SYZYGIES OF CANONICAL RIBBONS ON HIGHER GENUS CURVES

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ABSTRACT. We study the syzygies of the canonical embedding of a ribbon $\widetilde{C}$ on a curve C of genus $g \geq 1$. We show that the linear series Clifford index and the resolution Clifford index are equal for a general ribbon of arithmetic genus p_a on a general curve of genus g with $p_a \geq \max\{3g+7, 6g-4\}$. Among non-general ribbons, the case of split ribbons is particularly interesting. Equality of the two Clifford indices for a split ribbon is related to the gonality conjecture for C and it implies Green's conjecture for all double covers C' of C with $g(C') \geq \max\{3g+2, 6g-4\}$. We reduce it to the vanishing of certain Koszul cohomology groups of an auxiliary module of syzygies associated to C, which may be of independent interest.

1. INTRODUCTION

Let $X \subset \mathbb{P}^N$ be a projective variety. Let $I \subset k[X_0, \ldots, X_N] =: S$ be its homogeneous ideal, $R := S/I$ the homogeneous coordinate ring, and $L = \mathcal{O}_X(1)$. A fundamental problem is to understand the connection between geometric properties of (X, L) and homological/algebraic properties of R as an S module. In the mid 1980's, Mark Green used the Koszul cohomology groups $K_{p,q}(X, L)$ to formulate such a connection [G84, G84b]. Out of this work arose his conjecture about the Clifford index of a smooth projective curve C and the shape of the minimal free resolution of the homogeneous coordinate ring of its canonical embedding [G84, GL86]. This conjecture states that the Clifford index of C, a geometric quantity defined using linear series on C, is equal to its resolution Clifford index, an algebraic quantity related to the vanishing of the Koszul cohomology groups. In [BE95] and [EG95], the authors laid out an approach to prove Green's conjecture for a general curve by proving it for an everywhere non-reduced curve called a ribbon. We recall the definition.

Definition 1.1. A ribbon $\widetilde{Y}$ on a reduced connected scheme Y is a non-reduced scheme with $\widetilde{Y}_{\mathrm{red}} = Y$ such that

(1) the ideal sheaf $\mathcal{I}_{Y/\widetilde{Y}} \subset \mathcal{O}_{\widetilde{Y}}$ satisfies $\mathcal{I}_{Y/\widetilde{Y}}^2 = 0$ and
(2) viewed as an $\mathcal{O}_Y$ module, $\mathcal{I}_{Y/\widetilde{Y}}$ is a locally free sheaf L of rank one (called the conormal bundle of $\widetilde{Y}$).

The canonical ribbon conjecture, formulated in [BE95] and [EG95], states that the analogue of Green's conjecture holds for a rational ribbon, that is, a ribbon $\widetilde{C}$ with $\widetilde{C}_{\mathrm{red}} \cong \mathbb{P}^1$. By semi-continuity, it implies Green's conjecture for a general curve. In a series of pioneering papers, Voisin proved Green's conjecture for a general curve using different methods [V02], [V05]. Using

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the results of Voisin and Hirschowitz–Ramanan [HR98], the first author proved the canonical ribbon conjecture for rational ribbons [D18]. Eventually, Raicu–Sam [RS19] and Park [P22] gave independent proofs of the canonical ribbon conjecture, without using the results of Voisin and Hirschowitz–Ramanan, obtaining a proof of Green's conjecture for general curves via ribbons.

In this context, we study the minimal free resolutions of canonical ribbons over higher genus curves. It has been shown in [Gon06] and [GGP08], that in almost all cases such ribbons arise as flat limits of smooth curves. We see many compelling reasons to take up this study.

First, the canonical ribbon conjecture for ribbons is interesting for the same reasons as it is for smooth curves. We have a geometric notion of linear series on ribbons, formulated in [EG95]. By the same argument as for smooth curves, presence of linear series produces non-linear syzygies, or equivalently, non-vanishing $K_{p,2}$'s. It is natural to ask whether there are other reasons for non-linear syzygies, whose origins we do not understand. It is worthwhile to understand the answer in as many situations as we can. Ribbons on higher genus curves is one such setting.

Second, by semi-continuity Green's conjecture for split ribbons on C imply Green's conjecture for all double covers of C. Thus, understanding the minimal free resolution of the split ribbon has immediate payoff.

Third, ribbons provide one of the few ways of producing higher genus curves from lower genus ones–a kind of "induction". A ribbon $\widetilde{C}$ on a smooth curve C is given purely in terms of linear algebraic data: a rank 2 bundle obtained as an extension of Ω_C by the conormal bundle. Thus, ribbons allow us to connect questions about higher genus curves to questions about bundles on lower genus curves. The geometry of bundles on higher genus curves is much richer than that on $\mathbb{P}^1$. It is natural to ask if it leads to a richer connection.

Finally, the geography of the moduli space of ribbons over higher genus curves is much more interesting. In genus 0, the space of ribbons is naturally the ambient space of a rational normal curve. This space has many different statifications: (1) the stratification by the secant varieties of the rational normal curve, (2) the stratification by the blow-up index of the ribbon, (3) the stratification by the splitting type of the rank 2 bundle defining the ribbon, and (4) the stratification by the shape of the minimal free resolution of the canonically embedded ribbon. A consequence of the canonical ribbon conjecture is that *all* these stratifications agree. All stratifications have analogues for ribbons on higher genus curves. Understanding whether and how they differ is an intrinsically interesting question.

1.1. **Results.** Our first result is the canonical ribbon conjecture for a general ribbon of large genus on a general curve

Theorem 1.2 (Theorem 6.5, Theorem 6.6, Theorem 6.7 in the main text)**.** *Let C be a general curve of genus g. Fix a line bundle L on C. Let $\widetilde{C}$ be a general ribbon with conormal bundle L. Let $p_a(\widetilde{C})$ be the arithmetic genus, $\mathrm{gon}(\widetilde{C})$ the gonality, $\mathrm{LCliff}(\widetilde{C})$ the linear series Clifford index, and $\mathrm{RCliff}(\widetilde{C})$ the resolution Clifford index of $\widetilde{C}$. If $p_a(\widetilde{C}) \geq \max\{3g+7, 6g-4\}$, then we have*

$$\mathrm{LCliff}(\widetilde{C}) = \mathrm{gon}(\widetilde{C}) - 2 = \mathrm{RCliff}(\widetilde{C}) = \left\lfloor \frac{p_a(\widetilde{C}) - 1}{2} \right\rfloor.$$

To prove Theorem 1.2, we study two stratifications on the space $\mathbb{P}(H^0(2K_C - L)^*)$ of ribbons on C with conormal bundle L. The first is the stratification by the blow-up index of the ribbon,

and the second is the stratification by gonality. These two stratifications coincide with the secant stratification in the case when $C \cong \mathbb{P}^1$. In higher genus, the blow-up index stratification coincides with the secant stratification, but the gonality stratification is different and more interesting. Let $\mathrm{Sec}_k(C)$ denote the kth secant variety of C in the embedding $C \hookrightarrow \mathbb{P}H^0(2K_C - L)^*$.

Theorem 1.3 (Theorem 5.1 and Theorem 5.2)**.** *Let C be a smooth curve of genus g and gonality m. Then the subvariety W_d of $\mathbb{P}(H^0(2K_C - L)^*)$ parameterizing ribbons $\widetilde{C}$ with conormal bundle L containing a g_d^1 is the subvariety of $\mathrm{Sec}_{d+2g-2}(C)$ given by*

$$W_d = \bigcup_{e \le d/2} \left\{ \begin{array}{l} \text{Secant planes spanned by divisors of length } d + 2g - 2 \text{ that contain} \\ \text{a ramification divisor of a degree e map } C \to \mathbb{P}^1 \end{array} \right\}.$$

In particular, we have

$$\mathrm{Sec}_{d-2m}(C) \subset W_d \subset \mathrm{Sec}_{d+2g-2}(C).$$

We turn next to the split ribbon. This is a particularly interesting case—all smooth double covers isotrivially degenerate to a split ribbon, and hence, a proof of equality of the linear series and resolution Clifford index for a split ribbon establishes Green's conjecture for a smooth double cover. We first show that the linear series Clifford index of a split ribbon is $2m-2$, where m is the gonality of the underlying curve (Theorem 7.1). The rest of the section is devoted to proving that the following statements are equivalent to the fact that the resolution Clifford index of a split ribbon is $2m - 2$. We prove the following.

Theorem 1.4. *(see Theorem 7.5, Theorem 7.3) Let $\widetilde{C}$ be a split ribbon on a curve C of genus g and gonality m with conormal bundle L.*

(1) *The linear series Clifford index of $\widetilde{C}$ is $\mathrm{LCliff}(\widetilde{C}) = 2m - 2$.*
(2) *If $p_a(\widetilde{C}) \ge \max\{2g + 2m - 1, 6g - 4\}$, then the following are equivalent*

   (a) *The resolution Clifford index $\mathrm{RCliff}(\widetilde{C}) = 2m - 2$.*

   (b) *For all i, j with $i, j \ge 0$ and $i + j = 2m - 3$, the map*

   $$\Phi_{i,j} : \bigwedge^{i+1} H^0(K_C) \otimes K_{j,1}(C, K_C - L) \to \bigwedge^i H^0(K_C) \otimes K_{j,1}(C, K_C, K_C - L)$$

   is surjective.

   (c) *Set $M^j = \bigoplus_q K_{j,1}(C, qK_C, K_C - L)$. For all i, j with $i, j \ge 0$ and $i + j = 2m - 3$, we have*

   $$K_{i,1}(M^j, H^0(K_C)) = 0.$$

(3) *If any of the equivalent conditions of part (2) holds, then any smooth curve C' which is a double cover of C branched along $-2L$ satisfies Green's conjecture.*

We believe that it is interesting to study whether the equivalent statements are true for any smooth curve C (see Theorem 7.6). We show that the question has a positive answer when the curve is either elliptic or hyperelliptic. We end with Theorem 7.8, where we give an example of a split ribbon of low arithmetic genus, i.e, arithmetic genus 9 over a general curve of genus 3, which does *not* satisfy Green's conjecture.

1.2. **Organization and conventions.** In Section 2, we recall basic results about the canonical embedding of a ribbon. In Section 3, we recall and establish the basic properties of the Clifford index for ribbons. In Section 4 and Section 5, we study the blow-up index and gonality stratification of the space of ribbons. In Section 6, we use these stratifications to prove Green's conjecture for a general ribbon. In Section 7, we treat the case of a split ribbon.

All schemes are of finite type over an algebraically closed field of characteristic zero. For us, the projectivisation of V is $\mathbb{P}V = \mathrm{Proj}\,\mathrm{Sym}^*(V^*)$, which is the space of one-dimensional subspaces of V. Unless stated otherwise, a *curve* is a projective connected scheme of pure dimension 1.

2. CANONICAL EMBEDDING OF RIBBONS ON CURVES OF HIGHER GENUS

In this section, we study the canonical map of a ribbon on a higher genus curve. Let C be a smooth curve, L a line bundle on C, and $\widetilde{C}$ a ribbon on C with conormal bundle L.

We begin by analyzing sections of positive line bundles on $\widetilde{C}$.

Proposition 2.1. *In the setup above, let $\widetilde{M}$ be a line bundle on $\widetilde{C}$ and set $M := \widetilde{M}|_C$.*

(1) *Assume that M is very ample, $M \otimes L$ is base point free and $H^0(\widetilde{M}) \to H^0(M)$ surjects. Then $\widetilde{M}$ is very ample.*

(2) *Assume that M is projectively normal and for all $k \geq 1$, the maps*

$$H^0(k\widetilde{M}) \to H^0(kM) \text{ and } H^0(M \otimes L) \otimes H^0(kM) \to H^0((k+1)M \otimes L)$$

are surjective. Then the multiplication map $H^0(k\widetilde{M}) \otimes H^0(\widetilde{M}) \to H^0((k+1)\widetilde{M})$ is surjective for all $k \geq 1$.

Proof. Assuming (1), we have an exact sequence,

$$0 \to H^0(M \otimes L) \to H^0(\widetilde{M}) \to H^0(M) \to 0$$

Let $\widetilde{\zeta}$ be a length two subscheme of $\widetilde{C}$. We must show that the restriction $H^0(\widetilde{M}) \to H^0(\widetilde{M}|_{\widetilde{\zeta}})$ surjects. Let ζ be the intersection of $\widetilde{\zeta}$ with C, so that $\mathscr{O}_\zeta = \mathscr{O}_{\widetilde{\zeta}}|_C$. Then ζ is a subscheme of C of length one or two. If it is of length two, then $\widetilde{\zeta}$ is a subscheme of C. Since M is very ample, and $H^0(\widetilde{M}) \to H^0(M)$ surjects, we conclude that $H^0(\widetilde{M}) \to H^0(\widetilde{M}|_{\widetilde{\zeta}})$ surjects. we get a section of $\widetilde{M}$ separating $\mathscr{O}_{\widetilde{\zeta}}$. On the other hand, if $\mathscr{O}_\zeta$ is of length one, then we have the exact sequence

$$0 \to \mathscr{O}_\zeta \to \mathscr{O}_{\widetilde{\zeta}} \to \mathscr{O}_\zeta \to 0$$

Tensoring by $\widetilde{M}$ and taking global sections gives

$$0 \to H^0(M \otimes L \otimes \mathscr{O}_\zeta) \to H^0(\widetilde{M} \otimes \mathscr{O}_{\widetilde{\zeta}}) \to H^0(M \otimes \mathscr{O}_\zeta) \to 0,$$

and a commutative diagram of exact rows as follows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(M \otimes L) & \longrightarrow & H^0(\widetilde{M}) & \longrightarrow & H^0(M) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(M \otimes L \otimes \mathscr{O}_\zeta) & \longrightarrow & H^0(\widetilde{M} \otimes \mathscr{O}_{\widetilde{\zeta}}) & \longrightarrow & H^0(M \otimes \mathscr{O}_\zeta) & \longrightarrow & 0.
\end{array}
$$

Since both M and $M \otimes L$ are base point free, the flanking vertical maps surject and hence the middle vertical map surjects.

Assuming (2), we need to show that $H^0(k\widetilde{M}) \otimes H^0(\widetilde{M}) \to H^0((k+1)\widetilde{M})$ surjects. We have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(M \otimes L) \otimes H^0(k\widetilde{M}) & \longrightarrow & H^0(\widetilde{M}) \otimes H^0(k\widetilde{M}) & \longrightarrow & H^0(M) \otimes H^0(k\widetilde{M}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0((k+1)M \otimes L)) & \longrightarrow & H^0((k+1)\widetilde{M}) & \longrightarrow & H^0((k+1)M) & \longrightarrow & 0
\end{array}
$$

The conditions in (2) imply that the two flanking vertical maps surject, and hence the middle vertical map also surjects. □

We now apply the previous analysis to the canonical bundle.

Proposition 2.2. *Let $\widetilde{C}$ be a ribbon on a smooth irreducible curve C of genus $g \geq 1$ with conormal bundle L.*

(1) If $p_a \geq 2g + 2$, then $K_{\widetilde{C}}$ is very ample.
(2) If $p_a \geq 2g + 2$ and either $g = 1$ or $h^0(K_C + L) \leq g - 2$, then the map

$$
H^0(kK_{\widetilde{C}}) \otimes H^0(K_{\widetilde{C}}) \to H^0((k+1)K_{\widetilde{C}})
$$

is surjective for all $k \geq 1$. In particular if $g \geq 2$, the above surjectivity holds if $p_a \geq 4g - 2$.

Proof. The canonical bundle $K_{\widetilde{C}}$ of a ribbon sits in an exact sequence

$$
0 \to K_C \to K_{\widetilde{C}} \to K_{\widetilde{C}}|_C \to 0,
$$

and we have $K_{\widetilde{C}}|_C = K_C \otimes L^{-1}$. For $k \geq 2$, the space $H^1(k(K_C \otimes L^{-1}) \otimes L)$ vanishes while for $k = 1$, the map $H^1(K_C) \to H^1(K_{\widetilde{C}})$ is injective. Hence $H^0(kK_{\widetilde{C}}) \to H^0(k(K_C \otimes L^{-1}))$ is surjective for all $k \geq 1$. Observe that K_C is base point free and $K_C \otimes L^{-1}$ is very ample if $-\deg L \geq 3$. So (1) follows from Theorem 2.1. Further, under the conditions of (2) the map

$$
H^0(K_C) \otimes H^0(k(K_C \otimes L^{-1})) \to H^0((k+1)(K_C \otimes L^{-1}) \otimes L)
$$

is surjective by [G84, Theorem 4.e.1] Hence part (2) follows from the part (2) of Theorem 2.1. □

3. LINEAR SERIES CLIFFORD INDEX AND RESOLUTION CLIFFORD INDEX OF A RIBBON

Let C be a smooth curve of genus g and let $\widetilde{C}$ be a ribbon on C of arithmetic genus p_a and conormal bundle L. The exact sequence

$$
0 \to L \to \mathscr{O}_{\widetilde{C}} \to O_C \to 0
$$

implies
$$\chi\left(\mathcal{O}_{\widetilde{C}}\right) = \chi(\mathcal{O}_C) + \chi(L),$$

and hence
$$p_a = (2g - 1) - \deg L.$$

We recall some defintions as introduced in [EG95, Section 1]. A *generalized line bundle* on $\widetilde{C}$ is a torsion free coherent sheaf which is generically free of rank 1. We define the *degree* of a generalized line bundle $\widetilde{M}$ by
$$\deg(\widetilde{M}) = \chi(\widetilde{M}) - \chi(\mathcal{O}_{\widetilde{C}}).$$

The restriction of $\widetilde{M}$ to C may have torsion; let $\tau \subset \widetilde{M}|_C$ be the torsion subsheaf. We set $M = (\widetilde{M}|_C/\tau)$.

Sections of $\widetilde{M}$ are of two kinds, those that yield a non-zero section of M and those that do not. The sections of the first kind define an injective map $\mathcal{O}_{\widetilde{C}} \to \widetilde{M}$. The scheme-theoretic vanishing locus of such a section is a Cartier divisor on $\widetilde{C}$. We call such sections *Cartier* sections. The sections of the second kind do not give an injective map $\mathcal{O}_{\widetilde{C}} \to \widetilde{M}$. The entire reduced curve C is contained in their scheme-theoretic zero locus.

Let $\widetilde{M}$ be a generalised line bundle on $\widetilde{C}$. By [EG95, Theorem 1.1], there exists a unique divisor $\beta \subset C$ and a line bundle $\widetilde{M}'$ on the blow up $\widetilde{C}'$ of $\widetilde{C}$ along β such that $\widetilde{M}$ is the push-forward of $\widetilde{M}'$. Then $M = \widetilde{M}'|_C$.

A *generalized linear series* of rank r and degree d, or simply a g^r_d, on $\widetilde{C}$ is pair $\Phi = (V, \widetilde{M})$ where $\widetilde{M}$ is a generalized line bundle and $V \subset H^0(\widetilde{M})$ is of dimension $r + 1$, such that the restriction map $V \to H^0(M)$ is injective. The *Clifford index* of Φ is $d - 2r$. The *linear series Clifford index* of $\widetilde{C}$, denoted by $\text{LCliff}(\widetilde{C})$, is the minimum of the Clifford indices of all generalized linear series g^r_d such that $r \geq 1$ and $d \leq p_a - 1$. The *gonality* of $\widetilde{C}$, denoted by $\text{gon}(\widetilde{C})$, is the smallest d such that there exists a g^1_d on $\widetilde{C}$.

For a line bundle $\widetilde{H}$ on $\widetilde{C}$, we let $K_{p,q}(\widetilde{C}, \widetilde{H})$ be the Koszul cohomology group as defined in [G84]. For $\widetilde{H} = K_{\widetilde{C}}$, we know that $K_{p,q}(\widetilde{C}, K_{\widetilde{C}})$ is possibly non-zero only for $0 \leq p \leq p_a - 2$ and $0 \leq q3$. Within this range, the group vanishes for $(p > 0, q = 0)$ and $(p < p_a - 2, q = 3)$. So the most interesting cases are $q = 1$ and $q = 2$. Owing to the duality,
$$K_{p,q}(\widetilde{C}, K_{\widetilde{C}}) = K_{p_a - p - 2, 3 - q}(\widetilde{C}, K_{\widetilde{C}})^\vee,$$

understanding $K_{p,1}$'s is equivalent to understanding $K_{p,2}$'s. We also know that if $K_{p,2}(\widetilde{C}, K_{\widetilde{C}}) = 0$ then for all $i \geq p$, we have $K_{i,2}(\widetilde{C}, K_{\widetilde{C}}) = 0$. So it is important to understand the smallest p such that $K_{p,2}(\widetilde{C}, K_{\widetilde{C}}) \neq 0$. This p is called the *resolution Clifford index* of $\widetilde{C}$. We denote it by $\text{RCliff}(\widetilde{C})$. By duality, it is the smallest p such that $K_{p_a - 2 - p, 1}(\widetilde{C}, K_{\widetilde{C}}) \neq 0$.

3.1. Semicontinuity of gonality and Clifford index.
The resolution Clifford index is lower semicontinuous by the semi-continuity of cohomology. We now establish the lower semicontinuity of gonality and linear series Clifford index.

Fix a smooth curve C of genus g and a ribbon $\widetilde{C}$ on C of arithmetic genus p_a. Let $(T, 0)$ be a smooth pointed curve and $\mathscr{C} \to T$ a flat proper morphism of relative dimension 1. Suppose for all $t \in T$ with $t \neq 0$, the fiber $\mathscr{C}_t$ is a smooth curve of genus p_a and the fiber $\mathscr{C}_0$ is $\widetilde{C}$.

The following proposition constructs a limiting generalized g^r_d.

Proposition 3.1. *In the setup above, assume that for all $t \neq 0$, the curve $\mathscr{C}_t$ has a g_d^r. Assume that $p_a > d + 2g - 1$ and $r \geq 1$. Then $\widetilde{C}$ has a generalised $g_{d'}^r$, with $d' \leq d$.*

Proof. We follow the proof of [EG95, Theorem 2.1], replacing $\omega_{\mathscr{C}/T}$ by a more suitable line bundle.

Let η be the generic point of T. After a finite base change, we may assume that we have a line bundle $\mathscr{G}_\eta$ on $\mathscr{C}_\eta$ and an $(r + 1)$-dimensional subspace $V_\eta \subset H^0(\mathscr{C}_\eta)$. Choose a line bundle $\mathscr{E}$ on $\mathscr{C}$ of relative degree $2g - 4$, for example, by starting with a line bundle of degree $2g - 4$ on $\widetilde{C}$ and deforming it. Then $\mathscr{E}|_C$ is a line bundle of degree $g - 2$. Assume that it is a general line bundle of this degree.

Set $\mathscr{F} = \omega_{\mathscr{C}/T} \otimes \mathscr{E}^{-1}$.

$$\deg\left(\mathscr{F}_\eta \otimes \mathscr{G}_\eta^{-1}\right) = (2p_a - 2) - (2g - 4) - d$$
$$\geq p_a.$$

The last inequality follows from our assumption $p_a > d + (2g - 1)$. In particular, $\mathscr{F}_\eta \otimes \mathscr{G}_\eta^{-1}$ is effective. One of its section gives an inclusion $\mathscr{G}_\eta \subset \mathscr{F}_\eta$. Via this inclusion, we may think of V_η as a subspace of $H^0(\mathscr{F}_\eta)$.

By the theorem on cohomology and base change [Mum70, Section 5], there exists a map $K^0 \to K^1$ of locally free $\mathscr{O}_T$ modules of finite rank such that for every $S \to T$, we have a canonical isomorphism

$$\pi_*(\mathscr{C}_T \times S, \mathscr{F} \times S) = \ker(K_S^0 \to K_S^1).$$

We have the subspace $V_\eta \subset K_\eta^0$ of dimension $r + 1$. It extends to a locally free $\mathscr{O}_T$ module of the same rank $V \subset K^0$ such that the map $V|_0 \to K^0|_0$ remains injective. Since

$$V_\eta \subset \ker(K_\eta^0 \to K_\eta^1) = H^0(\mathscr{F}_\eta),$$

we have

$$V \subset \ker(K^0 \to K^1) = \pi_*(\mathscr{F})$$

and

$$V|_0 \subset \ker(K^0|_0 \to K^1|_0) = H^0(\mathscr{F}|_0)$$

by continuity.

We have the exact sequence

$$0 \to K_C \to K_{\widetilde{C}} \to K_{\widetilde{C}}|_C \to 0.$$

Tensoring by $\mathscr{E}^{-1}$ yields the exact sequence

(3.1) $$0 \to K_C \otimes \mathscr{E}^{-1} \to \mathscr{F}|_0 \to \mathscr{F}|_C \to 0.$$

We have

$$\deg(K_C \otimes \mathscr{E}^{-1}) = (2g - 2) - (g - 2) = g.$$

Since $\mathscr{E}|_C$ is general, so is $K_C \otimes \mathscr{E}^{-1}$, and hence $h^0(K_C \otimes \mathscr{E}^{-1}|_C) = 1$. By the long exact sequence, we see that

(3.2) $$\dim \ker\left(H^0(\mathscr{F}|_0) \to H^0(\mathscr{F}|_C)\right) = 1.$$

Let $\mathscr{G}$ be the subsheaf of $\mathscr{F}$ generated by V and let $G_0 \subset \mathscr{F}|_0$ be the image of $\mathscr{G}$. Equivalently, G_0 is the subsheaf of $\mathscr{F}|_0$ generated by V_0. Then G_0 is torsion free. Since $\dim V_0 = (r+1)$ is greater than the dimension 1 of the kernel of $H^0(\mathscr{F}|_0) \to H^0(\mathscr{F}|_C)$, the restriction map $V_0 \to H^0(\mathscr{F}|_C)$ is non-zero. Then it follows that the map $G_0 \subset \mathscr{F}|_0$ is an isomorphism at the generic point. Therefore, G_0 is a generalized line bundle. Since we have a surjection $\mathscr{G}|_0 \to G_0$, and $\deg \mathscr{G}|_0 = d$, we conclude that $\deg G_0 \le d$.

Set $G = G_0|_C/\text{torsion}$. We know that the map $V_0 \to H^0(G)$ is non-zero, so $\deg G \ge 0$. It remains to check that the map $V_0 \to H^0(G)$ is injective. Let $\beta \subset C$ be the divisor such that G_0 is the push-forward of a line bundle from the blow-up of $\widetilde{C}$ along β. Let L be the conormal bundle of $\widetilde{C}$. Then we have the exact sequence

$$0 \to G \otimes L(\beta) \to G_0 \to G \to 0;$$

see [EG95, § 1]. Let $d' = \deg G_0$. Then

$$\deg G + \deg \beta \le 2\deg G + \deg \beta = d',$$

so

$$\deg(G \otimes L(\beta)) \le d' + \deg L = d' + (2g-1) - p_a \le d + (2g-1) - p_a < 0,$$

where the last inequality follows from $p_a > d + (2g-1)$. As a result, $H^0(G \otimes L(\beta)) = 0$, and hence the map $H^0(G_0) \to H^0(G)$ is injective. Therefore, the composite $V_0 \subset H^0(G_0) \to H^0(G)$ is injective. $\qquad\square$

Corollary 3.2. *In the setup of Theorem 3.1, suppose the generic fibers $\mathscr{C}_t$ have gonality d. If $p_a > d + 2g - 1$, then $\mathscr{C}$ has gonality at most d. In particular, if $p_a > 4g + 1$, then $\mathscr{C}$ has gonality at most d.*

Proof. The first statement follows directly from Proposition 3.1. For the second statement, observe that $d \le (p_a + 3)/2$. So $p_a > 4g + 1$ implies $p_a > d + 2g - 1$. $\qquad\square$

In Theorem 3.2, the condition $p_a > d + 2g - 1$ is indeed necessary. See Theorem 5.5 for the failure of the existence of a limiting g^1_3 without this condition.

Corollary 3.3. *In the setup of Theorem 3.1, suppose the generic fibers $\mathscr{C}_t$ have Clifford index c and Clifford dimension r. If $p_a > c + 2r + (2g - 1)$, then $\mathrm{LCliff}(\widetilde{C}) \le c$. In particular, if $p_a > 4g - 3 + 4r$, then $\mathrm{LCliff}(\widetilde{C}) \le c$.*

Proof. The first statement follows directly from Proposition 3.1. For the second statement, observe that $c \le (p_a - 1)/2$. So $p_a > 4g - 3 + 4r$ implies $p_a > c + 2r + (2g - 1)$. $\qquad\square$

Using the results of [ELMS89], we eliminate the dependence of p_a on the Clifford dimension r in Corollary 3.3.

Corollary 3.4. *In the setup of Theorem 3.1, assume that the fibers $\mathscr{C}_t$ have Clifford index c and Clifford dimension r. Then $\mathrm{LCliff}(\widetilde{C}) \le c$ holds under any of the following hypotheses:*

(1) *$r = 1$ and $p_a > 4g + 1$,*
(2) *p_a is odd and $p_a > 8g + 1$,*
(3) *p_a is even, $(p_a, c, d) \ne (4r - 2, 2r - 3, 4r - 3)$, and $p_a > 8g + 1$.*

Proof. Under the first hypothesis, the conclusion is Corollary 3.2. Under the second or third hypothesis, [ELMS89, Corollary 3.5] says that $p_a \geq 8r - 7$. Then $p_a > 8g + 1$ implies $p_a > 4g - 3 + 4r$. Therefore, the conclusion follows from Corollary 3.3. □

3.2. **Green-Lazarsfeld non-vanishing theorem for ribbons.** In this section, we relate the linear series Clifford index and the resolution Clifford index of a ribbon. We show that, for p_a large compared to g, we have the inequality RCliff ≤ LCliff.

Theorem 3.5. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on smooth curve C of genus g with $h^0(\mathcal{O}_{\widetilde{C}}) = 1$. Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be line bundles on $\widetilde{C}$ and set*

$$\widetilde{M} = \widetilde{M}_1 \otimes \widetilde{M}_2.$$

For $i = 1, 2$, let $H^0(\widetilde{M}_i)$ be of dimension $r_i + 1$, with $r_i \geq 1$. Assume that

    (1) $\widetilde{M}_1$ is base point free, and
    (2) the zero locus of a general element of $H^0(\widetilde{M}_2)$ is zero dimensional.

Then

$$K_{r_1+r_2-1,1}(\widetilde{C}, \widetilde{M}) \neq 0$$

Proof. Let $s_1 \in H^0(\widetilde{M}_2)$ be a section whose scheme-theoretic zero locus D_1 is zero dimensional. Let $s_2 \in H^0(\widetilde{M}_1)$ be a section whose scheme-theoretic zero locus D_2 is disjoint from D_1. Note that both D_1 and D_2 are Cartier diviors on $\widetilde{C}$. Then, up to scaling, there is a unique section s_0 of $H^0(\widetilde{M})$ that vanishes on both D_1 and D_2; it is the section whose scheme-theoretic zero locus is $D_1 + D_2$.

The rest of the proof follows verbatim from [G84, Appendix]. We sketch it for the convenience of the reader. Recall that a Cartier section of a line bundle on $\widetilde{C}$ is a section that defines an injection from $\mathcal{O}_{\mathscr{C}}$. The scheme theoretic zero locus of a Cartier section is a Cartier divisor. Since $s \in H^0(\widetilde{M})$ is Cartier, a general element of $H^0(\widetilde{M})$ is Cartier. Consider $H^0(\widetilde{M}(-D_1))$ (resp. $H^0(\widetilde{M}(-D_2))$) seen as subspaces of $H^0(\widetilde{M})$. These two subspaces intersect along the one-dimensional subspace spanned by s. These subspaces contain Cartier sections s_2 and s_1, respectively, so their general section is Cartier. Choose bases of these subspaces consisting of Cartier sections as follows. Let

$$s_0, s_1, ..., s_{r_2}$$

be a basis of $H^0(\widetilde{M}(-D_2))$, and

$$s_0, s_{r-r_1+1}, ..., s_r$$

a basis of $H^0(\widetilde{M}(-D_1))$. Extend to a basis of $H^0(\widetilde{M})$ by adding Cartier sections

$$s_{r_2+1}, ..., s_{r-r_1}.$$

Let $\{e_0, \ldots, e_r\}$ be the dual basis of $H^0(\widetilde{M})^*$. Let $\iota = \sum_{i=1}^{r-r_1} e_i \otimes s_i$ and $s = \sum_{i=0}^{r} e_i \otimes s_i$. Let

$$\alpha = \iota \wedge e_{r_2+1} \wedge ... \wedge e_{r-r_1}$$

Then $\alpha \in \bigwedge^{r-r_1-r_2+1} H^0(\widetilde{M})^* \otimes H^0(\widetilde{M}(-D_2))$. Consider $s \wedge \alpha$. We have that

$$
\begin{aligned}
s \wedge \alpha \in \; & \bigwedge^{r-r_1-r_2+2} H^0(\widetilde{M})^* \otimes H^0(\widetilde{M}(-D_2) \otimes \widetilde{M}(-D_1)) \\
= \; & \bigwedge^{r-r_1-r_2+2} H^0(\widetilde{M})^* \otimes H^0(\widetilde{M}) \\
= \; & \bigwedge^{r_1+r_2-1} H^0(\widetilde{M}) \otimes H^0(\widetilde{M}).
\end{aligned}
$$

Since $s \wedge s \wedge \alpha = 0$, we see that s defines a Koszul cocycle, that is, an element of $K_{r_1+r_2-1,1}(\widetilde{C}, \widetilde{M})$. The fact that this element is non-zero follows exactly as in [G84, Appendix]. □

We now examine when line bundles residual to the canonical carry Cartier sections.

Lemma 3.6. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve C of genus g. Let $\widetilde{M_1}$ be a line bundle on $\widetilde{C}$ with $h^0(\widetilde{M_1}) \geq r+1$ and $r \geq 1$. Set $d = \deg \widetilde{M_1}$ and $c = d-2r$. Let $\widetilde{M_2} = K_{\widetilde{C}} \otimes \widetilde{M_1}^{-1}$. Assume that $d \leq p_a-1$ and $H^0(\widetilde{M_1})$ contains a Cartier section. If $p_a > 3g-2+c$ then $h^0(\widetilde{M_2}) \geq r+1$ and $H^0(\widetilde{M_2})$ contains a Cartier section.*

Proof. By Riemann–Roch, we have

$$
h^0(\widetilde{M_2}) = p_a - d - 1 + h^0(\widetilde{M_1}).
$$

Since $d \leq p_a - 1$ and $h^0(\widetilde{M_1}) \geq r+1$, we have $h^0(\widetilde{M_2}) \geq r+1$.

Set $\widetilde{M_2}|_C = M_2$. Let the conormal bundle of $\widetilde{C}$ be L. We have the exact sequence

$$
0 \to M_2 \otimes L \to \widetilde{M_2} \to M_2 \to 0,
$$

and hence

$$
\deg(M_2 \otimes L) = \frac{1}{2} \deg(\widetilde{M_2}) + \deg(L) = \frac{1}{2}(2p_a - 2 - d) - (p_a - 2g + 1) = 2g - 2 - \frac{d}{2}.
$$

If $d > 4g - 4$, then every section of $H^0(\widetilde{M_2})$ is Cartier. Assume that $d \leq 4g - 4$. Then

$$
r = \frac{1}{2}(d - c) \leq 2g - 2.
$$

Multiplication by a Cartier section of $\widetilde{M_1}$ gives an injection

$$
\widetilde{M_2} \hookrightarrow K_{\widetilde{C}}.
$$

Recall the short exact sequence

$$
0 \to H^0(K_C) \to H^0(K_{\widetilde{C}}) \to H^0(K_C \otimes L^{-1}).
$$

The inclusion $\widetilde{M_2} \hookrightarrow K_{\widetilde{C}}$ induces an injection

$$
H^0(M_2 \otimes L) \to H^0(K_C).
$$

In particular, $h^0(M_2 \otimes L) \leq g$. On the other hand, we know that

$$
h^0(\widetilde{M_2}) = p_a - d - 1 + h^0(\widetilde{M_1}) \geq p_a - d + r.
$$

From $p_a > 3g-2+c$ and $d \leq 4g-4$, it follows that $p_a - d + r > g$ and hence $h^0(\widetilde{M_2}) > h^0(M_2 \otimes L)$. So there exists a Cartier section of $\widetilde{M_2}$. □

We combine Theorem 3.5 and Theorem 3.6 to obtain the following non-vanishing result.

Theorem 3.7. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve C of genus g and linear series Clifford index $\mathrm{LCliff}(\widetilde{C})$. If $p_a > 3g - 2 + \mathrm{LCliff}(\widetilde{C})$, then $\mathrm{RCliff}(\widetilde{C}) \leq \mathrm{LCliff}(\widetilde{C})$, that is,*

$$K_{p_a - 2 - \mathrm{LCliff}(\widetilde{C}), 1}(\widetilde{C}, K_{\widetilde{C}}) \neq 0.$$

In particular, the non-vanishing holds if $p_a > 6g - 5$.

Proof. Let L be the conormal bundle of $\widetilde{C}$. Since $p_a > 2g - 1$, we have $\deg L = -(p_a - 2g + 1) < 0$, so $h^0(\mathcal{O}_{\widetilde{C}}) = 1$.

Let g_d^r be a generalized linear series on $\widetilde{C}$ with $d \leq p_a - 1$ and $r \geq 1$ and $d - 2r = c$. Recall that this is a pair $(V, \widetilde{M}_1)$ where $\widetilde{M}_1$ is a generalized line bundle of degree d and $V \subset H^0(\widetilde{M}_1)$ is an $(r + 1)$ dimensional vector subspace of global sections which injects into $H^0(M_1)$. We may assume that $\widetilde{M}_1$ is globally generated by V. Otherwise, we simply replace $\widetilde{M}_1$ by the subsheaf generated by V (see [EG95, Lemma 1.3]).

By [EG95, Theorem 1.1], there exists a divisor $\beta \subset C$ of degree b, such that after blowing up $\widetilde{C}$ along β we have a line bundle $\widetilde{M}_1'$ of degree $d' = d - b$ whose pushforward is $\widetilde{M}_1$. The blown up ribbon $\widetilde{C}'$ has conormal bundle $L(\beta)$ and arithmetic genus $p_a' = p_a - b$. Since $H^0(\widetilde{M}_1') = H^0(\widetilde{M}_1)$, we may treat V as a subspace of $H^0(\widetilde{M}_1')$. Since $d \leq p_a - 1$, we also have $d' \leq p_a' - 1$. Since $H^0(\widetilde{M}_1')$ contains V of dimension $r + 1$, we have $h^0(\widetilde{M}_1') \geq r + 1$. Since $V \subset H^0(\widetilde{M}_1)$ generates $\widetilde{M}_1$, it follows that $V \subset H^0(\widetilde{M}_1')$ generates $\widetilde{M}_1'$.

Set $c' = d' - 2r = c - b$. Since $p_a > 3g - 2 + c$, we have $p_a' > 3g - 2 + c'$. Set $\widetilde{M}_2' = K_{\widetilde{C}'} \otimes (\widetilde{M}_1')^{-1}$. By Theorem 3.6, $H^0(\widetilde{M}_2')$ contains a Cartier section. Write $h^0(\widetilde{M}_1') = r + 1 + a$ for some $a \geq 0$. By Riemann-Roch, we have $h^0(\widetilde{M}_2') = p_a' - d' + r + a$. By Theorem 3.5, we get

$$(3.3) \qquad\qquad\qquad K_{p_a' - d' + 2r - 2 + 2a, 1}(\widetilde{C}', K_{\widetilde{C}'}) \neq 0.$$

Since $a \geq 0$, we have

$$p_a' - d' + 2r + 2a \geq p_a' - d' + 2r - 2 = p_a' - c' - 2 = p_a - c - 2.$$

So from (3.3), we conclude

$$K_{p_a - c - 2, 1}(\widetilde{C}', K_{\widetilde{C}'}) \neq 0.$$

Then by [D18, Lemma 1], we deduce

$$K_{p_a - c - 2, 1}(\widetilde{C}, K_{\widetilde{C}}) \neq 0.$$

The proof is now complete. $\square$

4. Blow-up index stratification of the space of ribbons

Recall that a *split* ribbon $\widetilde{C}$ on C is one that admits a retraction map $\widetilde{C} \to C$. Every ribbon admits a blow-up that is a split ribbon. The *blow up index* of a ribbon is the minimum number of simple blow-ups necessary to make the ribbon split. In this section, we study the set of ribbons of a given blow-up index, its relationship with a secant variety, and with the linear series Clifford index.

4.1. **Blow-up index of a ribbon as a pushout and relations to secant variety.** Fix a smooth curve C. A ribbon $\widetilde{C}$ on C with conormal bundle L gives an extension

$$0 \to L \to \Omega_{\widetilde{C}}|_C \to K_C \to 0.$$

Conversely, given an extension

$$0 \to L \to E \to K_C \to 0,$$

there is a unique ribbon $\widetilde{C}$ on C with an isomorphism $E \cong \Omega_{\widetilde{C}}|_C$ compatible with the extension. Therefore, the space of ribbons on C with conormal bundle L is identified with $\mathbb{P}\operatorname{Ext}^1(\Omega_C, L)$ (see [BE95, Theorem 1.2]).

Fix a ribbon $\widetilde{C}$. Given a divisor $\beta \subset C$, we have the map $L \to L(\beta)$. Construct the push-out diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \Omega_{\widetilde{C}}|_C & \longrightarrow & K_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L(\beta) & \longrightarrow & E & \longrightarrow & K_C & \longrightarrow & 0.
\end{array}
$$

The extension in the second row corresponds to a ribbon $\widetilde{C}'$ with conormal bundle $L(\beta)$. By [BE95, Theorem 1.9], the ribbon $\widetilde{C}'$ is precisely the blow-up of $\widetilde{C}$ along the Weil divisor β of $\widetilde{C}$. Let e be the class of the extension in the first row and e' the extension in the second row. Let

(4.1) $$\mathbb{P}\operatorname{Ext}^1(K_C, L) \to \mathbb{P}\operatorname{Ext}^1(K_C, L(\beta))$$

be the map induced by $L \to L(\beta)$. Then e' is the image of e. If β is of sufficiently large degree, then the group on the right vanishes, and hence e' vanishes. Then $\widetilde{C}'$ is the split ribbon.

Definition 4.1. Let $\widetilde{C}$ be a ribbon on a smooth projective curve C. The *blow-up index $b(\widetilde{C})$* of $\widetilde{C}$ is the smallest b such that there exists a divisor β on C of degree b such that the blow-up of $\widetilde{C}$ along β is split.

Let $e \in \mathbb{P}\operatorname{Ext}^1(K_C, L)$ be the extension class corresponding to $\widetilde{C}$. Then $b(\widetilde{C})$ is the smallest such that there exists an effective divisor $\beta \subset C$ such that the image of e in $\mathbb{P}\operatorname{Ext}^1(K_C, L(\beta))$ vanishes.

Recall that C is a smooth curve. By Serre duality, we have the identification

$$\operatorname{Ext}^1(\Omega_C, L) = H^1(L \otimes K_C^{-1}) = H^0(K_C^2 \otimes L^{-1})^*.$$

Proposition 4.2. *Let L be a line bundle of negative degree on C. Let $i \colon C \hookrightarrow \mathbb{P}H^0(K_C^2 \otimes L^{-1})^*$ be the embedding given by the complete linear series of the very ample line bundle $K_C^2 \otimes L^{-1}$. Then the ribbons $\widetilde{C}$ on C with conormal bundle L and $b(\widetilde{C}) \le k$ correspond to the points of the k-secant variety of the embedding i.*

Proof. Let $\widetilde{C}$ be a ribbon on C with conormal bundle L and extension class

$$e \in H^0(K_C^2 \otimes L^{-1})^* = H^1(L \otimes K_C^{-1}).$$

Let $\beta \subset C$ be an effective divisor. Consider the map

$$H^1(L \otimes K_C^{-1}) \xrightarrow{f} H^1(L(\beta) \otimes K_C^{-1}).$$

The map above is Serre dual to the map

$$H^0(K_C^2 \otimes L^{-1})^* \to H^0(K_C^2 \otimes L^{-1}(-\beta))^*,$$

which in turn is dual to the map

$$H^0(K_C^2 \otimes L^{-1}(-\beta)) \xrightarrow{g} H^0(K_C^2 \otimes L^{-1})$$

induced by the inclusion $\mathcal{O}(-\beta) \to \mathcal{O}$. Observe that e lies in the kernel of f if and only if the composite

$$H^0(K_C^2 \otimes L^{-1}(-\beta)) \xrightarrow{g} H^0(K_C^2 \otimes L^{-1}) \xrightarrow{e} \Bbbk$$

vanishes. But the points $\lambda \in \mathbb{P} H^0(K^2 \otimes L^{-1})$ such that the composite $\lambda \circ g$ vanishes are precisely the points that lie on the span of β in the embedding i. It follows that the ribbons $\widetilde{C}$ of blow-up index at most k correspond the points lying on the span of an effective divisor of degree at most k, which is the k-secant variety of C. □

Corollary 4.3. *Let C be a smooth curve of genus g. Fix $p_a > 2g - 1$ and let L be a line bundle on C of degree $-p_a + 2g - 1$. Let $\widetilde{C}$ be a ribbon of arithmetic genus with conormal bundle L. Then*

(1) $0 \le b(\widetilde{C}) \le \lceil (p_a + g - 2)/2 \rceil$
(2) If $\widetilde{C} \in \mathbb{P}\operatorname{Ext}^1(K_C, L)$ is general, then $b(\widetilde{C}) = \lceil (p_a + g - 2)/2 \rceil$

Proof. The k−th secant variety of a curve $C \hookrightarrow \mathbb{P}^N$ is of expected dimension $\min(2k - 1, N)$ (see [Lan84]). So the result follows from Theorem 4.2. □

4.2. **Blow-up index as pullback and relations to stability of $\Omega_{\widetilde{C}|C}$.** There is another interpretation of the blow-up index, which is useful for measuring gonality. As usual, let $\widetilde{C}$ be a ribbon on C with conormal bundle L defined by an extension

$$0 \to L \to \Omega_{\widetilde{C}}|_C \to K_C \to 0.$$

An effective divisor $\beta \subset C$ gives a map of line bundle $K_C(-\beta) \to K_C$. This map yields the pullback diagram

(4.2)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & K_C(-\beta) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \Omega_{\widetilde{C}}|_C & \longrightarrow & K_C & \longrightarrow & 0.
\end{array}
$$

Denoting by $e \in \operatorname{Ext}^1(K_C, L)$ the class of the extension in the bottom row, the class of the extension in the top row is simply the image of e under the map

(4.3)
$$\operatorname{Ext}^1(K_C, L) \to \operatorname{Ext}^1(K_C(-\beta), L).$$

The observation leads to the following.

Proposition 4.4. *Let C be a smooth curve and L a line bundle on C. Let $\widetilde{C}$ be a ribbon on C of arithmetic genus with conormal bundle L. Let $e \in \operatorname{Ext}^1(K_C, L)$ be the extension class of*

$$0 \to L \to \Omega_{\widetilde{C}}|_C \to K_C \to 0.$$

The blow-up index of $\widetilde{C}$ is any of the following equal quantities:

(1) the minimum b such that there exists an effective divisor β on C of degree b such that the image of e under the push-out by $L \to L(\beta)$ vanishes,

(2) the minimum b such that there exists an effective divisor β on C of degree b such that the image of e under the pull-back by $K_C(-\beta) \to K_C$ vanishes.

If $p_a > g + 1$, then it is also equal to $2g - 2 - k$ where k is the maximum such that $\Omega_{\widetilde{C}}|_C$ has a sub line-bundle of degree k.

Proof. The quantity (1) is the definition of the blow-up index (Theorem 4.1). The equality of (1) and (2) follows because the push-out $L \to L(\beta)$ and pull-back $K_C(-\beta) \to K_C$ induce the same map

$$\mathrm{Ext}^1(K_C, L) \to \mathrm{Ext}^1(K_C, L(\beta)) = \mathrm{Ext}^1(K_C(-\beta), L).$$

We now prove the last statement. Note that the extension obtained by the pull-back along $K_C(-\beta) \to K_C$ splits if and only if there is a map $K_C(-\beta) \to \Omega_{\widetilde{C}}|_C$ such that the following diagram commutes

$$
\begin{array}{ccc}
K_C(-\beta) & & \\
\downarrow & \searrow & \\
\Omega_{\widetilde{C}}|_C & \longrightarrow & K_C
\end{array}
$$

In this case, $K_C(-\beta)$ is a sub-bundle of $\Omega_{\widetilde{C}}|_C$ of degree $2g - 2 - b$. So (3) $\leq$ (2). For the reverse inequality, let $M \subset \Omega_{\widetilde{C}}|_C$ be a sub line-bundle of largest degree k. By Theorem 4.3 we know that the blow-up index of $\widetilde{C}$ is at most $\lceil 1/2(p_a + g - 2) \rceil$. So $\Omega_{\widetilde{C}}|_C$ has a line sub-bundle of degree at least $2g - 2 - \lceil 1/2(p_a + g - 2) \rceil$. The condition $p_a > g + 1$ ensures that

$$2g - 2 - \lceil 1/2(p_a + g - 2) \rceil > \deg L = -p_a + 2g - 1.$$

Therefore $\deg M = k > \deg L$, and so M does not admit a non-zero map to L. It follows that the composite $M \to \Omega_{\widetilde{C}}|_C \to K_C$ is non-zero. So M has the form $K_C(-\beta)$ for some effective divisor β of degree $2g - 2 - k$. We conclude that (2) $\leq$ (3). $\qquad\square$

As a corollary, we relate the blow-up index with the stability of the rank 2 bundle $\Omega_{\widetilde{C}}|_C$.

Corollary 4.5. *Let $\widetilde{C}$ be a ribbon of arithmetic genus $p_a > g+1$ on a smooth curve of genus g. Then $\Omega_{\widetilde{C}}|_C$ is stable (resp. semistable) if and only if $b(\widetilde{C}) > 1/2(p_a + 3) - 2$ (resp. $b(\widetilde{C}) \geq 1/2(p_a + 3) - 2$).*

Proof. Note that

$$\deg(\Omega_{\widetilde{C}}|_C) = -\deg(L) + 2g - 2 = -(p_a - 2g + 1) + 2g - 2 = -(p_a + 3) + 4g.$$

So the slope of $\Omega_{\widetilde{C}}|_C$ is $-1/2(p_a + 3) + 2g$. The statement now follows from Theorem 4.4. $\qquad\square$

Remark 4.6. For ribbons of blow-up index between $-1/2(p_a + 3) - 2$ and the maximum, which is $\lceil 1/2(p_a + g - 2) \rceil$, the bundles $\Omega_{\widetilde{C}}|_C$ are semi-stable. But they are still distinguished by the maximum degree of sub line-bundles.

4.3. **Blow-up index, gonality, and linear series Clifford index.** We end this section with bounds on the gonality of a ribbon in terms of the blow-up index. Recall that the gonality of $\widetilde{C}$ is the minimum d such that there exists a generalized g^1_d on $\widetilde{C}$.

The following gives an upper bound on gonality.

Proposition 4.7. *Let C be a smooth curve of genus g and gonality m. Let $\widetilde{C}$ be a ribbon C of arithmetic genus p_a, blow-up index b, and gonality d. Assume there exist a smooth divisor in $|-2L|$ and that $p_a > 2g - 1 + 2m$, then*

$$d \le \min(b + 2m, \lfloor 1/2(p_a + 3)\rfloor).$$

Proof. Let $\widetilde{C}'$ be the split ribbon obtained by blowing up $\widetilde{C}$ along a divisor of degree b. Then we have a degree 2 map $\widetilde{C}' \to C$. The pull-back of a g^1_m on C yields a g^1_{2m} on $\widetilde{C}'$. Its push-forward to $\widetilde{C}$ gives a g^1_{b+2m} on $\widetilde{C}$. So $d \le b + 2m$.

Finally, in the prescribed range, all ribbons are smoothable [Gon06]. A smooth curve of genus p_a has gonality at most $\lfloor 1/2(p_a + 3)\rfloor$. So the bound $d \le \lfloor 1/2(p_a + 3)\rfloor$ follows from semi-continuity (Theorem 3.2). □

The following gives a lower bound on gonality.

Proposition 4.8. *Let C be a smooth curve of genus g and gonality m. Let $\widetilde{C}$ be a ribbon C of arithmetic genus p_a, blow-up index b, and gonality d. Then*

$$d \ge b - (2g - 2).$$

Proof. Let $\widetilde{M}$ denote the generalized line bundle associated to the g^1_d on $\widetilde{C}$. We observe that $\widetilde{M}$ must be generated by the 2 sections of the g^1_d. If not, the subsheaf generated by these two sections gives a generalised $g^1_{d'}$ with $d' < d$, contradicting the minimality of d.

Let us first treat the case that $\widetilde{M}$ is a line bundle. Since $\widetilde{M}$ is a line bundle, d is even. Then the g^1_d on $\widetilde{C}$ induces a finite map $\widetilde{C} \to \mathbb{P}^1$ of degree d, leading to the diagram

$$
\begin{array}{ccc}
 & C & \\
 & \big\downarrow\ \ \searrow & \\
 \widetilde{C} & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

The diagram above induces the diagram

$$
\begin{array}{ccc}
 & f^*(K_{\mathbb{P}^1}) & \\
 & \big\downarrow\ \ \searrow & \\
 \Omega_{\widetilde{C}}|_C & \longrightarrow & K_C.
\end{array}
$$

Note that $f \colon C \to \mathbb{P}^1$ is finite of degree $d/2$. We have the identification $f^*(K_{\mathbb{P}^1}) = K_C(-\beta)$, where β is the ramification divisor of f, which has degree $2g - 2 + d$. Owing to this diagram, the pull-back of the extension

$$0 \to L \to \Omega_{\widetilde{C}}|_C \to K_C \to 0$$

to $K_C(-\beta)$ is split. Therefore, by Theorem 4.4, we get $b \le 2g - 2 + d$.

Let us treat the general case, where $\widetilde{M}$ is only a generalized line bundle. Then there exists an effective divisor α of degree a such that $\widetilde{M}$ is the push-forward of a line bundle $\widetilde{M}'$ on the blow-up $\widetilde{C}'$ of $\widetilde{C}$ at a. Then we have a g^1_{d-a} on $\widetilde{C}'$ with the line bundle $\widetilde{M}'$. Plainly, the blow-up index b' of $\widetilde{C}'$ is bounded below by $b - a$. So we get

$$d - a \geq b' - (2g - 2) \geq b - a - (2g - 2),$$

and hence $d \geq b - (2g - 2)$, as required. $\square$

5. GONALITY STRATIFICATION OF THE SPACE OF RIBBONS

In this section we describe the stratification of the space of ribbons by gonality. Fix a smooth curve C of genus g and a line bundle L on C of negative degree. Then the space of ribbons on C with conormal bundle L is naturally the projective space $\mathbb{P} H^0(K_C^2 \otimes L^{-1})$. Let

$$W_d \subset \mathbb{P} H^0(K_C^2 \otimes L^{-1})$$

be the subset of ribbons that carry a generalized g^1_d. Let

$$\mathrm{Sec}_k(C) \subset \mathbb{P} H^0(K_C^2 \otimes L^{-1})$$

be the k-secant variety of $C \subset \mathbb{P} H^0(K_C^2 \otimes L^{-1})$.

Theorem 5.1. *With the notation above, $W_d \subset \mathrm{Sec}_{d+2g-2}(C)$ is the union of linear subspaces spanned by divisors of degree $d + 2g - 2$ that contain the ramification divisor of a map $C \to \mathbb{P}^1$ of degree at most $d/2$.*

Proof. Let $\widetilde{C} \in W_d$. Then $\widetilde{C}$ carries a g^1_d, say $\widetilde{M}$. We show that $\widetilde{C}$ lies on a linear subspace spanned by a divisor of degree $d + 2g - 2$ that contains a ramification divisor of a map $C \to \mathbb{P}^1$ of degree at most $d/2$. It suffices to treat the case when the two sections of the g^1_d genarate $\widetilde{M}$. (If not, we simply replace $\widetilde{M}$ by the subsheaf they generate).

The generalised line bundle $\widetilde{M}$ is the push-forward of a line bundle $\widetilde{M}'$ on a blow-up $\widetilde{C}'$ of $\widetilde{C}$ in a divisor $\beta \subset C$ of degree b. Note that the degree of $\widetilde{M}'$ is $d - b$. The two sections of $\widetilde{M}'$ generate $\widetilde{M}'$ and hence yield a map $\widetilde{C}' \to \mathbb{P}^1$ of degree $d - b$. Its restriction $f \colon C \to \mathbb{P}^1$ is a map of degree $e = (d - b)/2$. Let R be the ramification divisor of f. By [BE95, Theorem 1.6], we have a map $K_C(-R) \to \Omega_{\widetilde{C}'}|_C$ that makes the following diagram commute

(5.1)

$$
\begin{array}{ccc}
 & & f^*(K_{\mathbb{P}^1}) = K_C(-R) \\
 & \swarrow & \downarrow \\
0 \longrightarrow L(\beta) \longrightarrow \Omega_{\widetilde{C}'}|_C \longrightarrow K_C \longrightarrow 0.
\end{array}
$$

By Theorem 4.4, this means that $\widetilde{C}'$ splits after blowing it up in R. Then $\widetilde{C}$ splits after blowing it up in $\beta + R$. Note that $\beta + R$ has degree $d + 2g - 2$ and contains the ramification divisor R.

Conversely, let $R \subset C$ be the ramification divisor of a map $f \colon C \to \mathbb{P}^1$ of degree $e \leq d/2$, and let $\widetilde{C}$ lie in the linear span of a divisor of degree $d + 2g - 2$ of the form $\beta + R$, where β is effective. We produce a generalized g^1_d on $\widetilde{C}$. Since $\widetilde{C}$ lies in the span of $\beta + R$, the blow-up of $\widetilde{C}$ in $\beta + R$ is split (Theorem 4.2). Let $\widetilde{C}'$ be the blow-up of $\widetilde{C}$ in β, so that the blow-up of $\widetilde{C}'$ in R is split. Then

we have a map $K_C(-R) \to \Omega_{\widetilde{C}'}|_C$ making the diagram (5.1) commute. Applying [BE95, Theorem 1.6] again, we conclude that the map $f: C \to \mathbb{P}^1$ extends to a map $\widetilde{C}' \to \mathbb{P}^1$. Then the pull-back of $\mathcal{O}(1)$ is a g^1_{2e} on $\widetilde{C}'$, whose push-forward to $\widetilde{C}$ is a g^1_d on $\widetilde{C}$. $\qquad\square$

Corollary 5.2. *We have the inclusions*

$$\mathrm{Sec}_{d-2m}(C) \subset W_d \subset \mathrm{Sec}_{d+2g-2}(C).$$

Proof. The second inclusion is a part of Theorem 5.1. For the first, take $x \in \mathrm{Sec}_{d-2m}(C)$. Then x is in the span of a divisor α of degree $d - 2m$. Since m is the gonality of C, we have a map $C \to \mathbb{P}^1$ of degree m. Its ramification divisor R has degree $2g - 2 + 2m$. Then x is also in the span of $\alpha + R$. $\qquad\square$

Theorem 5.1 allows us to count the dimension of each gonality stratum. To do so, let us further stratify W_d. For a non-negative integer $e \le d/2$, set $b = d - 2e$. Let $W_{e,b} \subset W_d$ be the union of linear subspaces spanned by divisors of degree $d + 2g - 2$ that contain the ramification divisor of a map $C \to \mathbb{P}^1$ of degree e. Then, by definition, we have

$$W_d = \bigcup_{e=0}^{d/2} W_{e,b}.$$

From the proof of Theorem 5.1, we see that the points of $W_{e,b}$ correspond to ribbons $\widetilde{C}$ that carry a generalised g^1_d that is the push-forward of a line bundle from a blow-up of $\widetilde{C}$ in a divisor of degree b.

Let m be the gonality of C and let $W^1_e(C)$ be the moduli space of base-point free linear series of rank 1 and degree e on C.

Theorem 5.3. *Retain the setup above and assume that L is a line bundle of degree $\le -2m - 5$. Set $p_a = 2g - 1 - \deg L$. Then*

$$\dim(\overline{W}_{e,b}) \le \dim W^1_e(C) + 2g + 2e + 2b - 3.$$

In addition, if C is a general curve of genus g, the following hold.

(1) If $2e < g + 2$, then $\overline{W}_{e,b}$ is empty, and otherwise, we have

$$\dim(\overline{W}_{e,b}) \le g + 2d - 5.$$

(2) The gonality of a general ribbon on C with conormal bundle L is the maximum, namely $\lfloor 1/2(p_a + 3) \rfloor$.

Proof. Consider a point $(\phi, \beta) \in W^1_e(C) \times \mathrm{Sym}^b(C)$. Let R_ϕ be the ramification divisor of ϕ. Consider the span of $R + \beta$ in $\mathbf{P}H^0(K_C^2 \otimes L^{-1})$. For (ϕ, β) in a non-empty Zariski open subset $U \subset W^1_e(C) \times \mathrm{Sym}^b(C)$, this span has a constant dimension n. Since $\deg R = 2g + 2e - 2$ and $\deg \beta = b$, we have $n \le 2g + 2e + b - 3$. Consider the incidence variety

$$I \subset U \times \mathbf{P}H^0(K_C^2 \otimes L^{-1})$$

consisting of (ϕ, β, x) such that x lies on the span of $R_\phi + \beta$. Then I is a closed subvariety and its projection to $\mathbf{P}H^0(K_C^2 \otimes L^{-1})$ contains a Zariski open subset of $W_{e,b}$. The fibers of $I \to U$ are

projective spaces of dimension n. Therefore, we get

$$\dim \overline{W}_{e,b} \le \dim I = \dim U + n = \dim W_e^1(C) + b + n$$
$$\le \dim W_e^1(C) + b + (2g + 2e + b - 3),$$

as required.

If C is general, then $\dim W_e^1(C) = 2e - g - 2$ (and $W_e^1(C)$ is empty if $2e < g + 2$). Since $d = 2e + b$, we get

$$\dim \overline{W}_{e,b} \le g + 2d - 5.$$

The dimension of the ambient $\dim \mathbf{P}H^0(K_c^2 \otimes L^{-1})$ is $g + p_a - 3$. Therefore, if

$$g + 2d - 5 < g + p_a - 3,$$

then $\overline{W}_{e,b} \subset \mathbf{P}H^0(K_c^2 \otimes L^{-1})$ is a proper subset. That is, a generic point of the ambient space is contained in W_d only when $d \ge \lfloor (p_a + 3)/2 \rfloor$. In other words, a generic ribbon has gonality at least $\lfloor (p_a + 3)/2 \rfloor$. To see that a generic ribbon has gonality at most $\lfloor (p_a + 3)/2 \rfloor$, observe that such a ribbon is a limit of smooth curves and smooth curves have gonality at most $\lfloor (p_a + 3)/2 \rfloor$. Then apply the semi-continuity of gonality (Theorem 3.2). $\qquad \square$

We end the section with some examples of the gonality loci of ribbons. In all the examples, C will be a smooth curve of genus g and L a line bundle of negative degree on C. We set

$$p_a = -\deg L + 2g - 1.$$

We consider C embedded in $\mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^*$ by the complete linear series. Recall that the points of $\mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^*$ correspond to ribbons on C of arithmetic genus p_a and with conormal bundle L.

Example 5.4 (Ribbons on hyperelliptic curves)**.** Let C be a hyperelliptic curve of genus g. We describe $W_{e,b}$ explicitly for $e = 2$.

Assume for simplicity that $p_a \ge 2g + 5$. Let R be the ramification divisor of the degree 2 map $C \to \mathbf{P}^1$. Fix $e = 2$ and $d \ge 4$ and $b = d - 2e = d - 4$. By Theorem 5.1, $W_{2,d-4} \subset \mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^*$ is the union of the span of $R + \beta$ as β varies in $\mathrm{Sym}^b(C)$. Let $\langle R \rangle \subset \mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^*$ be the span of R. Then $\langle R \rangle$ is a projective space of dimension $2g + 1$. We have the linear projection

$$\mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^* \dashrightarrow \mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1}(-R))^*$$

with center $\langle R \rangle$. The projection of C yields an embedding $C \subset \mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1}(-R))^*$ by the complete linear series. Then $\overline{W}_{2,d-4} \subset \mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1})^*$ is the cone over the b-secant variety of C in $\mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1}(-R))^*$. Note that the ambient space $\mathbf{P}H^0(K_C^{\otimes 2} \otimes L^{-1}(-R))^*$ has dimension $p_a - g - 5$. Hence the b-secant variety has dimension $\min(2b - 1, p_a - g - 5)$. Therefore, we get

$$\dim \overline{W}_{2,d-4} = \min(2d + 2g - 7, p_a + g - 3).$$

Note that $2g + 2g - 7$ is the bound given by Theorem 5.3 and $p_a + g - 3$ is simply the dimension of the ambient projective space $\mathbf{P}(K_C^{\otimes 2} \otimes L^{-1})^*$. Observe that if $d \ge (p_a - g + 4)/2$, then $\overline{W}_{2,d-4}$ is the ambient projective space. In particular, the gonality of a generic ribbon over a hyperelliptic curve is less than the maximum $\lfloor (p_a + 3)/2 \rfloor$.

Example 5.5 (Non-existence of a limiting g_3^1)**.** Let C be an elliptic curve and take $p_a = 4$. Then C is a limit of smooth curves of genus 4, which have gonality 3. Nevertheless, from Theorem 5.1 it follows that W_3 is empty. So C does not admit a generalised g_3^1. This does not contradict the semi-continuity of gonality (Theorem 3.2), since the condition $p_a > d + 2g - 1$ is not satisfied.

Example 5.6 (An elliptic normal curve in $\mathbf{P}^5$)**.** Let C be a smooth curve of genus 1 and take $p_a = 7$. Then the conormal bundle L has degree -6. The complete linear series gives an embedding

$$C \subset \mathbf{P}(K_C^{\otimes 2} \otimes L^{-1})^* = \mathbf{P}^5.$$

Theorem 5.1 implies that W_d is empty for $d = 1, 2, 3$. We describe W_4. We have $W_4 = W_{2,0}$ and it is the union of the spans of the ramification divisors of degree 2 maps $C \to \mathbf{P}^1$. Fix one degree 2 map $f \colon C \to \mathbf{P}^1$; all others are translates of f. Let $R = p_1 + p_2 + p_3 + p_4$ be the ramification divisor of f and $P \subset \mathbf{P}^5$ the $\mathbf{P}^3$ spanned by R. Then W_4 is the union of the translates of P by the points of the Jacobian of C. This is a hypersurface of degree 6.

6. Green's conjecture and resolution Clifford index for a general ribbon on a general curve

In this section, we relate the linear series Clifford index to the resolution Clifford index. Specifically, we prove Green's conjecture for a general ribbon using Green's conjecture for a general smooth curve. The proof follows the ideas in [D18].

We begin with two results about embedded and abstract deformations of ribbons.

Proposition 6.1. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve C of genus g. Suppose $p_a \geq 4g - 2$. Let $\widetilde{C} \subset \mathbf{P}^N$ be the embedding by a complete linear series of a very ample line bundle of degree at least $2p_a$. Then the deformations of $\widetilde{C} \subset \mathbf{P}^N$ are unobstructed. That is, the Hilbert scheme of $\mathbf{P}^N$ is smooth at the point represented by $[\widetilde{C}]$.*

Proof. Let L be the conormal bundle of the ribbon $\widetilde{C}$.

We have the sequence

$$(6.1) \qquad\qquad 0 \to T_C \to T_{\mathbf{P}^N} \otimes \mathcal{O}_C \to N_{C/\mathbf{P}^N} \to 0$$

and the Euler exact sequence

$$(6.2) \qquad\qquad 0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus N} \to T_{\mathbf{P}^N} \otimes \mathcal{O}_C \to 0.$$

Note that $\deg \mathcal{O}_C(1) \geq p_a$ and $\deg \mathcal{O}_C(1) \otimes L \geq 2g - 1$. So both $\mathcal{O}_C(1)$ and $\mathcal{O}_C(1) \otimes L$ have vanishing H^1. From the long exact sequence in cohomology applied to (6.2) and its twist by L, we get

$$(6.3) \qquad\qquad H^1(T_{\mathbf{P}^N} \otimes \mathcal{O}_C) = 0 \text{ and } H^1(T_{\mathbf{P}^N} \otimes L) = 0.$$

As a result, from the long exact sequence in cohomology applied to (6.1) and its twist by L, we get

$$(6.4) \qquad\qquad H^1(N_{C/\mathbf{P}^N}) = 0 \text{ and } H^1(N_{C/\mathbf{P}^N} \otimes L) = 0.$$

We now turn to the normal bundle of $\widetilde{C}$. Since $\widetilde{C}$ is a local complete intersection, its normal bundle is locally free of rank $N - 1$. Restricting it to C yields the exact sequence

$$(6.5) \qquad\qquad 0 \to N_{\widetilde{C}/\mathbb{P}^N} \otimes L \to N_{\widetilde{C}/\mathbb{P}^N} \to N_{\widetilde{C}/\mathbb{P}^N} \otimes \mathcal{O}_C \to 0.$$

We have the exact sequence

$$0 \to I_C^2/(I_C \cdot I_{\widetilde{C}}) \to I_{\widetilde{C}}/(I_C \cdot I_{\widetilde{C}}) \to I_{\widetilde{C}}/I_C^2 \to 0,$$

whose terms are locally free $\mathscr{O}_C$-modules of ranks 1, $N-1$, and $N-2$, respectively. Specifically, the kernel is the line bundle L^2 and the middle term is the conormal bundle of $\widetilde{C}$ restricted to C. Applying $\mathrm{Hom}(-,\mathscr{O}_C)$ yields the sequence

(6.6) $$0 \to \mathrm{Hom}(I_{\widetilde{C}}/I_C^2,\mathscr{O}_C) \to N_{\widetilde{C}/\mathbb{P}^N} \otimes \mathscr{O}_C \to L^{-2} \to 0$$

Finally, we have the sequence

$$0 \to I_{\widetilde{C}}/I_C^2 \to I_C/I_C^2 \to I_C/I_{\widetilde{C}} \to 0,$$

whose terms are locally free $\mathscr{O}_C$-modules of ranks $N-2$, $N-1$, and 1, respectively. Specifically, the cokernel is the line bundle L and the middle term is the conormal bundle of C. Applying $\mathrm{Hom}(-,\mathscr{O}_C)$ yields the sequence

(6.7) $$0 \to L^{-1} \to N_{C/\mathbb{P}^N} \to \mathrm{Hom}(I_{\widetilde{C}}/I_C^2,\mathscr{O}_C) \to 0.$$

Using the vanishings (6.4), the long exact sequence in cohomology applied to (6.7) and its twist by L yields

(6.8) $$H^1(\mathrm{Hom}(I_{\widetilde{C}}/I_C^2,\mathscr{O}_C)) = 0 \text{ and } H^1(\mathrm{Hom}(I_{\widetilde{C}}/I_C^2,\mathscr{O}_C) \otimes L) = 0.$$

Note that $\deg L^{-2} \geq \deg L^{-1} = p_a - 2g + 1 \geq 2g - 1$. So $H^1(L^{-2}) = H^1(L^{-1}) = 0$. Combined with the vanishing (6.8), the long exact sequence in cohomology applied to (6.6) and its twist by L yields

(6.9) $$H^1(N_{\widetilde{C}/\mathbf{P}^N} \otimes \mathscr{O}_C) = 0 \text{ and } H^1(N_{\widetilde{C}/\mathbf{P}^N} \otimes \mathscr{O}_C \otimes L) = 0.$$

Finally, the long exact sequence in cohomology applied to (6.5) yields

$$H^1(N_{\widetilde{C}/\mathbf{P}^N}) = 0.$$

So the embedded deformations of $\widetilde{C} \subset \mathbf{P}^N$ are unobstructed. $\qquad\qquad\square$

We now turn to the abstract deformations of $\widetilde{C}$.

Proposition 6.2. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve C of genus g. Suppose $p_a \geq 4g - 2$. Then the deformations of $\widetilde{C}$ are unobstructed. That is, a versal deformation space of $\widetilde{C}$ is smooth.*

Proof. Choose an embedding $\widetilde{C} \subset \mathbf{P}^N$ by the complete linear series of a very ample line bundle of degree at least $2p_a$. We relate the abstract versus embedded deformations of $\widetilde{C}$.

Consider the conormal sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{\mathbb{P}^N}|_{\widetilde{C}} \to \Omega_{\widetilde{C}} \to 0.$$

Applying $\mathrm{Hom}(-,\mathscr{O}_{\widetilde{C}})$ yields

(6.10) $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2,\mathscr{O}_{\widetilde{C}}) \to \mathrm{Ext}^1(\Omega_{\widetilde{C}},\mathscr{O}_{\widetilde{C}}) \to \mathrm{Ext}^1(\Omega_{\mathbb{P}^N}|_{\widetilde{C}},\mathscr{O}_{\widetilde{C}}) \to \mathrm{Ext}^1(\mathscr{I}/\mathscr{I}^2,\mathscr{O}_{\widetilde{C}}) \to \mathrm{Ext}^2(\Omega_{\widetilde{C}},\mathscr{O}_{\widetilde{C}}).$

Consider $\mathrm{Ext}^1(\Omega_{\mathbb{P}^N}|_{\widetilde{C}},\mathscr{O}_{\widetilde{C}}) = H^1(T_{\mathbb{P}^N}|_{\widetilde{C}})$. We have the exact sequence

$$0 \to T_{\mathbf{P}^N}|_C \otimes L \to T_{\mathbf{P}^N}|_{\widetilde{C}} \to T_{\mathbf{P}^N|_C} \to 0.$$

By (6.3), the H^1 of the first and the third term vanishes. So H^1 of the middle term also vanishes. Now, (6.10) says that

$$H^0(N_{\widetilde{C}/\mathbf{P}^N}) \to \mathrm{Ext}^1(\Omega_{\widetilde{C}}, \mathscr{O}_{\widetilde{C}})$$

is surjective, and

$$H^1(N_{\widetilde{C}/\mathbf{P}^N}) \to \mathrm{Ext}^2(\Omega_{\widetilde{C}}, \mathscr{O}_{\widetilde{C}})$$

is injective. That is, the forgetful map from embedded to abstract deformations is surjective on tangent spaces and injective on obstruction spaces. As a result, the forgetful map is smooth. We know that the embedded deformation space is smooth (Theorem 6.1). We conclude that the abstract deformation space is also smooth. □

We examine the non-nodal locus in a versal deformation.

Proposition 6.3. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve of genus g with $p_a \geq 3g - 1$. Let $(U, 0)$ be a versal deformation space of $\widetilde{C}$ with a versal family of deformations $\mathscr{C} \to U$. Let $N \subset U$ be the closed subset containing $u \in U$ such that $\mathscr{C}_u$ has worse than nodal singularities. Then $N \subset U$ has codimension at least 2 at 0.*

Proof. We follow the proof of [D18, Proposition 3]. Since every ribbon $\widetilde{C}$ can be isotrivially deformed to a split ribbon, it is enough to prove the theorem for the split ribbon. So assume that $\widetilde{C}$ is the split ribbon. Let the conormal bundle be L and let $V = H^0(-2L)$. Every $v \in V$ gives a double cover of C whose branch divisor is the zero locus of v and whose structure sheaf is $O_C \oplus L$. Consider the family of curves $\mathscr{C}_V \to V$, whose fiber over $v \in V$ is the double of C defined by v. Note that the fiber over 0 is the split ribbon $\mathscr{C}$. By versality, possibly after passing to an étale neighborhood of 0, we have a map $(V, 0) \to (U, 0)$ and an isomorphism of $\mathscr{C}_V \to V$ with the pull-back of the versal family. Let $N_V \subset V$ be the pre-image of $N \subset U$. Then $N_V \subset V$ is the locus of $v \in V$ that define a worse than nodal $\mathscr{C}_v$. But the double cover $\mathscr{C}_v$ is has worse than nodal singularities if and only if v has a zero of multiplicity at least 3. Since $\deg(-L) = p_a - 2g + 1 \geq g$, we have $\deg(-2L) \geq 2g$. Then it follows that $N_V \subset V$ has codimension 2 at 0. Therefore, $N_U \subset U$ has codimension at least 2 at 0. □

As in [D18], let $\mathscr{U}$ denote the open sub-stack of the stack of all projective curves whose points corresond to curves X satisfying the following conditions:

  (1) X is Gorenstein of arithmetic genus p_a and $h^0(\mathscr{O}_X) = 1$.
  (2) K_X embeds X as a arithmetically Gorenstein subscheme of $\mathbb{P}^{p_a-1}$.
  (3) A versal deformation space of X is smooth.

Let M_{p_a} be the stack of smooth projective curves of genus p_a, and let $M_{p_a}^{nh} \subset M_{p_a}$ be the stack of non-hyperelliptic curves.

Let C be a smooth curve of genus g and let $\widetilde{C}$ be a ribbon on C of arithmetic genus p_a with $p_a \geq 4g - 2$. Then $X = \widetilde{C}$ satisfies the three conditions above: the first is clear; the second is Theorem 2.2; and the third is Theorem 6.2. Therefore, $\widetilde{C}$ defines a point $[\widetilde{C}]$ of $\mathscr{U}$.

Let p_a be odd, say $p_a = 2k + 1$. As shown in [D18, Proposition 4], we have a divisor $D \subset \mathscr{U}$ whose points represent curves X such that $K_{k-1,1}(X, K_X) \neq 0$. We also have a divisor in $M_{p_a}^{nh}$ whose points represent smooth curves of gonality $k + 1$. Let $D_{k+1} \subset \mathscr{U}$ be its closure.

Proposition 6.4. *Let $\widetilde{C}$ be a ribbon of arithmetic genus $p_a = 2k + 1$ on a smooth projective curve C of genus g with $p_a \geq 4g - 2$. Then $D_{k+1} = D$ on an open subset of $\mathcal{U}$ containing $[\widetilde{C}]$.*

Proof. The proof of [D18, Proposition 5] applies verbatim, thanks to Theorem 6.3. □

Proposition 6.5 (Green's conjecture for generic ribbons of odd genus). *Let $\widetilde{C}$ be a ribbon of arithmetic genus $p_a = 2k + 1$ on a smooth projective curve C of genus g with $p_a \geq \max\{4g + 2, 6g - 4\}$. Suppose $\widetilde{C}$ has gonality $k + 2$. Then its resolution Clifford index is k. That is, we have*

$$\mathrm{RCliff}(\widetilde{C}) = \mathrm{LCliff}(\widetilde{C}) = k.$$

In particular, the statement applies to a general ribbon of odd arithmetic genus $p_a \geq \max\{3g + 7, 6g - 4\}$ on a general curve of genus g.

Proof. By the semi-continuity of gonality (Theorem 3.2), we see that $[\widetilde{C}] \notin D_{k+1}$. Therefore, by Theorem 6.4, $[\widetilde{C}] \notin D$, so $K_{k-1,1}(\widetilde{C}, K_{\widetilde{C}}) = 0$. That is, $\mathrm{RCliff}(\widetilde{C}) \geq k$. But the lower bound on p_a further ensures that $\widetilde{C}$ is smoothable (see [Gon06] or [GGP08]). By semi-continuity, $\mathrm{RCliff}(\widetilde{C})$ is at most RCliff of a general curve of genus $2k + 1$; that is, $\mathrm{RCliff}(\widetilde{C}) \leq k$. So $\mathrm{RCliff}(\widetilde{C}) = k$.

Since $\widetilde{C}$ is $(k + 2)$-gonal, we have $\mathrm{LCliff}(\widetilde{C}) \leq k$. But we know that $\mathrm{LCliff}(\widetilde{C}) \geq \mathrm{RCliff}(\widetilde{C})$ (Theorem 3.7). So $\mathrm{LCliff}(\widetilde{C}) = k$.

The last assertion follows from Theorem 5.3, (2). □

The case of general ribbons of even arithmetic genus follows using blow-ups.

Proposition 6.6 (Green's conjecture for generic ribbons of even genus). *Let C be a general smooth curve of genus g. Fix an even non-negative integer $p_a = 2k \geq \max\{3g + 7, 6g - 4\}$ and a line bundle L of degree $-p_a + 2g - 1$ on C. Then a general ribbon $\widetilde{C}$ on C with conormal bundle L satisfies*

$$\mathrm{RCliff}(\widetilde{C}) = \mathrm{LCliff}(\widetilde{C}) = k - 1.$$

Proof. We have $\mathrm{RCliff}(\widetilde{C}) \leq \mathrm{LCliff}(\widetilde{C})$ by Theorem 3.7, and $\mathrm{LCliff}(\widetilde{C}) \leq k - 1$ by Theorem 3.2 (or Theorem 4.7) since $\widetilde{C}$ is smoothable under the condition on p_a. It remains to prove that $k - 1 \leq \mathrm{RCliff}(\widetilde{C})$, that is,

$$K_{k,1}(\widetilde{C}, K_{\widetilde{C}}) = 0.$$

By semi-continuity, it suffices to exhibit one $\widetilde{C}$ such that $K_{k,1}(\widetilde{C}, K_{\widetilde{C}}) = 0$.

Choose a point $x \in C$ and let $\widetilde{C}'$ be a generic ribbon on C with conormal bundle $L(-x)$. Let $\widetilde{C}$ be the blow-up of $\widetilde{C}'$ at x. Then $\widetilde{C}$ has conormal bundle L. Applying Theorem 6.5 to $\widetilde{C}'$, we get

$$K_{k,1}(\widetilde{C}', K_{\widetilde{C}'}) = 0.$$

By [D18, Lemma 1], we deduce

$$K_{k,1}(\widetilde{C}, K_{\widetilde{C}}) = 0.$$

The proof is now complete. □

The same idea allows us to obtain a lower bound on the resolution Clifford index of general ribbons of every blow-up index. But we have to relinquish control over the conormal bundle.

Proposition 6.7. *Let C be a general smooth curve of genus g. Fix a non-negative integer $\max\{3g + 7, 6g - 4\}$. Let $\widetilde{C}$ be a general ribbon on C of arithmetic genus p_a and blow-up index b. Set $m = \lfloor (g+3)/2 \rfloor$. Except in the case where p_a is even and g is odd and $b = \lceil (p_a + g - 2)/2 \rceil$ is the maximum possible, we have*

$$b - \lfloor g/2 \rfloor \leq \mathrm{RCliff}(\widetilde{C}) \leq \mathrm{LCliff}(\widetilde{C}) \leq \min(b + 2m - 2, \lfloor 1/2(p_a - 1) \rfloor).$$

Note that the excluded case is handled by Theorem 6.6.

Proof. Again, the second and the third inequalities follow from Theorem 3.7 and Theorem 4.7. So we are left to prove the first inequality. By semi-continuity, it suffices to exhibit one ribbon $\widetilde{C}$ of arithmetic genus p_a and blow-up index b such that $b - \lfloor g/2 \rfloor \leq \mathrm{RCliff}(\widetilde{C})$.

The proof depends on the parity of p_a and g. We fully explain the case when p_a is even and g is odd, and indicate the changes necessary in the other cases. Take

$$k = (p_a + g - 2) - 2b.$$

(If g is even, add 1 to the right hand side.) Observe that $k \geq 0$ (in the excluded case, we have $k = -1$). Set $p'_a = p_a + k$, and observe that it is odd. Let $\widetilde{C}'$ be a general ribbon on C of arithmetic genus p'_a. By Theorem 4.3, the blow-up index of $\widetilde{C}'$ is

$$b' = (p'_a + g - 2)/2 = b + k.$$

Consider the sequence of b' blow-ups that takes $\widetilde{C}'$ to the split ribbon. Let $\widetilde{C}$ be the ribbon obtained after k blow-ups in this sequence. Then $\widetilde{C}$ has arithmetic genus p_a and blow-up index b.

By Theorem 6.5 applied to $\widetilde{C}'$, we have

$$\mathrm{RCliff}(\widetilde{C}') = (p'_a - 1)/2.$$

That is,

$$K_{(p'_a-1)/2}(\widetilde{C}', K_{\widetilde{C}'}) = 0.$$

By [D18, Lemma 1], we conclude

$$K_{(p'_a-1)/2}(\widetilde{C}, K_{\widetilde{C}}) = 0,$$

and therefore

$$\begin{aligned}
\mathrm{RCliff}(\widetilde{C}) &\geq (p_a - 2) - (p'_a - 1)/2 + 1 \\
&= (p_a - k - 1)/2 \\
&= b - \lfloor g/2 \rfloor.
\end{aligned}$$

The proof is now complete. □

7. GONALITY AND CLIFFORD INDEX OF A SPLIT RIBBON

In this section we concentrate on the linear series Clifford index, gonality, and the resolution Clifford index of the split ribbon.

7.1. **Linear series Clifford index of a split ribbon.** We first find the linear series Clifford index.

Proposition 7.1. *Let $\widetilde{C}$ be a ribbon of arithmetic genus p_a on a smooth curve C of genus g and gonality m.*

(1) *If $\widetilde{C}$ has a g_d^1 then $d \geq 2m$.*
(2) *Suppose the Clifford index of C is $m - 2$. Then, if $\widetilde{C}$ has a g_d^r with $d \leq p_a - 1$, then $d - 2r \geq 2m - 2$*

Proof. We begin with some set-up for both statements. Let L be the conormal bundle of $\widetilde{C}$. Let $(V, \widetilde{M})$ be a g_d^r on $\widetilde{C}$. Then there exists a divisor $\beta \subset C$ such that $\widetilde{M}$ is the push-forward of a line bundle on the blow-up of $\widetilde{C}$ at β. Let b be the degree of β. Let $\widetilde{C}'$ be the blow-up of $\widetilde{C}$ at β and $\widetilde{M}'$ such a line bundle on $\widetilde{C}'$ whose push-forward is $\widetilde{M}$. Note that $\widetilde{C}'$ has arithmetic genus $p_a - b$ and conormal bundle $L(\beta)$. Set $M = \widetilde{M}|_C / \text{torsion}$. On $\widetilde{C}'$, we have the exact sequence

$$0 \to M \otimes L(\beta) \to \widetilde{M}' \to M \to 0.$$

Set $e = \deg M$, so that $\deg \widetilde{M}' = 2e = d - b$.

Having set-up the basic objects, we turn to the proof of (1). Assume that $r = 1$. Since $(V, \widetilde{M})$ is a generalised g_d^1, the restriction map $V \to H^0(M)$ is injective. Therefore, $h^0(M) \geq 2$. Since C has gonality m, we conclude that $e = \deg M \geq m$. As a result, $d = 2e + b \geq 2m$.

We now assume $r = 2$ and turn to the proof of (2). We make two cases:

(1) Suppose $e \geq 2g - 2$. Then $h^1(M) \leq 1$ and so $h^0(M) \leq 2 - g + e$. As a result, $r \leq 1 - g + e$, and

$$d - 2r = 2e + b - 2r \geq 2g - 2 + b \geq 2m - 2.$$

(2) Suppose $e < 2g - 2$. Then M contributes to the linear series Clifford index of C. That is, we have $e - 2r \geq \text{LCliff}(C)$. As a result, using the fact that $\text{LCliff}(C) \geq m - 3$, (see [MG91]) we get

$$d - 2r = 2e + b - 2r \geq 2\,\text{LCliff}(C) + 2r + b \geq 2m - 2.$$

$\square$

Corollary 7.2. *Let C be a smooth curve of gonality m. Let $\widetilde{C}$ be a split ribbon of arithmetic genus p_a.*

(1) *The gonality of $\widetilde{C}$ is $2m$.*
(2) *The linear series Clifford index of $\widetilde{C}$ is $2m - 2$.*

Proof. The pull-back of a g_m^1 on C yields a g_{2m}^1 on $\widetilde{C}$. Therefore, the gonality of $\widetilde{C}$ is at most $2m$ and the linear series Clifford index is at most $2m-2$. The lower bounds follow from Theorem 7.1.

$\square$

As a result of Theorem 7.2, we see that Green's conjecture for the split ribbon implies Green's conjecture for any smooth double cover of C.

Corollary 7.3. *Let D be a smooth curve of genus p_a and $f : D \to C$ a finite map of degree 2. Let L be the line bundle on C such that $f_* \mathcal{O}_D \cong \mathcal{O}_C \oplus L$. Let $\widetilde{C}$ be the split ribbon on C with conormal bundle L. If $\text{RCliff}(\widetilde{C}) = \text{LCliff}(\widetilde{C})$, then $\text{RCliff}(D) = \text{LCliff}(D)$.*

Proof. Let m be the gonality of C. By Theorem 7.2, we have

$$\mathrm{RCliff}(\widetilde{C}) = 2m - 2 = \mathrm{gon}(\widetilde{C}) - 2.$$

Note that $f \colon D \to C$ is specified by a global section of L^{-2}. Scaling this section by $t \in \mathbf{A}^1$ gives a family of double covers of C whose fiber at $t = 0$ is $\widetilde{C}$ and whose all other fibers are isomorphic to D. By the semi-continuity of Koszul cohomology, we have,

$$\mathrm{RCliff}(\widetilde{C}) \leq \mathrm{RCliff}(D).$$

Note that the pull-back of a g^1_m on C gives a g^1_{2m} on D, so

$$\mathrm{gon}(D) \leq 2m,$$

and hence

$$\mathrm{LCliff}(D) \leq 2m - 2.$$

The inequalities

$$2m - 2 = \mathrm{RCliff}(\widetilde{C}) \leq \mathrm{RCliff}(D) \leq \mathrm{LCliff}(D) \leq 2m - 2$$

yield the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.2. **Resolution Clifford index of a split ribbon.**

We now take up the resolution Clifford index of the split ribbon. Unsurprisingly, we can describe the Koszul complex of the split ribbon entirely in terms of the underlying curve and the conormal bundle.

We begin by defining a graded module that governs the Koszul complex of the split ribbon. Let C be a smooth curve of genus g and fix a line bundle L on C. Fix a non-negative integer p. Let $S = \mathrm{Sym}(H^0(C, K_C))$ with the usual grading where the elements of $H^0(C, K_C)$ have degree 1. The graded S-module M^p_L, or simply M^p if L is clear from the context, is defined by

$$M^p = \bigoplus_{q \geq 0} K_{p,1}\left(C, K_C^q, K_C \otimes L^{-1}\right).$$

The S-module structure is induced by the multiplication maps

$$K_{p,1}(C, K_C^q, K_C \otimes L^{-1}) \otimes H^0(C, K_C) \to K_{p,1}(C, K_C^{q+1}, K_C \otimes L^{-1}).$$

Consider the Koszul complex of the graded S-module M^p, namely

$$(7.1) \qquad \cdots \to \bigwedge^{i+1} H^0(K_C) \otimes M^p_{q-1} \to \bigwedge^i H^0(K_C) \otimes M^p_q \to \bigwedge^{i-1} H^0(K_C) \otimes M^p_{q+1} \to \cdots.$$

Let $\Phi_{i,p,q}$ be the first map shown above. Writing out the descriptions of the graded pieces of M^p, we have

$$(7.2) \qquad \Phi_{i,p,q} \colon \bigwedge^{i+1} H^0(K_C) \otimes K_{p,1}(C, K_C^{q-1}, K_C \otimes L^{-1}) \to \bigwedge^i H^0(K_C) \otimes K_{p,1}(C, K_C^q, K_C \otimes L^{-1}).$$

Since M^p_q's are themselves Koszul cohomology groups, the complex (7.1) arises as a row in a double complex after taking vertical cohomology. We now write this double complex. For brevity,

we write (A) for $H^0(C, A)$ and $V^{(i)}$ for $\bigwedge^i V$. The double complex is
(7.3)

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
\xrightarrow{\hspace{1cm}} (K_C)^{(i+1)} \otimes (K_C \otimes L^{-1})^{(p+1)} \otimes (K_C^{q-1}) \xrightarrow{\hspace{1.5cm}} (K_C)^{(i)} \otimes (K_C \otimes L^{-1})^{(p+1)} \otimes (K_C^{q}) \xrightarrow{\hspace{1cm}} \\
\downarrow & & \downarrow \\
\xrightarrow{\hspace{1cm}} (K_C)^{(i+1)} \otimes (K_C \otimes L^{-1})^{(p)} \otimes (K_C^{q} \otimes L^{-1}) \xrightarrow{d_{i,p,q}} (K_C)^{(i)} \otimes (K_C \otimes L^{-1})^{(p)} \otimes (K_C^{q+1} \otimes L^{-1}) \xrightarrow{\hspace{1cm}} \cdot \\
\downarrow & & \downarrow \\
\xrightarrow{\hspace{1cm}} (K_C)^{(i+1)} \otimes (K_C \otimes L^{-1})^{(p-1)} \otimes (K_C^{q+1} \otimes L^{-2}) \xrightarrow{\hspace{1.5cm}} (K_C)^{(i)} \otimes (K_C \otimes L^{-1})^{(p-1)} \otimes (K_C^{q+2} \otimes L^{-2}) \xrightarrow{\hspace{1cm}} \\
\downarrow & & \downarrow
\end{array}
$$

The map $\Phi_{i,p,q}$ in (7.2) is the map induced on vertical cohomology by the map $d_{i,p,q}$.

Lemma 7.4. *In the setup above, assume that $h^1(-L) = 0$ and $p \le 2g - 4$. Then $\Phi_{i,p,1}$ is surjective if and only if $K_{i,1}(M^p) = 0$.*

Proof. Note that $K_{i,1}(M^p)$ is the middle cohomology of the complex

$$(K_C)^{(i+1)} \otimes K_{p,1}(C, K_C \otimes L^{-1}) \xrightarrow{\Phi_{i,p,1}} (K_C)^{(i)} \otimes K_{p,1}(C, K_C, K_C \otimes L^{-1}) \to (K_C)^{(i-1)} \otimes K_{p,1}(C, K_C^2, K_C \otimes L^{-1}).$$

It suffices to prove that the last term vanishes. Let $r + 1 = h^0(K_C \otimes L^{-1})$. By duality [G84, 2.6.c], we have

$$K_{p,1}(C, K_C^2, K_C \otimes L^{-1}) = K_{r-1-p,1}(C, K_C^{-1}, K_C \otimes L^{-1})^*.$$

Shifting by 1 gives

$$K_{r-1-p,1}(C, K_C^{-1}, K_C \otimes L^{-1}) = K_{r-1-p,0}(C, L^{-1}, K_C \otimes L^{-1}).$$

Since $h^1(-L) = 0$, we get

$$h^0(-L) = h^0(K_C - L) - (2g - 2).$$

Since $p \le 2g - 4$, the right hand side is bounded above by $r - 1 - p$. By [G84, Theorem 3.a.1], we conclude that $K_{r-1-p,0}(C, L^{-1}, K_C \otimes L^{-1}) = 0$. $\square$

The following theorem relates the syzygies of the canonically embedded split ribbon with the Koszul cohomology of the module M^p.

Theorem 7.5. *Let C be a smooth curve of genus g and gonality m. Let*

$$p_a \ge \max(2g + 2m - 1, 6g - 4)$$

and let L be a line bundle on C of degree $-p_a + 2g - 1$. Let $\widetilde{C}$ be the split ribbon with conormal bundle L. The following are equivalent.

(1) $\widetilde{C}$ satisfies Green's conjecture, that is,

$$\mathrm{RCliff}(\widetilde{C}) = \mathrm{LCliff}(\widetilde{C}).$$

(2) *for all non-negative integers i, j such that $i + j = 2m - 3$, the map*

$$\Phi_{i,j,1} \colon \overset{i+1}{\bigwedge} H^0(K_C) \otimes K_{j,1}(C, K_C - L) \to \overset{i}{\bigwedge} H^0(K_C) \otimes K_{j,1}(C, K_C, K_C - L)$$

is surjective.

(3) *for all non-negative integers i, j such that $i + j = 2m - 3$, we have $K_{i,1}(M^j) = 0$.*

Proof. Let $\phi \colon \widetilde{C} \to C$ be the projection. For any line bundle A on C, we have a canonically split exact sequence of $\mathcal{O}_{\widetilde{C}}$-modules

$$0 \to A \otimes L \to \phi^* A \to A \to 0.$$

The splitting $A \to \phi^* A$ is given by the pull-back map. In particular, for the q-th tensor power of $K_{\widetilde{C}} = \phi^*(K_C \otimes L^{-1})$, we have the splitting

$$K_{\widetilde{C}}^q = (K_C^q \otimes L^{-q}) \oplus (K_C^q \otimes L^{-q+1}).$$

As a result, we have

(7.4) $$H^0(\widetilde{C}, K_{\widetilde{C}}^q) = H^0(C, K_C^q \otimes L^{-q}) \oplus H^0(K_C^q \otimes L^{-q+1}).$$

Let $\widetilde{S}$ and S be the graded rings

$$\widetilde{S} = \bigoplus_{q \geq 0} H^0(\widetilde{C}, K_{\widetilde{C}}^q), \text{ and}$$

$$S = \bigoplus_{q \geq 0} H^0(C, K_C^q \otimes L^{-q}),$$

and let J be the graded S-module

$$J = \bigoplus_{q \geq 0} H^0(C, K_C^q \otimes L^{-q+1}).$$

Let ϵ be a formal variable with $\epsilon^2 = 0$. Taking the direct sum over all q in (7.4) gives an isomorphism of graded rings

$$\widetilde{S} = S \oplus \epsilon J.$$

The Koszul complex that computes $K_{p,q}(\widetilde{C}, K_{\widetilde{C}})$ is

$$\mathcal{K} = \cdots \to \overset{p+1}{\bigwedge} \widetilde{S}_1 \otimes \widetilde{S}_{q-1} \to \overset{p}{\bigwedge} \widetilde{S}_1 \otimes \widetilde{S}_q \to \overset{p-1}{\bigwedge} \widetilde{S}_1 \otimes \widetilde{S}_{q+1} \to \cdots$$

Here, we take the three terms above to be in homological degres $q - 1, q, q + 1$, respectively. The compelx $\mathcal{K}$ has a subcomplex

$$\mathcal{S} = \cdots \to \overset{p+1}{\bigwedge} \widetilde{S}_1 \otimes \epsilon J_{q-1} \to \overset{p}{\bigwedge} \widetilde{S}_1 \otimes \epsilon J_q \to \overset{p-1}{\bigwedge} \widetilde{S}_1 \otimes \epsilon J_{q+1} \to \cdots,$$

and the quotient complex is

$$\mathcal{Q} = \cdots \to \overset{p+1}{\bigwedge} \widetilde{S}_1 \otimes S_{q-1} \to \overset{p}{\bigwedge} \widetilde{S}_1 \otimes S_q \to \overset{p-1}{\bigwedge} \widetilde{S}_1 \otimes S_{q+1} \to \cdots.$$

Observe that

$$\overset{p}{\bigwedge} \widetilde{S}_1 = \bigoplus_i \overset{i}{\bigwedge} S_1 \otimes \epsilon \overset{p-i}{\bigwedge} J_1.$$

The action of $\epsilon \wedge^{p-i} J_1$ on ϵJ_q and on S_q is zero. As a result, we see that

$$H^q(\mathscr{S}) = \bigoplus_i \overset{p-i}{\bigwedge} J_1 \otimes K_{i,q}(C, L, K_C \otimes L^{-1}), \text{ and}$$

$$H^q(\mathscr{Q}) = \bigoplus_i \overset{p-i}{\bigwedge} J_1 \otimes K_{i,q}(C, K_C \otimes L^{-1}).$$

The short exact sequence of complexes

$$0 \to \mathscr{S} \to \mathscr{K} \to \mathscr{Q} \to 0$$

induces a long exact sequence in cohomology

(7.5) $$\cdots \to H^{q-1}(\mathscr{Q}) \to H^q(\mathscr{S}) \to H^q(\mathscr{K}) \to \cdots.$$

Consider the connecting map

$$\bigoplus_i \overset{p+1-i}{\bigwedge} J_1 \otimes K_{i,q-1}(C, K_C \otimes L^{-1}) \to \bigoplus_i \overset{p-i}{\bigwedge} J_1 \otimes K_{i,q}(C, L, K_C \otimes L^{-1}).$$

It is easy to check that this map is diagonal, that is, the direct sum of maps

(7.6) $$\overset{p+1-i}{\bigwedge} J_1 \otimes K_{i,q-1}(C, K_C \otimes L^{-1}) \to \overset{p-i}{\bigwedge} J_1 \otimes K_{i,q}(C, L, K_C \otimes L^{-1}).$$

Note that $J_1 = H^0(C, K_C)$ and

$$K_{i,q}(C, L, K_C \otimes L^{-1}) = K_{i,q-1}(C, K_C, K_C \otimes L^{-1}),$$

so the map (7.6) becomes

(7.7) $$\overset{p+1-i}{\bigwedge} H^0(K_C) \otimes K_{i,q-1}(C, K_C \otimes L^{-1}) \to \overset{p-i}{\bigwedge} H^0(K_C) \otimes K_{i,q-1}(C, K_C, K_C \otimes L^{-1}).$$

For $q = 2$, this is precisely the map $\Phi_{p-i,i,1}$ from (7.2).

We are now ready to prove the equivalence of (1) and (2). By Theorem 7.2, we know that $\mathrm{LCliff}(\widetilde{C}) = 2m - 2$. By Theorem 3.7, we know that $\mathrm{RCliff}(\widetilde{C}) \leq \mathrm{LCliff}(\widetilde{C})$. So, (1) is equivalent to the vanishing

$$K_{2m-3,2}(\widetilde{C}, K_{\widetilde{C}}) = 0.$$

Taking $p = 2m - 3$ and $q = 2$, we have

$$K_{2m-3,2}(\widetilde{C}, K_{\widetilde{C}}) = H^q(\mathscr{K}).$$

Observe that we have

$$H^2(\mathscr{Q}) = \bigoplus_{i=0}^{2m-3} \overset{2m-3-i}{\bigwedge} H^0(K_C) \otimes K_{i,2}(C, K_C \otimes L^{-1}).$$

By our hypothesis, we have

$$\deg(L^{-1}) = p_a - 2g + 1 \geq 2m,$$

and therefore $\deg(K_C \otimes L^{-1}) \geq 2g + 2m - 2$. By [G84, Theorem 4.a.1], for $0 \leq i \leq 2m - 3$, we have

$$K_{i,2}(C, K_C \otimes L^{-1}) = 0.$$

So $H^2(\mathcal{Q}) = 0$. By the long exact sequence (7.5), the vanishing of $H^2(\mathcal{K})$ is equivalent to the surjectivity of

$$H^1(\mathcal{Q}) \to H^2(\mathcal{S}).$$

By (7.7), this surjectivity is equivalent to (2).

The equivalence of (2) and (3) is Theorem 7.4. $\qquad\square$

In light of Theorem 7.5, it is natural to ask the following.

Question 7.6. *Are the equivalent statements in Theorem 7.5 are true ?*

We observe that they are true for hyperelliptic curves.

Proposition 7.7. *The conditions in Theorem 7.5 hold for a hyperelliptic curve C.*

Proof. We begin with a more general statement. Let C be a smooth curve of genus g and let m be its gonality. We show that the map

$$\bigwedge^{i+1} H^0(K_C) \otimes K_{j,1}(C, K_C \otimes L^{-1}) \to \bigwedge^{i} H^0(K_C) \otimes K_{j,1}(C, K_C, K_C \otimes L^{-1})$$

is surjective for $i = 0$ and $j = 2m - 3$. To see this, we use the results and notation of [But94]. For line bundles A and B, the kernel of the map

$$\bigwedge^{j} H^0(A) \otimes H^0(B) \to \bigwedge^{j-1} H^0(A) \otimes H^0(A \otimes B)$$

is $H^0(\bigwedge^j M_A \otimes A \otimes B)$ (see, for example, [EL93, Lemma 1.4]). So it is enough to show that for $j = 2m - 3$, the map

$$H^0(K_C) \otimes H^0\left(\bigwedge^{j} M_{(K_C \otimes L^{-1})} \otimes K_C \otimes L^{-1}\right) \to H^0\left(\bigwedge^{j} M_{(K_C \otimes L^{-1})} \otimes K_C^2 \otimes L^{-1}\right)$$

is surjective. Since $K_C \otimes L^{-1}$ is semistable of degree (= slope) at least $2g$, the bundle $M_{K_C \otimes L^{-1}}$ is semistable ([But94, Theorem 1.2]). The surjection follows from [But94, Proposition 2.2]. $\qquad\square$

We end with an example of a split ribbon that does *not* satisfy Green's conjecture.

Example 7.8. Let C be a general curve of genus 3, and take $L = K_C^{-1}$. Note that C has gonality 3. Let $\widetilde{C}$ be the split ribbon with conormal bundle L. Then $p_a(\widetilde{C}) = 9$. By Theorem 7.2, we have LCliff($\widetilde{C}$) = 4. A `Macaulay2` computation—done over the base field $\mathbf{Z}/101\mathbf{Z}$ with a randomly chosen plane quartic C—shows that the minimal free resolution of the canonical embedding of $\widetilde{C}$ has the betti table

	0	1	2	3	4	5	6	7
total:	1	21	84	154	154	84	21	1
0 :	1	.	.	.	.	.	.	.
1 :	.	21	64	90	64	20	.	.'
2 :	.	.	20	64	90	64	21	.
3 :	.	.	.	.	.	.	.	1

which means RCliff($\widetilde{C}$) = 2.

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