A UNIVERSAL FORMULA FOR COUNTING CUBIC SURFACES

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Dedicated to our teacher, Joe Harris, on the occasion of his 70th birthday.

ABSTRACT. Using equivariant geometry, we find a universal formula that computes the number of times a general cubic surface arises in a family. As sample applications, we show that the PGL_4 orbit closure of a generic cubic surface has degree 96120, and that a general cubic surface arises 42120 times as a hyperplane section of a general cubic 3-fold.

1. INTRODUCTION

Cubic surfaces have fascinated geometers for centuries. Ever since Cayley and Salmon discovered the 27 lines, it has become something of a tradition to revisit the geometry of cubic surfaces using increasingly sophisticated tools and uncover more and more. Our contribution to this story is a universal answer to a varied collection of enumerative questions about cubic surfaces.

The isomorphism classes of cubic surfaces form a 4-dimensional moduli space. Therefore, we expect that in a 4-dimensional family a general isomorphism class should appear finitely many times. We offer a method for determining this number of occurrences. For example, one can ask: How many times does a general cubic surface appear, up to isomorphism, as

(A) a member of a general 4-dimensional linear system of cubic surfaces in $\mathbf{P}^3$?
(B) a hyperplane section of a general cubic threefold in $\mathbf{P}^4$?
(C) a blow up of 6 points in $\mathbf{P}^2$ varying in a general 4-dimensional linear system on a cubic curve?

Our universal formula answers all questions of this kind – the answers to these specific questions are 96120, 42120, and 25920, respectively.

To state our formula, we must fix a precise moduli space of cubics. Let us denote by $\mathcal{M}$ the GIT quotient of the set of semistable cubic surfaces in $\mathbf{P}^3$ by the action of PGL_4. By a *family of cubic surfaces*, we mean a flat, proper morphism whose geometric fibers are isomorphic to cubic surfaces in $\mathbf{P}^3$.

Theorem 1.0.1. *Let $\pi\colon \mathcal{X}\longrightarrow B$ be a good family of cubic surfaces over a proper base B. Let $\mathcal{V}$ denote the rank 4 vector bundle $\pi_*\left(\omega_\pi^{-1}\right)$, and let $v_i = c_i\left(\mathcal{V}\right)$ be its Chern classes. Assume that a general fiber of π is semistable, and let $\mu\colon B \dashrightarrow \mathcal{M}$ be the induced moduli map. Then,*

$$\deg \mu = 1080 \int_B \left(v_1^2 v_2 - v_1 v_3 + 9v_4\right). \qquad (1.1)$$

Definition 2.4.1 gives the precise meaning of a good family. This notion includes any family whose fibers are GIT-semistable or have a finite automorphism group.

1.1. Degree of the orbit closure.

Question (A) in § 1 fits in a broader ongoing story. Consider a hypersurface X in a projective space $\mathbf{P}V$ cut out by a homogeneous polynomial $F \in \mathrm{Sym}^d V^\vee$. In $\mathbf{P}\,\mathrm{Sym}^d V^\vee$, consider the orbit of $[F]$ under the action of $\mathrm{GL}(V)$, and let $\overline{\mathrm{Orb}}([F])$ be its closure. What is the degree of $\overline{\mathrm{Orb}}([F])$? To our knowledge, Enriques and Fano in 1897 were the first to address this question. They computed the degree for simple binary forms of all degrees [16]. In a series of seminal papers [3–5], Aluffi and Faber computed the degree of the orbit closure for *all* binary and ternary forms, finishing off the cases of points in $\mathbf{P}^1$ and curves in $\mathbf{P}^2$. The frontier case is that of cubic surfaces in $\mathbf{P}^3$. Theorem 1.0.1 applied to Question (A) settles it for general cubic surfaces—the degree of the orbit closure of a general cubic surface is 96120.

Question (A) appears as one of the "twenty-seven questions about the cubic surface" by Ranestad and Sturmfels [24]. In 2019, Brustenga i Moncusí, Timme, and Weinstein gave a "numerical proof" for the number 96120. In 2020, Cazzador and Skauli outlined an approach towards a proof by following the techniques of Aluffi and Faber [10, 12]. Our approach is completely different and novel, and the underlying change in perspective has the potential to solve other enumerative problems of a similar nature.

1.2. The equivariant class of the orbit closure.

The key idea is to consider not just the degree of an orbit closure, but rather the class of the orbit closure in the *equivariant Chow ring* $A^\bullet_{\mathrm{GL}\,V}(\mathrm{Sym}^d V^\vee)$. This equivariant class contains much more information than the degree of the orbit, yet, counter-intuitively, it is sometimes easier to find.

Let us explain how the equivariant class yields the degree. We have natural maps

$$A_{\mathrm{GL}\,V}^{\bullet}(\mathrm{Sym}^d\, V^{\vee}) \longrightarrow A_{\mathrm{GL}\,V}^{\bullet}(\mathrm{Sym}^d\, V^{\vee} \setminus \{0\}) \otimes \mathbf{Q}$$

$$\|$$

$$A_{\mathrm{PGL}\,V}^{\bullet}(\mathbf{P}\,\mathrm{Sym}^d\, V^{\vee}) \otimes \mathbf{Q} \longrightarrow A^{\bullet}(\mathbf{P}\,\mathrm{Sym}^d\, V^{\vee}) \otimes \mathbf{Q}.$$

Under the composite map, the class of $\overline{\mathrm{Orb}}(F) \in A_{\mathrm{GL}\,V}^{\bullet}(\mathrm{Sym}^d\, V^{\vee})$ is mapped to the class of $\overline{\mathrm{Orb}}([F]) \in A^{\bullet}(\mathbf{P}\,\mathrm{Sym}^d\, V^{\vee}) \otimes \mathbf{Q}$, which is simply the degree of $\overline{\mathrm{Orb}}([F])$.

So, what makes it possible to find the equivariant class? The ring $A_{\mathrm{GL}\,V}^{\bullet}(\mathrm{Sym}^d\, V^{\vee})$ is the polynomial ring in variables $c_1, \ldots, c_r$, where $r = \dim V$, and the variable c_i has degree i. The equivariant class of $\overline{\mathrm{Orb}}(F)$ lies in the graded component $A^m(\mathrm{Sym}^d\, V^{\vee})$, where $m = \dim \mathrm{Sym}^d\, V^{\vee} - \dim \mathrm{GL}\,V$. As a result, it can be expressed as a $\mathbf{Z}$-linear combination of monomials in c_i of total degree m. We can now use the method of undetermined coefficients. With quite some effort, we construct maps $B \longrightarrow [\mathrm{Sym}^d\, V^{\vee} / \mathrm{GL}\,V]$ where the pull-back of $\overline{\mathrm{Orb}}(F)$ as well as the classes c_i can be explicitly computed. Every such family gives a linear relationship between the coefficients in the expression of $\overline{\mathrm{Orb}}(F)$, which are to be determined. With sufficiently many families, we get enough information to find all the coefficients. For cubic surfaces, we construct more than enough 4-dimensional test families—there are 5 coefficients to be determined and we construct 8 families. The redundancy serves as extra confirmation of the end result.

Theorem 1.2.1. *Let V be a 4-dimensional vector space and let F be a general element of $\mathrm{Sym}^3\, V^{\vee}$. The equivariant class of the $\mathrm{GL}(V)$-orbit closure $\overline{\mathrm{Orb}}(F)$ of F in $A_{\mathrm{GL}\,V}^{\bullet}(\mathrm{Sym}^3\, V^{\vee})$ is given by*

$$[\overline{\mathrm{Orb}}\,F] = 1080(24c_1^4 + 12c_1^2 c_2 - 6c_1 c_3 + 9c_4),$$

where the $c_i = c_i(V)$ are the Chern classes of the tautological vector bundle V.

The formula in Theorem 1.2.1 appears to be different from the one in Theorem 1.0.1, but they are related by a simple change of variables; see § 4.2.

The technical heart of the paper is in the construction of *demonstrably* good families of cubic surfaces on which we can both calculate $\deg \mu$ and the Chern classes c_i. Because these test families are 4-dimensional, we encounter significant geometric challenges with some of the constructions. Just to give a teaser for our test families, we list their bases B here: $\mathbf{P}^4$, $\overline{M}_{0,7}$, a Hassett-weighted variant of $\overline{M}_{0,7}$, a blow up of the Hilbert scheme of two points on a quintic del Pezzo surface, and last but not least, the stack $B\mathbf{G}_m$. The curious reader can skip to § 3 for detailed constructions.

1.3. **The orbit closure problem.** To apply the method of undetermined coefficients, we have to ensure that no fibers of our family lie in the orbit closure of generic cubic surfaces. What is required is to prove statements of the form: "This particular cubic surface is not the flat limit of an isotrivial family of generic cubic surfaces." We develop a range of tools to address problems of this kind. To preserve the flow of the paper, and because these techniques are of independent interest, we assemble these tools in a separate section (§ 5).

1.4. **Further remarks.** The formula in Theorem 1.0.1 has some curious consequences.

Corollary 1.4.1. *If $\pi\colon \mathfrak{X}\longrightarrow B$ is a good family of cubic surfaces parametrized by a 4 dimensional proper variety B, then*

$$\int_B v_1^2 v_2 - v_1 v_3 + 9v_4 \geq 0$$

with equality holding if and only if a general cubic surface does not arise as a fiber.

Corollary 1.4.2. *Let $\pi\colon \mathfrak{X}\longrightarrow B$ be a good family of cubic surfaces parametrized by a 4 dimensional proper variety B. Assume that a general fiber of π is smooth and the induced map $\mu\colon B \dashrightarrow \mathcal{M}$ is dominant. Then the degree of μ is divisible by 1080.*

In particular, a moduli scheme of cubic surfaces that carries a *good* universal family must cover $\mathcal{M}$ by a map of degree divisible by 1080. Interestingly, the number 1080 is also the least common multiple of the orders of the finite groups that arise as automorphism groups of cubic surfaces. We do not know if 1080 is the minimal positive integer realizable as $\deg \mu$ as we vary across all good families of cubic surfaces.

1.5. **Future directions.** Theorem 1.2.1, and more importantly the method introduced in its proof, promises more enumerative vistas of exploration. Each *particular* cubic surface $F = 0$ registers its own equivariant orbit class $[\overline{\mathrm{Orb}}(F)]$. The methods developed in this paper have the potential to provide a *complete* understanding of how this class changes according to special changes in the geometry of $F = 0$, e.g. acquiring singularities or Eckardt points. We intend to conduct this detailed investigation in the future.

In a different direction, there should be analogous formulas counting Del Pezzo surfaces of degrees $1, 2$ and 4. We fully expect our methods to establish analogous counting formulas in these settings, with the case of degree 1 del Pezzo's being most challenging.

Finally, we add the equivariant version of a comment of Aluffi and Faber [3]: this paper computes the equivariant orbit class of a general vector in *one* representation of *one* group. A whole lot of work still needs to be done if we want a more general understanding, though the new equivariant flexibility will allow for further progress.

1.6. **Organization.** In § 2, we recall standard facts about cubic surfaces and their moduli. We also review the basic notions of equivariant Chow theory. The main new ingredient in this section is the definition of a good family (Definition 2.4.1). In § 3, we construct a number of test families of cubic surfaces and evaluate the linear relation on the coefficients of the class of the orbit closure given by each family. For each family, we first give an informal description. The first four families, taken up in the first four subsections, use variations on the theme of blowing up 6 points in $\mathbf{P}^2$. The next four families, taken up in the fifth subsection, are isotrivial; their base space is the stack $B\mathbf{G}_m$. Altogether, we produce 8 families for 5 undetermined coefficients. In § 4, we tie together the enumerative results and obtain the main theorems stated in the introduction. The last section, § 5 is devoted to developing methods to solve cases of the orbit closure problem.

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2. Preliminaries

In this chapter, we introduce the notion of a good family of cubic surfaces (Definition 2.4.1) and provide a tool (Proposition 2.5.5) to verify that a family is good. Before we do so, we first establish notation and recall some facts about cubic surfaces and their moduli.

2.1. **Notation and conventions.** Fix an algebraically closed field k of characteristic zero. In this paper, a 'scheme' means 'a scheme of finite type over k' and a 'point' means a 'k-point', unless explicitly stated otherwise. We do not distinguish between a vector bundle and the associated locally free sheaf of its sections.

Given a map of schemes $\pi\colon X \longrightarrow Y$ and a point $y \in Y$, we let X_y denote the fiber scheme $\pi^{-1}(y)$. If π is proper and Gorenstein, we write $\omega_{X/Y}$ or ω_π for the

dualizing line bundle. If X and Y are themselves Gorenstein and π is flat, then we have $\omega_{X/Y} = \omega_X \otimes \pi^* \omega_Y^{-1}$.

For a closed subscheme $Y \subset X$, the notation $\mathcal{I}_Y \subset \mathcal{O}_X$ denotes the ideal sheaf of $Y \subset X$ and if Y and X are smooth, then $N_{Y/X}$ denotes the normal bundle of $Y \subset X$.

Given a vector bundle $\mathcal{W}$, we let $\mathbf{P}\mathcal{W}$ be its projectivization. Contrary to Grothendieck's convention, $\mathbf{P}\mathcal{W}$ parametrizes 1-dimensional subspaces in the fibers of $\mathcal{W}$. In other words, we set

$$\mathbf{P}\mathcal{W} = \operatorname{Proj} \operatorname{Sym}(\mathcal{W}^\vee).$$

As a result, $\mathbf{P}\mathcal{W}$ comes equipped with a line bundle $\mathcal{O}(1)$ whose push-forward to the base is $\mathcal{W}^\vee$.

Let $Q \subset \mathbf{P}V$ be a non-degenerate quadric. The bilinear form associated to Q gives an isomorphism of $\mathbf{P}V$ with the dual projective space $\mathbf{P}V^\vee$. Given a point $p \in \mathbf{P}V$, we denote by $\operatorname{Polar}_Q(p) \subset \mathbf{P}V$ the hyperplane dual to p with respect to Q. Similarly, given a hyperplane $H \subset \mathbf{P}V$, we denote by $\operatorname{Pole}_Q(H) \in \mathbf{P}V$ the point dual to H with respect to Q. In almost all cases, we use these constructions when $\mathbf{P}V$ is $\mathbf{P}^2$.

2.2. **Moduli of cubic surfaces.** Let $\pi \colon \mathcal{X} \longrightarrow B$ be a family of cubic surfaces. That is, let π be a flat, proper morphism whose fibers are isomorphic to cubic hypersurfaces in $\mathbf{P}^3$. Since π is flat and its fibers are Gorenstein, it is a Gorenstein morphism, and hence admits a dualizing line bundle ω_π. By the adjunction formula, it follows that the restriction of ω_π^{-1} to every fiber is isomorphic to the restriction of $\mathcal{O}(1)$ with respect to an embedding of the fiber as a cubic hypersurface in $\mathbf{P}^3$. By standard theorems about cohomology and base-change, it follows that $R^i\pi_*(\omega_\pi^{-1})$ vanishes for $i > 0$, and is a vector bundle for $i = 0$. Set

$$\mathcal{V} = \pi_*\left(\omega_\pi^{-1}\right).$$

The natural map

$$\pi^*\mathcal{V} \longrightarrow \omega_\pi^{-1}$$

yields a closed embedding

$$\mathcal{X} \subset \mathbf{P}\mathcal{V}^\vee. \tag{2.1}$$

We know that

$$\omega_{\mathbf{P}\mathcal{V}^\vee/B} = \mathcal{O}(-4) \otimes \det \mathcal{V}^\vee,$$

and hence, by adjunction,

$$\omega_{\mathcal{X}/B} = \mathcal{O}(-4) \otimes \mathcal{O}(\mathcal{X}) \otimes \det \mathcal{V}^\vee|_{\mathcal{X}}.$$

By construction, we also have

$$\omega_{\mathfrak{X}/B} = \mathcal{O}(-1)|_{\mathfrak{X}},$$

and hence we obtain that

$$\mathcal{O}(\mathfrak{X}) = \mathcal{O}(3) \otimes \det \mathcal{V}. \tag{2.2}$$

In other words, $\mathfrak{X} \subset \mathbf{P}\mathcal{V}^\vee$ is the zero locus of a section of the line bundle $\mathcal{O}(3) \otimes \det \mathcal{V}^\vee$.

Consider the category $\mathscr{C}$ fibered in groupoids over the category of schemes whose objects over a scheme B are families of cubic surfaces $\pi \colon \mathfrak{X} \longrightarrow B$, and whose morphisms are pull-back diagrams. Consider the category $\mathscr{M}$ fibered in groupoids over the category of schemes whose objects over a scheme B are vector bundles $\mathcal{V}$ over B of rank 4 along with a nowhere zero section ξ of $\mathrm{Sym}^3 \mathcal{V}^\vee \otimes \det \mathcal{V}$, and whose morphisms are pull-back diagrams. Let V be a k-vector space of dimension 4. Note that $\mathscr{M}$ is simply the quotient stack

$$\mathscr{M} = \left[\mathrm{Sym}^3 V^\vee \otimes \det V \setminus \{0\} / \operatorname{GL} V\right].$$

Proposition 2.2.1. *The categories $\mathscr{C}$ and $\mathscr{M}$ are equivalent as categories fibered over* $\mathrm{Schemes}_k$.

Proof. Let $\pi \colon \mathfrak{X} \longrightarrow B$ be a family of cubic surfaces, and set $\mathcal{V} = \pi_*(\omega_\pi^{-1})$. Consider the relative canonical embedding $\mathfrak{X} \subset \mathbf{P}\mathcal{V}^\vee$. In this embedding, $\mathfrak{X}$ is cut out by a section of $\mathcal{O}(3) \otimes \det \mathcal{V}$ on $\mathbf{P}\mathcal{V}^\vee$, or equivalently, by a section of ξ of $\mathrm{Sym}^3 V^\vee \otimes \det \mathcal{V}$ on B. Since π is flat, ξ is nowhere zero on B. We thus have a natural transformation from $\mathscr{C}$ to $\mathscr{M}$.

To go from $\mathscr{M}$ to $\mathscr{C}$, given a rank 4 bundle $\mathcal{V}$ and a nowhere section ξ of $\mathrm{Sym}^3 \mathcal{V}^\vee \otimes \det \mathcal{V}$, define $\mathfrak{X} \subset \mathbf{P}\mathcal{V}^\vee$ as the zero-locus of the corresponding section of $\mathcal{O}(3) \otimes \det \mathcal{V}$. $\qquad\square$

We call $\mathscr{M}$ the *moduli stack of cubic surfaces*. Sometimes, it will be convenient to add in the zero section and consider the bigger stack

$$\mathscr{M}^* = \left[\mathrm{Sym}^3 V^\vee \otimes \det V / \operatorname{GL} V\right].$$

Note that the diagonal $\mathbf{G}_m \subset \operatorname{GL} V$ acts by weight -1 on $\mathrm{Sym}^3 V^\vee \otimes \det V$. In particular, its action is free. Therefore, the stack $\mathscr{M}$ is isomorphic to the quotient stack

$$\mathscr{M} = \left[\mathbf{P}\,\mathrm{Sym}^3 V^\vee / \operatorname{PGL} V\right].$$

We let $\mathcal{M}$ denote the GIT quotient of $\mathbf{P}\operatorname{Sym}^3 V^\vee$ by the action of $\operatorname{PGL} V$. The semistable and stable locus of the action were computed by Mumford; see [22, Chapter 4, §2] or [23, 1.14]. The stable locus consists of surfaces with at worst A_1 singularities, and the semistable locus includes surfaces with at worst A_2 singularities. The ring of invariants was computed much earlier, in the 1860s, by Salmon [27] and Clebsch [13, 14], and later by Beklemishev [7] using more modern methods. The invariant ring is freely generated by polynomials of degree 8, 16, 24, 32, and 40, and as a result, the coarse moduli space $\mathcal{M}$ is isomorphic to the weighted projective space $\mathbf{P}(1, 2, 3, 4, 5)$.

We recall perhaps the most famous fact about cubic surfaces, discovered in 1849.

Theorem 2.2.2 (Cayley [11], Salmon [26]). *Every smooth cubic surface contains exactly* 27 *lines.*

As we know, the 27 lines are key to understanding geometry of the cubic surface. Each line is a (-1)-curve, and hence contractible by Castelnuovo's theorem. A set of lines that are pairwise skew is particularly important, because they can be blown down simultaneously. The maximum number of such lines is 6, and they allow us to realize the cubic surface as a blow up of $\mathbf{P}^2$ at 6 points.

Following Schläfli [28], we call an ordered tuple of pairwise disjoint six lines $(\ell_1, \ldots, \ell_6)$ on a cubic surface a *six*. A *double six* is an unordered pair of sixes $\{(\ell_1, \ldots, \ell_6), (\ell'_1, \ldots, \ell'_6)\}$ such that

  – ℓ_i and ℓ'_i are disjoint for all i, and
  – ℓ_i and ℓ'_j meet at a point whenever $i \neq j$.

Theorem 2.2.3 (Schläfli [28]). *Each six on a smooth cubic surface has a unique extension to a double-six, and every smooth cubic surface contains* 36 *double-sixes.*

Thus, each smooth cubic surface admits 72 distinct maps to $\mathbf{P}^2$, up to automorphisms of $\mathbf{P}^2$, each of which blows down 6 lines.

The sixes on a cubic surface allow us to relate $\mathcal{M}$ to a more familiar moduli space. Let $\mathcal{M}^\dagger$ denote the moduli space of pairs (S, ℓ), where S is a smooth cubic surface and ℓ is a six on S. By Theorem 2.2.3, the natural forgetful map

$$\sigma \colon \mathcal{M}^\dagger \longrightarrow \mathcal{M}$$

$$(S, \ell) \mapsto S$$

is dominant and quasi-finite of degree

$$\deg \sigma = 72 \times 6! = 51840.$$

On the other hand, the space $\mathcal{M}^\dagger$ is isomorphic to the configuration space of 6 ordered points on $\mathbf{P}^2$

$$\mathcal{M}^\dagger \xrightarrow{\sim} \left((\mathbf{P}^2)^6 \setminus \text{Diagonals}\right) / \operatorname{Aut}(\mathbf{P}^2).$$

The isomorphism sends $(S, (\ell_i))$ to $(\mathbf{P}^2, (p_i))$, where $\beta \colon S \longrightarrow \mathbf{P}^2$ blows down the lines ℓ_i and $p_i = \beta(\ell_i)$.

2.3. **Equivariant Chow groups and equivariant orbits.** The standard reference for equivariant Chow theory is [15]. We recall a few relevant notions, mostly to set notation.

Given an algebraic space X with an action of an algebraic group G, denote by $A_G^i(X)$ the G-equivariant Chow group of codimension i cycles. Given a G-invariant subspace $Y \subset X$ of pure codimension i, denote by $[Y] \in A_G^i(X)$ the fundamental class of Y. If $Y \subset X$ has codimension c and $c > i$, then the restriction map

$$A_G^i(X) \longrightarrow A_G^i(X \setminus Y)$$

is an isomorphism; in general, it is surjective with kernel is $A_G^{i-c}(Y)$. A G-linearized vector bundle V on X admits equivariant Chern classes $c_i(V) \in A_G^i(X)$. The equivariant Chow groups are homotopy invariant: if $Y \longrightarrow X$ is a G-equivariant vector bundle, then the pull-back map $A_G^i(X) \longrightarrow A_G^i(Y)$ is an isomorphism. If X is smooth, then $\bigoplus_i A_G^i(X) = A_G^*(X)$ is a graded ring, where the multiplication is given by the intersection product.

The G-equivariant Chow groups of X are the same as the Chow groups of the quotient stack $[X/G]$:

$$A^i\left([X/G]\right) = A_G^i(X).$$

The equivariant Chow ring of a point with the action of $G = \operatorname{GL}_n$ is the polynomial ring

$$A_{\operatorname{GL}_n}(\bullet) = \mathbf{Z}[c_1, \ldots, c_n],$$

where c_i has degree i and can be identified with the i-th Chern class of the standard n-dimensional representation of GL_n. By homotopy invariance, for any GL_n representation W, we have

$$A_{\operatorname{GL}_n}(W) = \mathbf{Z}[c_1, \ldots, c_n].$$

If the group under consideration is $\operatorname{GL} V$, where V is an n-dimensional vector space, then we write v_i for the i-th Chern class of the tautological representation V. Then we have

$$A_{\operatorname{GL} V}(W) = \mathbf{Z}[v_1, \ldots, v_n].$$

Let V be a 4-dimensional vector space. Consider the $\mathrm{GL}\,V$ representation $W = \mathrm{Sym}^3 V^\vee \otimes \det V$. By Proposition 2.2.1, (non-zero) $\mathrm{GL}\,V$ orbits in W correspond canonically to cubic surfaces. Note that the action of $\mathrm{GL}\,V$ on W is generically free, and hence the codimension of a generic orbit is $20 - 16 = 4$. Given a point $w \in W$, let $\overline{\mathrm{Orb}}(w)$ denote the closure of its $\mathrm{GL}\,V$ orbit, and let

$$[\overline{\mathrm{Orb}}(w)] \in A^4_{\mathrm{GL}\,V}(W) = A^4(\mathscr{M})$$

denote the fundamental class. By a slight abuse of notation, if $X \subset \mathbf{P}^3$ is a cubic surface or $F \in k[X_0, \ldots, X_3]$ is a non-zero cubic form, we let $\overline{\mathrm{Orb}}(X)$ or $\overline{\mathrm{Orb}}(F)$ denote the closure of the orbit in W of the corresponding point w, and $[\overline{\mathrm{Orb}}(X)]$ or $[\overline{\mathrm{Orb}}(F)]$ the fundamental class. Since $\mathbf{A}^4(\mathscr{M})$ is generated by monomials in v_i of total degree 4, we have an expression

$$[\overline{\mathrm{Orb}}(X)] = a_{1^4} \cdot v_1^4 + a_{1^2 \cdot 2} \cdot v_1^2 v_2 + a_{1 \cdot 3} \cdot v_1 v_3 + a_{2^2} \cdot v_2^2 + a_4 \cdot v_4, \qquad (2.3)$$

where the coefficients a_i are integers. The coefficients (2.3) depend on X. We let

$$\partial\,\overline{\mathrm{Orb}}(X) = \overline{\mathrm{Orb}}(X) \setminus \mathrm{GL}\,V \cdot X,$$

denote the boundary of the orbit closure. Note that $\partial\,\overline{\mathrm{Orb}}(X) \subset W$ has codimension at least 5.

2.4. **Good families.** Our strategy is to compute $[\overline{\mathrm{Orb}}(X)]$ for a generic X by pulling it back to various families $B \longrightarrow \mathscr{M}$. If there is a point of B that lies in every $\overline{\mathrm{Orb}}(X)$, then we have the issue of excess intersection. A good family is one that does not suffer from this defect. Recall that V is a 4-dimensional k-vector space.

Definition 2.4.1 (Good family). A family of cubic surfaces $\pi \colon \mathcal{X} \longrightarrow B$ is *good* if the following holds: there exists a non-empty Zariski open subset $U \subset \mathrm{Sym}^3 V^\vee$ such that for every $F \in U$ and $b \in B$, the fiber $\mathcal{X}_b$ does not lie in $\partial\,\overline{\mathrm{Orb}}(F)$.

Example 2.4.2 (Fibers with finite automorphism groups). Suppose every $\mathcal{X}_b$ has a finite automorphism group. Then $\pi \colon \mathcal{X} \longrightarrow B$ is good. Indeed, let U be the subset consisting of elements with finite automorphism group. For every $F \in U$, the boundary $\partial\,\overline{\mathrm{Orb}}(F)$ consists of elements with a positive dimensional automorphism group, and hence it cannot contain $\mathcal{X}_b$.

Example 2.4.3 (GIT semistable fibers). Suppose every fiber $\mathcal{X}_b$ is GIT-semistable. Then the family $\pi \colon \mathcal{X} \longrightarrow B$ is good. Indeed, let U be the subset consisting of GIT-stable elements. For every $F \in U$, the orbit of F is closed in the semistable locus, and hence $\partial\,\overline{\mathrm{Orb}}(F)$ cannot contain $\mathcal{X}_b$.

Example 2.4.4 (A combination). Suppose every fiber $\mathcal{X}_b$ has a finite automorphism group or is GIT semistable. Let U be the subset corresponding to smooth cubic surfaces. Since smooth cubic surfaces have a finite automorphism group and are GIT stable, we see that $\pi\colon \mathcal{X}\longrightarrow B$ is good.

Proposition 2.4.5. *Let $\pi\colon \mathcal{X}\longrightarrow B$ be a good family of cubic surfaces. Assume that B is proper of dimension 4 and a general fiber $\mathcal{X}_b$ is semistable. Let $\mu\colon B\longrightarrow \mathcal{M}$ be the map induced by π and $m\colon B \dashrightarrow \mathcal{M}$ the rational map to the coarse moduli space. Then*

$$\deg m = \deg \mu^* \left(\overline{\mathrm{Orb}}(F) \right)$$

for a general cubic form F.

Proof. There exists an open $W \subset \mathcal{M}$ such that $m^{-1}(W)\longrightarrow W$ is a finite étale morphism. Then the degree of m is simply the number of points in $m^{-1}(u)$ for $u \in W$. By shrinking W if necessary, we may assume that the map

$$\mathcal{M}^{ss}\longrightarrow \mathcal{M}$$

is an isomorphism over W.

Let U be as in the definition a good family. Then, for every $u \in U$, we have the equality

$$\mathcal{X} \times_{\mathcal{M}} u = \mathcal{X} \times_{\mathcal{M}} \overline{u}.$$

By shrinking U if necessary, we may assume that U consists of semistable points and maps to $W \subset \mathcal{M}$ in the coarse space. Then, for every $u \in U$, we have

$$\begin{aligned}
\deg \mu^* \, \overline{\mathrm{Orb}}(u) &= \#\left(\mathcal{X} \times_{\mathcal{M}} u \right) \\
&= \#\left(\mathcal{X} \times_{\mathcal{M}} u \right) \\
&= \deg m.
\end{aligned}$$

$\square$

2.5. Cubic surfaces as blow-ups.

Cubic surfaces arise as blow ups of $\mathbf{P}^2$ at six points, or more generally, as blow ups of del Pezzo surfaces of degree d in $(d-3)$ points. Varying the center of blow up is a convenient source of families of cubic surfaces. The following proposition identifies certain nice sub-schemes to blow up.

Definition 2.5.1 (Admissible subscheme). Let S be a smooth del Pezzo surface of degree $d \geq 3$ and let $Z \subset S$ be a subscheme of length $(d-3)$. We say that Z is *admissible* if

    (1) Z is curvilinear (contained in a smooth curve),

(2) $h^0\left(\mathfrak{I}_Z \otimes \omega_S^{-1}\right) = 4$, and
(3) $\mathfrak{I}_Z \otimes \omega_S^{-1}$ is generated by its global sections.

The last condition means that Z is scheme-theoretically cut out by the sections of ω_S^{-1} that vanish on it.

Proposition 2.5.2. *Let $Z \subset S$ be an admissible length $(d-3)$ subscheme. Set $X = \mathrm{Bl}_Z\, S$. Then the following hold.*

(1) *X has at worst A_n singularities.*
(2) *$H^0\left(X, \omega_X^{-1}\right)$ is 4-dimensional and base-point free.*
(3) *$H^i(X, \omega_X^{-n}) = 0$ for all $n \geq 0$ and $i > 0$.*
(4) *The image of the map $X \longrightarrow \mathbf{P}^3$ induced by ω_X^{-1} is a cubic surface Y; the map $X \longrightarrow Y$ is birational.*
(5) *Let $S = \bigoplus_{n \geq 0} S_n$ be the homogeneous coordinate ring of Y. The pull-back map $S_n \longrightarrow H^0\left(X, \omega_X^{-n}\right)$ is an isomorphism.*
(6) *Y is normal and has rational double point (ADE) singularities.*

Proof. Let $\pi\colon X \longrightarrow S$ be the blow-up map. Since Z is curvilinear, X has at worst A_n singularities. Indeed, if Z is isomorphic to $\mathrm{Spec}\, k[t]/t^{n+1}$ at a point z in its support, then X is singular at a unique point of $\pi^{-1}(z)$; this singularity is of type A_n. Note that $E_z := \pi^{-1}(z)$ (taken with the reduced structure) is isomorphic to $\mathbf{P}^1$.

Let $E \subset X$ be the exceptional divisor, namely, the Cartier divisor defined by the ideal $\pi^{-1}\mathfrak{I}_Z$. Over the point z as above, E has multiplicity n along the component E_z. Furthermore, we have $(nE_z)^2 = -n$, and hence $E^2 = -(d-3)$. Also, see that we have

$$\omega_X = \pi^*\omega_S \otimes \mathcal{O}_X(E), \tag{2.4}$$

and therefore,

$$\omega_X^{-1} = \pi^*\omega_S^{-1} \otimes \mathcal{O}_X(-E).$$

By applying π_*, we get

$$\pi_*\left(\omega_X^{-1}\right) = \mathfrak{I}_Z \otimes \omega_S^{-1}.$$

By our assumption, $H^0\left(\mathfrak{I}_Z \otimes \omega_S^{-1}\right)$ is 4-dimensional and its base-locus is precisely Z. Hence $H^0\left(X, \omega_X^{-1}\right)$ is 4-dimensional and base-point free.

For the vanishing of higher cohomology, we induct on n. Since X is a rational surface with rational singularities, we have $h^1(\mathcal{O}_X) = h^1(\omega_X) = 0$. Let $C \subset X$ be the zero locus of a general section of ω_X^{-1}. By Bertini's theorem, C is smooth and by the adjunction formula, it has arithmetic genus 1. Taking global sections in the sequence

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

shows that $h^0(\mathcal{O}_C) = 1$, and hence C is connected. Twisting the above sequence by ω_X^{-n} gives

$$0 \longrightarrow \omega_X^{-n+1} \longrightarrow \omega_X^{-n} \longrightarrow \omega_X^{-n}|_C \longrightarrow 0.$$

The restriction $\omega_X^{-n}|_C$ has degree $3n$, which is positive if n is positive, and hence $h^i\left(\omega_X^{-n}|_C\right) = 0$ for $i > 0$. By applying the long exact sequence in cohomology, we see that if $i > 0$, then $h^i(\omega_X^{-n+1}) = 0$ implies $h^i(\omega_X^{-n}) = 0$. By induction on n, we conclude $h^i(\omega_X^{-n}) = 0$ for all $n \geq 0$ and $i > 0$.

To get the degree of the image of $X \longrightarrow \mathbf{P}^3$, we compute the self-intersection $c_1(\omega_X)^2$. Using (2.4) and $E^2 = -(d-3)$, we compute

$$c_1(\omega_X)^2 = c_1(\omega_S)^2 - d + 3 = 3.$$

Thus, either the image is a hyperplane Π and the map $X \longrightarrow \Pi$ has degree 3, or the image is a cubic surface Y and the map $X \longrightarrow Y$ is birational. We rule out the first possibility since the map $X \longrightarrow \mathbf{P}^3$ is defined by linearly independent sections, and hence does not factor through a hyperplane.

By Riemann–Roch, we get that $\chi(X, \omega_X^{-n}) = \chi(Y, \mathcal{O}(n))$. We also know that for $i > 0$ and $n \geq 0$, both $h^i(X, \omega_X^{-n})$ and $h^i(Y, \mathcal{O}(n))$ vanish. Hence, we get $h^0(X, \omega_X^{-n}) = h^0(Y, \mathcal{O}(n))$. Recall that, the n-th graded component S_n of the homogeneous coordinate ring of Y is precisely $H^0(Y, \mathcal{O}(n))$. Since $X \longrightarrow Y$ is surjective, the natural map $S_n \longrightarrow H^0(X, \omega_X^{-n})$ is injective. Since the dimensions of the source and the target are equal, it must be an isomorphism.

From the isomorphism $S \cong \bigoplus_{n \geq 0} H^0(X, \omega_X^{-n})$, we get

$$Y = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0\left(X, \omega_X^{-1}\right)\right).$$

Since X is normal, so is Y. It is known that an irreducible, normal, rational cubic surface, such as Y, only has rational double point (ADE) singularities [9]. $\qquad\square$

Definition 2.5.3 (Associated cubic surface). Let S be a smooth del Pezzo surface of degree d and let $Z \subset S$ be an admissible subscheme of length $d-3$. The *cubic surface associated to $Z \subset S$*, denoted by $X_Z \subset \mathbf{P}^3$, is the image of the map $\mathrm{Bl}_Z S \longrightarrow \mathbf{P}^3$ induced by the anti-canonical linear system $H^0\left(\mathrm{Bl}_Z S, \omega_{\mathrm{Bl}_Z S}^{-1}\right)$.

By Proposition 2.5.2, we may equivalently define X_Z as

$$X_Z = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0\left(\mathrm{Bl}_Z S, \omega_{\mathrm{Bl}_Z S}^{-n}\right)\right).$$

Let us make the construction $Z \rightsquigarrow X_Z$ in families. Let B be a scheme, $P \longrightarrow B$ a smooth proper family of del-Pezzo surfaces and $Z \subset P$ a closed subscheme, flat over B, whose fibers are admissible subschemes. Consider the blowup $\mathrm{Bl}_Z P$. Note that $Z \subset P$ is a local complete intersection, and hence the blow-up is B-flat and commutes with arbitrary base change. In particular, the fiber $(\mathrm{Bl}_Z P)_b$ is the blow up of P_b along Z_b. (The statement about blow-ups and base-change must be well-known, but we could not find a reference so we include a proof; see Proposition 2.5.4.) The family of cubic surfaces $X_Z \longrightarrow B$ associated to $Z \subset P \longrightarrow B$ is defined by

$$X_Z = \mathrm{Proj}_B \left(\bigoplus_{n \geq 0} \pi_* \left(\mathrm{Bl}_Z P, \omega_{\mathrm{Bl}_Z P/B}^{-n} \right) \right).$$

By Proposition 2.5.2, we see that X_Z is flat over B and its fiber over $b \in B$ is the cubic surface associated to $Z_b \subset P_b$, as in Definition 2.5.3.

Proposition 2.5.4. *Let* $\pi \colon P \longrightarrow B$ *be a smooth morphism and* $Z \subset P$ *a closed subscheme flat and lci over* B. *Then* $\mathrm{Bl}_Z P \longrightarrow B$ *is flat and for any* $C \longrightarrow B$, *the canonical map*

$$\mathrm{Bl}_{Z_C} P_C \longrightarrow (\mathrm{Bl}_Z P)_C.$$

is an isomorphism.

Proof. The statement is local on P. After passing to a suitable affine open, we are reduced to the following situation in algebra. We have a Noetherian ring B and a finitely generated smooth B-algebra A. We have an ideal $I \subset A$ such that A/I is B-flat and I is generated by a regular sequence. The blow-up is the Proj of the Rees algebra

$$R_I(A) = \bigoplus_{n \geq 0} I^n.$$

Since $I \subset A$ is generated by a regular sequence, the associated graded ring $\mathrm{gr}\, R_I(A)$ is a polynomial algebra over A/I, and hence flat over B. It follows that $R_I(A)$ is flat over B, and hence its Proj is flat over $\mathrm{Spec}\, B$.

For the statement about base-change, we show that the formation of the Rees algebra commutes with base-change. For a B-algebra C, let $J \subset A \otimes_B C$ be the image of $I \otimes_B C$. Consider the map

$$i \colon R_I(A) \otimes_B C \longrightarrow R_J(C).$$

We use again that the associated graded rings are polynomial algebras, on which the map induced by i is clearly an isomorphism. Hence i is an isomorphism. $\square$

Let $Z \subset \mathbf{P}^2$ be an admissible subscheme of length 6. An automorphism of the pair $(\mathbf{P}^2, Z)$ induces an automorphism of $\mathrm{Bl}_Z \mathbf{P}^2$ and hence an automorphism of the associated cubic surface X_Z.

Proposition 2.5.5. *Let $Z \subset \mathbf{P}^2$ be an admissible, length 6 subscheme and $X_Z \subset \mathbf{P}^3$ its associated cubic surface. The map*

$$\mathrm{Aut}(\mathbf{P}^2, Z) \longrightarrow \mathrm{Aut}(X_Z)$$

is injective and its image has finite index. In particular, $\mathrm{Aut}(\mathbf{P}^2, Z)$ is finite if and only if $\mathrm{Aut}(X_Z)$ is.

Proof. Since we have a birational isomorphism $\mathbf{P}^2 \dashrightarrow X_Z$, a non-trivial automorphism of $\mathbf{P}^2$ cannot induce a trivial automorphism of X_Z. Hence the map of automorphism groups is injective.

To show that the image has finite index, let $\tau \colon \widetilde{Y} \longrightarrow \mathrm{Bl}_Z \mathbf{P}^2$ be the minimal desingularization and set $Y = X_Z$. Since $\mathrm{Bl}_Z \mathbf{P}^2$ has only A_n singularities and such singularities are canonical, we conclude that

$$\tau^* \omega_{\mathrm{Bl}_Z \mathbf{P}^2} = \omega_{\widetilde{Y}}.$$

Consider the composite μ of

$$\widetilde{Y} \xrightarrow{\tau} \mathrm{Bl}_Z \mathbf{P}^2 \xrightarrow{\kappa} Y.$$

Note that

$$\mu^* \omega_Y = \omega_{\widetilde{Y}}.$$

We claim that $\mu \colon \widetilde{Y} \longrightarrow Y$ is the minimal desingularization of Y. To see this, let $Y' \longrightarrow Y$ be the minimal desingularization. Since $\widetilde{Y}$ is non-singular, the birational morphism $\mu \colon \widetilde{Y} \longrightarrow Y$ factors as a composite

$$\widetilde{Y} \xrightarrow{\alpha} Y' \longrightarrow Y.$$

Since α is a birational morphism between non-singular surfaces, it is a composite of blow-ups at points. In particular, if it is not an isomorphism, then there exists a (-1)-curve $E \subset \widetilde{Y}$ contracted by α. But then we have

$$E \cdot \omega_{\widetilde{Y}} = E \cdot \mu^* \omega_Y = 0,$$

which contradicts the adjunction formula

$$\left(\omega_{\widetilde{Y}} + E \right) \cdot E = -2.$$

Thus, we deduce that α is an isomorphism, and hence $\widetilde{Y} \longrightarrow Y$ is the minimal desingularization of Y.

Since every automorphism of a surface lifts uniquely to its minimal desingulariza-
tion, we have an injective map

$$\mathrm{Aut}(\mathbf{P}^3, X_Z) \longrightarrow \mathrm{Aut}(\widetilde{Y}).$$

It suffices to show that the image of $\mathrm{Aut}(\mathbf{P}^2, Z)$ has finite index in $\mathrm{Aut}(\widetilde{Y})$.

Let $H \in \mathrm{Pic}(\widetilde{Y})$ be the pullback of $\mathcal{O}(1)$ under the map $\widetilde{Y} \longrightarrow \mathbf{P}^2$. We claim that
the orbit of H under $\mathrm{Aut}(\widetilde{Y})$ is finite. To see this, note that we have

$$\omega_{\widetilde{Y}}^2 = 3 \text{ and } \omega_{\widetilde{Y}} \cdot H = -3,$$

and hence $(\omega_{\widetilde{Y}} + H)$ is orthogonal to $\omega_{\widetilde{Y}}$ under the intersection pairing. Since $\omega_{\widetilde{Y}}$
has positive self-intersection, its orthogonal complement $\omega_{\widetilde{Y}}^{\perp} \subset \mathrm{Pic}(\widetilde{Y}) = \mathrm{NS}(\widetilde{Y})$ is
a negative definite lattice by the Hodge index theorem. Thus, it has only finitely
many elements of a given self-intersection. Any automorphism of $\widetilde{Y}$ preserves $\omega_{\widetilde{Y}}$
and the self-intersection number. Therefore, under $\mathrm{Aut}(\widetilde{Y})$, the orbit of $\omega_{\widetilde{Y}} + H$ and
hence the orbit of H, must be finite.

Since the orbit of H is finite, its stabilizer is a finite index subgroup. We claim
that this subgroup is precisely the image of $\mathrm{Aut}(\mathbf{P}^2, Z)$. Let $\tau \colon \widetilde{Y} \longrightarrow \widetilde{Y}$ be an
automorphism fixing H. The map $\beta \colon \widetilde{Y} \longrightarrow \mathbf{P}^2$ is given by the complete linear system
$|H|$. Hence τ induces an automorphism $\tau' \colon \mathbf{P}^2 \longrightarrow \mathbf{P}^2$. It remains to show that τ'
preserves Z. Note that we have

$$\beta_* \left(\omega_{\widetilde{Y}}^{-1} \right) = \mathcal{O}(3) \otimes \mathcal{I}_Z,$$

and hence

$$\beta_* \left(\omega_{\widetilde{Y}}^{-1}(-3H) \right) = \mathcal{I}_Z.$$

Therefore, the induced map τ' preserves $\mathcal{I}_Z$, and hence Z. $\qquad\square$

3. Test families

3.1. **The first test family.** Fix a point z_0 and a conic $C \subset \mathbf{P}^2$ not containing z_0.
We vary 5 points $z_1, \ldots, z_5$ in a general 4-dimensional linear series on C. Blowing up
the six points $\{z_0, \ldots, z_5\}$ gives a 4-parameter family of cubic surfaces; see Figure 1.

Proposition 3.1.1. *Let C be a plane conic, $Z \subset C$ a subscheme of length 5, and
$z_0 \in \mathbf{P}^2$ a point not on C. Then the subscheme $Z \cup \{z_0\} \subset \mathbf{P}^2$ is admissible.*

Proof. The subscheme $Z' = Z \cup \{z_0\}$ is clearly curvilinear. Checking that $\mathcal{I}_{Z'} \otimes \mathcal{O}(3)$
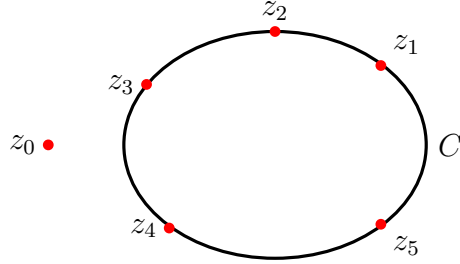is generated by its global sections is straightforward. $\qquad\square$

FIGURE 1. In the first test family, we blow up the plane along $\{z_0, \ldots, z_5\}$ as z_0 stays fixed and $z_1, \ldots, z_5$ vary in a general 4-dimensional series on a fixed conic C.

We now formally define the family. Let $H = C^{[5]}$ be the Hilbert scheme of 5 points in C, so that $H \cong \mathbf{P}^5$, and let $B \cong \mathbf{P}^4 \subset H \cong \mathbf{P}^5$ be a general hyperplane. Let $Z \subset B \times C$ be the restriction of the universal closed subscheme. Set

$$Z' = Z \sqcup (B \times \{z_0\}) \subset B \times \mathbf{P}^2.$$

Our family

$$\pi \colon X \longrightarrow B$$

is the family of cubic surfaces associated to $Z' \subset B \times \mathbf{P}^2$ (see Definition 2.5.3).

Proposition 3.1.2. *The family $\pi \colon X \longrightarrow B$ is good.*

Proof. We show that for every $b \in B$, the group $\mathrm{Aut}(X_b)$ is finite (see Example 2.4.2). By Proposition 2.5.5, it suffices to show that the group $\mathrm{Aut}(\mathbf{P}^2, Z'_b)$ is finite. Since the support of Z'_b is finite, it suffices to show that the subgroup of $\mathrm{Aut}(\mathbf{P}^2, Z'_b)$ that acts trivially on the support is finite. Let $\sigma \in \mathrm{Aut}(\mathbf{P}^2, Z'_b)$ act trivially on the support. Since C is the unique conic containing Z_b, the element σ must preserve C. Since $\sigma(z_0) = z_0$, and σ preserves C, it must fix the pair of points $\mathrm{Polar}_C(z_0) \cap C$. Since $B \subset \mathrm{Hilb}^5 C$ is general, we may assume that it does not contain the finitely many length 5 schemes supported on $\mathrm{Polar}_C(z_0) \cap C$. Then σ fixes at least 3 points of C, namely the two points $\mathrm{Polar}_C(z_0) \cap C$ and the points of $Z \subset C$. It follows that σ fixes all points of C and hence all points of $\mathbf{P}^2$. $\square$

We now compute with the Chern classes of the anti-canonical section bundle.

Proposition 3.1.3. *Let $\mathcal{V} = \pi_* \left(\omega_{X/B}^{-1} \right)$. Then we have*

$$\int_B v_1^4 = 16, \quad \int_B v_1^2 v_2 = 4, \quad \int_B v_1 v_3 = 0, \quad \int_B v_2^2 = 1, \quad \int_B v_4 = 0.$$

Proof. By the construction of the cubic surface, we see that $\mathcal{V}$ is the rank 4 bundle

$$\mathcal{V} = \pi_*(\mathcal{I}_{Z'} \otimes \omega_\pi^{-1}),$$

where $\pi \colon B \times \mathbf{P}^2 \longrightarrow B$ is the first projection. It contains the constant rank 2 bundle

$$\pi_* \left(\mathcal{I}_{B \times C} \otimes \mathcal{I}_{B \times z_0} \otimes \omega_\pi^{-1} \right) = H^0 \left(\mathcal{I}_C \otimes \mathcal{I}_{z_0} \otimes \omega_{\mathbf{P}^2}^{-1} \right) \otimes \mathcal{O}_B, \qquad (3.1)$$

and the quotient is the rank 2 bundle $\pi_* \left(I_{Z/B \times C} \otimes \omega_\pi^{-1} \right)$. To identify the quotient, note that $Z \subset B \times C$ is a divisor of class $(1,5)$ and $\omega_\pi^{-1}|_{B \times C}$ is isomorphic to $\mathcal{O}(0,6)$. Hence, $I_{Z/B \times C} \otimes \omega_\pi^{-1}$ is isomorphic to $\mathcal{O}(-1,1)$, and therefore

$$\pi_* \left(I_{Z/B \times C} \otimes \omega_\pi^{-1} \right) \cong \mathcal{O}_B(-1)^2. \qquad (3.2)$$

By combining (3.1) and (3.2), we see that $\mathcal{V}$ is an extension

$$0 \longrightarrow \mathcal{O}_B^2 \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_B(-1)^2 \longrightarrow 0. \qquad (3.3)$$

The Chern class calculation is now straightforward using the Whitney sum formula. $\qquad \square$

Let us find the degree of the map to moduli $B \dashrightarrow \mathcal{M}$.

Proposition 3.1.4. *The degree of $B \dashrightarrow \mathcal{M}$ is 4320.*

Proof. Consider the space of marked cubic surfaces $\mathcal{M}^\dagger$, which is isomorphic to the configuration space of 6 ordered points on $\mathbf{P}^2$. We have the action of S_5 on $\mathcal{M}^\dagger$ via permutations of the last 5 points. The map $B \dashrightarrow \mathcal{M}$ lifts to

$$B \longrightarrow \mathcal{M}^\dagger / S_5, \qquad (3.4)$$

where $b \in B$ is mapped to $(z_0, z_1 + \cdots + z_5)|_b$. Let us compute the degree of this map. Given a general $(y_0, y_1 + \cdots + y_5)$ in $\mathcal{M}^\dagger / S_5$, there is a unique conic Q through $y_1 + \cdots + y_5$. Note that there exists an automorphism of $\mathbf{P}^2$ that takes (C, z_0) to (Q, y_0), and such an automorphism is unique up to automorphisms preserving (C, z_0). Suppose, under one such automorphism, $y_1 + \cdots + y_5 \subset Q$ is taken to $z_1' + \cdots + z_5' \subset C$. We must find out how many points of B are equivalent to $z_1' + \cdots + z_5'$ up to an automorphism of $\mathbf{P}^2$ preserving (C, z_0). Let $w_0 + w_\infty \subset C$ be the scheme $\mathrm{Polar}_C(z_0) \cap C$. Then we have

$$\mathrm{Aut}(\mathbf{P}^2, C, z_0) = \mathrm{Aut}(C, w_0 + w_\infty) \cong \mathbf{G}_m \rtimes \mathbf{Z}/2\mathbf{Z}.$$

Choose an isomorphism $C \cong \mathbf{P}^1$ such that w_0 and w_∞ are identified with $[0:1]$ and $[1:0]$. Then the $\mathbf{G}_m$ corresponds to the diagonal $\mathbf{G}_m \subset \mathrm{PGL}_2$ and the $\mathbf{Z}/2\mathbf{Z}$ corresponds to the swapping of the two homogeneous coordinates. Write $z_1' + \cdots + z_5'$
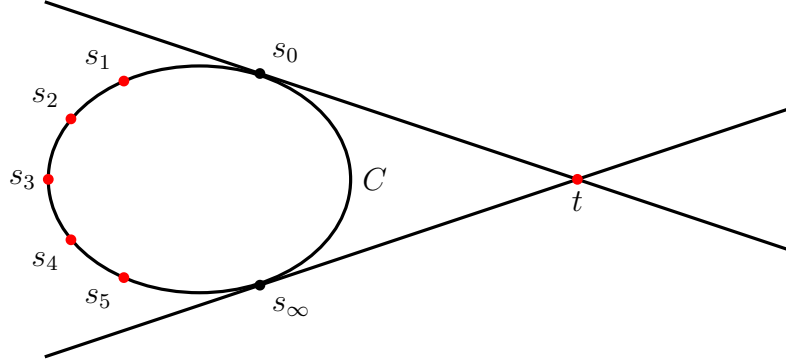
FIGURE 2. In the second test family, we blow up the plane at the points $\{s_1, \ldots, s_5, t\}$ as the seven points $\{s_1, \ldots, s_5, s_0, s_\infty\}$ vary freely on a conic C.

as the vanishing locus of a binary quintic form F. It is easy to see that under the action of the $\mathbf{G}_m$, the closure of the orbit of $V(F)$ in $C^{[5]} \cong \mathbf{P}^5$ is a rational normal curve. Hence, a general hyperplane B intersects the orbit in 5 points. Accounting for the additional $\mathbf{Z}/2\mathbf{Z}$, we conclude that there are precisely 10 points of B equivalent to $z_1' + \cdots + z_5'$ up to an automorphism of $\mathbf{P}^2$ preserving (C, z_0). Thus, the degree of the map in (3.4) is 10.

Since the degree of $\mathcal{M}^\dagger \longrightarrow \mathcal{M}$ is $72 \times 6!$, the degree of $B \longrightarrow \mathcal{M}$ is

$$72 \times 6!/5! \times 10 = 4320.$$

$\square$

Using Proposition 2.4.5 and the computation of the Chern classes and the degree, we obtain the following relation on the undetermined coefficients of $[\overline{\mathrm{Orb}}(X)]$ in (2.3) for a general cubic surface X:

$$16 \cdot a_{1^4} + 4 \cdot a_{1^2 2} + a_{2^2} = 4320. \tag{3.5}$$

3.2. **The second test family.** Forget the notation introduced in § 3.1. We take a smooth conic $C \subset \mathbf{P}^2$ and 7 distinct marked points $s_0, s_1, \ldots, s_5, s_\infty$ on C. Let $t \in \mathbf{P}^2$ be the pole, with respect to C, of the line joining s_0 and s_∞. We blow up the six points $s_1, \ldots, s_5, t$ to get a cubic surface (see Figure 2). As the 7 points vary on C, we get a 4-parameter family of cubic surfaces.

We now make the construction more precise, and show how to use it construct a family of cubic surfaces over $\overline{\mathcal{M}}_{0,7}$. Let $\phi \colon \mathcal{C} \longrightarrow \overline{\mathcal{M}}_{0,7}$ be the universal curve with the universal sections denoted by $\sigma_0, \sigma_1, \ldots, \sigma_5, \sigma_\infty$. Let $\mathcal{L}$ be the line bundle $\sigma_\infty^*(\omega_\phi)$ on $\overline{\mathcal{M}}_{0,7}$. As is customary, we denote its Chern class by ψ_∞. Consider the rank 2

vector bundle

$$\mathcal{A} = \phi_* \mathcal{O}_{\overline{\mathcal{C}}} \left(\sigma_\infty \right).$$

It is easy to check that the evaluation maps yield an isomorphism

$$\mathcal{A} \xrightarrow{\sim} \sigma_\infty^* \mathcal{O}_{\overline{\mathcal{C}}} \left(\sigma_\infty \right) \oplus \sigma_0^* \mathcal{O}_{\overline{\mathcal{C}}} \left(\sigma_\infty \right)$$

$$= \mathcal{L}^{-1} \oplus \mathcal{O}_{\overline{\mathcal{M}}_{0,7}}.$$

Set $\overline{\mathcal{C}} = \mathbf{P}\mathcal{A}^\vee$ with the structure map $\overline{\phi} \colon \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{M}}_{0,7}$. The evaluation map $\phi^* \mathcal{A} \longrightarrow \mathcal{O}_{\overline{\mathcal{C}}} \left(\sigma_\infty \right)$ yields a map $\gamma \colon \mathcal{C} \longrightarrow \overline{\mathcal{C}}$. The effect of γ on a stable curve $(C, s_0, \ldots, s_\infty)$ is to contract all irreducible components of C not containing the marked point s_∞. The unique remaining $\mathbf{P}^1$ is identified with the corresponding fiber of $\overline{\phi}$. Set $\overline{\sigma}_i = \sigma_i \circ \gamma$; these $\overline{\sigma}_i$ are sections of $\overline{\phi} \colon \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{M}}_{0,7}$. They satisfy the following properties

- $\overline{\sigma}_\infty$ is disjoint from all the other sections, and
- in each fiber of $\overline{\phi}$, the union of all seven sections is supported on at least three points.

We now invoke the relative 2-Veronese embedding

$$\iota \colon \overline{\mathcal{C}} \longrightarrow \mathcal{P} := \mathbf{P} \operatorname{Sym}^2 \mathcal{A}^\vee.$$

Let $\rho \colon \mathcal{P} \longrightarrow \overline{\mathcal{M}}_{0,7}$ be the structure morphism. Then $\overline{\mathcal{C}} \subset \mathcal{P}$ is a family of smooth conics with 7 marked points $\sigma_0, \ldots, \sigma_\infty$ satisfying the two properties above.

Let $\tau \colon \overline{\mathcal{M}}_{0,7} \longrightarrow \mathcal{P}$ be the section obtained by fiber-wise taking the pole with respect to $\overline{\mathcal{C}}$ of the line joining the points $\overline{\sigma}_0$ and $\overline{\sigma}_\infty$. Here is an alternate description of τ. The direct sum decomposition $\mathcal{A} = \mathcal{O} \oplus \mathcal{L}^{-1}$ gives the decomposition

$$\operatorname{Sym}^2 \mathcal{A} = \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2}.$$

In $\mathbf{P}\mathcal{A}^\vee$, the sections $\overline{\sigma}_0$ and $\overline{\sigma}_\infty$ correspond to the projections $\mathcal{A} \longrightarrow \mathcal{O}$ and $\mathcal{A} \longrightarrow \mathcal{L}^{-1}$, respectively. Hence, in the Veronese embedding, they correspond to the projections $\operatorname{Sym}^2 A \longrightarrow \mathcal{O}$ and $\operatorname{Sym}^2 A \longrightarrow \mathcal{L}^{-2}$, respectively. Using the quadratic form on $\operatorname{Sym}^2 \mathcal{A}$ given by the Veronese conic, it is easy to check that the section τ is given by the third projection $\operatorname{Sym}^2 A \longrightarrow \mathcal{L}^{-1}$.

We let $\Sigma \subset \overline{\mathcal{C}}$ denote the closed subscheme associated to the divisor $\overline{\sigma}_1 + \cdots + \overline{\sigma}_5$ and $\mathcal{Z} \subset \mathcal{P}$ the disjoint union $\Sigma \sqcup \tau$. By Proposition 3.1.1, $\mathcal{Z} \subset \mathcal{P}$ is a family of admissible subschemes. We let

$$\pi \colon \mathcal{X} \longrightarrow \overline{\mathcal{M}}_{0,7}$$

be the family of cubic surfaces associated to $\mathcal{Z} \subset \mathcal{P}$ (see Definition 2.5.3).

Proposition 3.2.1. *The family $\pi \colon \mathcal{X} \longrightarrow \overline{\mathcal{M}}_{0,7}$ is good.*

Proof. All fibers have finite automorphism groups by the same proof as Proposition 3.1.2. ∎

We now compute the Chern classes of the anti-canonical section bundle.

Proposition 3.2.2. *Let* $\mathcal{V} = \pi_* \omega_\pi^{-1}$ *and let* $v_i = c_i(\mathcal{V})$. *Then we have*

$$\int_{\overline{\mathcal{M}}_{0,7}} v_1^4 = 625, \quad \int_{\overline{\mathcal{M}}_{0,7}} v_1^2 v_2 = 125, \quad \int_{\overline{\mathcal{M}}_{0,7}} v_1 v_3 = -25,$$

$$\int_{\overline{\mathcal{M}}_{0,7}} v_2^2 = 25, \quad \int_{\overline{\mathcal{M}}_{0,7}} v_4 = -6.$$

For the proof, we use the following.

Lemma 3.2.3. *Let* $\pi\colon P \longrightarrow B$ *be a* $\mathbf{P}^1$*-bundle with a section* σ. *Let* $D \subset P$ *be a divisor of relative degree* d *with respect to* π *with* $d \geq -1$. *Then, in the Grothendieck group of* B, *we have*

$$[\pi_* \mathcal{O}(D)] = [\sigma^* \mathcal{O}(D)] + \cdots + [\sigma^* \mathcal{O}(D - (d-1)\sigma)].$$

Proof. We induct on d. Both sides are 0 for $d = -1$. The exact sequence

$$0 \longrightarrow \mathcal{O}(D - \sigma) \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D)|_\sigma \longrightarrow 0$$

pushed-forward to B provides the induction step. ∎

Proof of Proposition 3.2.2. Recall that $\mathcal{X} \longrightarrow \overline{\mathcal{M}}_{0,7}$ is the cubic surface associated to $\mathcal{Z} \subset \mathcal{P}$ in the $\mathbf{P}^2$-bundle $\rho\colon \mathcal{P} \longrightarrow \overline{\mathcal{M}}_{0,7}$. By construction, we have

$$\mathcal{V} = \rho_*(\mathcal{I}_{\mathcal{Z}} \otimes \omega_\rho^{-1}).$$

Hence, in the Grothendieck group of $\overline{\mathcal{M}}_{0,7}$, we have

$$[\mathcal{V}] = \rho_*(\omega_\rho^{-1}) - \rho_*(\omega_\rho^{-1}|_{\mathcal{Z}}). \tag{3.6}$$

By the relative Euler sequence, we see that ω_ρ^{-1} is isomorphic to $\mathcal{O}_{\mathcal{P}}(3) \otimes \rho^* \det(\mathrm{Sym}^2 \mathcal{A})^\vee$, and hence the first term in (3.6) is

$$\rho_*(\omega_\rho^{-1}) = \mathrm{Sym}^3(\mathrm{Sym}^2 \mathcal{A}) \otimes \mathcal{L}^3. \tag{3.7}$$

Since $\mathcal{Z}$ is the disjoint union $\Sigma \cup \tau$, the second term is a sum

$$\rho_*(\omega_\rho^{-1}|_{\mathcal{Z}}) = \rho_*(\omega_\rho^{-1}|_\tau) + \rho_*(\omega_\rho^{-1}|_\Sigma).$$

Since τ is defined by the surjection $\mathrm{Sym}^2 \mathcal{A} \longrightarrow \mathcal{L}^{-1}$, the restriction $\omega_\rho^{-1}|_\tau$ is trivial. To compute the restriction to Σ, we view Σ as a divisor in $\overline{\mathcal{C}} = \mathbf{P}A^\vee$ and let $D = \omega_\rho^{-1}|_{\overline{C}} = \mathcal{O}(6) \otimes \overline{\phi}^* \mathcal{L}^3$. Thus, in the Grothendieck group, we have

$$\rho_*(\omega_\rho^{-1}|_\Sigma) = \overline{\phi}_*(D) - \overline{\phi}_*(D(-\Sigma)). \tag{3.8}$$

We now use Lemma 3.2.3 with the section σ_∞ to compute the terms on the right hand side of (3.8). Putting this together with (3.6) and (3.7), we get

$$[\mathcal{V}] = \mathcal{L}^{-3} + \mathcal{L}^1 + \mathcal{L}^{-1} + \mathcal{L}^{-2}$$

in the Grothendieck group. The result is now a simple computation using $c_1(\mathcal{L}) = \psi_\infty$ and $\int_{\overline{\mathcal{M}}_{0,7}} \psi_\infty^4 = 1$.   □

Having computed the Chern classes, we find the degree of the map $\overline{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}$.

Proposition 3.2.4. *The degree of the map* $\overline{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}$, *induced by* $\pi \colon \mathcal{X} \longrightarrow \overline{\mathcal{M}}_{0,7}$ *is* $2 \times 72 \times 6!$.

Proof. The map lifts to $\overline{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}^\dagger$. Since the degree of $\mathcal{M}^\dagger \longrightarrow \mathcal{M}$ is $72 \times 6!$, we must prove that the degree of $\overline{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}^\dagger$ is 2.

Recall that $\mathcal{M}^\dagger$ is isomorphic to the configuration space of 6 points in $\mathbf{P}^2$. Given a general 6-tuple of points $(s_1, \ldots, s_5, y)$ in $\mathbf{P}^2$, there is a unique conic C passing through $s_1, \ldots, s_5$. Let x and y be the two points of $\mathrm{Polar}_C(t) \cap C$. Then the two pre-images of $(s_1, \ldots, s_5, t)$ in $\overline{\mathcal{M}}_{0,7}$ are $(C, x, s_1, \ldots, s_5, y)$ and $(C, y, s_1, \ldots, s_5, x)$. The proof is thus complete.   □

Using Proposition 2.4.5 and the computation of the Chern classes and the degree, we obtain the following relation on the undetermined coefficients of $[\overline{\mathrm{Orb}}(X)]$ in (2.3) for a general cubic surface X:

$$625a_{1^4} + 125a_{1^2 2} - 25a_{13} + 25a_{2^2} - 6a_4 = 2 \times 72 \times 6!. \tag{3.9}$$

3.3. **The third test family.** We take a cuspidal plane cubic $C \subset \mathbf{P}^2$ and let t_∞ be its unique flex point. We vary 5 points $\{t_1, \ldots, t_5\}$ on C and blow up the plane at $\{t_1, \ldots, t_5, t_\infty\}$ to get a cubic surface (see Figure 3). Since $\mathrm{Aut}(C)$ is one dimensional, this gives us a 4-parameter family of cubic surfaces.

We now convert the above construction into a good family of cubic surfaces over a complete base. We take our base to be $\widehat{\mathcal{M}}_{0,7}$, one of the compactifications of $\mathcal{M}_{0,7}$ constructed by Hassett [19]. Precisely, set

$$\mathbf{w} = (1 - \epsilon, \epsilon, \ldots, \epsilon, 1 - 3\epsilon), \text{ where } 0 < \epsilon \ll 1,$$

and let $\widehat{\mathcal{M}}_{0,7}$ be the space of $\mathbf{w}$-weighted stable pointed curves $(C, s_0, s_1, \ldots, s_5, s_\infty)$. Such curves are easy to describe:

  – C must be a smooth rational curve,
  – s_0 and s_∞ must be distinct,
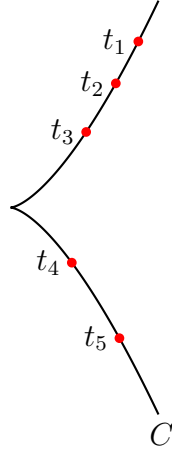 – at most one $s_1, \ldots, s_5$ can coincide with s_0,

FIGURE 3. In the family B_3, we blow up the points t_i for $i = 1, \ldots, 5$ and a fixed point t_∞ at infinity (not shown) as the points $\{t_1, \ldots, t_5\}$ move freely on a fixed cuspidal cubic C with a flex at t_∞.

– at most three of the $s_1, \ldots, s_5$ can coincide with s_∞.

There are no constraints on how many of the the points $s_1, \ldots, s_5$ coincide with each other. Let $\nu \colon C \longrightarrow \mathbf{P}^2$ be the map which is birational onto its image, whose image is a cuspidal cubic, and under which s_0 maps to the unique cusp point and s_∞ to the unique flex point. Let $Z \subset C$ be the subscheme corresponding to the divisor $s_1 + \cdots + s_5 + s_\infty$. Since at most one of the points $s_1, \ldots, s_5$ can coincide with s_0, it follows that $\nu \colon Z \longrightarrow \mathbf{P}^2$ is a closed embedding and the image is a curvilinear subscheme. It is also easy to check that the image is cut out by cubics. That is, $Z \subset \mathbf{P}^2$ is an admissible subscheme (see Definition 2.5.1). We let X_Z be the associated cubic surface (see Definition 2.5.3).

We now formalize the construction as a family over $\widehat{\mathcal{M}}_{0,7}$. Let $\phi \colon \mathcal{C} \longrightarrow \widehat{\mathcal{M}}_{0,7}$ be the universal curve with universal sections $\sigma_0, \sigma_1, \ldots, \sigma_5, \sigma_\infty$. Observe that ϕ is a $\mathbf{P}^1$-bundle. Consider the rank 2 bundle

$$\mathcal{A} = \phi_* \mathcal{O}_{\mathcal{C}}(\sigma_\infty). \tag{3.10}$$

As before, the evaluation maps yield an isomorphism

$$\mathcal{A} \xrightarrow{\sim} \sigma_\infty^* \mathcal{O}_{\mathcal{C}}(\sigma_\infty) \oplus \sigma_0^* \mathcal{O}_{\mathcal{C}}(\sigma_\infty)$$
$$= \mathcal{L}^{-1} \oplus \mathcal{O}_{\widehat{\mathcal{M}}_{0,7}},$$

where $\mathcal{L} = \sigma_\infty^*(\omega_\phi)$. We have a map $\mathcal{C} \longrightarrow \mathbf{P}\mathcal{A}^\vee$ induced by the surjection $\phi^* \mathcal{A} \longrightarrow \mathcal{O}_{\mathcal{C}}(\sigma_\infty)$, and since $\mathcal{C} \longrightarrow \widehat{\mathcal{M}}_{0,7}$ is a $\mathbf{P}^1$-bundle, this map is an isomorphism. Observe that the

sections σ_0 and σ_∞ correspond to the projections $\mathcal{A} \longrightarrow \mathcal{O}$ and $\mathcal{A} \longrightarrow \mathcal{L}^{-1}$, respectively. Now consider the summand

$$\mathcal{S} := \mathcal{O}_{\widehat{\mathcal{M}}_{0,7}} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \subset \mathrm{Sym}^3 \mathcal{A}, \qquad (3.11)$$

and denote by

$$\nu \colon \mathcal{C} \longrightarrow \mathbf{P}\mathcal{S}^\vee$$

the map induced by the surjection $\phi^* \mathcal{S} \longrightarrow \mathcal{O}_{\mathcal{C}}(3\sigma_\infty)$. Let $\rho \colon \mathbf{P}\mathcal{S}^\vee \longrightarrow \widehat{\mathcal{M}}_{0,7}$ be the structure map. Thanks to our choice of $\mathcal{S}$, fiber-wise the map ν maps $\mathcal{C}$ to cuspidal cubics with s_0 mapping to the cusp and s_∞ to the flex. Let $\mathcal{Z} \subset \mathcal{C}$ be the subscheme corresponding to the divisor $\sigma_1 + \cdots + \sigma_\infty$. Our family

$$\pi \colon \mathcal{X} \longrightarrow \widehat{\mathcal{M}}_{0,7}$$

is the family of cubic surfaces associated to the family of admissible subschemes $\nu \colon \mathcal{Z} \longrightarrow \mathbf{P}\mathcal{S}^\vee$.

Proposition 3.3.1. *The family $\pi \colon \mathcal{X} \longrightarrow \widehat{\mathcal{M}}_{0,7}$ is good.*

Proof. We show that no fiber $\mathcal{X}_b$ is in the closure of the orbit of any smooth cubic surface S without Eckardt points. Let $Z \subset \mathbf{P}^2$ be the image of $\nu \colon Z_b \longrightarrow \mathbf{P}\mathcal{S}_b^\vee$. By construction, Z does not have a point of multiplicity 6, and since Z lies on a cubic, no four distinct points of Z can be collinear. Suppose Z does not have a point of multiplicity 4. Then we use Proposition 5.2.1 and Proposition 5.2.2 to conclude that $\mathcal{X}_b$ is not in the orbit closure of S.

We now treat the cases where Z has a point of multiplicity 4 or 5. We repeatedly use the following fact about the geometry of the cuspidal cubic C: no line through the flex point is tangent to C, except the tangent line at the flex and the line joining the flex and the cusp.

Suppose $Z = p + q + 4r \subset C$, where p, q, r are distinct. Since r occurs with multiplicity greater than 1 in Z, it must be distinct from the cusp point (so $4r$ is unambiguous). If the tangent line $T_r Z = T_r C$ does not contain p or q, we appeal to Proposition 5.2.3 and Remark 5.2.4. Suppose $T_r Z$ contains p. Then $T_r Z$ cannot contain q, and hence p, q, r are not collinear. By the construction of Z, one of p, q, r must be the flex point s_∞. Using the fact about tangent lines through the flex, we see that the only possibility is $q = s_\infty$. Now $Z \subset C$ is uniquely determined up to an automorphism of $(\mathbf{P}^2, C)$. It is easy to write down such a Z and check directly that $\mathrm{Aut}(\mathbf{P}^2, Z)$ is finite. By Proposition 2.5.5, $\mathrm{Aut}(\mathcal{X}_b)$ is also finite, and $\mathcal{X}_b$ is not in the orbit closure of S. Alternatively, we see that the cubic surface associated to Z has a unique singular point, which is of type D_4. It is known that, up to isomorphism,

there are exactly two cubic surfaces with singularity D_4 (see [9, Lemma 4]). One has a finite automorphism group and the other one is treated by Proposition 5.1.3.

Suppose $Z = p + 5q$, where p and q are distinct. Then we must have $p = s_\infty$ and $q \notin \{s_0, s_\infty\}$. Let Q be the unique conic containing $5q$. We claim that C cannot contain p. If it did, then we get $5(q - p) = 0 \in \operatorname{Pic}(C) \cong \mathbf{G}_a$, forcing $p = q$. We also claim that the two points of $\operatorname{Polar}_Q(p) \cap Q$ are distinct from q. This is because Q and C share the tangent line at q; this line does not pass through p, but the tangent lines to Q at the points of $\operatorname{Polar}_Q(p)$ do pass through p. Now, any $\sigma \in \operatorname{Aut}(\mathbf{P}^2, Z)$ must preserve C the three points $\operatorname{Polar}_Q(p) \cap Q \cup \{q\}$. Then $\operatorname{Aut}(\mathbf{P}^2, Z)$ and hence $\operatorname{Aut}(\mathfrak{X}_b)$ is finite, and $\mathfrak{X}_b$ is not in the orbit closure of S. $\qquad\square$

It is time for enumerative computations.

Proposition 3.3.2. *Let $\mathcal{V} = \pi_* \omega_\pi^{-1}$ and let $v_i = c_i(\mathcal{V})$. Then on $\widehat{\mathcal{M}}_{0,7}$, we have*

$$\int v_1^4 = 3436, \quad \int v_1^2 v_2 = 1076, \quad \int v_1 v_3 = 116,$$

$$\int v_2^2 = 316, \quad \int v_4 = 0.$$

Proof. We begin by finding the class of $\mathcal{V}$ in the Grothendieck group of $\widehat{\mathcal{M}}_{0,7}$. The computation parallels the computation in the proof Proposition 3.2.2, so we will be brief. Recall that $\phi\colon \mathcal{C} \longrightarrow \widehat{\mathcal{M}}_{0,7}$ is the universal curve with sections $\sigma_0, \sigma_1, \ldots, \sigma_\infty$. Setting $\mathcal{L} = \sigma_\infty^* \omega_\phi$, recall the bundles

$$\mathcal{A} = \mathcal{O} \oplus \mathcal{L}^{-1} \text{ from (3.10), and } \mathcal{S} = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \text{ from (3.11).} \tag{3.12}$$

Recall that $\rho\colon \mathbf{P}\mathcal{S}^\vee \longrightarrow \widehat{\mathcal{M}}_{0,7}$ is the structure map and $\mathcal{Z} \subset \mathcal{C}$ the subscheme corresponding to the divisor $s_1 + \cdots + s_5 + s_\infty$. In the Grothendieck group of $\widehat{\mathcal{M}}_{0,7}$, we have

$$\begin{aligned}
\mathcal{V} &= \rho_* \omega_\rho^{-1} - \phi_* \left(\nu^* \omega_\rho^{-1} |_{\mathcal{Z}} \right) \\
&= (\operatorname{Sym}^3 \mathcal{S}) \otimes \det \mathcal{S}^\vee - \phi_* \left(\mathcal{O}\left(9\sigma_\infty\right) \otimes \phi^* \det \mathcal{S}^\vee |_{\mathcal{Z}} \right) \\
&= (\operatorname{Sym}^3 \left(\mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \right)) \otimes \mathcal{L}^5 - \phi_* \left(\mathcal{O}\left(9\sigma_\infty\right) \otimes \phi^* \mathcal{L}^5 \right) \\
&\quad + \phi_* \left(\mathcal{O}\left(9\sigma_\infty - \mathcal{Z}\right) \otimes \phi^* \mathcal{L}^5 \right) \\
&= \mathcal{L}^{-1} - \mathcal{L}^4 + (\mathcal{L}^2 + \mathcal{L}^3 + \mathcal{L}^4 + \mathcal{L}^5) \otimes \mathcal{O}\left(-\Delta_0\right).
\end{aligned} \tag{3.13}$$

Here $\Delta_0 \subset \widehat{\mathcal{M}}_{0,7}$ is the divisor $\sigma_0^* \mathcal{O}(\sigma_1 + \cdots + \sigma_5)$. This is the locus of marked curves where the marked point σ_0 coincides with σ_i for some $i = 1, \ldots, 5$. In the last step of the computation, we have used Lemma 3.2.3 with the section σ_0 and that

$$\sigma_0^* \mathcal{O}(\sigma_0) = \sigma_\infty^* \mathcal{O}(-\sigma_\infty) = \mathcal{L},$$

which holds since σ_0 and σ_∞ are two disjoint sections of a $\mathbf{P}^1$-bundle. The rest is a straightforward computation using the Whitney sum formula. $\qquad\square$

Let us indicate how to carry out the Chern class computation on $\widehat{\mathcal{M}}_{0,7}$. The boundary divisors of the Hassett spaces are products of smaller Hassett spaces. We can use this inductive structure to break down the computation, similar to how it is done on the usual $\overline{\mathcal{M}}_{0,7}$. Alternatively, we can use the map $\zeta\colon \overline{\mathcal{M}}_{0,7}\longrightarrow\widehat{\mathcal{M}}_{0,7}$ to pull back the classes to $\overline{\mathcal{M}}_{0,7}$, and do the computation there. Let $\widehat{\psi}_i = c_1(\sigma_i^*\omega_\phi)$ denote the ψ-classes on $\widehat{\mathcal{M}}_{0,7}$ and ψ_i its analogue on $\overline{\mathcal{M}}_{0,7}$. Let $\Delta_{i,j} \subset \overline{\mathcal{M}}_{0,7}$ be the boundary divisor whose general point corresponds to the nodal union of two smooth rational curves with one component only containing the marked points indexed i and j. It is easy to see that

$$\zeta^*\widehat{\psi}_0 = \psi_0 - \sum_{i=1}^{5}[\Delta_{0,i}], \text{ and}$$

$$\zeta^*[\Delta_0] = \sum_{i=1}^{5}[\Delta_{0,i}],$$

using which we can transport the entire Chern class computation to $\overline{\mathcal{M}}_{0,7}$.

We now compute the degree of the rational map from $\widehat{\mathcal{M}}_{0,7}$ to the moduli space $\mathcal{M}$ of cubic surfaces, induced by the family $\mathfrak{X}\longrightarrow\widehat{\mathcal{M}}_{0,7}$.

Proposition 3.3.3. *The degree of the map $\widehat{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}$ is $20 \times 72 \times 6!$.*

Proof. The map lifts to $\widehat{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}^\dagger$. The degree of this map is the answer to the following enumerative problem: given 6 general points in $\mathbf{P}^2$, how many cuspidal cubics contain the first five and have a flex point at the sixth? Fortunately for us, this question has been answered already—the answer is 20. The answers appears in a remarkable paper [21] of Miret and Xambó, who credit it to Schubert. The paper contains a staggering number of enumerative results about cuspidal cubics. Our answer is in the first column in the row labelled v^2 in the second table (titled "Order 2") in Section 12 of [21].

We briefly explain how to read off the result from the table in [21]. The letter v denotes the map from the (7-dimensional) space of cuspidal cubics to $\mathbf{P}^2$, the symbol X_0 the locus of cuspidal cubics passing through a general point in $\mathbf{P}^2$, and the symbol X_1 the locus of cuspidal cubics tangent to a general line in $\mathbf{P}^2$. The i-th column in the row labelled v^2 is the intersection number $(v^*H)^2 \cdot X_0^{6-i}X_1^{i-1}$, where $H \subset \mathbf{P}^2$ is a general line. For $i = 1$, this is precisely the number of cuspidal cubics flexed at a given point and passing through 5 general points.
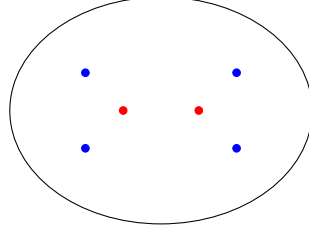
FIGURE 4. In the fourth test family, we blow up the plane at 4 fixed points (blue) and 2 variable points (red).

Since the degree of $\mathcal{M}^{\dagger} \longrightarrow \mathcal{M}$ is $72 \times 6!$, the degree of $\widehat{\mathcal{M}}_{0,7} \dashrightarrow \mathcal{M}$ is $20 \times 72 \times 6!$. $\quad\square$

Using Proposition 2.4.5 and the computation of the Chern classes and the degree, we obtain the following relation on the undetermined coefficients of $[\overline{\mathrm{Orb}}(X)]$ in (2.3) for a general cubic surface X:

$$3436 a_{1^4} + 1076 a_{1^2 2} + 116 a_{13} + 316 a_{2^2} = 20 \times 72 \times 6!. \tag{3.14}$$

3.4. **The fourth test family.** Our fourth family is perhaps the most obvious one. We blow up 6 points on $\mathbf{P}^2$, four of which remain fixed, and the remaining two vary freely (see Figure 4). A complication arises, however, while formalizing this construction. When the two varying points lie on the line joining two of the fixed points, the resulting configuration is no longer admissible. It takes some effort to resolve this issue.

Let S be the blow up of $\mathbf{P}^2$ at 4 general points. Then S is a quintic del Pezzo. Let $S^{[2]}$ denote the Hilbert scheme of 2 points on S. We have 10 exceptional curves $L_1, \ldots, L_{10}$ on S. Consider the planes $L_i^{[2]} \subset S^{[2]}$. Observe that a length 2 subscheme $Z \subset S$ is not admissible precisely when Z is contained in L_i for some i. For $i \neq j$, the lines L_i and L_j are either disjoint or intersect in a single (reduced) point. Therefore, $L_i^{[2]}$ and $L_j^{[2]}$ are disjoint subsets of $S^{[2]}$. Let Λ be the union of all $L_i^{[2]}$. The crux of this section is to show that the blow up $\mathrm{Bl}_\Lambda S^{[2]}$ hosts a good family of cubic surfaces.

Set

$$\widetilde{S^{[2]}} := \mathrm{Bl}_\Lambda S^{[2]}, \tag{3.15}$$

with $\beta \colon \widetilde{S^{[2]}} \longrightarrow S^{[2]}$ being the blow-down map. For $i = 1, \ldots, 10$, let $E_i = \beta^{-1}(L_i^{[2]})$ denote the components of the of the exceptional divisor of the blow-up. Consider

$$F_i := E_i \times L_i \subset \widetilde{S^{[2]}} \times S,$$

and let F be the (disjoint) union $F = \bigcup_i F_i$. Set

$$\mathcal{Y} := \mathrm{Bl}_F \left(\widetilde{S^{[2]}} \times S \right), \tag{3.16}$$

with $\eta \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}} \times S$ being the blow-down map. Consider the map

$$\widetilde{\pi} \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}}$$

obtained by composing η with the projection on to the first factor. It is easy to check that $\widetilde{\pi}$ is a flat family of surfaces, isomorphic to the constant family with fiber S over the complement of E in $\widetilde{S^{[2]}}$.

Let $\mathcal{Z} \subset S^{[2]} \times S$ be the universal closed subscheme of length 2. Let $\widetilde{\mathcal{Z}} \subset \widetilde{S^{[2]}} \times S$ be the fiber product

$$\widetilde{\mathcal{Z}} = \mathcal{Z} \times_{S^{[2]}} \widetilde{S^{[2]}}.$$

The next lemma is critical to our construction.

Lemma 3.4.1. *The closed embedding $i \colon \widetilde{\mathcal{Z}} \hookrightarrow \widetilde{S^{[2]}} \times S$ lifts to a closed embedding $j \colon \widetilde{\mathcal{Z}} \hookrightarrow \mathcal{Y}$.*

Proof. The subscheme $(L_i^{[2]} \times S) \cap \mathcal{Z}$ is contained in $(L_i^{[2]} \times L_i)$. Hence, we have an equality of schemes

$$(L_i^{[2]} \times L_i) \cap \mathcal{Z} = (L_i^{[2]} \times S) \cap \mathcal{Z}.$$

By pulling back both sides to $\widetilde{S^{[2]}} \times S$, we get

$$F_i \cap \widetilde{\mathcal{Z}} = (E_i \times S) \cap \widetilde{\mathcal{Z}}. \tag{3.17}$$

In particular, $F_i \cap \widetilde{\mathcal{Z}}$ is a Cartier divisor. By the universal property of the blow up, we get a lift $j \colon \widetilde{\mathcal{Z}} \longrightarrow \mathcal{Y}$. Since $i = \beta \circ j$ is a closed embedding, so is j. $\qquad\square$

The following diagram summarizes the situation so far:

$$
\begin{array}{ccccc}
 & & \mathcal{Y} & & \\
 & \overset{j}{\nearrow} & \downarrow{\scriptstyle \eta} & & \\
\widetilde{\mathcal{Z}} & \overset{i}{\hookrightarrow} & \widetilde{S^{[2]}} \times S & \longrightarrow & S^{[2]} \times S \\
 & & \downarrow & & \downarrow \\
 & & \widetilde{S^{[2]}} & \overset{\beta}{\longrightarrow} & S^{[2]}.
\end{array}
$$

We let $\mathcal{F} \subset \mathcal{Y}$ denote the exceptional divisor of η. It has 10 disjoint components, corresponding to each F_i, which in turn correspond to the ten lines L_i. To ease notation, we suppress the map $j \colon \widetilde{\mathcal{Z}} \longrightarrow \mathcal{Y}$, and write $\widetilde{\mathcal{Z}} \subset \mathcal{Y}$.

Let us study the family $\widetilde{\pi} \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}}$. Its fiber over a point in the complement of E is just S. The following proposition describes the fiber of $\mathcal{Y}$ over $e \in E_i$, along with the fiber of $\widetilde{\mathcal{Z}}$ in it; see Figure 5 for a picture.

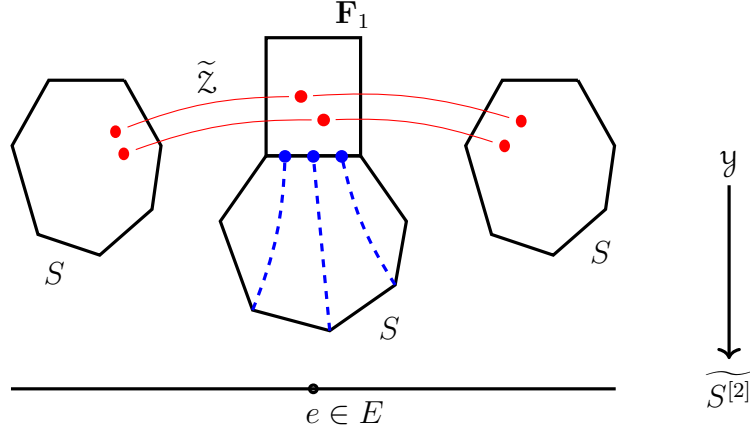**Proposition 3.4.2.** *Let $e \in E_i$ be any point, and let $\mathcal{Y}_e = \widetilde{\pi}^{-1}(e)$.*

FIGURE 5. The family of surfaces $\mathcal{Y} \longrightarrow \widetilde{S^{[2]}}$ and the subscheme $\widetilde{\mathcal{Z}} \subset \mathcal{Y}$. The dashed lines on the central fiber represent the three exceptional curves of S intersecting the double curve.

(1) $\mathcal{Y}_e$ is the union of two surfaces S and $\mathcal{F}_e$, where $\mathcal{F}_e$ is a copy of the Hirzebruch surface $\mathbf{F}_1$. They meet transversely along $L_i \subset S$ and a smooth curve of self-intersection 1 in $\mathbf{F}_1$.

(2) The subscheme $\widetilde{\mathcal{Z}}_e \subset \mathcal{Y}_e$ lies in $\mathcal{F}_e$ and is disjoint from S. It maps isomorphically onto its image in L_i under η.

Proof. We study the blowup $\mathcal{Y} = \mathrm{Bl}_F\left(\widetilde{S^{[2]}} \times S\right)$, beginning with the normal bundle $N_{F/\widetilde{S^{[2]}} \times S}$. We focus on the component $F_i = E_i \times L_i$. Denoting by $\pi_1 \colon F_i \longrightarrow E_i$ and $\pi_2 \colon F_i \longrightarrow L_i$ the two projections, we have

$$N_{F_i/\widetilde{S^{[2]}} \times S} = \pi_1^* N_{E_i/\widetilde{S^{[2]}}} \oplus \pi_2^* N_{L_i/S}.$$

The component of $\mathcal{F}$ lying above F_i is the projectivization $\mathbf{P}N_{F_i/\widetilde{S^{[2]}} \times S}$. Restricting the projectivization to $\{e\} \times L_i \simeq L_i$ yields $\mathbf{P}(\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(-1))$. Thus we get $\mathcal{F}_e \simeq \mathbf{F}_1$. Since $(\{e\} \times S) \cap F_i = \{e\} \times L_i$ is a Cartier divisor on $\{e\} \times S$, the proper transform of $\{e\} \times S$ in $\mathcal{Y}$ is again a copy of S. It meets $\mathcal{F}_e = \mathbf{P}(\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(-1)) \simeq \mathbf{F}_1$ along the section corresponding to the summand $N_{L_i/S} = \mathcal{O}_{L_i}(-1)$. Altogether, we get the description of $\mathcal{Y}_e$ provided in (1).

Since the embedding $i \colon \widetilde{\mathcal{Z}} \longrightarrow \widetilde{S^{[2]}} \times S$ is the composite

$$\widetilde{\mathcal{Z}} \xrightarrow{j} \mathcal{Y} \xrightarrow{\eta} \widetilde{S^{[2]}} \times S,$$

and $\mathcal{F} = \eta^{-1}(F)$, we have

$$\widetilde{\mathcal{Z}} \cap j^{-1}(\mathcal{F}) = \widetilde{\mathcal{Z}} \cap i^{-1}(F).$$

We have already seen in (3.17) that

$$F \cap \widetilde{\mathcal{Z}} = (E \times S) \cap \widetilde{\mathcal{Z}}.$$

Restricting the first coordinate of $E \times S$ to e implies $\widetilde{\mathcal{Z}_e} \subset \mathcal{F}_e$.

To see that $\widetilde{Z}_e$ is disjoint from S, consider the intersection product $[\widetilde{\mathcal{Z}}] \cdot [\mathcal{Y}_e]$. On the one hand, since $\widetilde{\mathcal{Z}} \longrightarrow \widetilde{S^{[2]}} \times S$ is finite, flat, and has degree 2, this intersection product is 2. On the other hand, by part (1), it is equal to $[\widetilde{\mathcal{Z}}] \cdot ([\mathcal{F}_e] + [S])$. Since we have $\widetilde{\mathcal{Z}}_e \subset \mathcal{F}_e$, we get $[\widetilde{\mathcal{Z}}] \cdot [S] = 0$, and hence $\widetilde{\mathcal{Z}}$ must be disjoint from S.

The last assertion follows immediately from the fact that $\widetilde{Z} \subset \mathcal{Y}$ maps isomorphically to its image in $\widetilde{S^{[2]}} \times S$. $\qquad\square$

The following proposition provides one more necessary detail about the position of $\widetilde{\mathcal{Z}}_e$ in the $\mathbf{F}_1$ component of $\mathcal{Y}_e$.

Proposition 3.4.3. *Maintain the notation of Proposition 3.4.2.*

(1) The length two subscheme $\widetilde{\mathcal{Z}}_e \subset \mathbf{F}_1$ is not contained in the directrix of $\mathbf{F}_1$.

(2) Set $\tau_i = T_i \cap \mathbf{F}_1$. The subscheme $\widetilde{\mathcal{Z}}_e \cup \tau_i$ is an admissible subscheme of $\mathbf{F}_1$.

Proof. The proof follows easily from unravelling the map

$$j \colon \widetilde{\mathcal{Z}}_e \longrightarrow \mathbf{F}_1,$$

which we now do. Set $Z = \widetilde{\mathcal{Z}}_e$ and let N_e be the 1-dimensional vector space

$$N_e = N_{E_i/\widetilde{S^{[2]}}}\big|_e.$$

From the proof of Proposition 3.4.2, we write the $\mathbf{F}_1$ component more canonically as

$$\mathbf{F}_1 = \mathbf{P}\left(N_e \otimes \mathcal{O}_{L_i} \oplus N_{L_i/S}\right).$$

The point $e \in E_i$ gives a (non-zero) map

$$d\beta \colon N_e \longrightarrow N_{L_i^{[2]}/S^{[2]}}\big|_{\beta(e)}.$$

The point $\beta(e) \in L_i^{[2]} \subset S^{[2]}$ corresponds to the length 2 subscheme $Z \subset L_i \subset S$. It is easy to see that we have a canonical identification

$$N_{L_i^{[2]}/S^{[2]}}\big|_{\beta(e)} = H^0\left(Z, N_{L_i/S}|_Z\right).$$

We thus get a map

$$d\beta \colon N_e \longrightarrow H^0\left(Z, N_{L_i/S}|_Z\right).$$

We leave it to the reader to verify that the map $j \colon Z \longrightarrow \mathbf{F}_1$ is induced by the map

$$N_e \otimes \mathcal{O}_Z \longrightarrow N_e \otimes \mathcal{O}_Z \oplus N_{L_i/S}|_Z \tag{3.18}$$

given by $(\mathrm{id}, \epsilon \circ d\beta)$, where

$$\epsilon \colon H^0\left(Z, N_{L_i/S}|_Z\right) \otimes \mathcal{O}_Z \longrightarrow N_{L_i/S}|_Z$$

is the evaluation map. Since ϵ is an isomorphism and $d\beta$ is non-zero, the second coordinate of (3.18) is non-zero. This non-vanishing equivalent to the proposition. (Incidentally, note that the first coordinate of (3.18) is invertible. This is equivalent to the fact that $Z \subset \mathbf{F}_1$ is disjoint from the curve where $\mathbf{F}_1$ meets the other component S. In the proof of Proposition 3.4.2, we concluded this fact using intersection numbers.)

The assertion about admissibility is straightforward. $\qquad\square$

We are now in a position to describe the family of cubic surfaces. Consider the family

$$\widetilde{\pi} \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}},$$

along with $\widetilde{\mathcal{Z}} \subset \mathcal{Y}$. Set $U = \widetilde{S^{[2]}} \setminus E$. Over U, the $\widetilde{\mathcal{Z}}$ is an admissible subscheme of $\mathcal{Y}$, and here our family will be the associated cubic surface as in Definition 2.5.3. We now describe our cubic surface over a point $e \in E_i$. In the S component of $\mathcal{Y}_e$, we have three exceptional curves that meet the double curve $L_i \subset S$. They intersect L_i in three distinct points. Let T_i be the union of these three exceptional curves (in Figure 5, these are indicated by the dashed lines). On the $\mathbf{F}_1$ component, we have a distinguished length 5 subscheme: the union of the length two subscheme $\widetilde{\mathcal{Z}}_e$ and the 3 points $\tau_i := T_i \cap L_i$. This subscheme of length 5 is admissible, and our cubic surface over e will be its associated cubic surface.

We construct the family that we described fiber-wise above using the linear series associated to the line bundle

$$\mathcal{L} = \omega_{\widetilde{\pi}}^{-1} \otimes \mathcal{O}_{\mathcal{Y}}(-\mathcal{F}). \tag{3.19}$$

Since $\mathcal{Y}$ is the blow up of $\widetilde{S^{[2]}} \times S$ with exceptional divisor $\mathcal{F}$, we can also write this line bundle as

$$\mathcal{L} = \omega_S^{-1} \otimes \mathcal{O}_{\mathcal{Y}}(-2\mathcal{F}). \tag{3.20}$$

The following proposition describes $\mathcal{L}$ on the degenerate fibers of $\widetilde{\pi}$.

Proposition 3.4.4. *Let $e \in E_i$ and $\mathcal{Y}_e = S \cup_{L_i} \mathbf{F}_1$ as in Proposition 3.4.2. Let $T_i \subset S$ be the union of the three exceptional curves meeting L_i and set $\tau_i = L_i \cap T_i$. Then*

(1) $\mathcal{L}|_S \simeq \mathcal{O}_S(T_i)$.
(2) $\mathcal{L}|_{\mathbf{F}_1} \simeq \omega_{\mathbf{F}_1}^{-1}$.

(3) The restriction map

$$H^0(\mathcal{Y}_e, \mathcal{L}|_{\mathcal{Y}_e}) \longrightarrow H^0(\mathbf{F}_1, \omega_{\mathbf{F}_1}^{-1}) \tag{3.21}$$

is an isomorphism onto the 6 dimensional vector space $H^0\left(\mathbf{F}_1, \omega_{\mathbf{F}_1}^{-1} \otimes \mathcal{I}_{\tau_i}\right)$.

(4) The base locus of the complete linear system $|\mathcal{L}|$ on $\mathcal{Y}_e$ is T_i.

Proof. We have $\omega_S^{-1} = \mathcal{O}(T_i + 2L_i)$. This and (3.20) gives $\mathcal{L}|_S = \mathcal{O}_S(T_i)$.

Let $\mathcal{S} \subset \mathcal{Y}$ be the proper transform of $E \times S \subset \widetilde{S^{[2]}} \times S$, so that the preimage in $\mathcal{Y}$ of $E \times S$ is the sum $\mathcal{F} + \mathcal{S}$. Then we have

$$\mathcal{O}(\mathcal{F})|_{\mathbf{F}_1} = \mathcal{O}(-\mathcal{S})|_{\mathbf{F}_1}.$$

Since $\omega_S^{-1} \cdot L_i = 1$, the pull-back of ω_S^{-1} to $\mathbf{F}_1$ is the class of a fiber. Since $\omega_{\mathbf{F}_1}^{-1}$ is the sum of a fiber class and 2 times the class of a section of self-intersection 1, we get $\mathcal{L}|_{\mathbf{F}_1} = \omega_{\mathbf{F}_1}^{-1}$.

The statements about the dimension and the base locus of $|\mathcal{L}|$ on $\mathcal{Y}_e$ are straightforward. The only key point is that $\mathcal{O}_S(T_i)$ is one-dimensional, and the vanishing locus of its non-zero sections is T_i. $\qquad\square$

Let

$$\mathcal{W} := \widetilde{\pi}_* \left(\mathcal{L} \otimes \mathcal{I}_{\widetilde{z}} \right).$$

It follows from Grauert's theorem that $\mathcal{W}$ is a vector bundle of rank 4. Consider the rational map

$$\kappa \colon \mathcal{Y} \dashrightarrow \mathbf{P}\mathcal{W}^\vee$$

induced by the evaluation

$$\widetilde{\pi}^* \mathcal{W} \longrightarrow \mathcal{L}.$$

It is easy to check that for every $e \in \widetilde{S^{[2]}}$, the image of $\kappa(\mathcal{Y}_e)$ is contained in a unique cubic surface in $\mathbf{P}\mathcal{W}_e^\vee$. Let $\mathcal{X} \subset \mathbf{P}\mathcal{W}^\vee$ be the resulting family of cubic surfaces. More precisely, consider the map

$$\mathrm{Sym}^3 \mathcal{W} \longrightarrow \pi_*(\mathcal{L}^3)$$

and verify (by looking at the fibers) that both sides are vector bundles and the map is surjective with kernel of rank 1. The rank 1 kernel defines the cubic $\mathcal{X} \subset \mathbf{P}\mathcal{W}^\vee$. Let $\pi \colon \mathcal{X} \longrightarrow \widetilde{S^{[2]}}$ be the obvious map.

The next proposition identifies the anti-canonical bundle of π.

Proposition 3.4.5. *We have $\omega_\pi^{-1} = \mathcal{O}(1) \otimes \pi^*\mathcal{O}(E)$.*

Proof. Let $\widehat{\mathcal{Y}}$ be the blow-up of $\mathcal{Y}$ along $\widetilde{\mathcal{Z}}$ and $\widehat{\pi}\colon \widehat{\mathcal{Y}} \longrightarrow \widetilde{S^{[2]}}$ the resulting map. Then it is easy to check that the rational map $\widehat{\kappa}\colon \widehat{\mathcal{Y}} \dashrightarrow \mathcal{X}$ (induced by κ) extends to a regular map away from the union of the curves T_i over the points of E_i. Recall that $\mathcal{S} \subset \mathcal{Y}$ is the proper transform of $E \times S$, so that $\widetilde{\pi}^*\mathcal{O}(E) = \mathcal{F} + \mathcal{S}$. The map

$$\widehat{\kappa}\colon \widehat{\mathcal{Y}} \setminus \mathcal{S} \dashrightarrow \mathcal{X}$$

is an isomorphism away from a locus of codimension at least 2 on both sides. If $\widehat{D} \subset \widehat{\mathcal{Y}}$ is the exceptional divisor of the blow-up, then we have

$$
\begin{aligned}
\widehat{\kappa}^*\mathcal{O}(1) &= \mathcal{L}(-\widehat{D}) \\
&= \omega_{\widetilde{\pi}}^{-1} \otimes \mathcal{O}(-\mathcal{F}) \otimes \mathcal{O}_{\widehat{Y}}(-\widehat{D}) \\
&= \omega_{\widehat{\pi}}^{-1} \otimes \mathcal{O}(-\mathcal{F}).
\end{aligned}
$$

We see that the line bundles $\widehat{\kappa}^*\mathcal{O}(1) \otimes \widehat{\pi}^*\mathcal{O}(E)$ and $\omega_{\widetilde{\pi}}^{-1}$ are isomorphic on $\widehat{\mathcal{Y}} \setminus \mathcal{S}$. Hence, they are isomorphic away from a locus of codimension at least 2 on $\mathcal{X}$. Since $\mathcal{X}$ is a hypersurface in a smooth variety, and it is smooth in codimension 1, it is normal. It follows that $\mathcal{O}(1) \otimes \pi^*\mathcal{O}(E)$ and ω_{π}^{-1} are isomorphic on $\mathcal{X}$. $\qquad\square$

Proposition 3.4.6. *The family* $\pi\colon \mathcal{X} \longrightarrow \widetilde{S^{[2]}}$ *is good.*

Proof. Over $e \notin E$, the cubic $\mathcal{X}_e$ is the one associated to $\mathcal{Z}_e \subset S$, and over $e \in E$, say $e \in E_i$, the cubic $\mathcal{X}_e$ is the one associated to $\widetilde{\mathcal{Z}}_e \cup \tau_i \subset \mathbf{F}_1$. In both cases, it is easy to verify using Proposition 2.5.5 that $\mathcal{X}_e$ has a finite automorphism group. $\quad\square$

Having constructed the family $\pi\colon \mathcal{X} \longrightarrow \widetilde{S^{[2]}}$, we take up the task of calculating the Chern classes of

$$\mathcal{V} = \pi_* \left(\omega_{\pi}^{-1} \right) = \mathcal{W} \otimes \mathcal{O}_{\widetilde{S^{[2]}}}(E).$$

Let $\mathcal{U} = \left(\omega_S^{-1} \right)^{[2]}$ be the rank 2 tautological bundle on $S^{[2]}$, defined by

$$\mathcal{U} = \pi_{1*}\pi_2^* \left(\omega_S^{-1} \right),$$

where π_i are the two projections on $S^{[2]} \times S$.

Proposition 3.4.7. *In the Grothendieck group of* $\widetilde{S^{[2]}}$, *we have*

$$\mathcal{V} = \mathcal{O}(E) + \mathcal{O}^3 + \mathcal{O}(-E)^2 - \mathcal{U} \otimes \mathcal{O}(-E),$$

where $\mathcal{O}$ denotes $\mathcal{O}_{\widetilde{S^{[2]}}}$.

Proof. We compute the class of $\mathcal{W}$ and obtain the class of $\mathcal{V}$ by tensoring by $\mathcal{O}(E)$. Recall the family $\widetilde{\pi} \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}}$ and the line bundle $\mathcal{L}$ on it defined in (3.19). Set

$$\mathcal{A} := \widetilde{\pi}_* \mathcal{L} \text{ and } \mathcal{B} := \widetilde{\pi}_* \left(\mathcal{L}|_{\widetilde{z}} \right).$$

Then $\mathcal{A}$ and $\mathcal{B}$ are vector bundles of rank 6 and 2, respectively. By the definition of $\mathcal{W}$, we have the exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{W} \longrightarrow 0,$$

and hence $\mathcal{W} = \mathcal{A} - \mathcal{B}$ in the Grothendieck group.

We first compute $\mathcal{B}$. By the push-pull formula and the description (3.20) of $\mathcal{L}$, we get

$$\mathcal{B} = \mathcal{U} \otimes \mathcal{O}_{\widetilde{S^{[2]}}}(-2E). \tag{3.22}$$

We now compute $\mathcal{A}$. Recall that $\eta \colon \mathcal{Y} \longrightarrow \widetilde{S^{[2]}} \times S$ is the blow-up at $F = \bigsqcup_i E_i \times L_i$. Using (3.20) again, we get

$$\eta_* \mathcal{L} = \omega_S^{-1} \otimes \mathcal{I}_F^2.$$

Using the pair of sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{I}_F^2 \longrightarrow \mathcal{O}_{\widetilde{S^{[2]}} \times S} \longrightarrow \mathcal{O}_{2F} \longrightarrow 0, \text{ and} \\ 0 \longrightarrow \mathcal{I}_F / \mathcal{I}_F^2 \longrightarrow \mathcal{O}_{2F} \longrightarrow \mathcal{O}_F \longrightarrow 0, \end{aligned} \tag{3.23}$$

and tensoring by ω_S^{-1} gives (in the Grothendieck group)

$$\eta_* \mathcal{L} = \omega_S^{-1} \otimes \left(\mathcal{O}_{\widetilde{S^{[2]}} \times S} - \mathcal{O}_F - \mathcal{I}_F / \mathcal{I}_F^2 \right). \tag{3.24}$$

On the component F_i of F, the conormal bundle $\mathcal{I}_F / \mathcal{I}_F^2$ splits as a direct sum

$$\pi_1^* \mathcal{O}_{E_i}(-E_i) \oplus \pi_2^* \mathcal{O}_{L_i}(1),$$

where $\pi_i \colon F_i \longrightarrow E_i \times L_i$ are the two projections. After substituting in (3.24) and pushing forward to $\widetilde{S^{[2]}}$, we get

$$\begin{aligned} \mathcal{A} &= \mathcal{O}^6 - \mathcal{O}_E^5 - \mathcal{O}_E(-E)^2 \\ &= \mathcal{O} + \mathcal{O}(-E)^3 + \mathcal{O}(-2E)^2. \end{aligned} \tag{3.25}$$

Subtracting (3.22) from (3.25) and tensoring by $\mathcal{O}(E)$ gives the result. $\qquad \square$

Having computed the class of $\mathcal{V}$, what remains is a computation in the Chow ring of $\widetilde{S^{[2]}}$. In the computation, we suppress pull-back symbols to ease notation. We have the following simplification.

Proposition 3.4.8. *Let $u_i = c_i(\mathcal{U})$ for $i = 1, 2$. Then $u_1 \cdot E = u_2 \cdot E = 0$.*

Proof. We show that $u_i \in A^i(S^{[2]})$ is supported on cycles disjoint from $\Lambda = \bigsqcup_i L_i^{[2]}$.

For u_2, fix a general $C \in |\omega_S^{-1}|$. Then C intersects each exceptional curve L_i transversely in a single point. Then u_2 is the class of $C^{[2]} \subset S^{[2]}$, which is clearly disjoint from Λ.

For u_1, fix a general pencil of curves $\{C_t \mid t \in \mathbf{P}^1\} \subset |\omega_S^{-1}|$. Then for every $t \in \mathbf{P}^1$, the curve C_t intersects each exceptional curve L_i transversely in a single point. Then u_1 is the class of

$$\{[Z] \in S^{[2]} \mid Z \subset C_t \text{ for some } t \in \mathbf{P}^1\},$$

which is clearly disjoint from Λ. $\qquad\square$

We now compute the non-trivial degree 4 intersection numbers.

Proposition 3.4.9. *On $\widetilde{S^{[2]}}$, we have*

$$\int u_1^4 = 36, \qquad \int u_1^2 u_2 = 15, \qquad \int u_2^2 = 10, \qquad \int E^4 = -30.$$

Proof. We know explicit cycles on $S^{[2]}$ that represent u_1 and u_2 (see the proof of Proposition 3.4.8). Using these, we convert the intersection numbers into enumerative problems, which we solve.

The number $\int u_1^4$ is the answer to the following problem. Choose 4 general anti-canonical pencils on S, and let $f \colon S \dashrightarrow (\mathbf{P}^1)^4$ be the rational map induced by them. Then u_1^4 is the number of double points of the map f (pairs of points in S that have the same image under f). We find that the number is 36 by applying the double point formula [17, Theorem 2] to a resolution of f.

The number $\int u_1^2 u_2$ is the answer to the following problem. Fix a general anti-canonical $C \subset S$ and two general anti-canonical pencils. Then $\int u_1^2 u_2$ is the number of double points of the map $C \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ induced by the two pencils. Using the genus of C, which is 1, and the arithmetic genus of $f(C)$, which is 16, we see that the number is 15.

The number $\int u_2^2$ is the number of pairs of points common to two general anti-canonical curves C_1 and C_2. Since there are 5 points in $C_1 \cap C_2$, the number of pairs is $\binom{5}{2} = 10$.

Finally, to compute $\int E^4$, observe that E is the disjoint union of E_i for $i = 1, \ldots, 10$. Each E_i is the $\mathbf{P}^1$-bundle over L_i defined by

$$E_i = \mathbf{P} N_{L_i^{[2]}/S^{[2]}},$$

and the restriction of E_i to E_i is $\mathcal{O}(-1)$. We have $L_i^{[2]} \cong \mathbf{P}^2$ and

$$N_{L_i^{[2]}/S^{[2]}} = N_{L_i/S}^{[2]} \cong \mathcal{O}_{\mathbf{P}^2}(-1) \oplus \mathcal{O}_{\mathbf{P}^2}(-1).$$

Now it is a straightforward to check that $E_i^4 = -3$, and hence $E^2 = -30$. $\quad\square$

Proposition 3.4.10. *Let $\mathcal{V} = \pi_* \left(\omega_\pi^{-1} \right)$ be the anti-canonical section bundle of $\pi \colon \mathcal{X} {\longrightarrow} \widetilde{S^{[2]}}$, and let $v_i = c_i(\mathcal{V})$. Then we have*

$$\int_{B_4} v_1^4 = 6, \qquad \int_{B_4} v_1^2 v_2 = 21, \qquad \int_{B_4} v_1 v_3 = 6, \qquad \int_{B_4} v_2^2 = 16, \qquad \int_{B_4} v_4 = 1.$$

Proof. Follows from the expression for $\mathcal{V}$ in the Grothendieck group we found in Proposition 3.4.7 and the Chern class computations we did in Proposition 3.4.9 and Proposition 3.4.10. $\quad\square$

Let $\mu \colon \widetilde{S^{[2]}} \dashrightarrow \mathcal{M}$ be the map to the moduli space of cubic surfaces induced by the family $\pi \colon \mathcal{X} {\longrightarrow} \widetilde{S^{[2]}}$.

Proposition 3.4.11. *The degree of $\mu \colon \widetilde{S^{[2]}} \dashrightarrow \mathcal{M}$ is $36 \times 6!$.*

Proof. Recall that the moduli space of marked cubic surfaces $\mathcal{M}^\dagger$ is the configuration space of 6 ordered points in $\mathbf{P}^2$. The map μ clearly lifts to a birational map

$$\widetilde{S^{[2]}} \dashrightarrow \mathcal{M}^\dagger / S_2,$$

where the S_2 permutes the last 2 points of the configuration. Since the degree of $\mathcal{M}^\dagger {\longrightarrow} \mathcal{M}$ is $72 \times 6!$, the degree of μ is half of that. $\quad\square$

Using Proposition 2.4.5 and the computation of the Chern classes and the degree, we obtain the following relation on the undetermined coefficients of $[\overline{\mathrm{Orb}}(X)]$ in (2.3) for a general cubic surface X:

$$6a_{1^4} + 21a_{1^2 2} + 6a_{13} + 16a_{2^2} + a_4 = 36 \times 6!. \tag{3.26}$$

3.5. **Isotrivial families.** In this section, we continue the theme of constructing good families, but with a twist. Let $X \subset \mathbf{P}^3$ be a cubic surface with an automorphism group G. We then get a family of cubic surfaces

$$\pi \colon [X/G] {\longrightarrow} BG,$$

and hence a map from BG to the moduli stack of cubic surfaces

$$\mu \colon BG {\longrightarrow} \mathscr{M}.$$

Suppose we know that the family π is good. In this case, this simply means that X is not in the closure of a general cubic surface S. Then, we get

$$\mu^* [\overline{\mathrm{Orb}}(S)] = 0,$$

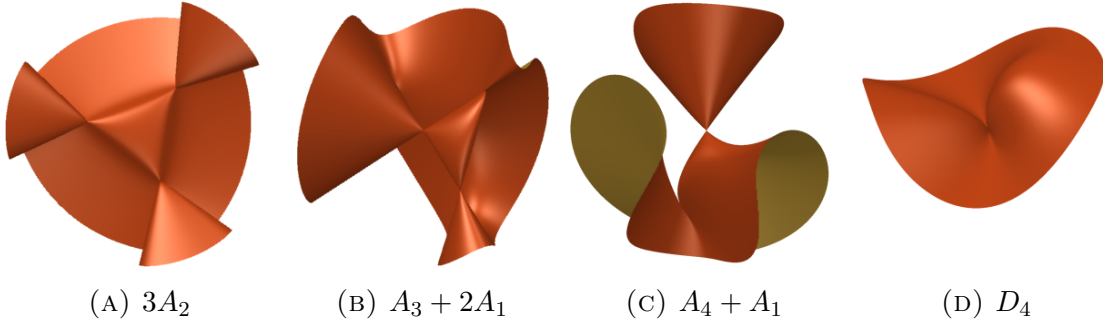(A) $3A_2$ (B) $A_3 + 2A_1$ (C) $A_4 + A_1$ (D) D_4

FIGURE 6. Some cubic surfaces with a $\mathbf{G}_m$-action that are not in the closure of the orbit of a generic cubic surface. We used the 3D grapher SURFER for these images.

which in turn gives a linear relation among the coefficients of the expression for $[\overline{\mathrm{Orb}}(S)]$ pulled back to $A^4(BG)$. To get a useful relation, the group $A^4(BG)$ must be rich. In particular, we must take G to be infinite; otherwise, $A^4(BG)$ is torsion, and we only get a congruence relation.

If we wish to obtain a family over a schematic base instead of BG, we can easily do so. We consider an arbitrary scheme B with the free action of G such that the quotient B/G is a scheme, and take the family to be

$$\pi \colon (X \times B)/G \longrightarrow B/G,$$

where G acts on $X \times B$ diagonally. In particular, for $G = \mathbf{G}_m$, the only group we use, we can take B to be an arbitrary $\mathbf{G}_m$-torsor (line bundle minus the zero section) over a scheme. See § 3.10 for an example.

To implement the strategy outlined above, we need to prove that the cubic surface X does not lie in the closure of the orbit of a general cubic surface. Following Definition 2.4.1, say that X is *good* if it has this property. Proposition 5.1.3 proves that the following cubic surfaces are good (the parenthesis describes the singularities):

(1) $x_0 x_1 x_3 = x_2^3$ $(3A_2)$
(2) $x_3(x_0 x_2 - x_1^2) = x_0 x_1^2$ $(A_3 + 2A_1)$
(3) $x_3(x_0 x_2 - x_1^2) = x_0^2 x_1$ $(A_4 + A_1)$
(4) $x_3 x_0^2 = x_1^3 + x_2^3$ (D_4)

For the reader's amusement, we include real pictures of the surfaces above in Figure 6.

Let $X \subset \mathbf{P}^3$ be a good cubic surface. Let G be the automorphism group of X and $\mathbf{G}_m \longrightarrow G$ a one-parameter subgroup. The family

$$[X/\mathbf{G}_m] \longrightarrow B\mathbf{G}_m$$

yields a relation

$$a_{1^4} v_1^4 + a_{1^2 \cdot 2} v_1^2 v_2 + a_{1 \cdot 3} v_1 v_3 + a_{2^2} v_2^2 + a_4 v_4 = 0, \tag{3.27}$$

where the v_i are the Chern classes of the push-forward of the anti-canonical bundle. Let us explain how to compute the push-forward of the anti-canonical bundle in this setting. For an integer a, let $\chi(a)$ denote the 1-dimensional $\mathbf{G}_m$ representation where the $\mathbf{G}_m$ action is given by

$$t \colon v \mapsto t^a v.$$

Suppose $X \subset \mathbf{P}^3$ is the zero-locus of a homogeneous cubic form $F \in k[x_0, x_1, x_2, x_3]$. Assume that the $\mathbf{G}_m$ acts on the variables x_i by scaling, say $t \in \mathbf{G}_m$ acts by

$$t \colon x_i \mapsto t^{w_i} x_i,$$

where $w_i \in \mathbf{Z}$. Since $V(F)$ is $\mathbf{G}_m$-fixed, there exists $w \in \mathbf{Z}$ such that

$$F(tx_0, tx_1, tx_2, tx_3) = t^w F(x_0, x_1, x_2, x_3).$$

Write $V^\vee$ for the k-vector space $\langle x_0, x_1, x_2, x_3 \rangle$. As a $\mathbf{G}_m$-representation, we have

$$V^\vee = \chi(w_1) \oplus \chi(w_2) \oplus \chi(w_3) \oplus \chi(w_4).$$

The cubic F defines a $\mathbf{G}_m$-invariant section of $\mathrm{Sym}^3 V^\vee \otimes \chi(-w)$. Thus, we can view X as the zero locus of the line bundle $\mathcal{O}(3) \otimes \pi^* \chi(-w)$ in the projective bundle $\pi \colon [\mathbf{P}V/\mathbf{G}_m] \longrightarrow B\mathbf{G}_m$. By the adjunction formula, we get

$$\omega_X^{-1} = \mathcal{O}_{\mathbf{P}V}(1) \otimes \det V \otimes \pi^* \chi(w)|_X,$$

and hence

$$\begin{aligned} \pi_* \left(\omega_X^{-1} \right) &= V^\vee \otimes \det V \otimes \chi(w) \\ &\cong \left(\bigoplus \chi(w_i) \right) \otimes \chi \left(-\sum w_i \right) \otimes \chi(w). \end{aligned} \tag{3.28}$$

3.6. The family $3A_2$. Consider

$$X = V(x_0 x_1 x_3 - x_2^3).$$

A $\mathbf{G}_m$ stabilizing X acts on the variables by weights $(a, b, 0, c)$ where $a, b, c \in \mathbf{Z}$ are any such that $a + b + c = 0$. Note that in this case, the weight w of the cubic equation defining X is 0. Hence, from (3.28), we get

$$\pi_* \left(\omega_X^{-1} \right) = \chi(a) \oplus \chi(b) \oplus \chi(0) \oplus \chi(c).$$

Letting $q = c_1(\chi(1)) \in A^1(B\mathbf{G}_m)$, we get

$$v_1 = 0, \quad v_2 = (ab + bc + ca) \cdot q^2, \quad v_3 = abc \cdot q^3, \quad v_4 = 0,$$

and hence

$$v_1^4 = 0, \quad v_1^2 v_2 = 0, \quad v_1 v_3 = 0, \quad v_2^2 \neq 0, \quad v_4 = 0.$$

Substituting in (3.27) yields the relation

$$a_{2^2} = 0. \tag{3.29}$$

3.7. The family $A_3 + 2A_1$. Consider

$$X = V(x_3(x_0 x_2 - x_1^2) - x_0 x_1^2).$$

A $\mathbf{G}_m$ stabilizing X acts on the variables by weights $(-3, 1, 5, -3)$. The weight w of the cubic equations defining X is -1. From (3.28), we get

$$\pi_* \left(\omega_X^{-1} \right) = \chi(-4) \oplus \chi(0) \oplus \chi(4) \oplus \chi(-4).$$

Setting $q = c_1(\chi(1)) \in A^1(B\mathbf{G}_m)$, we get

$$v_1 = -4q, \quad v_2 = -16q^2, \quad v_3 = 64q^3, \quad v_4 = 0,$$

and hence

$$v_1^4 = 256q^4 \quad v_1^2 v_2 = -256q^4, \quad v_1 v_3 = -256q^4, \quad v_2^2 = 256q^4, \quad v_4 = 0.$$

Substituting in (3.27) yields the relation

$$a_{1^4} - a_{1^2 2} - a_{13} + a_{2^2} = 0. \tag{3.30}$$

3.8. **The family $A_4 + A_1$.** Consider

$$X = V(x_3(x_0 x_2 - x_1^2) - x_0^2 x_1).$$

A $\mathbf{G}_m$ stabilizing X acts on the variables by weights $(1, -1, -3, 3)$. The weight w of the cubic defining X is 1. From (3.28), we get

$$\pi_* \left(\omega_X^{-1} \right) = \chi(2) \oplus \chi(0) \oplus \chi(-2) \oplus \chi(4).$$

Setting $q = c_1(\chi(1)) \in A^1(B\mathbf{G}_m)$, we get

$$v_1 = 4q, \quad v_2 = -4q^2, \quad v_3 = -16q^3, \quad v_4 = 0,$$

and hence

$$v_1^4 = 256q^4 \quad v_1^2 v_2 = -64q^4, \quad v_1 v_3 = -64q^4, \quad v_2^2 = 16q^4, \quad v_4 = 0.$$

Substituting in (3.27) yields the relation

$$16a_{1^4} - 4a_{1^2 2} - 4a_{13} + a_{2^2} = 0. \tag{3.31}$$

3.9. **The family D_4.** Consider

$$X = V(x_3 x_0^2 - x_1^3 - x_2^3).$$

A $\mathbf{G}_m$ stabilizing X acts on the variables by weights $(5, 1, 1, -7)$. The weight w of the cubic equation defining X is 3. By (3.28), we get

$$\pi_* \left(\omega_X^{-1} \right) = \chi(8) \oplus \chi(4) \oplus \chi(4) \oplus \chi(-4).$$

Setting $q = c_1(\chi(1))$, we get

$$v_1 = 12q, \quad v_2 = 16q^2, \quad v_3 = -192q^3, \quad v_4 = -512q^4,$$

and hence

$$v_1^4 = 20736q^4 \quad v_1^2 v_2 = 2304q^4, \quad v_1 v_3 = -2304q^4, \quad v_2^2 = 256q^4, \quad v_4 = -512q^4.$$

Substituting in (3.27) yields the relation

$$81a_{1^4} + 9a_{1^2 2} - 9a_{13} + a_{2^2} - 2a_4 = 0. \tag{3.32}$$

3.10. **Families over a base scheme.** By choosing a suitable base B and a map $B \longrightarrow B\mathbf{G}_m$, we can construct an isotrivial family over B by pulling back one of the families above. We describe this explicitly in an example.

Let $B = \mathbf{P}^4$. Set

$$V^\vee = \mathcal{O}(5) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-7).$$

Let x_0, x_1, x_2, x_3 be generators of the four summands on the standard $\mathbf{A}^4 \subset \mathbf{P}^4$. We have the section $(x_3 x_0^2 - x_1^3 - x_2^3)$ of $\mathrm{Sym}^3 V^\vee = \mathcal{O}(3)^{\oplus 20}$. Note that this section has a pole of order 3 along the hyperplane at infinity. As a result, it defines a section ξ of $\mathrm{Sym}^3 V^\vee \otimes \mathcal{O}(-3)$, which is nowhere vanishing. Equivalently, ξ is a section of $\mathcal{O}_{\mathbf{P}V}(3) \otimes \pi^* \mathcal{O}(-3)$. Our family $\mathcal{X} \longrightarrow B$ is defined by the zero-locus of ξ in $\mathbf{P}V \longrightarrow B$.

4. THE EQUIVARIANT ORBIT CLASS

We use the information provided by the test families to prove the theorems advertised in the introduction. Let V be a 4 dimensional vector space. Recall our notation W for the $\mathrm{GL}\,V$ representation

$$W = \mathrm{Sym}^3 V^\vee \otimes \det V$$

and

$$\mathcal{M} = [W \setminus \{0\} / \mathrm{GL}\,V]$$

for the moduli stack of cubic surfaces.

Theorem 4.0.1. *There exists a non-empty open subset $U \subset W$ such that for every cubic surface X represented by a point of U, the class of $\overline{\mathrm{Orb}}(X)$ in $A^4_{\mathrm{GL}\,V}(W)$ is given by*

$$[\overline{\mathrm{Orb}}(X)] = 1080 \cdot \left(v_1^2 v_2 - v_1 v_3 + 9 v_4\right),$$

where $v_i = c_i(V)$ are the Chern classes of the standard representation of $\mathrm{GL}\,V$.

Proof. By excision and homotopy invariance of equivariant Chow groups, we have

$$A^4(\mathcal{M}) = A^4_{\mathrm{GL}\,V}(W) = A^4_{\mathrm{GL}\,V}(\bullet).$$

The ring $A^*_{\mathrm{GL}\,V}(\bullet)$ is generated by the classes v_i, and hence $A^4_{\mathrm{GL}\,V}(\bullet)$ is a free $\mathbf{Z}$-module generated by the monomials in v_i of total degree 4. Therefore, for every cubic surface X, we have an expression

$$[\overline{\mathrm{Orb}}(X)] = a_{1^4} v_1^4 + a_{1^2 \cdot 2} v_1^2 v_2 + a_{1 \cdot 3} v_1 v_3 + a_{2^2} v_2^2 + a_4 v_4,$$

for some coefficients $a_i \in \mathbf{Z}$. For a generic X, the families in § 3 give the following linear relations among the coefficients

$$16 \cdot a_{1^4} + 4 \cdot a_{1^2 2} + a_{2^2} = 4320 \text{ from } (3.5)$$

$$625 a_{1^4} + 125 a_{1^2 2} - 25 a_{13} + 25 a_{2^2} - 6 a_4 = 103680 \text{ from } (3.9)$$

$$3436 a_{1^4} + 1076 a_{1^2 2} + 116 a_{13} + 316 a_{2^2} = 1036800 \text{ from } (3.14)$$

$$6 a_{1^4} + 21 a_{1^2 2} + 6 a_{13} + 16 a_{2^2} + a_4 = 25920 \text{ from } (3.26)$$

$$a_{2^2} = 0 \text{ from } (3.29)$$

$$a_{1^4} - a_{1^2 2} - a_{13} + a_{2^2} = 0 \text{ from } (3.30)$$

$$16 a_{1^4} - 4 a_{1^2 2} - 4 a_{13} + a_{2^2} = 0 \text{ from } (3.31)$$

$$81 a_{1^4} + 9 a_{1^2 2} - 9 a_{13} + a_{2^2} - 2 a_4 = 0 \text{ from } (3.32).$$

These equations uniquely determine all the coefficients a_i to be as stated.   □

Corollary 4.0.2. *The degree of the closure of the* PGL_4 *orbit of a generic cubic surface in* $\mathbf{P}^3$ *is* 96120.

Proof. Let $\mathbf{P}^3 = \mathbf{P}V$. Consider the universal cubic surface $X \subset \mathbf{P}V \times \mathbf{P} \operatorname{Sym}^3 V^\vee$. Let $\pi \colon \mathcal{X} \longrightarrow \mathbf{P} \operatorname{Sym}^3 V^\vee$ be the second projection and $\mu \colon \mathbf{P} \operatorname{Sym}^3 V^\vee \longrightarrow \mathcal{M}$ the map induced by π. Tautologically, for any cubic surface X, the pre-image under μ of $\overline{\mathrm{Orb}}(X)$ is the closure of the $\mathrm{PGL}(V)$ orbit of X. By adjunction, we have

$$\omega_\pi = \mathcal{O}(1) \boxtimes \mathcal{O}(1),$$

and hence

$$\mathcal{V} = \pi_* \omega_\pi \cong \mathcal{O}(1)^4.$$

For a generic X, we evaluate

$$[\overline{\mathrm{Orb}}(X)] = 1080(v_1^2 v_2 - v_1 v_3 + 9 v_4)$$

in $A^4(\mathbf{P} \operatorname{Sym}^3 V^\vee)$ and find that the answer is 96120 (times the class of a codimension 4 linear subspace).   □

Corollary 4.0.3. *Let* $X \subset \mathbf{P}^4$ *be a general cubic 3-fold, and let* $\pi \colon \mathcal{X} \longrightarrow \mathbf{P}^4$ *be the family of hyperplane sections of* X. *Given a generic cubic surface* S, *there are* 42120 *fibers of* π *isomorphic to* S.

Proof. Note that two cubic surfaces are isomorphic if and only if they are in the same PGL_4 orbit. Consider the pull-back of $\overline{\mathrm{Orb}}(S)$ to the base $\mathbf{P}^4$ of the family. By the genericity of X and S, this pull-back consists of distinct reduced points

corresponding to the points of $\overline{\mathrm{Orb}}(S) \setminus \partial\,\overline{\mathrm{Orb}}(S)$. The statement now follows from Theorem 4.0.1 and an easy Chern class computation, which we omit.   □

4.1. **Proof of Theorem 1.0.1 from the introduction.** The theorem follows immediately from Theorem 4.0.1 and Proposition 2.4.5.

4.2. **Proof of Theorem 1.2.1 from the introduction.** Let V be a 4-dimensional vector space. Set

$$\mathscr{N} = \left[\mathrm{Sym}^3 V^\vee \setminus \{0\} / \mathrm{GL}\, V\right].$$

Let $\mathcal{V}$ denote the vector bundle on $\mathscr{N}$ corresponding to the standard representation of $\mathrm{GL}\, V$. The tautological non-zero section of $\mathrm{Sym}^3 V^\vee$ on $\mathscr{N}$ defines a family of cubic surfaces

$$\pi \colon \mathscr{X} \longrightarrow \mathscr{N},$$

and hence a map $\mathscr{N} \longrightarrow \mathscr{M}$. Under this map, the pull-back of an orbit closure is an orbit closure. Note that $\mathscr{X} \subset \mathbf{P}\mathcal{V}$ is the zero locus of $\mathcal{O}(3)$. By adjunction, we have

$$\omega_\pi = \mathcal{O}(1) \otimes \det V|_{\mathscr{X}},$$

and hence

$$\pi_* \omega_\pi = V^\vee \otimes \det V.$$

Theorem 4.0.1 gives the class of the pull-back of a generic orbit closure in terms of the Chern classes $v_i = c_i(V^\vee \otimes \det V)$. Writing them in terms of $c_i = c_i(V)$ yields the result.

5. Addressing the orbit closure problem

We have repeatedly faced the problem of showing that a certain cubic surface is not in the orbit closure of a certain (generic) cubic surface. In this section, we develop two sets of techniques to address this question.

5.1. **Using GIT.** The first technique uses GIT and applies broadly to any orbit closure problem. Let W be a quasi-projective variety with a linearized action of a reductive group G. The basic observation is that if $x \in W$ is semi-stable and $s \in W$ is stable, then x does not lie in the orbit closure of s. We prove an extension of this idea that turns out to be highly effective.

Let G be a reductive group. Let W and H be smooth quasi-projective varieties along with an action of G. Consider the diagonal action of G on $W \times H$ and fix any G-linearization of this action.

Proposition 5.1.1. *With the setup above, let $x, s \in W$ be points satisfying the following conditions:*

    (1) x has a reductive stabilizer $T \subset G$.
    (2) for some $h \in H$ fixed by the identity component of T, the point $(x, h) \in W \times H$ is semi-stable.
    (3) for all $h \in H$, the point $(s, h) \in W \times H$ is stable.

Then x does not lie in the closure of the G-orbit of s.

The proof needs the following lemma. It is well-known to experts—it appears as a remark in [2] (see § 1.3, Immediate consequences: 5)—but we include a proof for completeness. Recall that k is an algebraically closed field of characteristic 0, all schemes considered are of finite type over k, and a point means a k-point.

Lemma 5.1.2. *Let U be a smooth scheme over k with an action of a linear algebraic group G. Let $x \in U$ be a point whose stabilizer group $T \subset G$ is reductive. If x lies in the closure of the G-orbit of a point $s \in U$, then there exists a one-parameter subgroup $\lambda \colon \mathbf{G}_m \longrightarrow T$ and a point s' in the G-orbit of s such that*

$$x = \lim_{t \longrightarrow 0} \lambda(t) \cdot s'.$$

Proof. By [1, Theorem 3], there exists a T-invariant locally closed affine $W \subset U$ containing the point x such that the map

$$\pi \colon [W/T] \longrightarrow [U/G]$$

is affine and étale. (Note that this result is substantially easier than the main theorem of [2].) Define Z as the fiber product

$$
\begin{array}{ccc}
Z & \longrightarrow & [W/T] \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec} k & \xrightarrow{\ s\ } & [U/G].
\end{array}
$$

Since π is representable, étale, and of finite type, Z is a finite disjoint of copies of $\operatorname{Spec} k$. Let $z_1, \ldots, z_n \in [W/T](k)$ be the points of Z. Since the point $x \in [U/G](k)$ lies in the closure of $s \in [U/G](k)$ and the map π is étale (and hence open), the point $x \in [W/T](k)$ lies in the closure of the set $\{z_1, \ldots, z_n\}$. But then x lies in the closure of z_i for some i. In other words, the point $x \in W(k)$ lies in the closure of the T-orbit of a lift $s_i \in W(k)$ of $z_i \in [W/T](k)$. By the Hilbert–Mumford criterion [20, Theorem 1.4], there exists a one-parameter subgroup $\lambda \colon \mathbf{G}_m \longrightarrow T$ such

that we have the equation

$$x = \lim_{t \longrightarrow 0} \lambda(t) \cdot s_i. \tag{5.1}$$

Since $s_i \in [W/T](k)$ maps to $s \in [U/G](k)$, the point $s_i \in W$ is in the same G-orbit as $s \in U$. The proof is now complete. $\square$

Proof of Proposition 5.1.1. Let $T_0 \subset T$ be the connected component of the identity. We prove the contrapositive. Assuming that x lies in the closure of the G-orbit of s, we show that for any T_0-fixed $h \in H$, the point $(x, h) \in W \times H$ is unstable.

By Lemma 5.1.2, there exists a one parameter subgroup $\lambda \colon \mathbf{G}_m \longrightarrow T_0$ and a point $s' \in W$ in the same G-orbit as s such that

$$x = \lim_{t \longrightarrow 0} \lambda(t) \cdot s'.$$

For any T_0-fixed $h \in H$ we have

$$(x, h) = \lim_{t \longrightarrow 0} \lambda(t) \cdot (s', h).$$

Since (s', h) is G-stable, (x, h) must be G-unstable. $\square$

We immediately get an application to cubic surfaces.

Proposition 5.1.3. *The following singular cubic surfaces are not in the closure of the* PGL_4*-orbit of any smooth cubic surface without an Eckardt point (the parenthesis describes the singularities):*

(1) $x_0 x_1 x_3 = x_2^3$ $(3A_2)$
(2) $x_3(x_0 x_2 - x_1^2) = x_0 x_1^2$ $(A_3 + 2A_1)$
(3) $x_3(x_0 x_2 - x_1^2) = x_0^2 x_1$ $(A_4 + A_1)$
(4) $x_3 x_0^2 = x_1^3 + x_2^3$ (D_4)

Proof. Let V be a 4-dimensional vector space. Set $W = \mathbf{P} \operatorname{Sym}^3 V^\vee$ and $H = \mathbf{P} V^\vee$; both have natural linearized actions of $\mathrm{PGL}(V)$. The product $W \times H$ parametrizes cubic surfaces along with a hyperplane. We use the GIT analysis for this space carried out by Gallardo and Martinez-Garcia [18]. We have a 1-parameter choice of linearizations on $W \times V$ indexed by a positive rational number t. The G-linearized line bundle corresponding to t is $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$, where a and b are (sufficiently divisible) positive integers satisfying $b/a = t$.

From [25, Theorem 3], we see that the listed cubic surfaces have a reductive stabilizer group. In fact, in all cases the connected component of the identity is a torus.

Let S be a smooth cubic surface without an Eckardt point corresponding to a point $s \in W$. Suppose we have $0 < t < 5/9$. From [18, Theorem 2], we see that for

any $h \in H$, the point (s, h) is t-stable. On the other hand, from [18, Table 2], we see that for some t in the range $0 < t < 5/9$, there exists a choice of $h \in H$, fixed by the connected component of the identity of the stabilizer of s, such that (s, h) is t-semistable. We now apply Proposition 5.1.1.

In the following table, we list the automorphism groups, the hyperplane h, and the value of t for which (s, h) is t-semistable. We denote the symmetric group on n letters by Σ_n. The automorphism groups are from [25] and the GIT semi-stability assertions are from [18].

X	Singularities of X	$\mathrm{Aut}(X)$	h	t
$x_0 x_1 x_3 = x_2^3$	$3A_2$	$\mathbf{G}_m^2 \rtimes \Sigma_3$	$x_2 = 0$	All $t \in (0, 1)$
$x_3(x_0 x_2 - x_1^2) = x_0 x_1^2$	$A_3 + 2A_1$	$\mathbf{G}_m \rtimes \Sigma_2$	$x_2 = 0$	$1/5$
$x_3(x_0 x_2 - x_1^2) = x_0^2 x_1$	$A_4 + A_1$	$\mathbf{G}_m$	$x_2 = 0$	$1/3$
$x_3 x_0^2 = x_1^3 + x_2^3$	D_4	$\mathbf{G}_m \rtimes \Sigma_3$	$x_3 = 0$	$3/7$

$\square$

5.2. **Using 6 points in $\mathbf{P}^2$.** The second technique is specific to cubic surfaces. We first reduce to the case of 6 points in $\mathbf{P}^2$, and then use the geometry of 1-parameter families of automorphisms of $\mathbf{P}^2$. This method can show, for example, that the cubic with the A_5 singularity does not lie in the closure of the orbit of a generic cubic. (This particular surface had been immune to all our previous strategies.)

Proposition 5.2.1. *Let $W \subset \mathbf{P}^2$ be an admissible length 6 subscheme and let $X = X_W$ be the cubic surface associated to W. Let $S \subset \mathbf{P}^2$ be a smooth cubic surface, and suppose X lies in the PGL_4-orbit closure of S. Then W lies in the PGL_3 orbit closure of a length 6 scheme $Z \subset \mathbf{P}^2$ such that $\mathrm{Bl}_Z \mathbf{P}^2$ is isomorphic to S.*

Proof. Since W is in the orbit closure of S, there exists a k-algebra DVR R with fraction field K and residue field k with a family of cubic surfaces $\pi \colon \mathcal{X} \longrightarrow \mathrm{Spec}\, R$ whose central fiber $\mathcal{X}_0$ is isomorphic to X and whose general fiber $\mathcal{X}_K$ is isomorphic to $S_K = S \times_k K$. We know that X has only ADE singularities (Proposition 2.5.2) and hence we can resolve the singularities of X in the family. That is, there exists a smooth and projective $\pi' \colon \mathcal{X}' \longrightarrow \mathrm{Spec}\, R$ and $\beta \colon \mathcal{X}' \longrightarrow \mathcal{X}$ such that β_K is an isomorphism and β_0 is the minimal resolution of singularities; see [8] or [29].

Set $X' = \mathcal{X}'_0$. We know from the proof of Proposition 2.5.5 that X' is also the minimal desingularization of $\mathrm{Bl}_Z \mathbf{P}^2$. Let $X' \longrightarrow \mathbf{P}^2$ be the composite $X' \longrightarrow \mathrm{Bl}_Z \mathbf{P}^2 \longrightarrow \mathbf{P}^2$, and let L' be pullback of $\mathcal{O}(1)$ to $\mathcal{X}'_0$. It is easy to see that $h^i(X', L') = 0$ for $i > 0$ and $h^0(X', L') = 3$. Since $H^2(X', \mathcal{O}_{X'}) = 0$, there are no obstructions to deforming

L'—it extends to compatible family of line bundles on $\mathcal{X}' \times_R \operatorname{Spec} R/t^n$ for every n, where $t \in R$ is the uniformizer. By Artin's approximation theorem [6] applied to the relative Picard functor of π', we deduce that L' extends to a line bundle $\mathcal{L}$ on $\mathcal{X}'$, after possibly replacing R by a finite étale cover. By cohomology and base change, we see that $\pi'_*\mathcal{L}$ is a free R-module of rank 3. It is also easy to see that the evaluation map $\pi'^*\pi'_*\mathcal{L} \longrightarrow \mathcal{L}$ is surjective. After choosing an isomorphism $\pi'_*\mathcal{L} \cong R^3$, we get a map $\alpha\colon \mathcal{X}' \longrightarrow \mathbf{P}^2_R$, which restricts to the original map $X' \longrightarrow \mathbf{P}^2$ on the central fiber.

Recall that $\mathcal{X}'_K = \mathcal{X}_K = S \times_k K$. Since $H^1(S, \mathcal{O}_S) = 0$, it follows that the line bundle $\mathcal{L}$ on $S \times_k K$ is constant, that is, isomorphic to $L \otimes_k K$ for some line bundle L on S. The complete linear system $|L|$ gives a map $a\colon S \longrightarrow \mathbf{P}^2$. It is easy to check that the map $\alpha\colon \mathcal{X}'_K \longrightarrow \mathbf{P}^2_K$ and $a_K\colon \mathcal{X}'_K \longrightarrow \mathbf{P}^2_K$ are equal, up to an automorphism of $\mathbf{P}^2_K$.

The natural map $\alpha^*\omega_{\mathbf{P}^2_R/R} \longrightarrow \omega_{\mathcal{X}'/R}$ gives a map $\omega_{\mathcal{X}'/R}^{-1} \longrightarrow \alpha^*\omega_{\mathbf{P}^2_R/R}^{-1}$ and hence an evaluation map

$$\pi'_*\left(\omega_{\mathcal{X}'/R}\right) \otimes_R \mathcal{O}_{\mathbf{P}^2_R} \longrightarrow \omega_{\mathbf{P}^2_R/R}^{-1}.$$

Tensor by $\omega_{\mathbf{P}^2_R/R}$ to get

$$\pi'_*\left(\omega_{\mathcal{X}'/R}\right) \otimes_R \omega_{\mathbf{P}^2_R/R} \longrightarrow \mathcal{O}_{\mathbf{P}^2_R}.$$

Then the cokernel is $\mathcal{O}_{\mathcal{Z}}$ for a closed subscheme $\mathcal{Z} \subset \mathbf{P}^2_R$. Note that both the general and the special fiber of $\mathcal{Z} \longrightarrow \operatorname{Spec} R$ have length 6, and hence $\mathcal{Z} \longrightarrow \operatorname{Spec} R$ is flat. The central fiber is the subscheme $W \subset \mathbf{P}^2$ and, up to an automorphism of $\mathbf{P}^2_K$, the general fiber is $Z \times_k K \subset \mathbf{P}^2_K$, where $Z \subset \mathbf{P}^2$ is such that $S \cong \operatorname{Bl}_Z \mathbf{P}^2$. The proof is now complete. $\qquad\square$

5.2.1. *The anatomy of linear rational maps* $\mathbf{P}^2_t \dashrightarrow \mathbf{P}^2_t$. To study orbit closures of collections of points in $\mathbf{P}^2$, we must understand one parameter families of automorphisms of $\mathbf{P}^2$.

Let R be a k-algebra DVR with residue field k, fraction field K, and uniformizer t. For an R-module M, we set $M_K = M \otimes_R K$ and $M_0 = M \otimes_R k$. Let $\phi\colon \mathbf{P}^2_K \longrightarrow \mathbf{P}^2_K$ be an isomorphism. We analyze the corresponding rational map $\mathbf{P}^2_R \dashrightarrow \mathbf{P}^2_R$. The analysis should hold for $\mathbf{P}^n$ for any n; we stick to $n = 2$ to keep us focused.

Let M and N be free R-modules of rank 3, and suppose $\phi\colon M_K \longrightarrow N_K$ is a K-linear isomorphism. By scaling ϕ by the correct power of the uniformizer, assume that ϕ induces a map $\phi\colon M \longrightarrow N$ such that $\phi_0\colon M_0 \longrightarrow N_0$ is non-zero. By the *rank* of ϕ, we mean the rank of ϕ_0.

By making a change of basis on M and N, we may assume that ϕ is given by a diagonal matrix

$$\phi \equiv \begin{pmatrix} 1 & & \\ & t^a & \\ & & t^b \end{pmatrix},$$

where $0 \leq a \leq b$ are integers.

If $a = b = 0$, then there is nothing to analyze.

Suppose $a = 0$ and $b > 0$, or equivalently, $\operatorname{rk} \phi_0 = 2$. In this case, the rational map ϕ is defined away from the point $[0 : 0 : 1]$ in $\mathbf{P}M_0$. It sends the point $[a : b : c]$ to the point $[a : b : 0]$, which lies on the line $Z = 0$. Said in a coordinate free manner, the rational map ϕ is defined away from a particular point $p \in \mathbf{P}M_0$. For $q \in \mathbf{P}M_0 \setminus \{0\}$, the image $\phi(q)$ lies on a particular line $L \subset \mathbf{P}N_0$, and the map

$$\phi \colon \mathbf{P}M_0 \setminus \{p\} \longrightarrow L \tag{5.2}$$

is the linear projection from p followed by an isomorphism.

Suppose $a > 0$, or equivalently $\operatorname{rk} \phi_0 = 1$. In this case, the rational map ϕ is defined away from the line $X = 0$ in $\mathbf{P}M_0$. It sends the point $[a : b : c]$ to the point $[1 : 0 : 0]$. Said in a coordinate free manner, the rational map ϕ is defined away from a particular line $L \subset \mathbf{P}M_0$ and the map

$$\phi \colon \mathbf{P}M_0 \setminus L \longrightarrow \mathbf{P}N_0 \tag{5.3}$$

is constant.

Using the matrix of ϕ, we can explicitly construct a resolution of the rational map $\phi \colon \mathbf{P}^2_R \dashrightarrow \mathbf{P}^2_R$ using a sequence of elementary transformations. We get the description of the maps (5.2) and (5.3) also from this resolution. Let $L \subset \mathbf{P}^2$ be the locus of indeterminacy of ϕ (with the reduced scheme structure). Let $\widetilde{P} \longrightarrow \mathbf{P}^2_R$ be the blow up of L. Then the central fiber of $\widetilde{P}$ is the union of $\operatorname{Bl}_L \mathbf{P}^2$ and the exceptional divisor E. If ϕ has rank 1, then L is a line, and these two components are $\mathbf{P}^2$ and the Hirzebruch surface $\mathbf{F}_1$, respectively, meeting along $L \subset \mathbf{P}^2$ and the unique (-1) curve on $\mathbf{F}_1$. If ϕ has rank 2, then L is a point, and these two components are $\mathbf{F}_1$ and $\mathbf{P}^2$, respectively, meeting along the (-1)-curve on $\mathbf{F}_1$ and a line in $\mathbf{P}^2$. In both cases, the first component can be contracted: to a point in the rank 1 case and to a line in the rank 2 case. The resulting threefold is again isomorphic to $\mathbf{P}^2_R$. The rational map $\mathbf{P}^2_R \dashrightarrow \mathbf{P}^2_R$ after this operation corresponds to the diagonal matrix with entries $(1, t^{a-1}, t^{b-1})$ in the rank 1 case and $(1, 1, t^{b-1})$ in the rank 2 case. Observe that the new locus of indeterminacy is disjoint from the image of the contracted component.
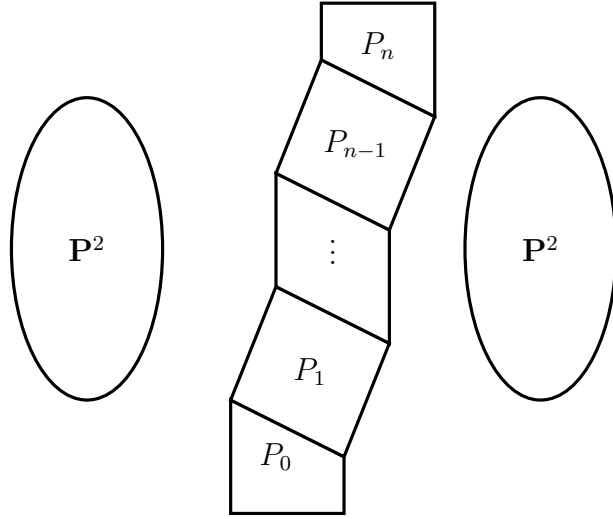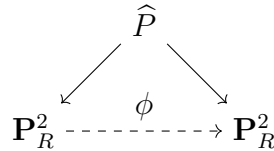
FIGURE 7. A family of surfaces with an accordion like central fiber arises while resolving a linear rational map $\mathbf{P}^2_R \dashrightarrow \mathbf{P}^2_R$.

We repeat the procedure until we reach the identity matrix. Alternatively, we can do all the blow-ups first, and then all the contractions. We then get a diagram

$$
\begin{array}{ccc}
 & \widehat{P} & \\
\swarrow & & \searrow \\
\mathbf{P}^2_R & \overset{\phi}{\dashrightarrow} & \mathbf{P}^2_R
\end{array}
$$

The central fiber of $\widehat{P}$ has an accordion like structure $P_0 \cup \cdots \cup P_n$ (see Figure 7), where each P_i is a copy of $\mathbf{F}_1$ for $i = 1, \ldots, n-1$, meeting the previous surface along the (-1)-curve and the next one along a $(+1)$-curve. The first surface P_0 is $\mathbf{F}_1$ if $a = 0$ (that is, ϕ has rank 2) or $\mathbf{P}^2$ if $a > 0$ (that is, ϕ has rank 1). Likewise, the last surface P_n is $\mathbf{F}_1$ if $a = b$ or $\mathbf{P}^2$ if $b > a$.

5.2.2. *Orbit closure of 6 general points in* $\mathbf{P}^2$. We use the analysis in § 5.2.1 to understand limits of a general set of six points in $\mathbf{P}^2$ under a one-parameter family of automorphisms. Let $Z \subset \mathbf{P}^2$ be a configuration of 6 distinct points, with no three collinear. A point $p \in \mathbf{P}^2 \setminus Z$ is a *star point* if, for some ordering $\{z_1, \ldots, z_6\}$ of Z, the lines $\langle z_1, z_2 \rangle$ and $\langle z_3, z_4 \rangle$ and $\langle z_5, z_6 \rangle$ are concurrent at p. A *star-free* configuration is one without any star points.

Proposition 5.2.2. *Let* $Z \subset \mathbf{P}^2$ *be a set of 6 points forming a star-free configuration. Suppose* $W \subset \mathbf{P}^2$ *is in the* PGL_3 *orbit closure of* Z. *Then there exists a line in* $\mathbf{P}^2$

containing at least 4 distinct points of W or there exists a point in $\mathbf{P}^2$ at which W has multiplicity at least 4.

Proof. There exists a k-algebra DVR R with fraction field K and $\phi \in \text{Aut}(\mathbf{P}^2_K)$ such that W is the flat limit of $\phi(Z_K)$. Consider the rational map

$$\phi \colon \mathbf{P}^2_R \dashrightarrow \mathbf{P}^2_R.$$

We use ϕ to understand W as a cycle. For $z \in Z$, let z_K denote the constant section $z \times_k K$ of $\mathbf{P}^2_K$, and let $\phi(z)_0$ be the flat limit of $\phi(z_K)$. Then the cycle $[W]$ of W is simply

$$[W] = \sum_{z \in Z} \phi(z)_0.$$

Suppose ϕ has rank 2. Then the indeterminacy locus of ϕ consists of a single point p in the central fiber, and the map $\phi_0 \colon \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ is the linear projection with center p onto a line $L \subset \mathbf{P}^2$ (see (5.2)). If $p \notin Z$, then ϕ_0 is defined for every $z \in Z$, and $\phi(z)_0$ is simply $\phi_0(z)$. Since Z is star-free, the linear projection from p can identify at most 2 pairs of points from Z. As a result, the set $\{\phi_0(z) \mid z \in Z\}$ consists of at least 4 distinct points on L. If $p \in Z$, consider $Z' = Z \setminus \{p\}$. Then ϕ_0 is defined on Z', and $\phi(z)_0$ is simply $\phi_0(z)$ for $z \in Z'$. Since no 3 points of Z are collinear, the linear projection from p maps the points of Z' to distinct points. As a result, the set $\{\phi_0(z) \mid z \in Z\}$ actually consists of 5 distinct points on L.

Suppose ϕ has rank 1. Then the indeterminacy locus of ϕ_0 is a line $L \subset \mathbf{P}^2$ and ϕ_0 is constant on $\mathbf{P}^2 \setminus L$. Let $Z' = Z \setminus L$. Since no three points of Z are collinear, L contains at most 2 points of Z, and hence Z' contains at least 4 points. By construction, ϕ_0 is defined on Z', and hence $\phi(z)_0 = \phi_0(z)$ for $z \in Z'$. Since ϕ_0 is constant, we conclude that W contains a point of multiplicity at least 4. $\square$

Proposition 5.2.3. *Let $W \subset \mathbf{P}^2$ be a curvilinear scheme of length 6 whose cycle is $p + q + 4r$, where p, q, r are distinct and $T_r W$ does not contain p or q. Then W does not lie in the PGL_3 orbit closure of a star-free configuration Z.*

Proof. We prove the contrapositive. Let $\phi \in \text{Aut}(\mathbf{P}^2_K)$ be an automorphism such that W is the flat limit of $\phi(Z_K)$. We show that $T_r W$ contains p or q. The key idea is to consider flat limits of lines in $\mathbf{P}^2$ under the action of ϕ. To do so, we use the same analysis as before but for the map between dual projective spaces

$$\phi^\vee \colon (\mathbf{P}^2)^\vee_R \dashrightarrow (\mathbf{P}^2)^\vee_R.$$

If A is the 3×3 matrix defining ϕ, then the matrix defining $\phi^\vee$ is $(A^T)^{-1}$.

If $\phi_0^\vee$ is defined for a line $\ell \in (\mathbf{P}^2)^\vee$, then the flat limit of the family of lines $\phi(\ell_K) \subset \mathbf{P}_K^2$ is the line corresponding to $\phi_0^\vee(\ell)$. The statements (5.2) and (5.3) for the dual projective space say the following. If $\phi^\vee$ has rank 1, then the indeterminacy locus of $\phi_0^\vee$ consists of a particular $\Lambda \in (\mathbf{P}^2)^\vee$. For $\ell \neq \Lambda$, the map $\phi_0^\vee$ is the composite of $\ell \mapsto \ell \cap \Lambda$ and a linear inclusion $\Lambda \longrightarrow (\mathbf{P}^2)^\vee$. In this case, a fiber of $\phi_0^\vee$ consists of a collection of concurrent lines, with the point of concurrency on Λ. If $\phi^\vee$ has rank 1, then the indeterminacy locus of $\phi_0^\vee$ consists of lines through a particular point $p \in \mathbf{P}^2$. For $\ell \in (\mathbf{P}^2)^\vee$ not containing p, the map $\phi_0^\vee$ is constant.

Consider a line ℓ joining pairs of points of Z. By semi-continuity, the flat limit of $\phi(\ell_K)$ must intersect W in a scheme of length at least 2. There are at most four lines in $\mathbf{P}^2$ that intersect W in a scheme of length at least 2: the lines $\overline{pq}$, $\overline{pr}$, $\overline{qr}$, and $T_r W$. In particular, if $\phi_0^\vee$ is defined at ℓ, then $\phi_0^\vee(\ell)$ must be one of these 4. This immediately implies that $\phi^\vee$ cannot have rank 1. To see this, suppose it has rank 1. Consider the $\binom{6}{2} = 15$ lines joining pairs of points of Z. If $\phi_0^\vee$ takes only 4 values on these lines (or may be undefined on one of them), we would be able to partition them into 4 collections of concurrent lines, such that the points of concurrency are collinear. But it is easy to check that this is impossible for a star-free configuration Z.

We have now concluded that $\phi^\vee$ has rank 2, and hence $\phi_0^\vee$ is constant on its domain. Suppose the constant value of $\phi_0^\vee$ corresponds to the line $L \subset \mathbf{P}^2$. Suppose $z_1 \in Z$ (resp. $z_2 \in Z$) is such that the flat limit of $\phi(z_{1K})$ is p (resp. q). Consider the 8 lines $\overline{z_i z}$ for $i = 1, 2$ and $z \in Z \setminus \{z_1, z_2\}$. Since these lines are not all concurrent, $\phi_0^\vee$ is defined for at least one of them, say $\ell = \overline{z_1 z}$. Then $\phi_0^\vee(\ell)$ corresponds to the line L. Since ℓ contains z_1, the flat limit L of $\phi(\ell_K)$ contains the flat limit p of $\phi(z_{1K})$.

Now consider the $\binom{4}{2} = 6$ lines joining pairs of points of $Z \setminus \{z_1, z_2\}$. Since the 6 lines are not all concurrent, $\phi_0^\vee$ is defined on at least one of them, say ℓ. Then the flat limit of $\phi(\ell_K)$ is also L. Since the flat limit of a point of $Z \setminus \{z_1, z_2\}$ is r, by semi-continuity, we have

$$\mathrm{mult}_r(L \cap W) \geq 2.$$

In other words, $L = T_r W$. But then $T_r W$ contains p. □

Remark 5.2.4. The same conclusion holds for a curvilinear W with cycle structure $2p + 4r$ such that $T_r W$ does not contain p. The same proof works with $p = q$.

5.2.3. *Applications.* Let $X \subset \mathbf{P}^3$ denote the cubic surface defined in homogeneous coordinates (x_0, x_1, x_2, x_3) by the equation
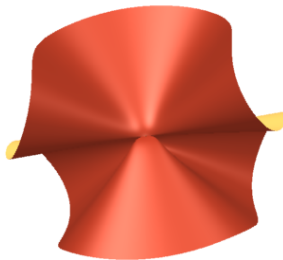
$$x_3 f_2 - f_3 = 0,$$

FIGURE 8. The unique cubic surface X with an A_5 singularity

where $f_2 = x_0x_1$ and $f_3 = x_0^3 + x_1^3 - x_1x_2^2$. This is the unique cubic surface possessing a single A_5 singularity, up to projective equivalence. For the reader's amusement, we include a real picture of X in Figure 8.

The surface X is the cubic surface associated to the length 6 subscheme $W \subset \mathbf{P}^2$ defined by $f_2 = 0$ and $f_3 = 0$. The scheme W has the cycle $p + q + 4r$, where p, q, r are distinct but collinear, and the tangent line T_rW is distinct from the line $\overline{pqr}$. The identity component of the automorphism group of X is $\mathbf{G}_a$.

Proposition 5.2.5. *Let S be a smooth cubic surface without an Eckardt point. Then X is not in the* PGL_4 *orbit closure of S.*

Proof. We know that S is isomorphic to $\mathrm{Bl}_Z \mathbf{P}^2$, where $Z \subset \mathbf{P}^2$ is a star-free configuration. The result now follows from Proposition 5.2.1 and Proposition 5.2.3. □

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