical operation called *localization* ([KS90, 1.6]). We will not go into the details of the construction of $D(\mathbf{A})$, but only say a few words about it. The objects of $D(\mathbf{A})$ are the same as objects of $C(\mathbf{A})$ and $K(\mathbf{A})$, namely the complexes in $\mathbf{A}$. A morphism in $D(\mathbf{A})$ from A to B is an equivalence class of diagrams (called 'roofs') of the form

$$A \overset{q}{\leftarrow} C \to B,$$

where q is a quasi isomorphism. Two roofs $A \leftarrow C \to B$ and $A \leftarrow C' \to B$ are equivalent if there is a third roof $A \leftarrow C'' \to B$ such that the following commutes

$$\begin{array}{ccccc} & \swarrow & C & \searrow & \\ & & \uparrow & & \\ A & \leftarrow & C'' & \rightarrow & B \\ & \nwarrow & \downarrow & \nearrow & \\ & & C' & & \end{array}$$

The salient properties of $D(\mathbf{A})$ are summarized in the following proposition.

Proposition A.8. *The derived category $D(\mathbf{A})$ is a triangulated category with a functor of triangulated categories $Q\colon K(\mathbf{A}) \to D(\mathbf{A})$ satisfying the following universal property: For all additive categories $\mathbf{B}$ and additive functors $F\colon K(\mathbf{A}) \to \mathbf{B}$, if F sends all quasi isomorphisms in $K(\mathbf{A})$ to isomorphisms in $\mathbf{B}$, then there exists a unique functor $Q_F\colon D(\mathbf{A}) \to \mathbf{B}$ such that $F = Q_F \circ Q$.*

The universal property implies that the cohomology functors $H^i\colon K(\mathbf{A}) \to \mathbf{A}$ factor through $D(\mathbf{A})$.

Isomorphisms in the derived category are sometimes called *weak isomorphisms.* We often abuse the term quasi isomorphisms to also mean weak isomorphisms. With this usage, a quasi isomorphism $A^\bullet \to B^\bullet$ need not come from an actual map of complexes $A^\bullet \to B^\bullet$.

One can construct the derived categories $D^*(\mathbf{A})$ from $K^*(\mathbf{A})$ for $* \in \{+, -, b\}$. It turns out that $D^*(\mathbf{A})$ is equivalent to the full subcategory of $D(\mathbf{A})$ consisting of objects from $K^*(\mathbf{A})$. Furthermore, $\mathbf{A}$ is equivalent to the full subcategory of $D(\mathbf{A})$ consisting of complexes A such that $A^n = 0$ for $n \neq 0$. In other words, we have

$$\mathrm{Hom}_{\mathbf{A}}(X, Y) = \mathrm{Hom}_{D(\mathbf{A})}(\cdots 0 \to X \to 0 \to \cdots, \cdots 0 \to Y \to 0 \cdots).$$

The derived category $D^+(\mathbf{A})$ takes a particularly simple form when $\mathbf{A}$ has 'enough injectives.' An object I of $\mathbf{A}$ is called *injective* if the functor $\mathrm{Hom}(-, I)$ is exact. The category $\mathbf{A}$ is said to have *enough injectives* if every object of $\mathbf{A}$ injects into an injective object.

Proposition A.9. [KS90, 1.7.10] *Let $\mathbf{A}$ be an abelian category with enough injectives and $\mathbf{I}$ be the full subcategory of $\mathbf{A}$ consisting of injective objects. Then, the natural functor $K^+(\mathbf{I}) \to D^+(\mathbf{A})$ is an equivalence of categories.*

It is convenient to have an analogous assertion in a more general setting. Let $\mathbf{C}$ be an abelian category and $\mathbf{J}$ a full additive subcategory such that

(1) every object in $\mathbf{C}$ injects into an object in $\mathbf{J}$;
(2) if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathbf{C}$ and A, B lie in $\mathbf{J}$, then C lies in J.

We call such a subcategory a *generating subcategory.* If $\mathbf{J}$ is a generating subcategory, then the map $D^+(\mathbf{J}) \to D^+(\mathbf{C})$ is an equivalence of categories [Noc, 1.4].

Having described derived categories, we come to derived functors. We focus on left exact functors and complexes bounded below. Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories and $F\colon \mathbf{A} \to \mathbf{B}$ an additive functor between them. Then F naturally induces a functor of triangulated categories $K^+(F)\colon K^+(\mathbf{A}) \to K^+(\mathbf{B})$. However, it need not do so between the derived categories. One can define a universal extension of it (in a precise sense, see [KS90, 1.8.1]) denoted by $RF\colon D^+(\mathbf{A}) \to D^+(\mathbf{B})$. The functor RF is called the *derived functor* of F. We do not go into the details of the universal property of the derived functor or its existence in general. We describe its construction in some special cases.

Assume that $\mathbf{A}$ is an abelian category with enough injectives, $\mathbf{B}$ an abelian category and $F\colon \mathbf{A} \to \mathbf{B}$ an additive, left exact functor. Let $\mathbf{I}$ be the full subcategory of $\mathbf{A}$ consisting of injective objects. Choose a quasi-inverse $q\colon D^+(\mathbf{A}) \to K^+(\mathbf{I})$. Define the *right derived functor* $RF\colon D^+(\mathbf{A}) \to D^+(\mathbf{B})$ as the composition

$$D^+(\mathbf{A}) \xrightarrow{q} K^+(\mathbf{I}) \xrightarrow{K^+(F)} K^+(\mathbf{B}) \to D^+(\mathbf{B}).$$

Concretely, the steps to compute RF of a complex $A^\bullet$ are the following. Choose a quasi isomorphism $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex of injectives. This is called taking an *injective resolution*. Then, we have $RF(A^\bullet) \cong F(I^\bullet)$.

One can define the derived functors in a more general setting. We say that F admits enough *F-injectives* if there is a generating subcategory $\mathbf{J}$ of $\mathbf{A}$ such that F restricted to $\mathbf{J}$ is exact. In such a case, the functor $K^+(F)\colon K^+(\mathbf{J}) \to D^+(\mathbf{B})$ sends quasi-isomorphisms to isomorphisms. Thus, it induces a functor $D^+(\mathbf{J}) \to D^+(\mathbf{B})$. The derived functor RF can now be constructed as the composition

$$D^+(\mathbf{A}) \xrightarrow{q} D^+(\mathbf{J}) \to D^+(\mathbf{B}), \tag{A.2}$$

where $q\colon D^+(\mathbf{A}) \to D^+(\mathbf{J})$ is a quasi-inverse of the equivalence $D^+(\mathbf{J}) \to D^+(\mathbf{A})$.

Definition A.10. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a left exact functor between abelian categories that admits enough F-injectives. Then, the n*th hypercohomology functor *$R^nF\colon D^+(A) \to B$ is defined by*

$$R^nF(A^\bullet) = H^n(RF(A^\bullet)).$$

An object A of $\mathbf{A}$ is called *F-acyclic* if $R^nF(A) = 0$ for $n > 0$. Suppose that F admits enough F-injectives. Let $\mathbf{J}$ be the full subcategory of $\mathbf{A}$ consisting of F-acyclic objects. Then $\mathbf{J}$ is a generating subcategory and F restricted to it is exact. Hence, we can use resolutions in $\mathbf{J}$ to compute the derived functors of F as described above.

Proposition A.11. [KS90, 1.8.7] *Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be abelian categories and $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ left exact, additive functors. Suppose both F and G admit enough injective objects and F sends F-injective objects to G-acyclic objects. Then $G \circ F$ admits enough injectives and we have*

$$R(G \circ F) = RG \circ RF.$$

For every object A in $\mathbf{A}$, there is a spectral sequence with $E_2^{p,q} = R^pG(R^qF(A))$ that abuts to $R^{p+q}(G \circ F)(A)$:

$$R^pG(R^qF(A)) \implies R^{p+q}(G \circ F)(A).$$

The spectral sequence in the proposition is called the Grothendieck spectral sequence.

We close this section with a brief discussion of bifunctors and their derived functors. Let $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}$ be three categories.

Definition A.12. *A bifunctor $F\colon \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}$ consists of the following data*

(1) A map of objects $F\colon Ob(\mathbf{C}_1) \times Ob(\mathbf{C}_2) \to Ob(\mathbf{C})$,
(2) for a pair of objects X_i, Y_i of $\mathbf{C}_i$ (for $i = 1, 2$), a map

$$F\colon \mathrm{Hom}_{\mathbf{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathbf{C}_2}(X_2, Y_2) \to \mathrm{Hom}_{\mathbf{C}}(F(X_1, X_2), F(Y_1, Y_2)),$$

such that for each object X in $\mathbf{C}_1$ (resp. Y in $\mathbf{C}_2$), the assignment $F(X, -)$ (resp. $F(-, Y)$) is a functor and the following commutes for all $X_1 \xrightarrow{f_1} Y_1$ in $\mathbf{C}_1$ and

$X_2 \xrightarrow{f_2} Y_2$ *in* $\mathbf{C}_2$*:*

$$\begin{array}{ccccc} & & F(X_1, X_2) & & \\ & \swarrow^{F(-,X_2)f_1} & & \searrow^{F(X_1,-)f_2} & \\ F(Y_1, X_2) & & & & F(X_1, Y_2) \\ & \searrow_{F(Y_1,-)f_2} & & \swarrow_{F(-,Y_2)f_1} & \\ & & F(Y_1, Y_2) & & \end{array} .$$

For example, the functor $\mathrm{Hom}(-,-)\colon \mathbf{C}^o \times \mathbf{C} \to \mathbf{Set}$ is a bifunctor. Other common example is $\otimes\colon \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$. A bifunctor F is called exact (resp. left exact, right exact), if $F(X,-)$ and $F(-,Y)$ are exact (resp. left exact, right exact) for all objects X in $\mathbf{C}_1$ and Y in $\mathbf{C}_2$.

Let $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}$ be abelian categories and $F\colon \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}$ a bifunctor. Let $A_i^\bullet$ be bounded below complexes in $K^+(\mathbf{C}_i)$. We construct the double complex $F(A_1^\bullet, A_2^\bullet)$, given by

$$F(A_1^\bullet, A_\cdot^\bullet)^{i,j} = F(A_1^i, A_2^j).$$

From the double complex, we construct the total complex $F(A_1^\bullet, A_2^\bullet)$ given by

$$Tot(F(A_1^\cdot, A_2^\bullet))^n = \oplus_{i+j=n} F(A_1^\bullet, A_2^\bullet)^{i,j}.$$

One can check that this gives a functor $K^+(F)\colon K^+(\mathbf{C}_1) \times K^+(\mathbf{C}_2) \to K^+(\mathbf{C})$. Under suitable conditions, one can construct a derived functor (satisfying a universal property, see [KS90, 1.10.4])

$$RF\colon D^+(\mathbf{C}_1) \times D^+(\mathbf{C}_2) \to D^+(\mathbf{C}).$$

One set of sufficient conditions is the following ([KS90, 1.10.8]):

(1) F is left exact;
(2) $\mathbf{C}_1$ has enough injectives;
(3) $F(I,-)$ is exact if $I \in Ob(\mathbf{C}_1)$ is injective.

Appendix B. Sheaf theory

In this section, we review some aspects of the theory of sheaves on a topological space. Most of the material can be found in [KS90]. A more simplified and concrete treatment is in [Ive86].

B.1. **Operations on sheaves.** Let X be a topological space. Denote by $\mathbf{Sh}(X)$ the category of sheaves of abelian groups on X. More generally, let R be a (commutative) ring and denote by $\mathbf{Sh}(X,R)$ the category of sheaves of R-modules on X. Then $\mathbf{Sh}(X,R)$ is an abelian category with enough injectives ([Ive86, II, 7.3]. We abbreviate $D^+(\mathbf{Sh}(X,R))$ by $D^+(X,R)$ and $D^+(\mathbf{Sh}(X))$ by $D^+(X)$. Although most of the definitions apply to an arbitrary ring R, we will sometimes specialize to the case of a field to avoid dealing with issues of flatness.

Let X and Y be topological spaces and $f\colon X \to Y$ a continuous map. For a sheaf $\mathscr{F}$ of R modules on X, the *direct image* or *pushforward* $f_*\mathscr{F}$ is a sheaf on Y given by

$$f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)).$$

The pushforward is a left exact functor $f_*\colon \mathbf{Sh}(X,R) \to \mathbf{Sh}(Y,R)$. The global sections functor Γ is the special case of the pushforward where Y is a point.

For a sheaf $\mathscr{G}$ of R modules on Y, the *inverse image* or *pullback* is the sheaf $f^*\mathscr{G}$ on X associated to the presheaf

$$f^*\mathscr{G}(U) = \varinjlim_{V\supset f(U)} \mathscr{G}(V).$$

The pullback is an exact functor $f^*\colon \mathbf{Sh}(Y,R) \to \mathbf{Sh}(X,R)$.

The functors f^* is left adjoint to f_*. In other words, for a sheaf $\mathscr{F}$ on X and $\mathscr{G}$ on Y we have a natural identification

$$\mathrm{Hom}(f^*\mathscr{G},\mathscr{F}) = \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}).$$

Let $i\colon Z \hookrightarrow X$ be the inclusion of a subspace and $\mathscr{F}$ a sheaf on X. We abbreviate $i^*\mathscr{F}$ by $\mathscr{F}|_Z$ and $\Gamma(Z, i^*\mathscr{F})$ by $\Gamma(Z,\mathscr{F})$. It is easily checked that this agrees with the existing notation in case of an open subset Z. If X is paracompact and $Z \subset X$ a closed subset then we have ([KS90, 2.5.1])

$$\varinjlim_{U\supset Z} \mathscr{F}(U) = \Gamma(Z,\mathscr{F}).$$

From now on, we restrict all the topological spaces to be Hausdorff, paracompact and locally compact. All the spaces considered in this paper satisfy these properties. In this case, we define an additional important functor called direct image with proper support.

Let X and Y be Hausdorff, paracompact and locally compact and $f\colon X \to Y$ be a continuous map. Recall that f is called *proper* if the inverse image under f of a compact subset of Y is compact. For a sheaf $\mathscr{F}$ on X, we define a subsheaf of $f_*\mathscr{F}$ called the *direct image with proper support* and denoted by $f_!\mathscr{F}$ as follows. For an open subset U of Y, we set

$$f_!\mathscr{F}(U) = \{s \in \Gamma(f^{-1}(U),\mathscr{F}) \mid f|_{\mathrm{Supp}(s)} \to Y \text{ is proper.}\}.$$

It can be checked that this assignment is a sheaf. Thus, we obtain a functor $f_!\colon \mathbf{Sh}(X,R) \to \mathbf{Sh}(Y,R)$, which is checked to be left exact. In the special case where Y is a point, $f_!$ is denoted by Γ_c. Note that $\Gamma_c(\mathscr{F})$ is simply the group of global sections of $\mathscr{F}$ that have compact support.

Let $Z \subset X$ be a closed subspace. Define the *sections of $\mathscr{F}$ supported on Z* as follows. For an open set $U \subset X$, set

$$\Gamma_Z(U,\mathscr{F}) = \{s \in \Gamma(U,\mathscr{F}) \mid \mathrm{Supp}(s) \subset Z\}.$$

The assignment $U \to \Gamma_Z(U,\mathscr{F})$ defines as subsheaf of $\mathscr{F}$ denoted by $\Gamma_Z(\mathscr{F})$. The functor $\Gamma_Z\colon \mathbf{Sh}(X,R) \to \mathbf{Sh}(X,R)$ is left exact.

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of R-modules on X. The tensor product $\mathscr{F} \otimes_R \mathscr{G}$ is defined as the sheaf associated to the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_R \mathscr{G}(U).$$

The hom sheaf $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is defined by

$$U \mapsto \mathrm{Hom}_{\mathbf{Sh}(U,R)}(\mathscr{F}|_U,\mathscr{G}|_U).$$

$\mathscr{H}om$ is a left exact bifunctor $\mathbf{Sh}(X,R)^o \times \mathbf{Sh}(X,R) \to \mathbf{Sh}(X,R)$. Likewise, $\otimes_R$ is a right exact bifunctor $\mathbf{Sh}(X,R) \times \mathbf{Sh}(X,R) \to \mathbf{Sh}(X,R)$. They satisfy the adjoint property

$$\mathrm{Hom}(\mathscr{F} \otimes_R \mathscr{G},\mathscr{H}) = \mathrm{Hom}(\mathscr{F},\mathscr{H}om(\mathscr{G},\mathscr{H})).$$

Since $\mathbf{Sh}(X, R)$ has enough injectives, one can construct the derived functors of all the left exact functors. Also, the category $\mathbf{Sh}(X, R)$ satisfies the conditions at the end of Appendix B, and hence we have the derived bifunctor

$$R\mathscr{H}om\colon D^-(X, R) \times D^+(X, R) \to D^+(X, R).$$

The existence of the right derived functors of $\otimes_R$ depends on the behavior of R. However, in the case that R is a field, the functor $\otimes_R$ is exact, and hence readily descends to the derived category $\otimes_R\colon D^+(X, R) \times D^+(X, R) \to D^+(X, R)$.

Definition B.1. *The derived functors $R^i\Gamma$ on $D^+(X, R)$ are called the* hypercohomology functors *and are denoted by $\mathbb{H}^i$.*

The derived functors $R^i\Gamma_c$ on $D^+(X, R)$ are called the hypercohomology functors with compact support *and are denoted by $\mathbb{H}^i_c$.*

For a sheaf $\mathscr{F}$ on X, we denote by $H^i(X, \mathscr{F})$ the hypercohomology $\mathbb{H}^i(\mathscr{F})$ and by $H^i_c(X, \mathscr{F})$ the hypercohomology $\mathbb{H}^i_c(X, \mathscr{F})$. Here $\mathscr{F}$ is seen as a complex concentrated in degree 0.

The Grothendieck spectral sequence (Proposition A.11) applied to the identity on $D^+(X, R)$ and Γ, gives the often used spectral sequence

$$H^p(X, H^q(\mathscr{A}^\bullet)) \implies \mathbb{H}^{p+q}(\mathscr{A}^\bullet). \tag{B.1}$$

Let $f\colon X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathscr{F}$ on X, we have $\Gamma(X, \mathscr{F}) = \Gamma(Y, f_*\mathscr{F})$. Hence, we have

$$R\Gamma(X, -) = R\Gamma(Y, -) \circ Rf_*. \tag{B.2}$$

In particular, for the inclusion $j\colon Z \hookrightarrow X$, we have

$$\mathbb{H}^i(X, j_*\mathscr{F}^\bullet) = \mathbb{H}^i(Z, \mathscr{F}^\bullet).$$

Analogous results hold for $f_!$ and Γ_c.

Definition B.2. *A sheaf $\mathscr{F}$ on X is called* soft *if the natural map*

$$\Gamma(X, \mathscr{F}) \to \Gamma(Z, \mathscr{F})$$

is surjective for all closed subsets $Z \subset X$.

The soft sheaves are acyclic with respect to the functors Γ, Γ_c, f_* and $f_!$.

Proposition B.3. *Let $A^\bullet$ be a bounded above complex of soft sheaves. Let F be one of the functors Γ, Γ_c, f_* or $f_!$. Then we have*

$$R^iF(A^\bullet) = H^i(FA^\bullet).$$

This follows either from the more general construction of the derived functors given in (A.2) or from the degeneration of the spectral sequence

$$E_1^{ij} = R^jF(A^i) \implies R^{i+j}F(A^\bullet).$$

B.2. **Poincaré-Verdier duality.** The sheaf theoretic machinery lets us generalize Poincaré duality for manifolds to a more general duality for finite dimensional locally compact spaces. Before we state the theorem, let us define what dimension means in this context.

Definition B.4. *(*[Ive86, III,9.4]*) Let X be a locally compact space. The dimension of X, denoted by* $\dim X$*, is the smallest integer n for which*

$$H_c^{n+1}(X, \mathscr{F}) = 0,$$

for all sheaves $\mathscr{F}$ on X.

One can check that $\dim \mathbb{R}^n = n$ and the (topological) dimension of a manifold is the same as the one in Definition B.4.

Let X be an n-dimensional locally compact space and k a field. Denote by $D^+(k)$ be the derived category of bounded below complexes of k-vector spaces[5].

Theorem B.5 (Verdier duality I)**.** *(*[Ive86, V.2]*) Let X be a locally compact finite dimensional space and k a field. The functor $R\Gamma_c \colon D^+(\mathbf{Sh}(X,k)) \to D^+(k)$ admits a right adjoint: There exists a complex $\mathbf{D}_X$ in $D^+(\mathbf{Sh}(X,k))$ such that for all complexes $\mathscr{A}^\bullet$ in $D^+(X,k)$, we have*

$$\mathrm{Hom}_{D^+(X,k)}(\mathscr{A}^\bullet, \mathbf{D}_X) = \mathrm{Hom}_{D^+(k)}(R\Gamma_c(\mathscr{A}^\bullet), k).$$

The complex $\mathbf{D}_X$ is called the *dualizing* complex. It can be represented by a complex of injective sheaves concentrated in degrees $-\dim X$ to 0. For a complex of sheaves $\mathscr{A}^\bullet$ in $D^b(X,k)$, the complex $R\mathscr{H}om(\mathscr{A}^\bullet, \mathbf{D}_X)$ is called the *Verdier dual* of $\mathscr{A}^\bullet$ and denoted by $\mathcal{D}_X\mathscr{A}^\bullet$. As a consequence of Verdier duality, we have the following.

Theorem B.6. *Let $\mathscr{A}^\bullet$ be an object of $D^b(X,k)$. We have*

$$\mathbb{H}^i(\mathcal{D}_X\mathscr{A}^\bullet) \cong \mathrm{Hom}(\mathbb{H}_c^{-i}(\mathscr{A}^\bullet), k).$$

The cohomology sheaves of $\mathbf{D}_X$ can be readily described.

Proposition B.7. *Let X be a finite dimensional locally compact space and $\mathbf{D}_X$ the dualizing complex on X. The cohomology sheaf $H^p(\mathbf{D}_X)$ is the sheaf associated to the presheaf*

$$U \mapsto \mathrm{Hom}(H_c^p(U, k_U), k).$$

In particular, if X is an oriented n dimensional manifold, then $\mathbf{D}_X \cong k_X[n]$. Therefore, we have $\mathcal{D}_X k_X = k_X[n]$. Taking $\mathscr{A}^\bullet = k_X$ in Theorem B.6, we obtain the classical Poincaré duality:

$$H^{n-i}(X,k) = \mathbb{H}^{-i}(k_X[n]) \cong \mathrm{Hom}(\mathbb{H}_c^i(k_X), k) = \mathrm{Hom}(H_c^i(X,k), k).$$

Theorem B.5 can be generalized to a relative situation.

Theorem B.8 (Verdier duality II)**.** *Let $f \colon X \to Y$ be a continuous map of finite dimensional locally compact spaces. Then $Rf_! \colon D^+(X,k) \to D^+(Y,k)$ admits a right adjoint $f^!$. In other words, for an object $\mathscr{F}^\bullet$ of $D^+(X,k)$ and $\mathscr{G}^\bullet$ of $D^+(Y,k)$ we have an isomorphism (natural in $\mathscr{F}^\bullet$, $\mathscr{G}^\bullet$):*

$$\mathrm{Hom}_{D^+(X,k)}(\mathscr{F}^\bullet, f^!\mathscr{G}^\bullet) \cong \mathrm{Hom}_{D^+(Y,k)}(Rf_!\mathscr{F}^\bullet, \mathscr{G}^\bullet).$$

The upper shriek functor can be described easily in the case of the inclusion $j \colon Z \hookrightarrow X$ of closed subset. In this case, we have $j^! = j^* R\Gamma_Z$.

We close the section by describing a distinguished triangle in $D^+(\mathbf{Sh}(X,R))$ that is used frequently. Let $j \colon Z \hookrightarrow X$ be the inclusion of a closed subset and $\mathscr{F}$ a sheaf

[5] This is just the homotopy category of bounded below complexes of k-vector spaces

on X. Set $U = X \backslash Z$ and let $i\colon U \hookrightarrow X$ be the inclusion. We have an exact sequence of sheaves on X:

$$0 \to \Gamma_Z(\mathscr{F}) \to \mathscr{F} \to i_*i^*\mathscr{F}. \tag{B.3}$$

The map $\mathscr{F} \to i_*i^*\mathscr{F}$ is surjective if $\mathscr{F}$ is an injective sheaf. Therefore, for a complex $\mathscr{F}^\bullet$ in $D^+(X, R)$, we have a distinguished triangle

$$\begin{array}{ccc} R\Gamma_Z(\mathscr{F}^\bullet) & \longrightarrow & \mathscr{F}^\bullet \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & Ri_*i^*\mathscr{F}^\bullet & \end{array}. \tag{B.4}$$

Using $j^*R\Gamma_Z = j^!$, the distinguished triangle above can be written as

$$\begin{array}{ccc} j_*j^!\mathscr{F}^\bullet & \longrightarrow & \mathscr{F}^\bullet \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & Ri_*i^*\mathscr{F}^\bullet & \end{array}. \tag{B.5}$$

Acknowledgements

I am grateful to Prof. Mike Hopkins for supervising my minor thesis. I thank him for discussing the material at length, asking key questions and suggesting computational examples. I am thankful to Prof. Joe Harris for listening to my explanation of the material and making valuable comments. I also thank Prof. Dennis Gaitsgory for a helpful discussion. Thanks to Rex Cheung for answering the many questions I had about sheaves, derived categories and Verdier duality.

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