VECTOR BUNDLES AND FINITE COVERS

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ABSTRACT. Motivated by the problem of finding algebraic constructions of finite coverings in commutative algebra, the Steinitz realization problem in number theory, and the study of Hurwitz spaces in algebraic geometry, we investigate the vector bundles underlying the structure sheaf of a finite flat branched covering. We prove that, up to a twist, every vector bundle on a smooth projective curve arises from the direct image of the structure sheaf of a smooth, connected branched cover.

1. INTRODUCTION

Associated to a finite flat morphism $\phi\colon X \to Y$ is the vector bundle $\phi_*\mathcal{O}_X$ on Y. This is the bundle whose fiber over $y \in Y$ is the vector space of functions on $\phi^{-1}(y)$. In this paper, we address the following basic question: which vector bundles on a given Y arise in this way? We are particularly interested in cases where X and Y are smooth projective varieties.

Our main result is that, up to a twist, every vector bundle on a smooth projective curve Y arises from a branched cover $X \to Y$ with smooth projective X. Let d be a positive integer and let k be an algebraically closed field with $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Theorem 1.1 (Main). *Let Y be a smooth projective curve over k and let E be a vector bundle of rank $(d-1)$ on Y. There exists an integer n (depending on E) such that for any line bundle L on Y of degree at least n, there exists a smooth curve X and a finite map $\phi\colon X \to Y$ of degree d such that $\phi_*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y \oplus E^\vee \otimes L^\vee$.*

The reason for the $\mathcal{O}_Y$ summand is as follows. Pull-back of functions gives a map $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$, which admits a splitting by $1/d$ times the trace map. Therefore, every bundle of the form $\phi_*\mathcal{O}_X$ contains $\mathcal{O}_Y$ as a direct summand. The dual of the remaining direct summand is called the *Tschirnhausen bundle* and is denoted by $E = E_\phi$ (the dual is taken as a convention.) Theorem 1.1 says that on a smooth projective curve, a sufficiently positive twist of every vector bundle is Tschirnhausen.

The reason for needing the twist is a bit more subtle, and arises from some geometric restrictions on Tschirnhausen bundles. For $Y = \mathbf{P}^n$ and a smooth X, the Tschirnhausen bundle E is ample by a result of Lazarsfeld [24]. For more general Y and smooth X, it enjoys several positivity properties as shown in [29, 30]. The precise necessary and sufficient conditions for being Tschirnhausen (without the twist) are unknown, and seem to be delicate even when $Y = \mathbf{P}^1$.

The attempt at extending Theorem 1.1 to higher dimensional varieties Y presents interesting new challenges. We discuss them through some examples in § 4. As it stands, the analogue of Theorem 1.1 for higher dimensional varieties Y is false. We end the paper by posing modifications for which we are unable to find counterexamples.

1.1. **Motivation and related work.**

The question of understanding the vector bundles associated to finite covers arises in many different contexts. We explain three main motivations below.

1.1.1. *The realization problem for finite covers.* Given a space Y and a positive integer d, a basic question in algebraic geometry is to find algebraic constructions of all possible degree d branched coverings of Y. The prototypical example occurs when $d = 2$. A double cover $X \to Y$ is given as

$X = \mathrm{Spec}(\mathcal{O}_Y \oplus L^\vee)$ where L is a line bundle on Y, and the algebra structure on $\mathcal{O}_Y \oplus L^\vee$ is specified by a map $L^{\otimes -2} \to \mathcal{O}_Y$ of $\mathcal{O}_Y$-modules. In other words, the data of a double cover consists of a line bundle L and a section of $L^{\otimes 2}$. In general, a degree d cover $X \to Y$ is given as $X = \mathrm{Spec}(\mathcal{O}_Y \oplus E^\vee)$ where E is a vector bundle on Y of rank $(d-1)$. The specification of the algebra structure, however, is much less obvious. For higher d, it is far from clear that simple linear algebraic data determines an algebra structure. In fact, given an E it is not clear whether there exists a (regular/normal/Cohen-Macaulay) $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus E^\vee$, that is, whether E can be realized as the Tschirnhausen bundle of a cover $\phi\colon X \to Y$ for some (regular/normal/Cohen-Macaulay) X. We call this the *realization problem* for Tschirnhausen bundles.

For $d = 3, 4$, and 5, theorems of Miranda, Casnati, and Ekedahl provide a linear algebraic description of degree d coverings of Y in terms of vector bundles on Y [11,25]. These descriptions give a direct method for attacking the realization problem for d up to 5. For $d \geq 6$, however, no such description is known, and finding one is a difficult open problem. Theorem 1.1 solves the realization problem for all d up to twisting by a line bundle, circumventing the lack of effective structure theorems.

The realization problem has attracted the attention of several mathematicians, even in the simplest non-trivial case, namely where $Y = \mathbf{P}^1$ [3, 13, 27, 32]. Historically, this problem for $Y = \mathbf{P}^1$ is known as the problem of classifying *scrollar invariants*. Recall that every vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles. Suppose $\phi\colon X \to Y = \mathbf{P}^1$ is a branched cover with X smooth and connected. Writing $E_\phi = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$, the scrollar invariants of ϕ are the integers $a_1, \dots, a_{d-1}$. For $d = 2$, any positive integer a_1 is realized as a scrollar invariant of a smooth double cover. For $d = 3$, a pair of positive integers (a_1, a_2) with $a_1 \leq a_2$ is realized as scrollar invariants of smooth triple coverings if and only if $a_2 \leq 2a_1$ [25, § 9]. Though it may be possible to use the structure theorems to settle the cases of $d = 4$ and 5, such direct attacks are infeasible for $d \geq 6$. Nevertheless, the picture emerging from the collective work of several authors [13,27], and visible in the $d = 3$ case, indicates that if the a_i are too far apart, then they cannot be scrollar invariants.

Theorem 1.1 specialized to $Y = \mathbf{P}^1$ says that the picture is the cleanest possible if we allow twisting by a line bundle.

Corollary 1.2. *Let $a_1, \dots, a_{d-1}$ be integers. For every sufficiently large c, the integers $a_1+c, \dots, a_{d-1}+c$ can be realized as scrollar invariants of $\phi\colon X \to \mathbf{P}^1$ where X is a smooth projective curve.*

Before our work, the work of Ballico [3] came closest to a characterization of scrollar invariants up to a shift. He showed that one can arbitrarily specify the smallest $d/2$ of the $(d-1)$ scrollar invariants. Corollary 1.2 answers the question completely: one can in fact arbitrarily specify *all* of them.

1.1.2. *Arithmetic analogues.* The realization problem of Tschirnhausen bundles is a well-studied and difficult open problem in number theory. When $\phi\colon \mathrm{Spec}\,\mathcal{O}_L \to \mathrm{Spec}\,\mathcal{O}_K$ is the map corresponding to the extension of rings of integers of number fields L/K, the isomorphism class of E_ϕ is encoded by its Steinitz class, which is the ideal class $\det E \in \mathrm{Cl}(K)$. Indeed, the structure theorem of projective modules over a Dedekind domain [34] says that every projective module E of rank $(d-1)$ is isomorphic to $\mathcal{O}_K^{d-2} \oplus \det E$ as an $\mathcal{O}_K$-module. A long-standing unsolved problem in number theory is to prove that, for each fixed degree $d \geq 2$, every element of the class group is realized as the Steinitz class of some degree d extension of K. The first cases ($d \leq 5$) of this problem follow from the work of Bhargava, Shankar, and Wang [8, Theorem 4]. In general, the realization problem for Steinitz classes is open, with progress under various conditions on the Galois group; see [9] and the references therein.

Theorem 1.1 completely answers the complex function field analogue of the realization problem for Steinitz classes.

Corollary 1.3. *Suppose Y is a smooth affine curve, and $I \in \operatorname{Pic}(Y)$. Then I is realized as the Steinitz class of a degree d covering $\phi : X \to Y$, with X smooth and connected. That is, there exists $\phi : X \to Y$ with X smooth and connected such that*

$$E_\phi \cong \mathcal{O}_Y^{d-2} \oplus I.$$

Proof. Extend E to a vector bundle E' on the smooth projective compactification Y' of Y. Apply Theorem 1.1 to E', twisting by a sufficiently positive line bundle L on Y' whose divisor class is supported on the complement $Y' \setminus Y$. We obtain a smooth curve X' and a map $\phi : X' \to Y'$ whose Tschirnhausen bundle is $E' \otimes L$; letting $X = \phi^{-1}(Y)$, we obtain the corollary. □

We note that the affine covers in the above corollary have full (S_d) monodromy groups, as can easily be deduced from the method of proof of Theorem 1.1.

The analogy between the arithmetic and the geometric realization problems discussed above for affine curves extends further to projective curves, provided we interpret the projective closure of an arithmetic curve like $\operatorname{Spec} \mathcal{O}_K$ in the sense of Arakelov geometry [36]. For simplicity, take $K = \mathbf{Q}$ and $Y = \mathbf{P}^1$. A vector bundle on a "projective closure" of $\operatorname{Spec} \mathbf{Z}$ in the Arakelov sense is a free $\mathbf{Z}$-module E with a Hermitian form on its complex fiber $E \otimes \mathbf{C}$. Let $L/\mathbf{Q}$ be an extension of degree d. The Tschirnhausen bundle E_ϕ of $\phi\colon \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ is naturally an Arakelov bundle, where the Hermitian form is induced by the trace. Thus, the realization problem has a natural interpretation in the Arakelov sense. An Arakelov bundle over $\operatorname{Spec} \mathbf{Z}$ of rank r is just a lattice of rank r, and the set of such lattices (up to isomorphism and scaling) forms an orbifold (a double quotient space), denoted by $\mathcal{S}_r$. A theorem of Bhargava and Harron says that for $d \leq 5$, the (Arakelov) Tschirnhausen bundles are equidistributed in $\mathcal{S}_{d-1}$ [7, Theorem 1]. Again, one crucial ingredient in their proof is provided by the structure theorems for finite covers. We may view Corollary 1.2 as a (complex) function field analogue, but for all d.

1.1.3. *Geometry of Hurwitz spaces.* Another source of motivation for Theorem 1.1 concerns the geometry of moduli spaces of coverings, known as Hurwitz spaces. For simplicity, take $k = \mathbf{C}$ and let Y be a smooth projective curve over k. Denote by $H_{d,g}(Y)$ the coarse moduli space that parametrizes primitive covers $\phi : X \to Y$ where ϕ is a map of degree d and X is a smooth curve of genus g (the cover ϕ is primitive if $\phi_*\colon \pi_1(X) \to \pi_1(Y)$ is surjective). The space $H_{d,g}(Y)$ is an irreducible algebraic variety [17, Theorem 9.2].

The association $\phi \rightsquigarrow E_\phi$ gives rise to interesting cycles on $H_{d,g}(Y)$, called the Maroni loci. For a vector bundle E on Y, define the *Maroni locus* $M(E) \subset H_{d,g}(Y)$ as the locally closed subset that parametrizes covers with Tschirnhausen bundle isomorphic to E. This notion generalizes the classical Maroni loci for $Y = \mathbf{P}^1$, which play a key role in describing the cones of various cycles classes on $H_{d,g}(Y)$ in [14] and [28]. It would be interesting to know if the cycle of $\overline{M(E)}$ has similar distinguishing properties, such as rigidity or extremality, more generally than for $Y = \mathbf{P}^1$. A first step towards this study is to determine when these cycles are non-empty and of the expected dimension. As a consequence of the method of proof of the main theorem, we obtain the following.

Theorem 1.4. *Set $b = g - 1 - d(g_Y - 1)$. Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L of degree $b - e$, the Maroni locus $M(E \otimes L) \subset H_{d,g}(Y)$ contains an irreducible component having the expected codimension $h^1(\operatorname{End} E)$.*

Theorem 1.4 is Theorem 3.13 in the main text. Going further, it would be valuable to know whether all the components of $M(E\otimes L)$ are of the expected dimension or, even better, if $M(E\otimes L)$ is irreducible. The results of [15, § 2] imply irreducibility for $Y=\mathbf{P}^1$ and some vector bundles E. But the question remains open in general.

More broadly, the association $\phi \rightsquigarrow E_\phi$ allows us to relate $H_{d,g}(Y)$ to the moduli space of vector bundles on Y. Denote by $M_{r,k}(Y)$ the moduli space of semi-stable vector bundles of rank r and degree k on Y. It is well-known that $M_{r,k}(Y)$ is an irreducible algebraic variety [35]. Note that the Tschirnhausen bundle of a degree d and genus g cover of Y has rank $d-1$ and degree $b=g-1-d(g_Y-1)$. One would like to say that $\phi \rightsquigarrow E_\phi$ yields a rational map

$$H_{d,g}(Y) \dashrightarrow M_{d-1,b},$$

but to say so we must know the basic fact that a general element $\phi\colon X\to Y$ of $H_{d,g}(Y)$ gives a semi-stable vector bundle E_ϕ. We obtain this as a consequence of our methods.

Theorem 1.5. *Suppose $g_Y\geq 2$, and set $b=g-1-d(g_Y-1)$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle of a general degree d and genus g branched cover of Y is stable. Moreover, the rational map $H_{d,g}(Y)\dashrightarrow M_{d-1,b}(Y)$ defined by $\phi\mapsto E_\phi$ is dominant.*

The same statement holds for $g_Y=1$, with "stable" replaced with "regular poly-stable."

Theorem 1.5 is Theorem 3.11 in the main text.

The low degree cases ($d\leq 5$) of Theorem 1.5 were proved by Kanev [21, 22, 23] using the structure theorems. The crucial new ingredient in our approach is the use of deformation theory to circumvent such direct attacks. The validity of Theorem 1.5 for low g is an interesting open problem. It would be nice to know whether $\phi\mapsto E_\phi$ is dominant as soon as we have $\dim H_{d,g}(Y)\geq \dim M_{d-1,b}(Y)$.

We also draw the reader's attention to results, similar in spirit to Theorem 1.5, proved by Beauville, Narasimhan, and Ramanan [4]. Motivated by the study of the Hitchin fibration, they study not the pushforward of $\mathcal{O}_X$ itself but the pushforwards of general line bundles on X.

1.2. **Strategy of proof.** The proof of Theorem 1.1 proceeds by degeneration. To help the reader, we first outline our approach to a weaker version of Theorem 1.1. In the weaker version, we consider not the vector bundle E itself, but its projectivization $\mathbf{P}E$, which we call the *Tschirnhausen scroll*. A branched cover with Gorenstein fibers $\phi\colon X\to Y$ with Tschirnhausen bundle E factors through a *relative canonical embedding* $\iota\colon X\hookrightarrow \mathbf{P}E$ by the main theorem in [10].

Theorem 1.6. *Let E be any vector bundle on a smooth projective curve Y. Then the scroll $\mathbf{P}E$ is the Tschirnhausen scroll of a finite cover $\phi\colon X\to Y$ with X smooth.*

The following steps outline a proof of Theorem 1.6 which parallels the proof of the stronger Theorem 1.1. We omit the details, since they are subsumed by the results in the paper.

(1) First consider the case

$$E=L_1\oplus\cdots\oplus L_{d-1},$$

where the L_i are line bundles on Y whose degrees satisfy

$$\deg L_i \ll \deg L_{i+1}.$$

For such E, we construct a nodal cover $\psi\colon X\to Y$ such that $\mathbf{P}E_\psi=\mathbf{P}E$. For example, we may take X to be a nodal union of d copies of Y, each mapping isomorphically to Y under ψ, where the ith copy meets the $(i+1)$th copy along nodes lying in the linear series $|L_i|$.

(2) Consider $X \subset \mathbf{P}E$, where X is the nodal curve constructed above. We now attempt to find a smoothing of X in $\mathbf{P}E$. However, the normal bundle $N_{X/\mathbf{P}E}$ may be quite negative. Fixing this negativity is the most crucial step.

To overcome the negativity, we draw motivation from Mori's idea of making curves flexible by attaching free rational tails. If we view a cover $X \to Y$ as a map from Y to the classifying stack BS_d as done in [1], then attaching rational tails can be interpreted as attaching general rational normal curves to X in the fibers of $\mathbf{P}E \to Y$. Of course, the classifying stack BS_d is not a projective variety, so the above only serves as an inspiration.

(3) Given a general point $y \in Y$, the d points $\psi^{-1}(y) \subset \mathbf{P}E_y \simeq \mathbf{P}^{d-2}$ are in linear general position, and therefore they lie on many smooth rational normal curves $R_y \subset \mathbf{P}E_y$. Choose a large subset $S \subset Y$, and attach general rational normal curves R_y for each $y \in S$ to X, obtaining a new nodal curve $Z \subset \mathbf{P}E$.

(4) The key technical step is showing that the new normal bundle $N_{Z/\mathbf{P}E}$ is sufficiently positive. Using this positivity, we get that Z is the flat limit of a family of smooth, relatively-canonically embedded curves $X_t \subset \mathbf{P}E$. The generic cover $\phi : X_t \to Y$ in this family satisfies $E_\phi \cong L_1 \oplus \cdots \oplus L_{d-1}$.

(5) We tackle the case of an arbitrary bundle E as follows.
  (a) We note that every vector bundle E degenerates *isotrivially* to a bundle of the form $E_0 = L_1 \oplus \cdots \oplus L_{d-1}$ treated in the previous steps.
  (b) We take a cover $X_0 \to Y$ with Tschirnhausen bundle E_0 constructed above. Using the abundant positivity of $N_{X_0/\mathbf{P}E_0}$, we show that $X_0 \subset \mathbf{P}E_0$ deforms to $X \subset \mathbf{P}E$. The cover $\phi : X \to Y$ satisfies $\mathbf{P}E_\phi \cong \mathbf{P}E$.

We need to refine the strategy above to handle the vector bundle E itself, and not just its projectivization. Therefore, we work with the *canonical affine embedding* of X in the total space of E. The proof of Theorem 1.1 involves carrying out the steps outlined above for the embedding $X \subset E$ relative to the divisor of hyperplanes at infinity in a projective completion of E.

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1.4. **Conventions.** We work over an algebraically closed field k. All schemes are of finite type over k. Unless specified otherwise, a point is a k-point. The projectivization $\mathbf{P}V$ of a vector bundle V refers to the space of 1-dimensional *quotients* of V. We identify vector bundles with their sheaves of sections. An injection is understood as an injection of sheaves. Given a quasi-coherent sheaf F on a scheme X and a point p of X, we use $F|_p$ to denote the fiber of F at p. As a convention, we use $=$ to denote canonical isomorphisms and $\cong$ to denote non-canonical ones.

2. VECTOR BUNDLES, THEIR INFLATIONS, AND DEGENERATIONS

This section contains standard results about vector bundles, their degenerations, and finite covers. The only new ingredient is a careful but straightforward study of elementary transformations ("inflations") that decrease higher cohomology (Proposition 2.5). Throughout, Y is a smooth, projective, connected curve over k, an algebraically closed field of arbitrary characteristic.

2.1. **Inflations.** Let E be a vector bundle on Y. A *degree n inflation* of E is a vector bundle E^+ along with an injective map of sheaves $E \to E^+$ whose cokernel is finite of length n. If the cokernel is supported on a subscheme $S \subset Y$, then we say that $E \to E^+$ is an inflation of E at S.

Remark 2.1. Let $E \to E^+$ be a degree one inflation. In standard parlance, E and E^+ are said to be related by an *elementary transformation*. We use "inflation" only to emphasize the asymmetry in the relationship.

Fix a point $y \in Y$. Consider an inflation $E \to E^+$ whose cokernel is supported (scheme-theoretically) at y. Then we have an exact sequence

$$0 \to E \to E^+ \to B \to 0, \tag{2.1}$$

where the cokernel is annihilated by the maximal ideal $m_y \subset \mathcal{O}_Y$.

The dual of (2.1) is the sequence

$$0 \to E^{+\vee} \to E^\vee \to A \to 0,$$

where the cokernel is supported at y. Note that we have a canonical identification

$$A = \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_Y}(B, \mathcal{O}_Y) = B^\vee \otimes N_{y/Y}.$$

The inflation $E \to E^+$ is determined by the surjection

$$E^\vee|_y \to A. \tag{2.2}$$

We call (2.2) the *defining quotient* of the inflation $E \to E^+$.

2.2. **Inflations and higher cohomology.** We now study how inflations increase positivity.

Proposition 2.2. *Let $E \to E^+$ be an inflation. Then $h^1(Y, E^+) \le h^1(Y, E)$. In particular, if $H^1(Y, E) = 0$, then $H^1(Y, E^+) = 0$. If, furthermore, E is globally generated, then so is E^+.*

Proof. For the statement about h^1, apply the long exact sequence on cohomology to

$$0 \to E \to E^+ \to E^+/E \to 0,$$

and use that E^+/E has zero-dimensional support. For global generation, consider the sequence

$$0 \to E(-y) \to E \to E|_y \to 0, \tag{2.3}$$

and the associated long exact sequence on cohomology. It follows that if E is globally generated and $H^1(Y, E) = 0$, then $H^1(Y, E(-y)) = 0$ for every $y \in Y$. But then we also have $H^1(Y, E^+) = 0$ and $H^1(Y, E^+(-y)) = 0$ for every $y \in Y$. From the sequence (2.3) for E^+, we conclude that E^+ is also globally generated. □

Let $V \subset E^\vee \otimes \Omega_Y|_y$ be the image of the evaluation map

$$H^0\left(E^\vee \otimes \Omega_Y\right) \to E^\vee \otimes \Omega_Y|_y.$$

Let $q \colon E^\vee|_y \to k^n$ be a surjection and let $E \to E^+_q$ be the inflation with q as its defining quotient.

Proposition 2.3. *With the notation above, let $q_V \colon V \to k^n \otimes \Omega_Y|_y$ be the restriction of $q \otimes \mathrm{id}$ to $V \subset E^\vee \otimes \Omega_Y|_y$. Then we have*

$$h^0(Y, E^+_q) = h^0(Y, E) + n - \mathrm{rk}\, q_V, \text{ and}$$
$$h^1(Y, E^+_q) = h^1(Y, E) - \mathrm{rk}\, q_V.$$

Proof. We have the exact sequence $0 \to E_q^{+\vee} \to E^\vee \xrightarrow{q} k^n \to 0$, where the cokernel is supported at y. Tensoring by Ω_Y, taking the long exact sequence in cohomology, and using Serre duality yields the proposition. □

Proposition 2.4. *Suppose E is such that $h^1(Y,E) \neq 0$. Let $E \to E^+$ be a degree 1 inflation at a general point of Y with a general defining quotient. Then*

$$h^1(Y,E^+) = h^1(Y,E) - 1 \text{ and } h^0(Y,E^+) = h^0(Y,E).$$

Proof. If $h^1(E) = h^0(E^\vee \otimes \Omega_Y) \neq 0$, the space $V \subset E^\vee \otimes \Omega_Y|_y$ defined above is non-zero if $y \in Y$ is general. Then, for a general choice of $q \colon E_q^\vee \to k$, we have $\operatorname{rk} q_V = 1$. The statement now follows from Proposition 2.3. □

The following sharpens the meaning of "general" in Proposition 2.4.

Proposition 2.5. *Let $y \in Y$ be such that the image V of the evaluation map*

$$H^0(E^\vee \otimes \Omega_Y) \to E^\vee \otimes \Omega_Y|_y$$

is non-zero. Suppose we have a set Q of surjections $E_y^\vee \to k$ such that the linear span of Q is the entire projective space $\mathbf{P}E^\vee|_y$. Then for some $q \in Q$ we have

$$h^1(Y,E_q^+) \leq h^1(Y,E) - 1$$

Proof. Since $V \neq 0$ and Q spans $\mathbf{P}E^\vee|_y$, we must have $q_V \neq 0$ for some $q \in Q$. Then the statement follows from Proposition 2.3. □

Corollary 2.6. *Let $n \geq h^1(Y,E)$ be a non-negative integer. Choose n general points $y_1, \dots, y_n \in Y$ and general surjections $q_i \colon E^\vee|_{y_i} \to k$. Let $E \to E^+$ be the inflation at $y_1, \dots, y_n$ whose defining quotients are $q_1, \dots, q_n$. Then*

1. *$H^1(Y,E^+) = 0$,*
2. *furthermore, if $n \geq 2h^1(Y,E(-y))$ for all $y \in Y$, then E^+ is also globally generated.*

Proof. For the first part, apply Proposition 2.4 repeatedly. For the second part, take an arbitrary $y \in Y$. Let E' be obtained from E by $n/2$ general inflations, as in the first part. Then $H^1(Y,E'(-y)) = 0$. By upper semi-continuity, there are only finitely many $z \in Y$ for which $H^1(Y,E'(-z)) \neq 0$. Let E^+ be obtained from E' by $n/2$ more general inflations. Then we have $H^1(Y,E^+(-z)) = 0$ for all $z \in Y$, which implies that E^+ is globally generated. □

Proposition 2.2 and Corollary 2.6 together imply the following.

Corollary 2.7. *Let E be a vector bundle on Y and let n be large enough. Suppose E' is a coherent sheaf of the same generic rank as E and E' contains an inflation of E at n general points with n general defining quotients. Then $H^1(Y,E') = 0$ and E' is globally generated.*

Remark 2.8. The generality requirement on the quotients q_i in Corollary 2.6 and Corollary 2.7 is only in the sense of not satisfying any *linear* equations. That is, it is satisfied as long as the q_i are chosen from a set that linearly spans the space of quotients (see Proposition 2.5).

2.3. **Nodal curves and inflations of the normal bundle.** A common setting for inflations in the paper is the following. Let P be a smooth variety. Let X and R be smooth curves in P that intersect at a point p so that their union Z has a node at p. We analyze the relationship of the normal bundle $N_{Z/P}$ with $N_{X/P}$ and $N_{R/P}$, following a similar analysis in [18].

We first recall a natural map

$$N_{Z/P}|_p \to N_{p/X} \otimes N_{p/R}. \tag{2.4}$$

To define (2.4), consider the multiplication map

$$I_{R/P} \otimes_{\mathcal{O}_P} I_{X/P} \to I_{Z/P}.$$

The restriction maps $\mathcal{O}_P \to \mathcal{O}_X$ and $\mathcal{O}_P \to \mathcal{O}_R$ yield surjections $I_{R/P} \to I_{p/X}$ and $I_{X/P} \to I_{p/R}$. The multiplication map above induces a map $a\colon I_{p/X} \otimes_{\mathcal{O}_P} I_{p/R} \to I_{Z/P}|_p$ fitting in the diagram

$$\begin{array}{ccc} I_{R/P} \otimes_{\mathcal{O}_P} I_{X/P} & \longrightarrow & I_{Z/P} \\ \downarrow & & \downarrow \\ I_{p/X} \otimes_{\mathcal{O}_P} I_{p/R} & \xrightarrow{a} & I_{Z/P}|_p. \end{array}$$

To describe a explicitly, suppose we are given $f \in I_{p/X}$ and $g \in I_{p/R}$. Then we have

$$a\colon f \otimes g \mapsto \widetilde{f} \cdot \widetilde{g}, \tag{2.5}$$

where $\widetilde{f} \in I_{R/P}$ is a lift of f and $\widetilde{g} \in I_{X/P}$ is a lift of g. The product $\widetilde{f} \cdot \widetilde{g} \in I_{Z/P}$ depends on the chosen lifts, but it is easy to check that its image in $I_{Z/P}|_p$ depends only on f and g. The source and target of the map a are supported at p, and hence can be treated as $k = \mathcal{O}_P/m_p$ vector spaces. The map in (2.4) is the k-linear dual of the map a.

Together with the natural map $T_P|_Z \to N_{Z/P}$, the map in (2.4) yields a right exact sequence

$$T_P|_Z \to N_{Z/P} \to N_{p/X} \otimes N_{p/R} \to 0. \tag{2.6}$$

The sequence in (2.6) identical to the sequence considered before Proposition 1.1 in [18]. In [18], the cokernel of $T_P|_Z \to N_{Z/P}$ is identified with the sheaf T^1_Z, which is indeed isomorphic to $N_{p/X} \otimes N_{p/R}$ [2, Chapter XI, equation (3.8)]. Observe that the kernel of the map $T_P|_p \to N_{Z/P}|_p$ is the two dimensional space spanned by the subspaces $T_X|_p$ and $T_R|_p$ of $T_P|_p$.

Let us restrict the exact sequence (2.6) to X. We see that the composite $T_X \to T_P|_X \to N_{Z/P}|_X$ is zero, and hence the map $T_P|_X \to N_{Z/P}|_X$ factors as

$$T_P|_X \to N_{X/P} \to N_{Z/P}|_X.$$

As a result, we obtain the diagram

$$\begin{array}{ccccccccc} & & T_P|_X & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & N_{X/P} & \longrightarrow & N_{Z/P}|_X & \longrightarrow & N_{p/X} \otimes N_{p/R} & \longrightarrow & 0. \end{array} \tag{2.7}$$

The exact row in (2.7) exhibits $N_{Z/P}|_X$ as a degree 1 inflation of $N_{X/P}$ at p.

Let us understand the defining quotient of the inflation in (2.7). To do so, we dualize—apply $\mathrm{Hom}_X(-, \mathcal{O}_X)$—to the exact sequence in (2.7). The sheaves $N_{X/P}$ and $N_{Z/P}|_X$ are locally free on X with duals $I_{X/P}|_X$ and $I_{Z/P}|_X$, respectively. Since the cokernel $N_{p/X} \otimes N_{p/R}$ is supported at p, it vanishes under $\mathrm{Hom}_X(-, \mathcal{O}_X)$, but contributes an $\mathscr{E}\mathrm{xt}^1_X(-, \mathcal{O}_X)$ term. To identify this $\mathscr{E}\mathrm{xt}^1$ term, recall that for a vector space A considered as an $\mathcal{O}_X$-module supported at p, we have

$$\mathscr{E}\mathrm{xt}^1_{\mathcal{O}_X}(A, \mathcal{O}_X) = N_{p/X} \otimes A^\vee, \tag{2.8}$$

where $A^\vee = \mathrm{Hom}_k(A, k)$ is the k-linear dual of A. Applying (2.8) to $A = N_{p/X} \otimes N_{p/R}$ yields

$$\mathscr{E}\mathrm{xt}^1_{\mathcal{O}_X}(N_{p/X} \otimes N_{p/R}, \mathcal{O}_X) = N_{p/X} \otimes I_{p/X} \otimes I_{p/R}|_p = I_{p/R}|_p.$$

Thus, the result of applying $\mathrm{Hom}_X(-, \mathcal{O}_X)$ to (2.7) is the short exact sequence

$$0 \longleftarrow I_{p/R}|_p \longleftarrow I_{X/P}|_X \longleftarrow I_{Z/P}|_X \longleftarrow 0. \tag{2.9}$$

In the exact sequence (2.9), the map $I_{Z/P}|_X \to I_{X/P}|_X$ is induced by the inclusion $I_{Z/P} \subset I_{X/P}$ and the map $I_{X/P}|_X \to I_{p/R}|_p$ is induced by the map $I_{X/P} \to I_{p/R}$ given by restriction of functions from P to R. From (2.9), we see that the defining quotient of the inflation in (2.7) is the restriction map

$$I_{X/P}|_p \to I_{p/R}|_p. \tag{2.10}$$

Observe that $I_{p/R}|_p$ is a one-dimensional k-vector space.

Restricting the exact sequence (2.6) to R yields an analogous picture. We may write the two inflations obtained in this way together as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{X/P} & \longrightarrow & N_{Z/P}|_X & \xrightarrow{\alpha} & N_{p/X} \otimes N_{p/R} & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & N_{R/P} & \longrightarrow & N_{Z/P}|_R & \xrightarrow{\beta} & N_{p/X} \otimes N_{p/R} & \longrightarrow & 0. \end{array} \tag{2.11}$$

By construction, the maps α and β are induced from the same map $N_{Z/P} \to N_{p/X} \otimes N_{p/R}$.

Finally, note that the discussion above extends naturally to the case of two smooth curves attached nodally at a finite set of points instead of a single point.

2.4. **Isotrivial degenerations.** We say that a bundle E *isotrivially degenerates* to a bundle E_0 if there exists a pointed smooth curve $(\Delta, 0)$ and a bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y \times \{0\}} \cong E_0$ and $\mathcal{E}\big|_{Y \times \{t\}} \cong E$ for every $t \in \Delta \setminus \{0\}$.

Proposition 2.9. *Let E be a vector bundle on Y, and let N be a non-negative integer. Then E isotrivially degenerates to a vector bundle E_0 of the form*

$$E_0 = L_1 \oplus \cdots \oplus L_r,$$

where the L_i are line bundles and $\deg L_i + N \leq \deg L_{i+1}$ for all $i = 1, \ldots, r-1$.

For the proof of Proposition 2.9, we need a lemma.

Lemma 2.10. *There exists a filtration*

$$E = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0,$$

satisfying the following properties.

(1) *For every $i \in \{0, \ldots, r-1\}$, the sub-quotient F_i/F_{i+1} is a line bundle.*
(2) *Set $L_i = F_i/F_{i+1}$ for $i \in \{1, \ldots, r-1\}$ and $L_r = F_0/F_1$. For every $i \in \{1, \ldots, r-1\}$, we have*

$$\deg L_i + N \leq \deg L_{i+1}.$$

Proof. The statement is vacuous for $r = 0$ and 1. So assume $r \geq 2$. Note that if $F_\bullet$ is a filtration of E satisfying the two conditions, and if L is a line bundle, then $F_\bullet \otimes L$ is such a filtration of $E \otimes L$. Therefore, by twisting by a line bundle of large degree if necessary, we may assume that $\deg E \geq 0$.

Let us construct the filtration from right to left. Let $L_{r-1} \subset E$ be a line bundle with $\deg L_{r-1} \leq -N$ and with a locally free quotient. Set $F_{r-1} = L_{r-1}$. Next, let $L_{r-2} \subset E/F_{r-1}$ be a line bundle with $\deg L_{r-2} \leq \deg L_{r-1} - N$ and with a locally free quotient. Let $F_{r-2} \subset E$ be the preimage of L_{r-2}. Continue in this way. More precisely, suppose that we have constructed

$$F_j \supset F_{j+1} \supset \cdots \supset F_{r-1} \supset F_r = 0$$

such that $L_i = F_i / F_{i+1}$ satisfy

$$\deg L_i \leq \deg L_{i+1} - N,$$

and suppose $j \geq 2$. Then let $L_{j-1} \subset E/F_j$ be a line bundle with $\deg L_{j-1} \leq \deg L_j - N$ with a locally free quotient. Let $F_{j-1} \subset E$ be the preimage of L_{j-1}. Finally, set $F_0 = E$.

Condition 1 is true by design. Condition 2 is true by design for $i \in \{1, \ldots, r-2\}$. For $i = r-1$, note that $\deg L_{r-1} \leq -N$ by construction. On the other hand, we must have $\deg L_r \geq 0$. Indeed, we have $\deg E \geq 0$ but every sub-quotient of $F_\bullet$ except F_0/F_1 has negative degree. Therefore, condition 2 holds for $i = r-1$ as well. □

Proof of Proposition 2.9. Let $F_\bullet$ be a filtration of E satisfying the conclusions of Lemma 2.10. It is standard that a coherent sheaf degenerates isotrivially to the associated graded sheaf of its filtration. The construction goes as follows. Consider the $\mathcal{O}_Y[t]$-module

$$\bigoplus_{n \in \mathbf{Z}} t^{-n} F_n,$$

where $F_n = 0$ for $n > r$ and $F_n = E$ for $n < 0$. The corresponding sheaf $\mathcal{E}$ on $Y \times \mathbf{A}^1$ is coherent, $k[t]$-flat, satisfies $\mathcal{E}_{Y \times \{t\}} \cong E$ for $t \neq 0$, and $\mathcal{E}_{Y \times \{0\}} \cong L_1 \oplus \cdots \oplus L_r$. □

2.5. **The canonical affine embedding.** We end the section with a basic construction that relates finite covers and their Tschirnhausen bundles. Let d be a positive integer and assume that $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Let X be a curve of arithmetic genus g_X; let $\phi \colon X \to Y$ be a finite flat morphism of degree d; and let E be the associated Tschirnhausen bundle. Then we have a decomposition $\phi_* \mathcal{O}_X = \mathcal{O}_Y \oplus E^\vee$. The map $E^\vee \to \phi_* \mathcal{O}_X$ induces a surjection $\operatorname{Sym}^* E^\vee \to \phi_* \mathcal{O}_X$. Taking the relative spectrum gives an embedding of X in the total space $\operatorname{Tot}(E)$ of the vector bundle associated to E; we often denote $\operatorname{Tot}(E)$ by E if no confusion is likely. We call $X \subset E$ the *canonical affine embedding*. Note that the degree of E is half of degree of the branch divisor of ϕ, namely

$$\deg E = g_X - 1 - d(g_Y - 1).$$

For all $y \in Y$, the subscheme $X_y \subset E_y$ is in affine general position (not contained in a translate of a strict linear subspace of E_y).

The canonical affine embedding is characterized by the properties above.

Proposition 2.11. *Retain the notation above. Let F be a vector bundle on Y of the same rank and degree as E, and let $\iota \colon X \to F$ be an embedding over Y such that for a general $y \in Y$, the scheme $\iota(X_y) \subset F_y \cong \mathbf{A}^{d-1}$ is in affine general position. Then we have $F \cong E$, and up to an affine linear automorphism of F/Y, the embedding ι is the canonical affine embedding.*

Proof. The restriction map $\operatorname{Sym}^* F^\vee \to \phi_* \mathcal{O}_X = \mathcal{O}_Y \oplus E^\vee$ induces a map

$$\lambda \colon F^\vee \to E^\vee.$$

Since a general fiber $X_y \subset F_y$ is in affine general position, the map λ is an injective map of sheaves. But the source and the target are locally free of the same degree and rank. Therefore, λ is an isomorphism.

Recall that the affine canonical embedding is induced by the map

$$(0, \mathrm{id}) \colon E^\vee \to \mathcal{O}_Y \oplus E^\vee = \phi_* \mathcal{O}_X.$$

Suppose ι induces the map

$$(\alpha, \lambda)\colon F^\vee \to \mathcal{O}_Y \oplus E^\vee.$$

Compose ι with the affine linear isomorphism of $T_\alpha\colon \operatorname{Tot}(F) \to \operatorname{Tot}(F)$ over Y defined by the map $\operatorname{Sym}^* F^\vee \to \operatorname{Sym}^* F^\vee$ induced by

$$(-\alpha, \mathrm{id})\colon F^\vee \to \mathcal{O}_Y \oplus F^\vee.$$

Then $T_\alpha \circ \iota\colon X \to F$ is the affine canonical embedding, as desired. □

3. Proof of the main theorem

Let d be a positive integer, and assume that $\operatorname{char} k = 0$ or $\operatorname{char} k > d$. Throughout, Y is a smooth, projective, connected curve over k.

3.1. **The split case with singular covers.** As a first step, we treat the case of a suitable direct sum of line bundles and allow the source curve X to be singular.

Proposition 3.1. *Let $E = L_1 \oplus \cdots \oplus L_{d-1}$, where the L_i are line bundles on Y with $\deg L_1 \geq 2g_Y - 1$ and $\deg L_{i+1} \geq \deg L_i + (2g_Y - 1)$ for $i \in \{1, \ldots, d-2\}$. There exists a nodal curve X and a finite flat map $\phi\colon X \to Y$ of degree d such that $E_\phi \cong E$.*

The proof is inductive, based on the following "pinching" construction. Let $\psi\colon Z \to Y$ be a finite cover of degree r. Let X be the reducible nodal curve $Z \cup Y$, where Z and Y are attached nodally at distinct points (see Figure 1). More explicitly, let $y_i \in Y$ and $z_i \in Z$ be points such that $\psi(z_i) = y_i$. Define R as the kernel of the map

$$\psi_* \mathcal{O}_Z \oplus \mathcal{O}_Y \to \bigoplus_i k_{y_i},$$

defined around y_i by

$$(f, g) \mapsto f(z_i) - g(y_i).$$

Then $R \subset \psi_* \mathcal{O}_Z \oplus \mathcal{O}_Y$ is an $\mathcal{O}_Y$-subalgebra and $X := \operatorname{Spec}_Y R$ is a nodal curve. Let $\phi\colon X \to Y$ be the natural finite flat map. Set $D = \sum y_i$.

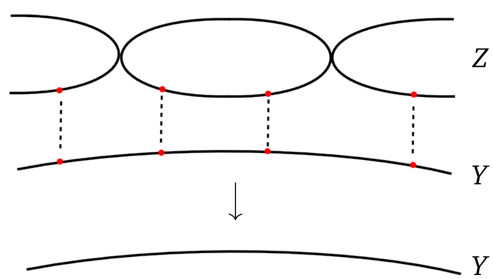

Figure 1. The pinching construction, in which pairs of points indicated by dotted lines are identified to form nodes.

Lemma 3.2. *In the setup above, we have an exact sequence*

$$0 \to E_\psi \to E_\phi \to \mathcal{O}_Y(D) \to 0.$$

Proof. The closed embedding $Z \to X$ gives a surjection

$$\phi_* \mathcal{O}_X \to \psi_* \mathcal{O}_Z$$

whose kernel is $\mathcal{O}_Y(-D)$. Factoring out the $\mathcal{O}_Y$ summand from both sides and taking duals yields the claimed exact sequence. □

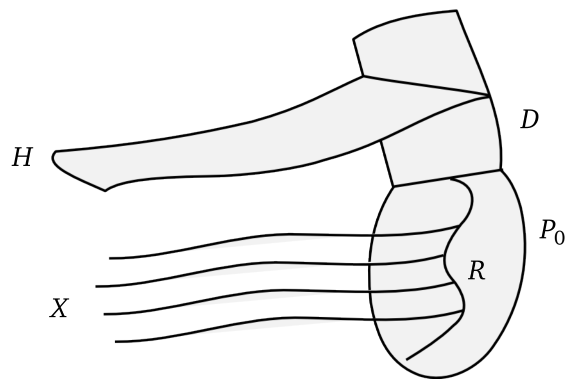

FIGURE 2. Attaching rational normal curves to X to make the normal bundle positive

Proof of Proposition 3.1. We use induction on d, starting with the base case $d = 1$, which is vacuous.

By the inductive hypothesis, we may assume that there exists a nodal curve Z and a finite cover $\psi: Z \to Y$ of degree $(d-1)$ such that $E_\psi \cong L_2 \oplus \cdots \oplus L_{d-1}$. Let $X = Z \cup Y \to Y$ be a cover of degree d obtained from $Z \to Y$ by a pinching construction such that $\mathcal{O}_Y(D) = L_1$. By Lemma 3.2, we get an exact sequence

$$0 \to L_2 \oplus \cdots \oplus L_{d-1} \to E_\phi \to L_1 \to 0. \tag{3.1}$$

But we have $\operatorname{Ext}^1(L_1, L_i) = H^1(L_i \otimes L_1^\vee) = 0$ since $\deg(L_i \otimes L_1^\vee) \geq 2g_Y - 1$. Therefore, the sequence (3.1) is split, and we get $E_\phi = L_1 \oplus \cdots \oplus L_{d-1}$. The induction step is then complete. □

3.2. **Attaching rational curves.** We now describe a procedure to make a finite cover more flexible, so that it can be deformed easily. The procedure is local on Y, so we zoom in to a cleaner local situation.

Let Y be a smooth curve and $0 \in Y$ an arbitrary point. Set $P = \mathbf{P}^{d-1} \times Y$, and let $H \subset P$ be a divisor, flat over Y, which restricts to hyperplanes on the fibers. Let $X \subset P$ be finite and étale of degree d over Y and disjoint from H. Assume that the fibers of $X \to Y$ are in linear general position in $\mathbf{P}^{d-1}$. Use the subscript 0 to denote the fiber over 0.

Let $R \subset P_0$ be a rational normal curve that contains X_0 and is transverse to H_0. Let $\widetilde{P} \to P$ be the blow-up along H_0. We use the same notation to denote the proper transforms of R, H, X, and P_0 in $\widetilde{P}$ (these are isomorphic copies). Denote by D the exceptional divisor of the blow-up. Let Z be the nodal curve $Z = X \cup R$. Set $\delta = X \cap R = X_0$. See Figure 2 for a sketch of the setup. Denote by ϕ the projection to Y.

Our goal is to establish $Z \subset \widetilde{P}$ as a more flexible replacement of $X \subset P$. (The reason for the blow up is to keep the curve away from the divisor H). For this goal, we must relate the normal bundle $N_{Z/\widetilde{P}}$ with $N_{X/P}$. Establishing this relationship takes some effort; the upshot is Corollary 3.7, which is the only statement we use in the later sections.

The following identifies the restriction of $N_{Z/\widetilde{P}}$ to R.

Lemma 3.3. *We have the following diagram with exact rows and columns*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{R/P_0} & \longrightarrow & N_{R/\widetilde{P}} & \longrightarrow & N_{P_0/\widetilde{P}}\big|_R & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(d+1)^{d-2} & \longrightarrow & N_{Z/\widetilde{P}}\big|_R & \longrightarrow & F & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & N_{\delta/X} \otimes N_{\delta/R} & = & N_{\delta/X} \otimes N_{\delta/R} & & \end{array}$$

where the maps in the first row and middle column are standard, and the others are induced from them. The sheaf F is canonically isomorphic to $\phi^* N_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} \mathcal{O}_R(\delta - D)$, and thus isomorphic to $\mathcal{O}_R(1)$.

Proof. The first row is standard. Augment it by considering the natural map $N_{R/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_R$ recalled in § 2.3. Since $R \subset P_0 = \mathbf{P}^{d-1}$ is a rational normal curve, we have an isomorphism $N_{R/P_0} \cong \mathcal{O}(d+1)^{d-2}$ (see [31, II] or [33, Example 4.6.6]). A local calculation shows that the map $N_{R/P_0} \to N_{Z/\widetilde{P}}\big|_R$ remains an injection when restricted to any point of R, and hence its cokernel F is locally free, and plainly, of rank 1. From § 2.3, we know that the cokernel of the natural map $N_{R/\widetilde{P}} \to N_{Z/\widetilde{P}}\big|_R$ is $N_{\delta/X} \otimes N_{\delta/R}$. By the snake lemma, the cokernel of the map $N_{P_0/\widetilde{P}}\big|_R \to F$ is the same. Since both $N_{P_0/\widetilde{P}}\big|_R$ and F are line bundles, and the map between them degenerates exactly at δ, we obtain an isomorphism

$$F = N_{P_0/\widetilde{P}}\big|_R(\delta).$$

Combined with the isomorphism

$$N_{P_0/\widetilde{P}} = N_{P_0/P}(-D),$$

and the isomorphism $N_{P_0/P} = \phi^* N_{0/Y}$, we get the canonical isomorphism $F = \phi^* N_{0/Y} \otimes \mathcal{O}_R(\delta - D)$, as claimed. Since the degree of δ is d and that of $D|_R$ is $d-1$, we see that $F \cong \mathcal{O}_R(1)$. □

Let us describe the three maps in (3.3) involving the bundle F. For all three, it is easier to describe the duals.

The $\mathcal{O}_R$-dual of $N_{P_0/\widetilde{P}}|_R \to F$ is the map

$$\phi^* I_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} \mathcal{O}_R(D - \delta) \to I_{P_0/\widetilde{P}}|_R \tag{3.2}$$

given as follows. Let t be a section of $I_{0/Y}$ and let f be a section of $\mathcal{O}_R(D - \delta)$ on some open subset of $\widetilde{P}$, interpreted as a rational function on R vanishing along δ with at most simple poles along D. On this open subset, let $\widetilde{f}$ be any section of $\mathcal{O}_{\widetilde{P}}(D)$ that restricts to f on R. Then the map (3.2) sends the section represented by $t \otimes f$ to the section represented by $t \cdot \widetilde{f}$. Observe that the possible poles of $\widetilde{f}$ along D are cancelled by the vanishing of t along D. Since t vanishes on P_0, so does the product $t \cdot \widetilde{f}$, and represents a section of $I_{P_0/\widetilde{P}}$. It is easy to check that its image in $I_{P_0/\widetilde{P}}|_R$ depends only on f, and not on the lift.

The $\mathcal{O}_R$-dual of $N_{Z/\widetilde{P}}|_R \to F$ is the map

$$\phi^* I_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} \mathcal{O}_R(D - \delta) \to I_{Z/\widetilde{P}}|_R \tag{3.3}$$

given as follows. Let t be a section of $I_{0/Y}$ and f be a section of $\mathcal{O}_R(D - \delta)$ on some open subset of $\widetilde{P}$. On this open subset, let $\widetilde{f}$ be any section of $I_{X/\widetilde{P}} \otimes \mathcal{O}_{\widetilde{P}}(D)$ that restricts to f on R. Then the map (3.3) sends the section represented by $t \otimes f$ to the section represented by $t \cdot \widetilde{f}$. As before, observe that $t \cdot \widetilde{f}$ has no poles, it vanishes along Z, and its image in $I_{Z/\widetilde{P}}|_R$ does not depend on the lift.

The $\mathcal{O}_\delta$-dual of $F|_\delta \to N_{\delta/X} \otimes N_{\delta/R}$ is the map

$$I_{\delta/X} \otimes_{\mathcal{O}_{\widetilde{P}}} I_{\delta/R} = \phi^* I_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} I_{\delta/R}|_\delta \to \phi^* I_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} \mathcal{O}_R(D-\delta)|_\delta, \tag{3.4}$$

given as follows. Let t be a section of $I_{0/Y}$ and f a section of $I_{\delta/R}$ on some open subset of $\widetilde{P}$. On this open set, lift f to a section $\widehat{f}$ of $\mathcal{O}_R(D-\delta)$. Then the map in (3.4) sends the element represented by $t \otimes f$ to the element represented by $t \otimes \widehat{f}$. Dually, the map

$$F = \phi^* N_{0/Y} \otimes_{\mathcal{O}_{\widetilde{P}}} \mathcal{O}_R(\delta - D) \to N_{\delta/X} \otimes N_{\delta/R} = \phi^* N_{0/Y} \otimes N_{\delta/R} \tag{3.5}$$

is simply $\mathrm{id} \otimes \mathrm{res}$, where

$$\mathrm{res}\colon \mathcal{O}_R(D-\delta) \to \mathcal{O}_R(D-\delta)|_\delta = N_{\delta/R}$$

is the restriction map.

To see why the map in (3.4) is as claimed above, consider the following diagram, obtained by restricting the bottom-right sqaure in the diagram in Lemma 3.3 to δ and taking $\mathcal{O}_\delta$-duals:

$$\begin{array}{ccc}
I_{Z/\widetilde{P}}|_\delta & \xleftarrow[b]{} & F^\vee|_\delta = I_{\delta/X} \otimes \mathcal{O}_R(D-\delta)|_\delta \\
{\scriptstyle a}\uparrow & & \uparrow{\scriptstyle c} \\
I_{\delta/X} \otimes I_{\delta/R} & = & I_{\delta/X} \otimes I_{\delta/R}.
\end{array}$$

Let $t \in I_{0/Y}$, and $f \in I_{\delta/R}$, and $\widehat{f} \in \mathcal{O}_R(D-\delta)$ be as in the definition of (3.5). To see that c indeed maps $t \otimes f$ to $t \otimes \widehat{f}$, it suffices to observe that

$$b(t \otimes \widehat{f}) = a(t \otimes f).$$

To compute the left hand side, we use the description of b from (3.3). We choose a lift $\widetilde{f} \in I_{X/\widetilde{P}} \otimes \mathcal{O}_{\widetilde{P}}(D)$ of $\widehat{f} \in \mathcal{O}_R(D-\delta)$. Then $b(t \otimes \widehat{f}) \in I_{Z/\widetilde{P}}|_\delta$ is the element represented by $t\widetilde{f}$. To compute the right hand side, we use the description of a from (2.5). We choose the lift $t \in I_{R/\widetilde{P}}$ of $t \in I_{\delta/X}$ and $\widetilde{f} \in I_{X/\widetilde{P}}(D)$ of $f \in I_{\delta/R}$, observing that $\widetilde{f}$ is indeed a section of $I_{X/\widetilde{P}}$ in a neighorbood of δ. Then $a(t \otimes f) \in I_{Z/\widetilde{P}}|_\delta$ is also the element represented by $t\widetilde{f}$.

Lemma 3.3 and the discussion following it gives us a good understanding of the relationship between $N_{Z/\widetilde{P}}$ and $N_{R/P}$. Next, we must relate $N_{Z/\widetilde{P}}$ and $N_{X/P}$. For this purpose, we need an auxiliary bundle M, which we now define. Set $N^+ = N_{Z/\widetilde{P}}$ and $N = N_{X/P}$. Consider the diagram

$$\begin{array}{ccccc}
 & & N^+|_R & & \\
 & & \downarrow & & \\
N^+|_X & \longrightarrow & N^+|_\delta & \longrightarrow & 0.
\end{array}$$

Define M by the sequence

$$0 \longrightarrow M \longrightarrow \phi_*\left(N^+|_X\right) \longrightarrow \operatorname{coker}\left(\phi_*\left(N^+|_R\right) \to \phi_*\left(N^+|_\delta\right)\right) \to 0. \tag{3.6}$$

The following explains how M is related to $N^+ = N_{Z/\widetilde{P}}$.

Lemma 3.4. *We have $R^1\phi_* N^+ = 0$, and an isomorphism of sheaves on Y*

$$M = \phi_* N^+/\mathit{torsion}.$$

Proof. By Lemma 3.3, we see that $H^1(R, N^+|_R) = 0$. Since Z is the union $X \cup R$ and $X \to Y$ is finite, $H^1(R, N^+|_R) = 0$ implies that $R^1\phi_* N^+ = 0$.

To see the isomorphism $M = \phi_* N^+/\text{torsion}$, consider the sequence on Z

$$0 \to N^+ \to N^+|_X \oplus N^+|_R \to N^+|_\delta \to 0,$$

where the last map sends (f, g) to $f|_\delta - g|_\delta$. Its push-forward to Y and the defining sequence of M fit in the diagram

$$\begin{array}{ccccccccc}
 & & & & & & \phi_*\left(N^+|_R\right) & & \\
 & & & & & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & \phi_* N^+ & \longrightarrow & \phi_*\left(N^+|_X\right) \oplus \phi_*\left(N^+|_R\right) & \longrightarrow & \phi_*\left(N^+|_\delta\right) & \longrightarrow & 0 \\
 & & \vdots\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \phi_*\left(N^+|_X\right) & \longrightarrow & \operatorname{coker}(p) & \longrightarrow & 0
\end{array},$$

where the middle vertical map is the projection on the first coordinate. (In the diagram, the dashed arrow is induced from the others.) By the snake lemma, we see that the map $\phi_* N^+ \to M$ is surjective. Since M is torsion free, and of the same generic rank as $\phi_* N^+$, we conclude that

$$M = \phi_* N^+/\text{torsion}.$$

□

Having related M and $N^+ = N_{Z/\widetilde{P}}$, we now relate M and $N = N_{X/P}$.

Lemma 3.5. *The bundle M is an inflation of $\phi_* N$ of degree 2 at $0 \in Y$. More precisely, we have an exact sequence*

$$0 \to \phi_* N \to M \to \phi_* F \to 0.$$

Proof. The proof involves some standard diagram chases. From Lemma 3.3, we obtain the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{R/P_0} & \longrightarrow & N^+|_R & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{R/P_0}\big|_\delta & \longrightarrow & N^+|_\delta & \longrightarrow & N_{\delta/X} \otimes N_{\delta/R} & \longrightarrow & 0.
\end{array}$$

From the isomorphism $N_{R/P_0} \cong \mathcal{O}(d+1)^{d-2}$ in Lemma 3.3, we get that the map $N_{R/P_0} \to N_{R/P_0}\big|_\delta$ is surjective on global sections. By the snake lemma, we get

$$\operatorname{coker}\left(\phi_*\left(N^+|_R\right) \to \phi_*\left(N^+|_\delta\right)\right) = \operatorname{coker}\left(\phi_* F \xrightarrow{e} \phi_*\left(N_{\delta/X} \otimes N_{\delta/R}\right)\right).$$

Substituting in the defining sequence (3.6) of M, we obtain

$$0 \longrightarrow M \longrightarrow \phi_*\left(N^+|_X\right) \longrightarrow \operatorname{coker}(e) \longrightarrow 0. \tag{3.7}$$

Recall from (2.11) in § 2.3 the sequence of sheaves on X

$$0 \to N \to N^+|_X \to N_{\delta/X} \otimes N_{\delta/R} \to 0.$$

Since $X \to Y$ is finite, the sequence remains exact after applying ϕ_*. We thus have the diagram

$$\begin{array}{ccccccccc} & & & & & & \phi_* F & & \\ & & & & & & \downarrow e & & \\ 0 & \longrightarrow & \phi_* N & \longrightarrow & \phi_*\left(N^+|_X\right) & \longrightarrow & \phi_*\left(N_{\delta/X} \otimes N_{\delta/R}\right) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \phi_*\left(N^+|_X\right) & \longrightarrow & \operatorname{coker}(e) & \longrightarrow & 0, \end{array} \tag{3.8}$$

where the dashed arrow is induced from the others. By the snake lemma, we get

$$0 \to \phi_* N \to M \to \phi_* F \to 0, \tag{3.9}$$

as asserted. □

Let us explicitly compute the defining quotient of the inflation $\phi_* N \to M$. We dualize—apply $\operatorname{Hom}_Y(-, \mathcal{O}_Y)$—to the diagram (3.8). The first two columns consist of locally free sheaves on Y. The last column consists of skyscraper sheaves supported at $0 \in Y$, which contribute only $\mathcal{E}\mathrm{xt}^1$ terms. Recall that for a vector space A considered as a sheaf supported at 0, we have

$$\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_Y}(A, \mathcal{O}_Y) = \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_Y}(k, \mathcal{O}_Y) \otimes A^\vee = N_{0/Y} \otimes A^\vee. \tag{3.10}$$

By applying (3.10) to $A = \phi_* F = N_{0/Y} \otimes \phi_* \mathcal{O}_R(\delta - D)$, we get

$$\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_Y}(\phi_* F, \mathcal{O}_Y) = H^0(\mathcal{O}_R(\delta - D))^\vee \text{ supported at } 0 \in Y.$$

By applying (3.10) to $A = \phi_*\left(N_{\delta/X} \otimes N_{\delta/R}\right) = N_{0/Y} \otimes \phi_* N_{\delta/R}$, we get

$$\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_Y}(\phi_*\left(N_{\delta/X} \otimes N_{\delta/R}\right), \mathcal{O}_Y) = H^0(N_{\delta/R})^\vee \text{ supported at } 0 \in Y.$$

Thus, the dual of (3.8) gives the following diagram (unimportant details suppressed)

$$\begin{array}{ccccccccc} & & H^0(\mathcal{O}_R(\delta - D))^\vee & & & & & & \\ & & \uparrow e^\vee & & & & & & \\ 0 & \longleftarrow & H^0\left(N_{\delta/R}\right)^\vee & \longleftarrow & (\phi_* N)^\vee & \longleftarrow & \left(\phi_*\left(N^+|_X\right)\right)^\vee & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longleftarrow & \cdots & \longleftarrow & M^\vee & \longleftarrow & \cdots & \longleftarrow & 0. \end{array} \tag{3.11}$$

In this diagram, the vector spaces in the leftmost column should be thought of as skyscraper sheaves at $0 \in Y$. The defining quotient we seek is thus the composite

$$(\phi_* N)^\vee|_0 \to H^0\left(N_{\delta/R}\right)^\vee \xrightarrow{e^\vee} H^0(\mathcal{O}_R(\delta - H))^\vee. \tag{3.12}$$

Observe that the first term in (3.12) is equal to $H^0(N_{X_0/P_0})^\vee = H^0(N_{\delta/P_0})^\vee$, since $X_0 = \delta$. The first map in (3.12), namely the map $H^0(N_{\delta/P_0})^\vee \to H^0(N_{\delta/R})^\vee$ is just H^0 applied to the defining quotient of the inflation $N \to N^+|_X$. We have studied this map in § 2.3 (see (2.10)). To recall it, write

$$H^0(N_{\delta/P_0})^\vee = H^0(I_{\delta/P_0}\big|_\delta), \text{ and}$$
$$H^0(N_{\delta/R})^\vee = H^0(I_{\delta/R}\big|_\delta).$$

Then the first map in (3.12) is H^0 applied to the natural restriction map

$$I_{X/\widetilde{P}}|_\delta = I_{\delta/P_0}|_\delta \to I_{\delta/R}|_\delta.$$

The second map in (3.12) is dual to the map

$$e\colon H^0(F) \to H^0(N_{\delta/X} \otimes N_{\delta/R})$$

obtained by applying H^0 to the map $F \to N_{\delta/X} \otimes N_{\delta/R}$, which we identified as id$\otimes$res in (3.5). Hence, we can explicitly describe the composite map in (3.12)

$$\alpha\colon H^0(I_{\delta/P_0}|_\delta) \to H^0(\mathcal{O}_R(\delta - D))^\vee \tag{3.13}$$

as follows. Consider an element $g \in H^0(I_{\delta/P_0}|_\delta)$. In a neighorhood of δ, choose a lift $\widetilde{g} \in I_{\delta/P_0}$ of g. Then $\alpha(g)$ is the function $H^0(\mathcal{O}_R(\delta - D)) \to k$ defined by

$$\alpha(g)\colon f \mapsto \sum_{x\in\delta} (f \cdot \widetilde{g})|_x.$$

Here $(f \cdot \widetilde{g})|_x$ is the evaluation at x of the function $(f \cdot \widetilde{g})$. Observe that $f \cdot \widetilde{g}$ is a regular function on R in a neighorbood of δ, and its evaluation at $x \in \delta$ depends only on g, and not on the lift $\widetilde{g}$.

The following proposition shows that a general R gives an M that contains a *general* degree 1 inflation of $\phi_* N$.

Proposition 3.6. *Let $Q \subset \mathbf{P}H^0\left(I_{\delta/P_0}|_\delta\right)$ be the set consisting of projections $q\colon H^0\left(I_{\delta/P_0}|_\delta\right) \to k$ that factor through the map $\alpha\colon H^0\left(I_{\delta/P_0}|_\delta\right) \to H^0(\mathcal{O}_R(\delta - D))^\vee$ for some rational normal curve $R \subset P_0$ containing δ. Then Q spans $\mathbf{P}H^0\left(I_{\delta/P_0}|_\delta\right)$.*

Proof. We begin by explicitly writing the curves R. Without loss of generality, $\delta \subset P_0 = \mathbf{P}^{d-1}$ consists of the d coordinate points, and the hyperplane $H_0 \subset \mathbf{P}^{d-1}$ is cut out by the equation $\sum X_i = 0$. We can write rational curves $R \subset P_0$ that contain δ as follows. Let $b_1, \ldots, b_d \in k^\times$ and $a_1, \ldots, a_d \in k$ be arbitrary constants with $a_i \neq a_j$ for $i \neq j$. Let x be a variable and let Π be the product $(x - a_1)\cdots(x - a_d)$. Consider the map $\mathbf{A}^1 \to P_0 = \mathbf{P}^{d-1}$ defined by

$$x \mapsto \left[\frac{b_1\Pi}{x - a_1} : \cdots : \frac{b_d\Pi}{x - a_d}\right]. \tag{3.14}$$

Let $R \subset P_0$ be the closure of the image; this is a rational normal curve. To see that R contains δ, simply observe that the map(3.14) sends a_i to the ith coordinate point. The divisor $D \subset R$ is cut out by $\sum X_i$, which pulls back under (3.14) to the polynomial

$$\Gamma = \sum b_i \frac{\Pi}{x - a_i}.$$

We now choose a basis of $H^0\left(I_{\delta/P_0}|_\delta\right)$. Let $Y_1, \ldots, Y_d$ be homogeneous coordinates on $P_0 = \mathbf{P}^{d-1}$ and let $\delta_j \in \delta$ be the jth coordinate point (where only $Y_j \neq 0$). For $i, j \in \{1, \ldots, d\}$ with $i \neq j$, define $g(i,j) \in H^0\left(I_{\delta/P_0}|_\delta\right)$ by

$$g(i,j)\big|_{\delta_\ell} = \begin{cases} Y_i/Y_j & \text{if } \ell = j,\\ 0 & \text{if } \ell \neq j.\end{cases}$$

Plainly, $\langle g(i,j)\rangle$ forms a basis of $H^0\left(I_{\delta/P_0}|_\delta\right)$.

The parametrization of R in (3.14) gives a basis of $H^0(\mathcal{O}_R(\delta - D))$ as follows. Identifying $H^0(\mathcal{O}_R(\delta - D))$ with the set of rational functions with zeros along $D = H_0 \cap R$ and possible poles along δ, we get a basis of $H^0(\mathcal{O}_R(\delta - D))$ given by $\langle \Gamma/\Pi, x\Gamma/\Pi\rangle$. Let $\langle u, v\rangle$ be the dual basis.

Let us use the description of α after (3.13) to compute the map

$$\alpha\colon \langle g(i,j)\rangle \to \langle u,v\rangle.$$

We see that $\alpha(g(i,j))$ is the functional on $\langle \Gamma/\Pi, x\Gamma/\Pi\rangle$ given by

$$\begin{aligned}\frac{\Gamma}{\Pi} &\mapsto \left(\frac{Y_i}{Y_j}\cdot\frac{\Gamma}{\Pi}\right)\Big|_{a_j}\\ &= \left(\frac{b_i(x-a_j)}{b_j(x-a_i)}\cdot\sum_{\ell}\frac{b_\ell}{x-a_\ell}\right)\Big|_{a_j}\\ &= \frac{b_i}{a_j-a_i}, \text{ and}\\ \frac{x\Gamma}{\Pi} &\mapsto \left(\frac{Y_i}{Y_j}\cdot\frac{x\Gamma}{\Pi}\right)\Big|_{a_j}\\ &= \frac{a_jb_i}{a_j-a_i}.\end{aligned}$$

Thus, the map α is

$$g(i,j)\mapsto \frac{b_i}{a_j-a_i}\cdot u + \frac{a_jb_i}{a_j-a_i}\cdot v. \tag{3.15}$$

The maps $q\colon \langle g(i,j)\rangle \to k$ in Q are precisely the maps (3.15) with u and v replaced by arbitrary elements of k. It is easy to verify that, as rational functions in the variables $a_1,\dots,a_d$, $b_1,\dots,b_d$, u, and v, the $d(d-1)$ functions $\frac{b_i(u+a_jv)}{a_j-a_i}$ are k-linearly independent. In other words, there is no k-linear equation that is satisfied by the maps q for all values of the a's, the b's, u, and v. The proposition follows. ☐

Corollary 3.7. *We have $R^1\phi_*N_{Z/\widetilde{P}} = 0$, and if R is general, then the sheaf $\phi_*N_{Z/\widetilde{P}}$ contains an inflation of $\phi_*N_{X/P}$ with a general defining quotient (away from any prescribed linear subspace).*

Proof. The vanishing of $R^1\phi_*N_{Z/\widetilde{P}}$ is from Lemma 3.4. Let $\Lambda \subset \mathbf{P}H^0(N_{X_0/P_0}) = \mathbf{P}H^0(I_{\delta/P_0}|_\delta)$ be any proper linear subspace. Choose $q\colon H^0(I_{\delta/P_0}|_\delta)\to k$ that is not contained in Λ and that factors through the map $\alpha\colon H^0(I_{\delta/P_0}|_\delta)\to H^0(\mathcal{O}_R(\delta-D))^\vee$ for some rational normal curve $R\subset P_0$ containing X_0. Such a q exists by Proposition 3.6. Let $N^\dagger$ be the inflation of $\phi_*N_{X/P}$ with defining quotient q, namely

$$N^\dagger = \ker(\phi_*N_{X/P}\xrightarrow{q} k),$$

where the k is supported at 0. By the exact sequence in Lemma 3.5, we see that $N^\dagger$ is a subsheaf of $\phi_*N_{Z/\widetilde{P}}$/torsion and hence of $\phi_*N_{Z/\widetilde{P}}$. ☐

3.3. **Smoothing out.**

Proposition 3.8 (Key)**.** *Let X be a nodal curve with a finite map $\phi\colon X\to Y$ of degree d. Let E be the Tschirnhausen bundle of ϕ. There exists a finite set $S\subset Y$, a smooth curve X', and a finite map $X'\to Y$ of degree d such that the following hold.*

1. *The Tschirnhausen bundle of $X'\to Y$ is $E' = E\otimes\mathcal{O}_Y(S)$.*
2. *Consider X' as embedded in (the total space of) E' by the canonical affine embedding. Then we have $H^1(X', N_{X'/E'}) = 0$.*

Furthermore, if n is large enough (determined by $X \to Y$), we may take S to have size n and an arbitrary divisor class of degree n.

Proof. Consider X embedded in the total space of E by the canonical affine embedding. Compactify the total space of E to the projective bundle $P = \mathbf{P}\left(E^\vee \oplus \mathcal{O}_Y\right)$, and let $H \subset \mathbf{P}$ be the hyperplane at infinity. Then we have $X \subset P$, disjoint from H, and $N_{X/E} = N_{X/P}$.

Set $N = \max\{h^1(N_{X/P}(-y)), y \in Y\}$ and let $n \geq 2N$. Choose a general $S \subset Y$ of size n and over every $y \in S$, perform the surgery described in § 3.2. Explicitly, let $\widetilde{P} \to P$ be the blow-up at $\sqcup_{y \in S} H_y \subset P$ and let R_y be a rational normal curve in the proper transform of P_y passing through X_y. Let $Z \subset \widetilde{P}$ be the curve

$$Z = X \bigcup \cup_{y \in S} R_y.$$

Note that Z is a connected nodal curve with arithmetic genus

$$\rho_a(Z) = \rho_a(X) + (d-1)n.$$

Thanks to Corollary 3.7, if we choose the rational curves R_y generically, then $\phi_* N_{Z/\widetilde{P}}$ contains a degree n inflation of $\phi_* N_{X/P}$ at S with general defining quotients. By Corollary 2.7, we conclude that $H^1(N_{Z/\widetilde{P}}) = 0$ and $N_{Z/\widetilde{P}}$ is globally generated.

Consider the Hilbert scheme of curves in $\widetilde{P}$. Since $H^1(N_{Z/\widetilde{P}}) = 0$, the Hilbert scheme is smooth at $[Z \subset \widetilde{P}]$ (see [33, Theorem 3.2.12]). Furthermore, since $N_{Z/\widetilde{P}}$ is globally generated, for every node $z \in Z$, the surjective map

$$N_{Z/\widetilde{P}} \to \mathscr{E}\mathrm{xt}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)\big|_z$$

is also surjective surjective on global sections. As a result, Z is the flat limit of a family of smooth curves in $\widetilde{P}$ (see [18, Proposition 1.1]). Let $X' \subset \widetilde{P}$ be a general member of such a family. By semi-continuity, the vanishing of H^1 and global generation continues to hold for $N_{X'/\widetilde{P}}$.

Let $\pi \colon \widetilde{P} \to P'$ be the blow-down of all the $P_y \subset \widetilde{P}$ (it is helpful to refer to Figure 2 again). We now check that $X' \subset \widetilde{P}$ maps isomorphically to its image in P'. Indeed, see that $P_y \cdot Z = 1$, and hence $P_y \cdot X' = 1$. Since X' is smooth and connected, $P_y \cap X'$ consists of a single (reduced) point, and hence, the blow-down of P_y does not change X'.

Let $H' \subset P'$ be the proper transform of $H \subset P$. Plainly, $X' \subset P'$ stays away from H'.

We claim that $P' \setminus H'$ is the total space of $E' = E \otimes \mathcal{O}_Y(S)$. Granting this claim, it is easy to see that $X' \subset E'$ is the canonical affine embedding and $H^1(N_{X'/E'}) = 0$. Indeed, E' has the correct degree:

$$\begin{aligned}\deg E' &= \deg E + n(d-1)\\ &= \rho_a(X) + d - 1 + n(d-1)\\ &= \rho_a(X') + d - 1,\end{aligned}$$

and a general fiber of $X' \to Y$ is in affine general position in E', so Proposition 2.11 applies. To see the vanishing of H^1, observe that we have an injection

$$N_{X'/\widetilde{P}} \xrightarrow{d\pi} N_{X'/P'}.$$

Since $H^1(N_{X'/\widetilde{P}}) = 0$, we get $H^1(N_{X'/P'}) = H^1(N_{X'/E'}) = 0$.

The claim that $E' = P' \setminus H'$ remains to be proved. Plainly, $P' \setminus H' \to Y$ is an $\mathbf{A}^d$-bundle. If we produce a section $\sigma' \colon Y \to P' \setminus H'$, then we can conclude that it is a vector bundle; the section acts as the zero section. Start with the zero section $\sigma \colon Y \to E \subset P$ and let $\sigma' \colon Y \to P'$ be its proper transform. Then σ' stays away from H' and gives a section $\sigma' \colon Y \to P' \setminus H'$. So we have proved that $P' \setminus H' \to Y$ is a vector bundle. To identify which vector bundle it is, it suffices to identify the normal bundle of the zero section σ'. By construction, we have $N_{\sigma/P} = E$ and hence also have $N_{\sigma/\widetilde{P}} = E$. A simple local calculation shows that the map $d\pi\colon N_{\sigma/\widetilde{P}} \to N_{\sigma'/P'}$ degenerates fully (becomes 0) at every $y \in S$, and gives an isomorphism

$$N_{\sigma/\widetilde{P}} = N_{\sigma'/P'} \otimes \mathcal{O}_Y(-S).$$

We deduce that $N_{\sigma'/P'} = E \otimes \mathcal{O}_Y(S)$, as required.

Finally, instead of $n \geq 2N$, if we take $n \geq 2N + g(Y)$, then the additional freedom to choose the $g(Y)$ points allows us to put S in any prescribed divisor class of degree n. $\square$

3.4. **The general case.** We now use the results of § 3.1 and § 3.3 to deduce the main theorem. Recall that Y is a connected, projective, and smooth curve over k, an algebraically closed field with $\operatorname{char} k = 0$ or $\operatorname{char} k > d$.

Theorem 3.9. *Let E be a vector bundle on Y of rank $(d-1)$. There exists an n (depending on E) such that for any line bundle L of degree at least n, there exists a smooth curve X and a finite flat morphism $\phi\colon X \to Y$ of degree d such that $E_\phi \cong E \otimes L$. Furthermore, we have $H^1(X, N_{X/E\otimes L}) = 0$, where $X \subset E \otimes L$ is the canonical affine embedding.*

Proof. Choose an isotrivial degeneration E_0 of E of the form

$$E_0 = L_1 \oplus \cdots \oplus L_{d-1},$$

where the L_i's are line bundles with $\deg L_i + (2g_Y - 1) \leq \deg L_{i+1}$. That is, let $(\Delta, 0)$ be a pointed curve and $\mathcal{E}$ a vector bundle on $Y \times \Delta$ such that $\mathcal{E}|_0 = E_0$ and $\mathcal{E}|_t \cong E$ for all $t \in \Delta \setminus \{0\}$. Such a degeneration exists by Proposition 2.9. Let $\pi\colon Y \times \Delta \to Y$ be the first projection. After replacing $\mathcal{E}$ by $\mathcal{E} \otimes \pi^*\lambda$ for a line bundle λ on Y of large degree, we may also assume that $\deg L_1 \geq 2g_Y - 1$.

By Proposition 3.1, there exists a nodal curve W and a finite flat morphism $W \to Y$ with Tschirnhausen bundle E_0. By the key proposition (Proposition 3.8), there exists an n such that for any line bundle L of degree at least n, we can find a smooth curve X_0 and a finite map $X_0 \to Y$ with Tschirnhausen bundle $E_0' = E_0 \otimes L$. Furthermore, we can make X_0 satisfy $H^1(N_{X_0/E_0'}) = 0$. Set $\mathcal{E}' = \mathcal{E} \otimes \pi^* L$. Let $\mathcal{H}$ be the component of the relative Hilbert scheme of $\operatorname{Tot}(\mathcal{E}') \to \Delta$ containing the point $[X_0 \subset E_0']$. Since $H^1(N_{X_0/E_0'}) = 0$, the map $\mathcal{H} \to \Delta$ is smooth at $[X_0 \subset E_0']$ by [33, Theorem 3.2.12]. In particular, $\mathcal{H} \to \Delta$ is dominant. As a result, there exists a point $[X \subset \mathcal{E}_t'] \in \mathcal{H}$, where X is smooth and $t \in \Delta$ is generic. By the choice of $\mathcal{E}$, we have $\mathcal{E}_t' = E \otimes L$. Since $H^1(N_{X_0/E_0 \otimes L}) = 0$, get that $H^1(N_{X/E\otimes L}) = 0$ by semi-continuity. Let $\phi\colon X \to Y$ be the projection. Since $X_0 \subset E_0 \otimes L$ is the canonical affine embedding, Proposition 2.11 implies that $X \subset E \otimes L$ is also the canonical affine embedding. The proof is now complete. $\square$

Remark 3.10. Theorem 3.9 can be stated in terms of moduli stacks of covers and bundles in the following way. Denote by $\mathcal{H}_d(Y)$ the stack whose S points are finite flat degree d morphisms $\phi\colon C \to Y \times S$, where $C \to S$ is a smooth curve. Let $\mathrm{Vec}_{d-1}(Y)$ be the stack whose S points are vector bundles of rank $(d-1)$ on $Y \times S$. Both $\mathcal{H}_d(Y)$ and $\mathrm{Vec}_{d-1}(Y)$ are algebraic stacks, locally of finite type, and smooth over k. The rule

$$\tau\colon \phi \mapsto E_\phi$$

defines a morphism $\tau\colon \mathcal{H}_d(Y) \to \mathrm{Vec}_{d-1}(Y)$. Then Theorem 3.9 says that given $E \in \mathrm{Vec}_{d-1}(Y)$ and given any line bundle L on Y of large enough degree, there exists a point $[\phi\colon X \to Y]$ of $\mathcal{H}_d(Y)$ such that $\tau(\phi) = E \otimes L$, and furthermore, such that the map τ is smooth at $[\phi]$.

3.5. **Hurwitz spaces and Maroni loci.** We turn to the proof of Theorem 1.5 stated in the introduction. First we establish notation and conventions regarding the various Hurwitz spaces. Throughout § 3.5, take the base field $k = \mathbf{C}$.

Let $\mathcal{H}^{\text{all}}_{d,g}(Y)$ be the stack whose objects over S are S-morphisms $\phi\colon C \to Y \times S$, where $C \to S$ is a smooth, proper, connected curve of genus g, and ϕ is a finite morphism of degree d. Observe that $\mathcal{H}^{\text{all}}_{d,g}(Y)$ is an open substack of the Kontsevich stack of stable maps $\overline{\mathcal{M}}_g(Y, d[Y])$ constructed, for example, in [16] or in [5]. As a result, $\mathcal{H}^{\text{all}}_{d,g}(Y)$ is a separated Deligne–Mumford stack of finite type over k. Using the deformation theory of maps [33, Example 3.4.14], it follows that $\mathcal{H}^{\text{all}}_{d,g}(Y)$ is smooth and equidimensional of dimension $2b = (2g-2) - d(2g_Y - 2)$. Denote by $\mathcal{H}^{\text{simple}}_{d,g}(Y) \subset \mathcal{H}^{\text{all}}_{d,g}(Y)$ the open substack of simply branched maps, namely the substack whose S-points correspond to maps $\phi\colon C \to Y \times S$ whose branch divisor $\operatorname{br}\phi \subset Y \times S$ is étale over S (the branch divisor is defined as the vanishing locus of the discriminant [37, Tag 0BVH]). The transformation $\phi \mapsto \operatorname{br}\phi$ gives a morphism

$$\mathcal{H}^{\text{all}}_{d,g}(Y) \to \operatorname{Sym}^{2b} Y$$

with finite fibers. Since the source is equidimensional of the same dimension as the target and the map is quasi-finite, each component of $\mathcal{H}^{\text{all}}_{d,g}(Y)$ maps dominantly on $\operatorname{Sym}^{2b}(Y)$. In particular, $\mathcal{H}^{\text{simple}}_{d,g}(Y)$ is dense in $\mathcal{H}^{\text{all}}_{d,g}(Y)$. By a celebrated theorem of Clebsch [12], if $g_Y = 0$, then $\mathcal{H}^{\text{simple}}_{d,g}(Y)$ is connected (equivalently, irreducible). More generally, by [17, Theorem 9.2], the connected (= irreducible) components of $\mathcal{H}^{\text{all}}_{d,g}(Y)$ are classified by the subgroup $\phi_*\pi_1(C)$ of $\pi_1(Y)$. Recall that ϕ is called *primitive* if $\phi_*\pi_1(C) = \pi_1(Y)$, or equivalently, if ϕ does not factor through an étale covering $\widetilde{Y} \to Y$. Denote by $\mathcal{H}^{\text{primitive}}_{d,g}(Y) \subset \mathcal{H}^{\text{all}}_{d,g}(Y)$ the connected (= irreducible) component whose points correspond to primitive covers.

The connection between primitive and simply branched covers is the following. By [6, Proposition 2.5], if $\phi\colon C \to Y$ is a simply branched covering, then ϕ is primitive if and only if the monodromy map

$$\pi_1(Y \setminus \operatorname{br}\phi) \to S_d$$

is surjective. Therefore, we can view $\mathcal{H}^{\text{primitive}}_{d,g}(Y)$ as a partial compactification of the stack of simply branched covers of Y with full monodromy group S_d. By convention, $\mathcal{H}_{d,g}(Y)$ (without any superscript) denotes the component $\mathcal{H}^{\text{primitive}}_{d,g}(Y)$ of $\mathcal{H}^{\text{all}}_{d,g}(Y)$.

Being open substacks of the Kontsevich stack, the Hurwitz stacks described above admit quasi-projective coarse moduli spaces, which we denote by the roman equivalent $H_{d,g}$ of $\mathcal{H}_{d,g}$. Denote by $M_{r,k}(Y)$ the moduli space of vector bundles of rank r and degree k on Y. Let $\mathcal{U} \subset \mathcal{H}_{d,g}(Y)$ be the (possibly empty) open substack consisting of points $[\phi] \in \mathcal{H}_{d,g}(Y)$ such that E_ϕ is semi-stable. We have a morphism $\mathcal{U} \to M_{d-1,b}(Y)$ defined functorially as follows. An object $\phi\colon C \to Y \times S$ of $\mathcal{U}$ maps to the unique morphism $S \to M_{d-1,b}(Y)$ induced by the bundle E_ϕ on $Y \times S$. Let $U \subset H_{d,g}(Y)$ be the coarse space of $\mathcal{U}$. By the universal property of coarse spaces, the morphism $\mathcal{U} \to M_{d-1,b}(Y)$ descends to a morphism $U \to M_{d-1,b}(Y)$. If U is non-empty, then we can think of $U \to M_{d-1,b}(Y)$ as a rational map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$.

Recall that Y is a smooth, projective, connected curve over $\mathbf{C}$.

Theorem 3.11. *Let $g_Y \geq 2$. If g is sufficiently large (depending on Y and d), then the Tschirnhausen bundle associated to a general point of $H_{d,g}(Y)$ is stable. Moreover, the rational map*

$$H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$$

given by $[\phi] \mapsto E_\phi$ *is dominant.*

The same statement holds for $g_Y = 1$ *with "stable" replaced by "regular poly-stable."*

Proof. Let $g_Y \geq 2$; the proof for $g_Y = 1$ is identical with "stable" replaced by "regular poly-stable."

Let $\phi_0 \colon X_0 \to Y$ be an element of the primitive Hurwitz space $H_{d,g_0}(Y)$ with Tschirnhausen bundle E_0. For some line bundle L of sufficiently large degree, there exists $\phi \colon X \to Y$ with Tschirnhausen bundle $E = E_0 \otimes L$ with $H^1(N_{X/E}) = 0$ by Proposition 3.8. From the proof of Proposition 3.8, we know that $X \to Y$ is obtained as a deformation of the singular curve formed by attaching vertical rational curves to X_0. Recall that in a deformation, the π_1 of a general fiber surjects on to the π_1 of the special fiber. Hence, since $\pi_1(X_0) \to \pi_1(Y)$ is surjective, so is $\pi_1(X) \to \pi_1(Y)$. That is, $X \to Y$ is primitive.

We know that the moduli stack of vector bundles on Y is irreducible [20, Appendix A] and therefore, the locus of stable bundles forms a dense open substack. So, we can find a vector bundle $\mathcal{E}$ on $Y \times \Delta$ such that $\mathcal{E}_{Y \times \{0\}} = E$ and $\mathcal{E}_{Y \times \{t\}}$ is stable for $t \in \Delta \setminus \{0\}$. As $H^1(N_{X/E}) = 0$, the curve $X \subset E$ deforms to the generic fiber of $\mathcal{E} \to \Delta$, by the same relative Hilbert scheme argument as used in the proof of Theorem 3.9. Let $X_t \subset \mathcal{E}_t$ be such a deformation. Then $X_t \to Y$ is a primitive cover with a stable Tschirnhausen bundle. We conclude that for sufficiently large g, the Tschirnhausen bundle of a general element of $H_{d,g}(Y)$ is stable.

Let $\phi \colon X \to Y$ be an element of $H_{d,g}(Y)$ with stable Tschirnhausen bundle E such that $H^1(N_{X/E}) = 0$. The above argument shows that such coverings exist if g is sufficiently large. Let S be a versal deformation space for E and $\mathcal{E}$ a versal vector bundle on $Y \times S$. See [26, Lemma 2.1] for a construction of S in the analytic category. In the algebraic category, we can take S to be a suitable Quot scheme (see, for example, [20, Proposition A.1]). Let $\mathcal{H}$ be the component of the relative Hilbert scheme of $\operatorname{Tot}(\mathcal{E})/S$ containing the point $[X \subset E]$, and let $\mathcal{H}^{\mathrm{sm}} \subset \mathcal{H}$ be the open subset parametrizing $[X_t \subset \mathcal{E}_t]$ with smooth X_t. Since $H^1(N_{X/E}) = 0$, the map $\mathcal{H}^{\mathrm{sm}} \to S$ is smooth at $[X \subset E]$ by [33, Theorem 3.2.12]. In particular, it is dominant. By Proposition 2.11, we know that for $[X_t \subset \mathcal{E}_t] \in \mathcal{H}^{\mathrm{sm}}$, the bundle $\mathcal{E}_t$ is indeed the Tschirnhausen bundle of $X_t \to Y$. We conclude that the map $H_{d,g}(Y) \dashrightarrow M_{d-1,b}(Y)$ is dominant. □

Remark 3.12. It is natural to ask for an effective lower bound on g in Theorem 3.11. By studying our proof, we get lower bounds of order $d^3 g_Y$. It may be interesting to obtain sharper results.

Recall that the Maroni locus $M(E)$ is the locally closed subset of $H_{d,g}(Y)$ defined by

$$M(E) = \left\{ [\phi] \in H_{d,g}(Y) \mid E_\phi \cong E \right\}.$$

Theorem 3.13. *Let E be a vector bundle on Y of rank $(d-1)$ and degree e. If g is sufficiently large (depending on Y and E), then for every line bundle L on Y of degree $b-e$, the Maroni locus $M(E \otimes L)$ contains an irreducible component of the expected codimension $h^1(Y, \operatorname{End} E)$.*

Proof. Set $E' = E \otimes L$. Let H^{sm} be the open subset of the Hilbert scheme of curves in $\operatorname{Tot}(E')$ parametrizing $[X \subset E']$ with X smooth of genus g embedded so that for all $y \in Y$, the scheme $X_y \subset E'_y$ is in affine general position. By Proposition 2.11, the Tschirnhausen bundle map

$$\tau \colon H^{\mathrm{sm}} \to M(E')$$

is a surjection. Furthermore, the fibers of τ are orbits under the group A of affine linear transformations of E' over Y. Plainly, the action of the group is faithful.

By Proposition 3.8, there exists $[X \subset E'] \in H^{\mathrm{sm}}$ with $H^1(N_{X/E'}) = 0$. We can now do a dimension count. Note that $N_{X/E'}$ is a vector bundle on X of rank $(d-1)$ and degree $(d+2)b$, where

$b = g_X - 1 - d(g_Y - 1)$. Then the dimension of H^{sm} at $[X \subset E']$ is given by

$$\begin{aligned}\dim_{[X]} H^{\mathrm{sm}} &= \chi(N_{X/E'})\\ &= (d+2)b - (g_X - 1)(d-1)\\ &= 3b - d(d-1)(g_Y - 1)\end{aligned}$$

The dimension of the fiber of τ is given by

$$\begin{aligned}\dim A &= \hom(E'^\vee, \mathcal{O}_Y \oplus E'^\vee)\\ &= b - d(d-1)(g_Y-1) + h^1(\operatorname{End} E).\end{aligned}$$

As a result, the dimension of $M(E')$ at $[\phi]$ is given by

$$\begin{aligned}\dim_{[\phi]} M(E') &= \dim_{[X]} H^{\mathrm{sm}} - \dim A\\ &= 2b - h^1(\operatorname{End} E).\end{aligned}$$

Since $\dim H_{d,g}(Y) = 2b$, the proof is complete. □

4. Higher dimensions

In this section, we discuss the possibility of having an analogue of Theorem 1.1 for higher dimensional Y. For simplicity, take $k = \mathbf{C}$.

Let us begin with the following question.

Question 4.1. *Let Y be a smooth projective variety, L an ample line bundle on Y, and E a vector bundle of rank $(d-1)$ on Y. Is $E \otimes L^n$ a Tschirnhausen bundle for all sufficiently large n?*

The answer to Question 4.1 is “No”, at least without additional hypotheses.

Example 4.2. Take $Y = \mathbf{P}^4$, and $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then a sufficiently positive twist E' of E cannot be the Tschirnhausen bundle of a smooth branched cover X.

To see this, recall that the data of a Gorenstein triple cover $X \to Y$ with Tschirnhausen bundle E' is equivalent to the data of a nowhere vanishing global section of $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ (see [25] or [11]). For $E' = E \otimes L^n$ with large n, the rank 4 vector bundle $\operatorname{Sym}^3 E' \otimes (\det E')^\vee$ is very ample. Thus, its fourth Chern class is nonzero. Therefore, a general global section must vanish at some points.

In fact, it is easy to see by direct calculation that the fourth Chern class of $\operatorname{Sym}^3 E \otimes (\det E)^\vee$ can vanish if and only if $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $b = 2a$. Conversely, $E = \mathcal{O}(a) \oplus \mathcal{O}(2a)$ is the Tschirnhausen bundle of a cyclic triple cover of $\mathbf{P}^4$. Thus, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ can be a Tschirnhausen bundle of a smooth triple cover of $\mathbf{P}^4$ if and only if $b = 2a$.

Example 4.2 illustrating the failure of Theorem 1.1 can be generalized to all degrees ≥ 3, provided the base Y is allowed to be high dimensional.

Proposition 4.3. *Let $d \geq 3$. The answer to Question 4.1 is “No” for all Y of dimension at least $d\binom{d}{2}$.*

Proof. Let $\phi : X \to Y$ be a finite, flat, degree d map. Then the sheaf $\phi_* \mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras, and it splits as $\phi_* = \mathcal{O}_Y \oplus E^\vee$.

Suppose over some point $y \in Y$, the multiplication map

$$m : \operatorname{Sym}^2 E^\vee \to \phi_* \mathcal{O}_X$$

is identically zero. Then, we have a k-algebra isomorphism

$$(\phi_* \mathcal{O}_X)|_y \cong k[x_1, \dots, x_{d-1}]/(x_1, \dots, x_{d-1})^2.$$

That is, $\phi^{-1}(y)$ is isomorphic to the length d "fat point", defined by the square of the maximal ideal of the origin in an affine space. When $d \geq 3$, these fat points are not Gorenstein. Since Y is smooth, this implies X can not even be Gorenstein, let alone smooth.

Now, if E is a vector bundle on Y and L is a sufficiently positive line bundle, then the bundle

$$M := \operatorname{Hom}(\operatorname{Sym}^2(E \otimes L)^\vee, \mathcal{O}_Y \oplus (E \otimes L)^\vee)$$

is very ample. A general global section $m \in H^0(Y, M)$ will vanish identically at some points $y \in Y$ provided

$$\dim Y \geq \operatorname{rk} M = d\binom{d}{2}.$$

We conclude that if $\dim Y \geq d\binom{d}{2}$, then Question 4.1 has a negative answer. □

Observe that Proposition 4.3 remains true even if we relax the requirement that X be smooth to X be Gorenstein.

The following result due to Lazarsfeld suggests the possibility that Proposition 4.3 may be true with a much better lower bound than $d\binom{d}{2}$.

Proposition 4.4. *Let E be a vector bundle of rank $(d-1)$ on $\mathbf{P}^r$, where $r \geq d+1$. Then $E(n)$ is not a Tschirnhausen bundle of a smooth, connected cover for sufficiently large n.*

Proof. The proof relies on [24, Proposition 3.1] which states that for a branched cover $\phi\colon X \to \mathbf{P}^r$ of degree $d \leq r-1$ with X smooth and connected, the pullback map

$$\phi^*\colon \operatorname{Pic}(\mathbf{P}^r) \to \operatorname{Pic} X$$

is an isomorphism. In particular, the dualizing sheaf ω_ϕ is isomorphic to $\phi^*\mathcal{O}(l)$ for some l. Note that ω_ϕ is represented by an effective divisor (the ramification divisor), so $l > 0$. Therefore, we get

$$\mathcal{O}_{\mathbf{P}^r} \oplus E = \phi_*\omega_\phi = \phi_*\mathcal{O}(l) = \mathcal{O}_{\mathbf{P}^r}(l) \oplus E^\vee(l).$$

Since X is connected, $E^\vee$ has no global sections. Using this, it is easy to conclude from the above sequence that $\mathcal{O}_{\mathbf{P}^r}(l)$ is a summand of E.

Suppose $E(n)$ is a Tschirnhausen bundle of a smooth connected cover for infinitely many n. Applying the reasoning above with E replaced by $E(n)$ shows that E must have line bundle summands of infinitely many degrees. Since this is impossible, the proposition follows. □

The reasoning in Example 4.2 implies the following.

Proposition 4.5. *For degree 3, Question 4.1 has an affirmative answer if and only if* $\dim Y < 4$.

Proof. Let $\phi\colon X \to Y$ be a Gorenstein finite covering of degree 3 with Tschirnhausen bundle E. Then by the structure theorem of triple covers in [25] or [11], we get an embedding $X \subset \mathbf{P}E$ as a divisor of class $\mathcal{O}_{\mathbf{P}E}(3)$. Thus, X is given by a global section on $\mathbf{P}E$ of $\mathcal{O}_{\mathbf{P}E}(3)$, or equivalently a global section on Y of $\operatorname{Sym}^3 E \otimes \det E^\vee$. Note that since $X \to Y$ is flat, the global section of $\operatorname{Sym}^3 E \otimes \det E^\vee$ is nowhere vanishing.

Suppose we are given an arbitrary rank 2 vector bundle E on Y. Set $D = \mathcal{O}_{\mathbf{P}E}(3)$ and $V = \operatorname{Sym}^3 E \otimes \det E^\vee$. If we twist E by L^n, then $\mathbf{P}E$ is unchanged but D changes to $D + 3nL$ and V changes to $V \otimes L^n$. For sufficiently large n, the bundle $V \otimes L^n$ is ample. If $\dim Y < 4$, then a general section of $V \otimes L^n$ is nowhere zero on Y. Furthermore, the divisor $X \subset \mathbf{P}E$ cut out by the corresponding section of $\mathcal{O}(D + 3nL)$ is smooth by Bertini's theorem. By construction, the resulting $X \to Y$ has Tschirnhausen bundle $E \otimes L^n$.

On the other hand, if $\dim Y \geq 4$, then every global section of $V \otimes L^n$ must vanish at some point in Y. Thus, $E \otimes L^n$ cannot arise as a Tschirnhausen bundle. □

4.1. **Modifications of the original question.** Following the discussion in the previous section, natural modified versions of Question 4.1 emerge. The first obvious question is the following.

Question 4.6. *Is the analogue of Theorem 1.1 true for all Y with* $\dim Y \leq d$*?*

We can also relax the finiteness assumption on ϕ.

Question 4.7. *Let Y be a smooth projective variety, E a vector bundle in Y. Is E isomorphic to $(\phi_* \mathcal{O}_X / \mathcal{O}_Y)^\vee$, up to a twist, for a* generically *finite map $\phi : X \to Y$ with smooth X?*

Remark 4.8. A similar question is addressed in work of Hirschowitz and Narasimhan [19], where it is shown that any vector bundle on Y is the direct image of *some* line bundle on a smooth variety X under a generically finite morphism.

Alternatively, we can keep the finiteness requirement on ϕ in exchange for the smoothness of X. We end the paper with the following open-ended question.

Question 4.9. *What singularity assumptions on X (or the fibers of ϕ) yield a positive answer to Question 4.1?*

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