ANTICANONICAL TROPICAL CUBIC DEL PEZZOS CONTAIN EXACTLY 27 LINES

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1. Introduction

Supplementary material

Many of the results in this paper rely on calculations performed using `Sage` [4]. Several of them required new implementations within this platform using `Python`. We have created supplementary files so that the reader can reproduce all the claimed assertions done via explicit computations. These files can be found at:

https://people.math.osu.edu/cueto.5/anticanonicalTropDelPezzoCubics/

In addition to all `Sage` scripts, the website contains all input and output files both as `Sage` object files and in plain text. We have also included the supplementary files on the `arXiv`'s submission of this paper. They can be obtained by downloading the source.

All computations are performed symbolically using either our own implementation of group-actions on polynomial rings, tropical operations (min for tropical addition and usual product for tropical multiplication), or build-in functions for computations with Weyl groups, polyhedra and factorizations of rational functions over the rational numbers. They were performed on a 2.4 GHz Intel(R) Core 2 Duo with 3MB cache and 2GB RAM. The implementation of the construction of the Bergman complex of a general matroid from its nested sets is new and exploits symmetries whenever possible. The computation of the Bergman complex for the arrangement associated to the root system E_6 takes about 1 hour to finish. The time is split evenly between the calculation of adjacencies and the whole Bergman complex.

The most time-demanding calculations are those in Section 8. The computation of the tropicalization of each 5×45 matrix of rational functions associated to each cone in the Naruki fan takes about 20 minutes. The computation-time required to search for tropically singular 3×3-minors for each such matrix is not uniform. With the exception of a single cone, each calculation takes about 30 seconds. For the problematic cone and each choice of three rows in the corresponding matrices, the computation takes 10 minutes since several extremal curves have no tropical non-singular 3×3 minors, and the calculation exhausts all 3-element subsets from $\{0, \dots, 44\}$.

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2. Moduli of cubic del Pezzo surfaces and the action of $\mathrm{W}(\mathrm{E}_6)$

In this section, we review the classical construction of the moduli space of marked del Pezzo surfaces, originating in the work of Coble. Our main references are [1] and [2], which describe classical and tropical moduli spaces of del Pezzo surfaces of arbitrary degree. To simplify and focus our exposition, we restrict ourselves to cubic del Pezzos.

A *cubic del Pezzo surface* is a smooth projective surface with semi-ample anti-canonical bundle whose class has self-intersection 3. It follows that the anti-canonical bundle is also base-point free, and the map given by its sections is birational onto a cubic surface in $\mathbb{P}^3$. If the anti-canonical bundle is ample, then the map is a closed embedding. In this case, we say that the surface is a *Fano* cubic surface. The convention of using *del Pezzo* for semi-ample anti-canonical bundle and *Fano* for ample anti-canonical is not standard in the literature, but it agrees with the one followed in [1].

A cubic del Pezzo surface X is obtained by 6 successive blow-ups of $\mathbb{P}^2$. Conversely, any surface obtained in this way is a cubic del Pezzo provided that no 3 centers of the blow up lie on a line and the 6 do not lie on a conic. As a result, the Picard group of X is isomorphic to $\mathbb{Z}^7$, generated by the canonical class K_X and the classes of the 6 exceptional divisors. The Picard group contains 27 *exceptional divisor classes*, namely classes E with $K_X \cdot E = 1$ and $E^2 = -1$. On a Fano surface, these are precisely the classes of the 27 lines. More generally, each exceptional class is represented by a unique effective divisor. An ordered collection of 6 exceptional classes $E_1, \dots, E_6$ with $E_i \cdot E_j = -\delta_{ij}$ is called a *marking* of X. For example, if X is obtained from $\mathbb{P}^2$ by blowing up 6 distinct points, then the 6 exceptional divisors give a marking. It turns out that there are $72 \cdot 6!$ markings of a del Pezzo surface (72 if we disregard the ordering). So there are essentially 72 ways in which a general cubic surface arises as a blow up of $\mathbb{P}^2$.

The blow-up construction shows that the moduli space of marked cubic del Pezzos contains an open subset isomorphic to a dense open subset U of $(\mathbb{P}^2)^6$. The set U consists of 6 tuples of distinct points, no 3 of which lie on a line, and the 6 do not lie on a conic. We denote this set by $M^\circ_{m,3}$.

The group of automorphisms of the lattice $\langle \operatorname{Pic} X, \cdot \rangle$ that fix K_X is the Weyl group $\mathrm{W}(\mathrm{E}_6)$. This group acts transitively not only on the exceptional classes, but also on the markings. Consequently, it also acts on $M^\circ_{m,3}$. The quotient is an open subset of the moduli of (unmarked) del Pezzo cubics, which we denote by M°_3. This will only play an auxiliary role in the sequel.

2.1. **The Naruki space and Coble covariants.** The space M°_3 admits a natural compactification M^*_3 using Geometric Invariant Theory (GIT). The compactified moduli space M^*_3 is the GIT quotient $\mathbb{P}\operatorname{Sym}^3(\mathbb{C}^4)/\operatorname{SL}(4)$. The line bundles $\mathcal{O}(n)$ on $\mathbb{P}\operatorname{Sym}^3(\mathbb{C}^4)$ descend to rank one sheaves on M^*_3 (they are line bundles if M^*_3 is considered as a stack rather than a coarse space, but we will ignore this point). It is convenient to denote by $\mathcal{O}_{M^*_3}(1)$ the sheaf descended from $\mathcal{O}(4)$. The sheaf $\mathcal{O}_{M^*_3}(1)$ has the following geometric interpretation. Consider the rank 6 bundle $R^1 f_* \omega^{-1}$, where ω is the relative dualizing sheaf of the universal family over M^*_3. Then there is an isomorphism

$$\mathcal{O}_{M^*_3}(1) = \det R^1 f_* \omega^{-1}.$$

The compactification M^*_3 of M°_3 gives a natural compactification $M^*_{m,3}$ of $M^\circ_{m,3}$ called the *Naruki space*. Let $M^*_{m,3}$ be the normalization of M^*_3 in $M^\circ_{m,3}$. The action of $\mathrm{W}(\mathrm{E}_6)$ on $M^\circ_{m,3}$ extends to an action on $M^*_{m,3}$ and the map $M^*_{m,3} \to M^*_3$ is the quotient. Denote by $\mathcal{O}_{M^*_{m,3}}(1)$ the pullback of $\mathcal{O}_{M^*_3}(1)$.

Global sections of $\mathcal{O}^*_{M_{m,3}}(1)$ are called *Coble covariants*. The space $H^0(\mathcal{O}_{M^*_{m,3}}(1))$ is a 10 dimensional irreducible representation of $\mathrm{W}(\mathrm{E}_6)$. It yields a very ample linear series on $M^*_{m,3}$ and gives

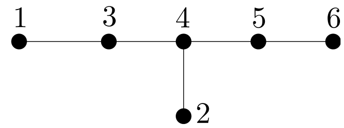

FIGURE 1. The labelled Dynkin diagram of the Root system E_6. Each label i corresponds to the simple root α_i.

an embedding of $M^*_{m,3}$ in $\mathbb{P}^9$. It was this particular projective model $M^*_{m,3}$ that Naruki discovered. We will describe the Coble covariants and the Naruki space explicitly using an equivariant uniformization of $M^*_{m,3}$.

2.2. **An equivariant uniformization.** Let $\mathfrak{h}_6$ be the lattice of type E_6 tensored with $\mathbb{K}$. More explicitly, $\mathfrak{h}_6$ is the $\mathbb{K}$-vector space spanned by $d_1, \dots, d_6$ with a bilinear form given by

$$d_i \cdot d_j = \begin{cases} -1/9 & \text{if } i \neq j, \\ \ \ 8/9 & \text{if } i = j. \end{cases}$$

The elements $(d_j - d_i)$ for $i < j$ and $(d_i + d_j + d_k)$ for $i < j < k$, and $(d_1 + \cdots + d_6)$ together form the set of 36 positive roots of E_6 associated to the simple roots $\alpha_i = (di + 1 - di)$ for $i \neq 2$ and $\alpha_2 = d_1 + d_2 + d_3$. It induces Bourbaki's convention for labelling the Dynkin diagram of type E_6, as in Figure 1.

We have a map $\mathfrak{h}_6^* \to (\mathbb{P}^2_{\mathbb{K}})^6$ given by

$$p \mapsto \left([1 : d_1(p) : d_1^3(p)], \dots, [1 : d_6(p) : d_6^3(p)]\right).$$

Denote by $\mathfrak{h}_6^\circ$ the complement in $\mathfrak{h}_6^*$ of the zeros of the roots. For $p \in \mathfrak{h}_6^\circ$, the points $[1 : d_i(p) : d_i^3(p)]$ for $1 \leq i \leq 6$ are distinct, no three of them lie on a line, and the six do not lie on a conic. Therefore, the blow up of $\mathbb{P}^2_{\mathbb{K}}$ at these points gives a marked cubic Fano surface, where the marking is given by the six exceptional divisors. Scaling p by $t \in \mathbb{K}$ produces a different set of six points, but they are related to the original six by the automorphism of $\mathbb{P}^2_{\mathbb{K}}$ defined by $[X : Y : Z] \mapsto [X : tY : t^3 Z]$. As a result, the resulting marked surfaces are canonically isomorphic. We thus get a morphism

$$\mathbb{P}(\mathfrak{h}_6^\circ) \to M^\circ_{m,3}. \tag{2.1}$$

It is easy to check that the map in (2.1) is equivariant with respect to the action of $\mathrm{W(E_6)}$. Furthermore, it is surjective and flat [1, Theorem 3.1] of relative dimension 1.

We digress momentarily to describe the fibers of the morphism (2.1). Note that the points $[d^3 : d : 1]$ lie on the cuspidal plane cubic with affine equation $x^2 = y^3$. Conversely, there exists a cuspidal cubic passing through six distinct points on $\mathbb{P}^2_{\mathbb{K}}$. In fact, such cubics form a one-parameter family. Having fixed such a cuspidal cubic, it can be brought to the standard one given by $x^2 = y^3$ by an automorphism of $\mathbb{P}^2_{\mathbb{K}}$ unique up to a $\mathbb{G}_m$ acting by $[X : Y : Z] \mapsto [X : tY : t^3 Z]$. Therefore, we may think of $\mathbb{P}\mathfrak{h}_6^*$ as the moduli space of 6 distinct points on $\mathbb{P}^2$ along with a choice of a cuspidal cubic passing through them. Then the map $\mathbb{P}(\mathfrak{h}_6^\circ) \to M^\circ_{m,3}$ simply forgets the cuspidal cubic.

Let $Z \subset \mathbb{P}(\mathfrak{h}_6^*)$ be the union of the $\mathrm{W(E_6)}$ orbits of the linear subspace defined by $d_1 = d_2 = d_3$. More intrinsically, Z is the union of the linear subspaces whose points are fixed by a Weyl subgroup of $\mathrm{W(E_6)}$ of type A_3. The map $\mathbb{P}(\mathfrak{h}_6^\circ) \to M^\circ_{m,3}$ extends to a regular map $\mathbb{P}(\mathfrak{h}_6^*) \setminus Z \to M^*_{m,3}$ [1, Proposition 4.10]. There is a natural isomorphism of the pullback of $\mathcal{O}_{M^*_{m,3}}(1)$ to $\mathbb{P}(\mathfrak{h}_6^*)$ with $\mathcal{O}_{\mathbb{P}(\mathfrak{h}_6^*)}(9)$. As a result, we can write (the pullbacks of) the Coble covariants as homogeneous polynomials of degree 9 in $d_1, \dots, d_6$.

2.3. **Yoshida and Cross functions.** We now describe two sets of Coble covariants that play a key role in the paper. Both sets are $\mathrm{W(E_6)}$ invariant and have a beautiful description as homogeneous polynomials of degree 9 in $d_1, \dots, d_6$ in terms of root subsystems of the root system E_6. We give this description as well as explicit formulas which we will use heavily for computation. By the *discriminant* Δ of a root system, we mean the square root of the product of all the roots, both positive and negative. This is a polynomial, well-defined up to a sign. Prescribing a set of positive roots R^+ pins down the sign – we simply take the product of all the positive roots.

The first set of Coble covariants are the *Yoshida functions.* These are are the discriminants of sub root systems of $\mathfrak{h}_6$ of type $A_2^{\oplus 3}$. For example, the subsystem

$$\begin{aligned} S =&\langle d_5 - d_6, d_1 + d_2 + d_6, d_1 + d_2 + d_5\rangle \oplus \\ &\langle d_3 - d_4, d_4 + d_5 + d_6, d_3 + d_5 + d_6\rangle \oplus \\ &\langle d_1 - d_2, d_2 + d_3 + d_4, d_1 + d_3 + d_4\rangle \end{aligned} \tag{2.2}$$

yields the Yoshida function

$$\begin{aligned} \mathcal{Y}(S) =&(d_5 - d_6)(d_1 + d_2 + d_6)(d_1 + d_2 + d_5) \\ &(d_3 - d_4)(d_4 + d_5 + d_6)(d_3 + d_5 + d_6) \\ &(d_1 - d_2)(d_2 + d_3 + d_4)(d_1 + d_3 + d_4). \end{aligned} \tag{2.3}$$

The group $\mathrm{W}(\mathrm{E}_6)$ acts transitively on the Yoshida functions, so the others can be computed using the group action. There are 80 Yoshida functions (40 up to sign). Since the Yoshida functions are products of the roots, they are invertible on $\mathbb{P}(\mathfrak{h}_9^\circ)$. Equivalently, the corresponding Coble covariants are invertible on $M_{m,3}^\circ$. The Yoshida functions span the 10 dimensional space of Coble covariants. Since the linear system defined by the Coble covariants is very ample on $M_{m,3}^*$, we can recover $M_{m,3}^* \subset \mathbb{P}^{39}$ as the closure of image of the map

$$\mathbb{P}\mathfrak{h}_6 \dashrightarrow \mathbb{P}^{39}$$

defined by the 40 Yoshida functions (up to sign).

The second set of Coble covarints are the *Cross functions.* Let $S \subset \mathfrak{h}_6$ be a sub root system of type $A_2^{\oplus 3}$ and let $\alpha \in \mathfrak{h}_6$ be a root not orthogonal to any of the summands of S. Let S^+ be a set of positive roots for S and denote by s_α the reflection in the plane orthogonal to α. The Cross associated to the pair (S^+, α) is the difference

$$\mathrm{Cross}(S^+, \alpha) = \Delta(S^+) - s_\alpha \Delta(S^+).$$

Note that the Cross is a difference of two Yoshidas. Furthermore, due to the linear relations between the Yoshida functions, each cross can be expressed in 4 distinct ways as a difference of Yoshidas. The data of (S, α) (without the choice of positive roots) determines the cross function up to a sign; we denote it by $\mathrm{Cross}(S, \alpha)$. Observe that there are three roots in S^+ orthogonal to α, say α_1, α_2, and α_3. Clearly, these three roots divide the cross $\Delta(S^+) - s_\alpha \Delta(S^+)$. Furthermore, it is easy to see that α also divides $\mathrm{Cross}(S^+, \alpha)$. Therefore, $\mathrm{Cross}(S^+, \alpha)$ factors as

$$\mathrm{Cross}(S, \alpha) = \alpha\alpha_1\alpha_2\alpha_3 Q, \tag{2.4}$$

where Q is a quintic polynomial. It turns out that Q is irreducible. For example, taking S as in (2.2) (with the listed roots taken to be positive) and $\alpha = d_2 + d_4 + d_5$ yields the Cross function

$$\begin{aligned} \mathrm{Cross}(S^+, \alpha) =&(d_3 + d_5 + d_6)(d_2 + d_4 + d_5)(d_1 + d_3 + d_4)(d_1 + d_2 + d_6) \\ &(d_1^2 d_2^2 d_3 + d_1^2 d_2 d_3^2 - d_2^2 d_3 d_4^2 - d_2 d_3^2 d_4^2 - d_1^2 d_2^2 d_5 - d_1^2 d_2 d_3 d_5 + d_1 d_2^2 d_3 d_5 - d_1^2 d_3^2 d_5 \\ &\quad + d_1 d_2 d_3^2 d_5 + d_1^2 d_4^2 d_5 - d_1 d_2^2 d_5^2 - d_1 d_2 d_3 d_5^2 + d_2^2 d_3 d_5^2 - d_1 d_3^2 d_5^2 + d_2 d_3^2 d_5^2 + d_1 d_4^2 d_5^2 \\ &\quad - d_2^2 d_3 d_4 d_6 - d_2 d_3^2 d_4 d_6 - d_1^2 d_4^2 d_6 + d_2^2 d_4^2 d_6 + d_2 d_3 d_4^2 d_6 + d_3^2 d_4^2 d_6 + d_1^2 d_4 d_5 d_6 \\ &\quad - d_1 d_4^2 d_5 d_6 + d_1 d_4 d_5^2 d_6 - d_4^2 d_5^2 d_6 - d_2^2 d_3 d_6^2 - d_2 d_3^2 d_6^2 - d_1^2 d_4 d_6^2 + d_2^2 d_4 d_6^2 \\ &\quad + d_2 d_3 d_4 d_6^2 + d_3^2 d_4 d_6^2 + d_1^2 d_5 d_6^2 - d_1 d_4 d_5 d_6^2 + d_1 d_5^2 d_6^2 - d_4 d_5^2 d_6^2). \end{aligned}$$

The group $\mathrm{W}(\mathrm{E}_6)$ acts transitively on the Cross functions, so the others can be computed using the group action.

The combinatorics of the factorization of a Cross function into 4 linear factors and 1 quintic factor is the following. Note that the four linear factors form four mutually orthogonal roots. Given any four mutually orthogonal roots, there is a unique (up to sign) Cross function divisible by all four. This yields a bijection between the set of the Cross functions (up to sign) and sub root systems of type $A_1^{\oplus 4}$ of $\mathfrak{h}_6$. Abusing notation, we denote the Cross associated to a subsystem S of type $A_1^{\oplus 4}$ by $\mathrm{Cross}(S)$. The Weyl group $\mathrm{W}(\mathrm{E}_6)$ acts transitively also on the quintic factors of the Cross functions. Given a quintic in this orbit, there are three sub root systems of type $A_1^{\oplus 4}$ of $\mathfrak{h}_6$ whose Cross is divisible by the quintic.

The locus of vanishing of the cross functions has the following geometric interpretation. Recall that an *Eckhard point* on a Fano cubic surface is the point of concurrency of three exceptional curves. The locus of Fano cubic surfaces with an Eckhard point forms a divisor in M_3°, called the *Eckhard divisor*.

Proposition 2.1. *The vanishing locus of the product of all Cross functions in M_3° is the Eckhard divisor.*

Proof. It suffices to prove the statement on $\mathbb{P}(\mathfrak{h}_6^\circ)$, where we can do a direct computation. Let $p_i = [1 : d_i : d_i^3]$ for $i = 1, \ldots, 6$ and distinct d_i and let X be the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_6$. Consider the triple of exceptional curves on X given by the proper transforms of the lines $L_{ij} = \overline{p_i p_j}$ for $(i, j) = (1, 2)$, $(3, 4)$, and $(5, 6)$. The equation of L_{ij} is

$$d_i d_j (d_i + d_j) X - (d_i^2 + d_i d_j + d_j^2) Y + Z = 0.$$

The three lines L_{12}, L_{34}, and L_{56} are concurrent if and only if the determinant of the matrix

$$M = \begin{pmatrix} d_1 d_2 (d_1 + d_2) & d_1^2 + d_1 d_2 + d_2^2 & 1 \\ d_3 d_4 (d_3 + d_4) & d_3^2 + d_3 d_4 + d_4^2 & 1 \\ d_5 d_6 (d_5 + d_6) & d_5^2 + d_5 d_6 + d_6^2 & 1 \end{pmatrix}. \tag{2.5}$$

vanishes. Note that $\det M$ is an irreducible homogeneous quintic polynomial, which is a quintic factor of a suitable Cross function. In the example above it is obtained from the quintic in (2.4) by applying a permutation in $\mathfrak{S}_6$. $\square$

Let $(X, E_1, \ldots, E_6)$ be a marked Fano cubic surface. The marking yields a marking of the 27 exceptional curves on X. Express X as the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_6$ such that the E_i is the exceptional divisor over i. Then the 27 exceptional curves are

- E_i: The exceptional divisor over p_i, for $1 \le i \le 6$;
- F_{ij}: The proper transform of the line through $p_i p_j$ for $1 \le i \ne j \le 6$;
- G_j: The proper transform of the conic through $\{p_1, \ldots, p_6\} \setminus \{p_j\}$.

Note that the two indices in F_{ij} are *unordered*, namely $F_{ij} = F_{ji}$. While doing computations, we choose the indices so that $i < j$.

An *anticanonical triangle* in X is a triple of exceptional curves whose pairwise intersection numbers are 1. In this case, their sum is an anticanonical divisor (zero locus of a section of the anticanonical bundle). There are 45 anticanonical triangles on a cubic surface. On a marked Fano cubic surface as above, they are

- $x_{ij} = \{E_i, F_{ij}, G_j\}$ for $1 \le i \ne j \le 6$.
- $y_{ijklmn} = \{F_{ij}, F_{kl}, F_{mn}\}$ for distinct i, j, k, l, m, n in $\{1, \ldots, 6\}$.

Note that the two indices of X are *ordered*, namely $x_{ij} \ne x_{ji}$. The ordering of the six indices of Y is important only to the extent of the resulting tripartition of $\{1, \ldots, 6\}$. That is, we consider two sextuple (i, j, k, l, m, n) and (i', j', k', l', m', n') if we have the equality of sets

$$\{\{i, j\}, \{k, l\}, \{m, n\}\} = \{\{i', j'\}, \{k', l'\}, \{m', n'\}\}.$$

While doing computations, we choose the indices so that $i < j$, $k < l$, $m < n$, and $i < k < m$. The action of $\mathrm{W}(\mathrm{E}_6)$ on the markings induces an action on both sets $\{E_i, F_{ij}, G_j\}$ and $\{x_{ij}, y_{ijklmn}\}$.

The proof of Proposition 2.1 gives a bijection between the 45 anticanonical triangles and the 45 quintic factors of the cross functions (up to sign), equivariant with respect to the action of $\mathrm{W}(\mathrm{E}_6)$. Explicitly, this correspondence is defined by

$$y_{123456} \longleftrightarrow \det M \tag{2.6}$$

where M is the matrix in (2.5). More intrinsically, the bijection is characterized by the property that the vanishing locus of the quintic associated to an anticanonical triangle is the locus of marked cubic surfaces where the triangle degenerates to three concurrent lines.

3. The anticanonical embedding

In this section, we give an explicit $\mathrm{W}(\mathrm{E}_6)$ equivariant description of the anti-canonical map of the universal cubic surface, and use this to describe the tropicalized anti-canonically embedded universal cubic surface. The following field extensions of $\mathbb{K}$ will play a prominent role:

$$F := \mathbb{K}(\mathcal{Y}_i \colon 0 \leq i \leq 39) \subset L := \mathbb{K}(\mathfrak{h}_6) = \mathbb{K}(d_1, \ldots, d_6). \tag{3.1}$$

Here, the parameters $d_1, \ldots, d_6$ are algebraically independent over $\mathbb{K}$ and give explicit expressions for the lattice $\mathfrak{h}_6$, as discussed in 2.2. Geometrically, the extensions in (3.1) are characterized as follows: F is the fraction field of $M_{m,0}$ and L is the fraction field of $\mathfrak{h}_6$. The inclusion $F \hookrightarrow L$ is induced by the uniformization map $\mathbb{P}(\mathfrak{h}_6) \dashrightarrow M_{m,3}$ defined in Subsection 2.2.

The key input in our description is the explicit presentation and tropicalization of the Cox ring of the universal del Pezzo surface in [3]. We begin by recalling this description for cubic surfaces. Let $X_F \to \operatorname{Spec} F$ the universal marked cubic surface. Let E_i, F_{ij}, and G_j be the 27 exceptional curves of X_F as defined in Subsection 2.3. We have an isomorphism

$$\operatorname{Pic}(X_F) = \mathbb{Z}\langle H, E_1, \ldots, E_6\rangle,$$

where H is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the map $X_F \to \mathbb{P}^2_F$ that blows down $E_1, \ldots, E_6$.

Recall that the *Cox ring* of X_F is the $\mathbb{Z}^7$ graded F-algebra

$$\operatorname{Cox}(X_F) = \bigoplus_{(n_0,\ldots,n_6)\in\mathbb{Z}^7} H^0(X_F, n_0K_X + n_1E_1 + \cdots + n_6E_6).$$

Each effective divisor in X_F gives an element of $\operatorname{Cox}(X_F)$, well-defined up to scaling. Let S be the 27 element set consisting of variables E_i for $1 \leq i \leq 6$ and F_{ij} for $1 \leq i \neq j \leq 6$ (following the ordering conventions in Subsection 2.3), which we think of as the set of markings of the 27 exceptional curves on X_F. This identification endows S with an $\mathrm{W}(\mathrm{E}_6)$ action.

Theorem 3.1 ([3, Proposition 2.2])**.** *We have a* $\mathrm{W}(\mathrm{E}_6)$ *equivariant surjection*

$$L[S] \to \operatorname{Cox}(X_L)$$

that sends a variable $E \in S$ *to a generator of* $H^0(X_L, \mathcal{O}(E))$*. The kernel is generated by a* $\mathrm{W}(\mathrm{E}_6)$ *invariant set of 270 quadratic trinomials (up to sign). Explicitly, the generators are the* $\mathrm{W}(\mathrm{E}_6)$ *conjugates of the following:*

$$(d_3 - d_4)(d_1 + d_3 + d_4)E_2F_{12} - (d_2 - d_4)(d_1 + d_2 + d_4)E_3F_{13} + (d_2 - d_3)(d_1 + d_2 + d_3)E_4F_{14}.$$

Note that $L[S]$ has two natural gradings. The first one is $\mathbb{Z}$-valued, and each variable has degree 1. The other one is $\mathbb{Z}^7$-valued, and is induced from the $\mathbb{Z}^7$-grading on $\operatorname{Cox}(X_L)$:

$$\deg E_i := e_i \ ; \quad \deg G_i := 2e_0 - \sum_{l=1}^{6} e_l + e_i \ \ (1 \leq i \leq 6) \ ; \ \ \deg F_{ij} := e_0 - e_i - e_j \quad (1 \leq i < j \leq 6). \tag{3.2}$$

Notice that both grading inducee natural torus actions over L by the corresponding lattice. The choice of grading used will be clear from context.

The *anti-canonical* ring of X_F is the $\mathbb{Z}$-graded F algebra

$$A(X_F) = \sum_{n\in\mathbb{Z}} H^0(X_F, -nK_X).$$

Since K_X is anti-ample, the nonzero graded components of $A(X_F)$ are in non-negative degrees.

Following the order conventions in Subsection 2.3 we let T be the 45 element set consisting of 30 variables x_{ij} for $1 \leq i \neq j \leq 6$ and 15 variables y_{ijklmn} for distinct tripartitions $\{\{i,j\},\{k,l\},\{m,n\}\}$ of $\{1,\dots,6\}$. We view T as the set of markings of the 45 anticanonical triangles on X_F. This identification yields an action of $\mathrm{W}(\mathrm{E}_6)$-action on T.

Theorem 3.2. *We have an* $\mathrm{W}(\mathrm{E}_6)$ *equivariant surjection*

$$L[T] \to A(X_L).$$

The ideal is generated by an $\mathrm{W}(\mathrm{E}_6)$ *equivariant set of 270 linear trinomials (up to sign) and 120 cubic binomials (up to sign). Explicitly, the linear trinomials are the* $\mathrm{W}(\mathrm{E}_6)$ *conjugates of the following*

$$(d_3-d_4)(d_1+d_3+d_4)x_{21} - (d_2-d_4)(d_1+d_2+d_4)x_{31} + (d_2-d_3)(d_1+d_2+d_3)x_{41},$$

and the cubic binomials are the $\mathrm{W}(\mathrm{E}_6)$*-conjugates of the following*

$$y_{123456}y_{142536}y_{162345} - y_{123645}y_{162534}y_{142356}.$$

Proof. Since the anti-canonical map embeds X_L as a cubic hypersurface in $\mathbb{P}^3_L$, we know that the anti-canonical ring is isomorphic (abstractly) to $L[X,Y,Z,W]/Q$, where Q is a cubic polynomial. In particular, it is generated by the degree 1 graded component.

Denote the images of E_i, F_{ij}, G_j by e_i, f_{ij}, g_j, respectively. Denote graded components by subscripts and set $a = (3,-1,-1,-1,-1,-1,-1)$. We have

$$L[S]_a = L\langle E_iF_{ij}G_j, F_{ij}F_{kl}F_{mn}\rangle.$$

From Theorem 3.1, we have a surjection $L[S]_a \to \mathrm{Cox}(X_L)_a$, and by definition we have an equality $\mathrm{Cox}(X_L)_a = A(X_L)_1$. Since $A(X_L)_1$ generates $A(X_L)$, we conclude that the elements $e_if_{ij}g_j$ and $f_{ij}f_{kl}f_{mn}$ generate $A(X_L)$. In other words, the map $L[T] \to A(X_L)$ is surjective.

Let I be the kernel of $L[T] \to A(X_L)$ and J the kernel of $L[S] \to \mathrm{Cox}(X_L)$. We have the diagram

$$\begin{tikzcd} L[T]_1 \arrow[r, equal] \arrow[d] & L[S]_a \arrow[d] \\ A(X_L)_1 \arrow[r, equal] & \mathrm{Cox}(X_L)_a, \end{tikzcd} \tag{3.3}$$

and therefore we have $I_1 = J_a$. But for each of the 270 quadric generators q of J, there is a unique variable v such that vq lies in J_a. For example, for the quadric generator listed in Theorem 3.1, it is the variable G_2. Therefore, the component J_a is spanned by the $\mathrm{W}(\mathrm{E}_6)$ conjugates of

$$(d_3-d_4)(d_1+d_3+d_4)E_2F_{12}G_2 - (d_2-d_4)(d_1+d_2+d_4)E_3F_{13}G_2 + (d_2-d_3)(d_1+d_2+d_3)E_4F_{14}G_2.$$

These turn into the linear equations claimed above.

It is easy to check that the cubic $y_{123456}y_{142536}y_{162345} - y_{123645}y_{162534}y_{142356}$ is in the kernel of $L[T] \to A(X_L)$. Therefore, so are its conjugates. Since we know that the anti-canonical ideal is principal modulo the linear polynomials, any cubic that is nonzero modulo the linear polynomials generates the ideal. By evaluating at a generic choice of d_i, we check that this is indeed the case for $y_{123456}y_{142536}y_{162345} - y_{123645}y_{162534}y_{142356}$. ∎

Anand: Strengthen the above to a statement on an explicit open subset of the moduli space? Complement of roots and quintics?

Remark 3.3. A simple inspection at the defining equations of X_L confirms that the equivariant map from Theorem 3.2 is compatible with the $\mathbb{Z}$- and $\mathbb{Z}^7$-grading on $L[T]$ and $A(X_L)$, respectively.

Theorem 3.1 and Theorem 3.2 describe the universal Cox ring and the universal anti-canonical ring after a base change from F to L. We now make a simple change of variables that allows us to describe the anti-canonical ring over F, without the need to base change to L. Recall from (2.6) that we have a $\mathrm{W}(\mathrm{E}_6)$ equivariant bijection between the set T and the set of quintic factors of Cross functions (up to sign). Make a choice of signs once and for all. Denote the quintic corresponding to x_{ij} by Q_{ij} and the quintic corresponding to y_{ijklmn} by Q_{ijklmn}. Set

$$X_{ij} = x_{ij}/Q_{ij} \text{ and } Y_{ijklmn} = y_{ijklmn}/Q_{ijklmn}. \tag{3.4}$$

Let us abuse notation slightly and use the same symbol T for set the variables X_{ij} and Y_{ijklmn}. Note that $\mathrm{W}(\mathrm{E}_6)$ acts on the old set of variables $\{x_{ij}, y_{ijklmn}\}$ by permutations and it acts on the set of 45 quintics $\{Q_{ij}, Q_{ijklmn}\}$ by signed permutations. Therefore, it acts on the new set of variables $\{X_{ij}, Y_{ijklmn}\}$ also by signed permutations. The signs depend on the signs chosen in the bijection betwen T and the set of quintics.

After rescaling the variables as above, the linear polynomial in Theorem 3.2 takes the form

$$(d_3 - d_4)(d_1 + d_3 + d_4)Q_{21}X_{21} - (d_2 - d_4)(d_1 + d_2 + d_4)Q_{31}X_{31} + (d_2 - d_3)(d_1 + d_2 + d_3)Q_{41}X_{41}.$$

After multiplying throughout by $(d_5 - d_6)(d_1 + d_5 + d_6)$, it turns out that all three coefficients become Cross functions. We can then write the linear polynomial as

$$\mathrm{Cross}(S_1)X_{21} - \mathrm{Cross}(S_2)X_{31} + \mathrm{Cross}(S_3)X_{41}, \tag{3.5}$$

where the Cross functions are those associated to the root subsystems of type $A_1^{\oplus 4}$ given by the four factors of

$$\begin{aligned} S_1 &= (d_3 - d_4)(d_5 - d_6)(d_1 + d_3 + d_4)(d_1 + d_5 + d_6),\\ S_2 &= (d_2 - d_4)(d_5 - d_6)(d_1 + d_2 + d_4)(d_1 + d_5 + d_6),\\ S_3 &= (d_2 - d_3)(d_5 - d_6)(d_1 + d_2 + d_3)(d_1 + d_5 + d_6). \end{aligned} \tag{3.6}$$

The cubic polynomial in Theorem 3.2 undergoes similar transformation. After rescaling the variables we obtain the cubic

$$Q_{123456}Q_{142536}Q_{162345}y_{123456}y_{142536}y_{162345} - Q_{123645}Q_{162534}Q_{142356}y_{123645}y_{162534}y_{142356}.$$

We now mutiply throughout by the following product of degre 12

$$\begin{aligned} P = &(d_1 + d_2 + d_3 + d_4 + d_5 + d_6)^3(d_1 - d_2)(d_3 - d_4)(d_5 - d_6)\\ &(d_1 - d_4)(d_2 - d_5)(d_3 - d_6)(d_1 - d_6)(d_2 - d_3)(d_4 - d_5). \end{aligned}$$

This P can be written as a product of 3 quartics in two ways:

$$\begin{aligned} P &= F_1F_2F_3\\ &= (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_2)(d_3 - d_4)(d_5 - d_6)\\ &\quad (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_4)(d_2 - d_5)(d_3 - d_6)\\ &\quad (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_6)(d_2 - d_3)(d_4 - d_5), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} P &= G_1G_2G_3\\ &= (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_2)(d_3 - d_6)(d_4 - d_5)\\ &\quad (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_6)(d_2 - d_5)(d_3 - d_4)\\ &\quad (d_1 + d_2 + d_3 + d_4 + d_5 + d_6)(d_1 - d_4)(d_2 - d_3)(d_5 - d_6) \end{aligned} \tag{3.8}$$

with the property that each of the degree 9 polynomials F_1Q_{123456}, F_2Q_{142356}, F_3Q_{162345}, G_1Q_{123645}, G_2Q_{162534}, G_3Q_{142356} is a Cross function. As a result, we can write the cubic polynomial as

$$\mathrm{Cross}(F_1)\,\mathrm{Cross}(F_2)\,\mathrm{Cross}(F_3)Y_{123456}Y_{142356}Y_{123645}-\mathrm{Cross}(G_1)\,\mathrm{Cross}(G_2)\,\mathrm{Cross}(G_3)Y_{123645}Y_{162534}Y_{142356}, \tag{3.9}$$

where $\mathrm{Cross}(P)$ denotes the Cross associated to the sub root system of type $A_1^{\otimes 4}$ spanned by the four factors of the quartic P.

We can now describe the universal anticanonical ring on the moduli space, without having to pass to a uniformization.

Theorem 3.4. *We have an* $\mathrm{W}(\mathrm{E}_6)$ *equivariant surjection*

$$F[T] \to A(X_F)$$

whose ideal is generated by an $\mathrm{W}(\mathrm{E}_6)$ *equivariant set of 270 linear trinomials (up to sign) and 120 cubic binomials (up to sign). Explicitly, the linear trinomials are the* $\mathrm{W}(\mathrm{E}_6)$ *conjugates of the following*

$$\mathrm{Cross}(S_1)X_{21} - \mathrm{Cross}(S_2)X_{31} + \mathrm{Cross}(S_3)X_{41},$$

and the cubic binomials are the $\mathrm{W}(\mathrm{E}_6)$ *conjugates of the following*

$$\mathrm{Cross}(F_1)\,\mathrm{Cross}(F_2)\,\mathrm{Cross}(F_3)Y_{123456}Y_{142356}Y_{123645}-\mathrm{Cross}(G_1)\,\mathrm{Cross}(G_2)\,\mathrm{Cross}(G_3)Y_{123645}Y_{162534}Y_{142356}.$$

Above, $\mathrm{Cross}(P)$ *denotes the Cross function associated to the root system of type* $A_1^{\oplus 4}$ *spanned by the four linear factors of the quartic polynomial* P*. The quartics* S_i *are defined in* (3.6)*,* F_j *in* (3.7)*, and* G_k *in* (3.8)*.*

Note that the Cross functions and the variables X_{ij} and Y_{ijklmn} both involve the choice of a sign, which has been suppressed in the statement of Theorem 3.4.

We compute the Yoshida functions, the Cross functions, the universal anti-canonical ring, and the $\mathrm{W}(\mathrm{E}_6)$ action on these objects explicitly using `Sage`; the results are collected in **??**. In the notation used there, the linear polynomial in Theorem 3.4 is

$$\mathrm{Cross}_{116}\,X_{21} - \mathrm{Cross}_2\,X_{31} + \mathrm{Cross}_{19}\,X_{41},$$

which can be written in terms of the Yoshida functions as

$$(\mathcal{Y}_3 - \mathcal{Y}_{37})X_{21} - (\mathcal{Y}_{20} - \mathcal{Y}_8)X_{31} + (\mathcal{Y}_3 - \mathcal{Y}_5)X_{41}. \tag{3.10}$$

The cubic polynomial in Theorem 3.4 is

,

which can be written in terms of the Yoshida functions as

.

From these two, the anti-canonical ideal can be computed by applying the $\mathrm{W}(\mathrm{E}_6)$ action. The action is also given explicitly in **??**.

4. The Bergman fan of E_6 and the tropical Naruki space

The monomial Yoshida map

$$\Upsilon\colon \mathbb{G}_m^{36}/\mathbb{G}_m \to \mathbb{G}_m^{40}/\mathbb{G}_m. \tag{4.1}$$

Maria: Can you add a remark along these lines, with a reference to the Supplementary material?

Remark 4.1. rank of the Yoshida matrix is 16. The exponents of the Yoshida functions give a sublattice of $\mathbb{Z}^{36}$ of index 3.

Maria: Write sample vector in the left kernel.

5. Combinatorial types of anticanonical tropical del Pezzo cubic surfaces

In this section, we study the combinatorial types of smooth tropical del Pezzo cubics in $\mathbb{P}^{44}$ without Eckardt points induced by the anticanonical embedding. Our main results says that these types agree with the types induced by Cox embedding. Furthermore, they are determined by the polyhedral structure of the Naruki fan [3, Table 1].

In Section 3 we presented the anticanonical map of the universal cubic surface over the field L from (3.1). Theorem 3.2 relates the Cox and anticanonical embeddings of the universal cubic surface X_L over L by means of an $\mathrm{W}(\mathrm{E}_6)$-equivariant monomial map with non-negative exponents. Each smooth del Pezzo cubic X with no Eckardt points defined over the field $\mathbb{K}$ is obtained by specialization of the parameters $d_1, \ldots, d_6$ at elements in $\mathbb{K}$ outside the vanishing locus of the product of all Yoshida and Cross functions from (2.3) and (2.4). Our choice of markings T and S for the 45 anticanonical triangles and the 27 extremal curves together with Theorem 3.2 yield a degree 3 map

$$\alpha\colon \operatorname{Spec}(\mathbb{K}[S]) \to \operatorname{Spec}(\mathbb{K}[T]). \tag{5.1}$$

The $\mathrm{W}(\mathrm{E}_6)$-monomial map α is defined by a rank 21 matrix A of size 45×27 over $\mathbb{Z}$ having only 0/1 entries and three nonzero entries per row. The matrix is recorded in the Supplementary material. By Remark 3.3, this map is compatible with the gradings on the coordinate rings, which we now recall.

The natural gradings on $L[T]$ and $A(X_F)$ discussed in Section 3 induce natural torus actions on $\mathbb{K}[S]$ and $\mathbb{K}[T]$. More precisely, the action of the 7-dimensional multiplicative split torus $\mathbb{G}_m^7$ over $\mathbb{K}$ on $\operatorname{Spec}(\mathbb{K}[T])$ is determined by the $\mathbb{Z}^7$-grading (3.2) as follows:

$$\underline{t} * E_i = t_i E_i\ ; \quad \underline{t} * G_i = t_0^2 \Big(\prod_{k \neq i} t_k\Big)^{-1} G_i \quad (1 \leq i \leq 6)\ ; \quad \underline{t} * F_{ij} = t_0 (t_i t_j)^{-1} F_{ij}\ (1 \leq i < j \leq 6). \tag{5.2}$$

The rank 7 sublattice Λ of $\mathbb{Z}^{27}$ inducing this action is saturated and contains the all-ones vector. At the cocharacter level, the all-ones vector action is obtained by the cocharacter $t_0^3 t_1 \cdots t_5$. The grading induced by the latter identifies $\operatorname{Proj}(\mathbb{K}[T])$ with $\mathbb{P}^{26}$. It inherits an action by the torus $\mathbb{G}_m^7/\mathbb{G}_m$. The action of t_0 on the 45 anticanonical coordinates in $\mathbb{K}[T]$ is given by scaling by t_0^3. The projectivization $\operatorname{Proj}(\mathbb{K}[S])$ is a $\underline{3}$-weighted projective space, which we denote by $\mathbb{P}^{44}$.

The monomial map α from (5.1) is compatible with the torus actions discussed above and induces a monomial degree 3 map on the quotient space.

$$\overline{\alpha}\colon \mathbb{P}^{26}/(\mathbb{G}_m^7/\mathbb{G}_m) \simeq \operatorname{Proj}(\mathbb{K}[S])/(\mathbb{G}_m^7/\mathbb{G}_m) \to \operatorname{Proj}(\mathbb{K}[T]) \simeq \mathbb{P}^{44}. \tag{5.3}$$

Furthermore, for any choice of generic values of $d_1, \ldots, d_6$ in $\mathbb{K}$, the corresponding del Pezzo cubic X embeds in $\mathbb{P}^{26}/(\mathbb{G}_m^7/\mathbb{G}_m)$. Anand: Should we cite [RSS] and say that we take a compactification of their embedding? The map $\overline{\alpha}$ yield an embedding of the quotient del Pezzo cubic

$$\overline{\alpha}\colon X \hookrightarrow \mathbb{P}^{26}/(\mathbb{G}_m^7/\mathbb{G}_m) \to \mathbb{P}^{44}. \tag{5.4}$$

Remark 5.1. Notice that the choice of coordinates on $\mathbb{P}^{44}$ is given by the marking T, rather than by the Yoshida-adapted variables (3.4). The tropicalization of X in $\mathbb{TP}^{44}$ induced by the latter will be obtained by translating the tropicalization of X induced by the marking T by the image of the vector $(\operatorname{val}(Q_{ij}), \operatorname{val}(Q_{ijklmn})\colon ij,\, ijklmn) \in \mathbb{R}^{45}/\mathbb{R}\cdot\mathbf{1}$. This simple operation preserves the combinatorial types.

We now turn our attention to the combinatorial types of anticanonical tropical del Pezzo cubics in $\mathbb{TP}^{44}$, answering a question of Ren, Shaw and Sturmfels [3, Section 5]. The following statement shows that these types match the classification for the Cox embedding described in [3, Table 1]. The remainder of this section will be devoted to its proof.

Theorem 5.2. *The combinatorics of tropical smooth del Pezzo cubics without Eckardt points in* $\mathbb{TP}^{44}$ *is completely determined by the line arrangement at infinity. They agree with the types induced by the Cox embedding and hence are classified by the Naruki fan.*

Proof. By Remark 5.1 it suffices to show the statement for the embedding induced by the marking T. The result follows by translating the classification of tropical cubics induced by the Cox embeeding [3, Table 1] to the anticanonical embedding in $\mathbb{P}^{44}$ using the map $\overline{\alpha}$ from (5.3). Proposition 5.5 shows the combinatorics of both tropicalizations is the same. Lemma 5.4 describes the boundary of the tropical surface as a line arrangement at infinity. The characterization of $\mathcal{T}X$ in terms of the line arrangement at infinity follows from [3, Lemma 3.3] and Proposition 5.5. □

We start by discussing the structure of the boundary of $\mathcal{T}X \subset \mathbb{TP}^{44}$. Our embedding characterizes it as an arrangement of tropical lines, as we now explain. By construction, the intersection of $\mathcal{T}X$ and the hyperplane indexed by an anticanonical triangle is the union of the tropicalization of the lines associated to the three constituent vertices of the triangle in the Schläfli graph. For example, $\mathcal{T}X \cap \{X_{12} = \infty\} = \mathcal{T}E_1 \cup \mathcal{T}F_{12} \cup \mathcal{T}G_2$. Each of these tropical lines at infinity is a metric balanced tree with prescribed directions for its leaf edges. We refer to Definition 7.1 for more precisions. Thus, the boundary of $\mathcal{T}X$ is an *arrangement of trees*.

Remark 5.3. In the absence of Eckardt points, any point in a fixed classical line in the cubic surface $X \subset \mathbb{P}^{44}$ lies in exactly 5 or 9 coordinate hyperplanes in $\mathbb{TP}^{44}$. The number depends on the nature of the point. For example, the non-nodal points in E_1 lie in exactly 5 hyperplanes, whereas the node $E_1 \cap F_{12}$ lies in the intersection of 9 hyperplanes indexed by $\{X_{1k} \colon k \neq 1\} \cup \{X_{21}\} \cup \{Y_{123456}, Y_{123546}, Y_{123645}\}$. The action of $\mathrm{W(E_6)}$ allows us to extend the count from E_1 to the remaining 26 lines.

Our first result shows that the combinatorics of the tree arrangement at infinity in $\mathcal{T}X$ matches that of the intersection complex of the 27 lines in X. For this reason, we refer to an intersection point between two boundary tropical lines as a *nodal point* of the boundary of $\mathcal{T}X$.

Lemma 5.4. *Let X be a smooth cubic del Pezzo surface without Eckardt points viewed in $\mathbb{P}^{44}$ via the anticanonical embedding. Then, the 27 classical lines in X tropicalize to distinct trees in $\mathbb{TP}^{44}$. Furthermore, two such trees intersect if and only if their classical counterparts do.*

Proof. Each tropicalization of a classical line lies in the intersection of the 5 hyperplanes determined by the anticanonical triangles containing the corresponding line. For example, the line E_1 is contained in precisely the 5 hyperplanes determined by the vanishing of $X_{12}, \ldots, X_{16}$. The data corresponding to the remaining 26 extremal curves can be obtained from the action of $\mathrm{W(E_6)}$. These 27 5-tuples of hyperplanes are distinct, and therefore, so are the 27 trees.

The statement regarding the pairwise intersection of all trees follows from the fact that if two classical lines do not meet, then the set of anticanonical triangles containing each one of them is disjoint. Our previous discussion characterizing a tree in terms of the 5 hyperplanes containing it implies that any intersection point between the tropicalization of two disjoint lines will have at least 10 infinity coordinates. This count dissagrees with the exact number of hyperplanes containing any point in the boundary of $\mathcal{T}X$ in the absence of Eckardt points stated in Remark 5.3. We conclude that the intersection complex of the boundary tropical lines and the 27 classical lines agree. □

Lemma 5.4 shows that the intersection complex of the tree arrangement at infinity is encoded in the *Schläfli graph*, just as it happened with tropicalization with respect to the Cox embedding [3, Section 2]. In particular, it allows us to label the leaves of each tree by the collection of hyperplanes containing it. Each node lies in 9 hyperplanes by Remark 5.3. The topological type of each tree depends on the polyhedral structure of the Naruki fan [3, Section 5]. We analyze their combinatorics in Section 6.

We end this section by comparing the combinatorial types of tropical smooth tropical del Pezzo cubics with no Eckardt points induced by the Cox and anticanonical embeddings. Following 5.1, we assume the anticanonical embedding of the classical smooth del Pezzo cubic X in $\mathbb{P}^{44}$ is associated to the marking T.

Under tropicalization, the action of $\mathbb{G}_m^7/\mathbb{G}_m$ on $\mathbb{P}^{26}$ induced by the lattice $\overline{\Lambda} := \Lambda/\mathbb{Z}{\cdot}\mathbf{1}$ gives rise to the quotient space $\mathbb{TP}^{44}/\overline{\Lambda}_{\mathbb{R}}$, where $\overline{\Lambda}_{\mathbb{R}} := \overline{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}$. By functoriality with respect to monomial maps, the tropicalization of $\overline{\alpha}$ yields a linear map

$$\operatorname{trop}(\overline{\alpha})\colon \mathbb{TP}^{26}/\overline{\Lambda}_{\mathbb{R}} \to \mathbb{TP}^{44} \tag{5.5}$$

having the same associated 45×27-matrix A as (5.1). The map $\operatorname{trop}(\overline{\alpha})$ is well defined since the preimage of $\mathbb{Z}{\cdot}\mathbf{1}$ under A is the lattice Λ.

The following result ensures that the combinatorics of $\mathcal{T}X \subset \mathbb{TP}^{26}/\overline{\Lambda}_{\mathbb{R}}$ are preserved under the tropical map:

Proposition 5.5. *The tropical map* $\operatorname{trop}(\overline{\alpha})$ *from* (5.5) *is is injective on each compact tropical del Pezzo cubic* $\mathcal{T}X$.

Proof. The definition of the map α is compatible with the boundary structure on the source and target spaces by Remark 5.3 and Lemma 5.4. This compatibility is preserved under tropicalization. In particular, the preimages of distinct strata of $\mathbb{TP}^{44}$ are disjoint strata of $\mathcal{T}X \subset \mathbb{TP}^{26}/\overline{\Lambda}_{\mathbb{R}}$. Thus, it suffices to check injectivity on each strata of $\mathcal{T}X$ induced by the structure of $\mathbb{TP}^{26}/\overline{\Lambda}_{\mathbb{R}}$.

We start by discussing injectivity on the big open cell $(\mathbb{R}^{27}/\mathbb{R}\cdot 1)/\overline{\Lambda}_{\mathbb{R}}$. This follows from the preceeding discussion and (5.1) because

$$\ker(\operatorname{trop}(\alpha^{\#})) \cap (\mathbb{R}^{27}/\mathbb{R}{\cdot}1) = \ker(A) \cap (\mathbb{R}^{27}/\mathbb{R}{\cdot}1) = \overline{\Lambda}_{\mathbb{R}}.$$

This identity is checked by a simple matrix multiplication in the Supplementary material.

By Lemma 5.4 the recession fan of $\mathcal{T}X$ in both $\mathbb{TP}^{26}/\overline{\Lambda}_{\mathbb{R}}$ and $\mathbb{TP}^{44}$ is the cone over the Schläfli graph. Hence, points in the boundary of $\mathcal{T}X$ lies in at most two hyperplanes at infinity. Points in two hyperplanes at infinity correspond to the tropicalization of the intersection of two extremal curves in X. Up to the action of $\mathrm{W}(\mathrm{E}_6)$-there are two types of boundary strata, determined by the number of ∞-coordinates. We choose our two representatives as those associated to the curve E_1 and the pair (E_1, F_{12}), respectively. Keeping the notation from the proof of Theorem 3.2, they are defined inside $\mathcal{T}X$ by the conditions $e_1 = \infty$ and $e_1 = f_{12} = \infty$, respectively. The latter consists of a single point, so injectivity follows automatically.

On the relative interior of the strata of $\mathcal{T}X$ defined by $e_1 = \infty$, a point is defined by the remaining 26 coordinates. Its image under $\operatorname{trop}(\overline{\alpha})$ will have ∞ coordinates precisely at the 5 anticanonical triangles x_{1i} for $i = 2, \ldots, 6$. Thus, the image on the strata will be completely determined by the corresponding 36×26 submatrix A' of A associated to the remaining coordinates. A simple calculation available in the Supplementary material shows that the projection of the lattice Λ to $\mathbb{Z}^{26}$ is a saturated rank 7 lattice Λ' containing the all-ones vector. Furthermore, A' has rank 20 and it contains the $\mathbb{R}$-span of Λ'. We conclude that A' is injective on the quotient space $\mathbb{R}^{26}/\Lambda'_{\mathbb{R}}$. The same holds for $\operatorname{trop}(\overline{\alpha})$ and the boundary strata of $\mathcal{T}X$ induced by e_1, as we wanted to show. □

6. Combinatorial types of tree arrangements

Maria: Put a labeling lemma for the tree arrangement with a local picture here Maybe we can do it for the 2 generic types?

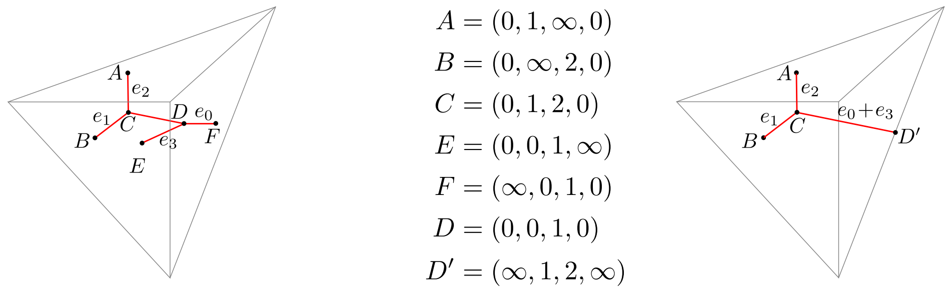

FIGURE 2. A generic tropical line in $\mathbb{TP}^3$ and a non-generic one associated to the partition $\{0, 3\} \sqcup \{1\} \sqcup \{2\}$ of $\{0, 1, 2, 3\}$.

7. Tropical lines in $\mathbb{TP}^{n-1}$

In this section we study tropical lines in $\mathbb{TP}^{n-1}$ and extend characterizations of collinearity from the tropical projective torus to the compact setting. This result will be crucial in Section 8. We start by recalling the definition of a tropical line in tropical projective space:

Definition 7.1. A *generic tropical line* in tropical projective space $\mathbb{TP}^{n-1}$ meeting the interior of $\mathbb{TP}^{n-1}$ is an embedded metric tree which is balanced with multiplicity 1 edges and has n leaf edges (those adjacent to a leaf of the tree) pointing into the coordinate directions. *Non-generic tropical lines* meeting the interior of $\mathbb{TP}^{n-1}$ are tree with $m \leq n$ leaves, and the leaf edges have directions $e_{B_j} := \sum_{i \in B_j} e_i$ for $j = 1, \ldots, m$, where the non-empty sets $B_1, \ldots, B_m$ partition the set $\{0, \ldots, n\}$.

The previous definition is nothing but the compactification of tropical lines in the tropical projective torus $\mathbb{R}^n/\mathbb{R}\cdot\mathbf{1}$. All leaves of such trees will be at the boundary of $\mathbb{TP}^{n-1}$. The collections $B_1, \ldots, B_m$ correspond to the coordinates with values ∞ of each of the m leaves. In particular, all leaves lie in the relative interior of distinct cells. Figure 2 gives an example of two tropical lines in $\mathbb{TP}^3$ meeting its interior.

Remark 7.2. We can extend Definition 7.1 to tropical lines in the boundary of $\mathbb{TP}^{n-1}$ by viewing an ambient boundary cell as a $\mathbb{TP}^{s-1} \subset \mathbb{TP}^{n-1}$ where the remaining coordinates are taken to be ∞.

Every tropical line in $\mathbb{TP}^{n-1}$ is realizable by a classical line in $\mathbb{P}^{n-1}$ (the proof in [**?**, Theorem 3.8] can be extended from the dense torus to the coordinate hyperplanes using Remark 7.2). Classically, we can easily determine when a finite set r of points in $\mathbb{P}^{n-1}$ is collinear. It suffices to build an $r \times n$-matrix and check that all 3×3 minors vanish. Foundational work on tropical linear algebra [**?**] shows that analogous statement holds for characterizing tropical lines in the tropical projective torus. The determinant of each minor is replaced by its tropical permanent, as we now define:

Definition 7.3. The *tropical permanent* of a matrix $S \in \overline{\mathbb{R}}^{d \times d}$ is defined by

$$\operatorname{perm}(S) = \min_{\sigma \in \mathfrak{S}_d} \{s_{1\sigma(1)} + \ldots + s_{d\sigma(d)}\}, \tag{7.1}$$

where $\mathfrak{S}_d$ denotes the set of permutations of $[d]$. The matrix S is called *tropically singular* if the minimum in (7.1) is achieved at least twice. Otherwise, we say S is *tropically non-singular.*

To simplify notation, we sometimes refer to the tropical permanent (as opposed to the matrix) as tropically singular or non-singular. The following lemma extends the characterization of tropical lines meeting the tropical projective torus to the compact setting.

Lemma 7.4. *Fix a collection $\mathcal{C}$ of r points in $\mathbb{TP}^{n-1}$ with pairwise disjoint ∞-entries. The collection $\mathcal{C}$ is tropically collinear if and only if all 3×3-minors of the associated $r \times n$-matrix with entries in $\overline{\mathbb{R}}$ are tropically singular.*

Proof. We let $\{p_1, \dots, p_r\}$ be the collection $\mathcal{C}$ of points and call M the tropical $r \times n$-matrix in the statement. When the points in $\mathcal{C}$ lie in the tropical projective torus, the matrix M has entries in $\mathbb{R}$, and the statement follows from [**?**, Corollary 3.8 and Theorem 6.5]. If we allow some of the points to lie in the boundary of $\mathbb{TP}^{n-1}$ the argument needs to be slightly modified. Our hypothesis on the ∞-entries of each p_i ensures that each coordinate hyperplane contains at most one point of $\mathcal{C}$. In particular, any tropical line containing them must meet the interior of $\mathbb{TP}^{n-1}$.

Suppose the collection $\mathcal{C}$ is tropically collinear, and let $\mathcal{T}L$ be the tropical line through its points. To use the criterion over the torus, we must replace $\mathcal{C}$ by r points with no ∞-coordinates. Every point p_i in the boundary of $\mathbb{TP}^{n-1}$ will be a leaf of our tree $\mathcal{T}L$. We replace each such p_i by a point p'_i in the leaf edge ending at p_i. If a point p_i has only real coefficients we set $p'_i = p_i$.

The new collection $\mathcal{C}'$ is contained in $\mathcal{T}L$ and the corresponding tropical matrix M' given by the entrywise valuations has only real entries. Therefore, all 3×3-minors of M' are tropically singular. As the points p'_i approach the original points, the tropical permanents of the submatrices of M' approach the corresponding tropical permanents for M, hence they are also tropically singular.

For the converse, we approximate our collection $\mathcal{C}$ by a collection $\mathcal{C}'$ with only real entries and with only tropically singular 3×3-minors. We do so as follows. After permutation of columns, we may assume that M has the form:

$$M = \left(\begin{array}{c|c|c|c|c}
\infty \ \dots \ \infty & * \ \dots \ * & \dots & * \ \dots \ * & * \ \dots \ * \\
\cline{1-2}
* \ \dots \ * & \infty \ \dots \ \infty & * \ \dots \ * & * \ \dots \ * & * \ \dots \ * \\
\cline{2-2}
* \ \dots \ \dots & \dots & \ddots & * \ \dots \ * & * \ \dots \ * \\
\cline{4-4}
* \ \dots \ \dots & \dots & * \ \dots \ * & \infty \ \dots \ \infty & * \ \dots \ * \\
\hline
* \ \dots \ * & * \ \dots \ * & * \ \dots \ * & * \ \dots \ * & * \ \dots \ * \\
\vdots \quad\ \vdots & \vdots \quad\ \vdots & \vdots \quad\ \vdots & \vdots \quad\ \vdots & \vdots \quad\ \vdots \\
* \ \dots \ * & * \ \dots \ * & * \ \dots \ * & * \ \dots \ * & * \ \dots \ *
\end{array}\right),$$

where the bottom right block matrix has size $s \times l$, with $s, l \geq 0$. The entries labeled with $*$ in M take only real values, as opposed to the entries marked with ∞'s.

We let B_i be the columns corresponding to the ∞-entries on the i-th row of M with $i = 1, \dots, r-s$. Since every row is an element of $\mathbb{TP}^{n-1}$ we can find an s_i in $\{0, \dots, n-1\}$ with $M_{is_i} \in \mathbb{R}$. If $|B_i| \geq 2$, working with the 3×3-tropical permanents involving 2 columns of B_i and the column s_i, we conclude that the $(r-1) \times |B_i|$-submatrix of M with rows in $[n] \smallsetminus \{i\}$ and columns in B_i has tropical rank 2, that is, all its 2×2-minors are tropically singular. In particular, the difference of any two columns in this submatrix is a multiple of the all-ones vector.

By removing all but the first column from each B_i and remembering the corresponding $\lambda \mathbf{1}$ differences with respect to the first column of B_i (for reconstruction purposes), we may assume $|B_i| = 1$ for all $i \leq r-s$ and $|B_i| = 0$ for $i > r-s$. We claim that the points in $\mathcal{C}$ will be tropically collinear if and only if the projection to the chosen $r-s$ coordinates is. Indeed, the tropical rank 2 condition on the columns indexed by B_i will guarantee that we can reconstruct a tropical line in $\mathbb{TP}^{n-1}$ through $\mathcal{C}$ from the projection to $\mathbb{TP}^{r-s-1}$, namely, that preimage will satisfy the required the balancing condition with multiplicity 1 and all leaf edges will have the mandatory directions.

We fix a large integer N and we let M' be the matrix obtained by replacing every ∞-entry of M with the number N, and consider $\mathcal{C}' := \{p'_i \colon 1 \leq i \leq r\}$ the corresponding set of r points in $\mathbb{TP}^{n-1}$. Our choice of N and the condition that $|B_i| = 0$ or 1 for all i guarantees that every 3×3-tropical permanent of M' takes the same value that the corresponding tropical permanent of M'. In particular, by our hypotheses, all the 3×3-tropical permanents of M' are also singular. Since M' has only real entries, we conclude that the configuration $\mathcal{C}'$ is tropically collinear.

We let $\mathcal{C}''$ be the configuration obtained by adding to $\mathcal{C}'$ all $r-s$ points $p'_i + \mu e_i$ where $\mu \geq 0$ and e_i is the ith canonical basis vector in $\mathbb{K}^n$. Our choice of N also ensures that all the 3×3-tropical permanents of $\mathcal{C}''$ are singular as well. Hence, the expanded configuration will remain collinear.

As we let μ go to ∞, the added $r-s$ points $p_i' + \mu e_i$ approach p_i for $i \leq r-s$ and furthermore, the sequence of tropical lines containing $\mathcal{C}''$ is ultimately constant. We conclude that $\mathcal{C}$ lies in the limiting tropical line. This concludes our proof. ☐

8. Tropical lines on anticanonical tropical cubic del Pezzo surfaces

In this section, we esturn our attention to the second guiding question in the paper, namely, the number of tropical lines on anticanonically embedded tropical cubic surfaces. Our main proof technique exploits the rigid structure of teh boundary of the tropical surface and builds candidate points in the boundary of potential tropical lines meeting the interior of the tropical surface. A tropical computation of ranks shows that in the non-trivially valued case, these points are never tropically collinear. In the trivially valued case, which we discuss in Section 9, this construction yields 27 additional tropical lines.

As in the previous sections, we let X be a smooth cubic del Pezzo surface without Eckardt points embedded in $\mathbb{P}^{44}$ via the anticanonical map, and we consider its induced tropicalization $\mathcal{T}X \subset \mathbb{TP}^{44}$. Our discussion in Section 5 reveals a key property of this embedding: the surface X intersects the 45 coordinate hyperplanes exactly at its 27 lines. Since the boundary of $\mathcal{T}X$ is the tropicalization of the boundary of X, we conclude that any tropical line in the boundary of $\mathcal{T}X$ must be supported on the arrangement of trees determined by the tropicalizations of the 27 classical lines.

The following is the main result in this section. To simplify the exposition, its proof will be provided by a series of auxiliary technical lemmas and propositions.

Theorem 8.1. *Let X be a smooth del Pezzo cubic without Eckardt points. Assume some Yoshida function on X has non-trivial valution. Then, the anticanonically embedded tropical del Pezzo cubic $\mathcal{T}X$ has exactly 27 tropical lines.*

Proof. Lemma 5.4 ensures that the boundary of $\mathcal{T}X$ contains exactly 27 tropical lines. Propositions 8.2, 8.9 and Lemma 8.10 imply that there are no tropical lines in $\mathcal{T}X$ meeting its interior. ☐

Since every tropical line in $\mathbb{TP}^{44}$ is realizable, we write our potential tropical line as $\mathcal{T}L$ for some line L in $\mathbb{P}^{44}$. Sine any extra line on $\mathcal{T}X$ meets its interior, our earlier discussion ensures that L meets the dense torus. The tropical line satisfies the following two properties:

(i) $\mathcal{T}L$ meets all 45 boundary hyperplanes in $\mathbb{TP}^{44}$, each one of them indexed by an anticanonical triangle. Each such intersection consists of one point.
(ii) $\mathcal{T}L$ intersects each boundary tropical line in at most one point.

Since $\mathcal{T}L$ is a tree with at most 45 leaves and the leaf edges have directions with disjoint support, there is at most one point in the intersection between $\mathcal{T}L$ and a given boundary hyperplane. Equality holds by construction. Condition (ii) is a direct consequence of $\mathcal{T}L$ meeting the interior of $\mathcal{T}X$ and the description of leaf-edge directions of $\mathcal{T}L$.

Our first technical result shows that any potential tropical line meeting the interior of $\mathcal{T}X$ has exactly 5 points in its boundary and they correspond to the intersection points between pairs of boundary tropical lines meeting a common third one. We refer to them as *nodal points*.

Proposition 8.2. *Let X be a smooth cubic del Pezzo surface without Eckardt points. Consider its tropicalization with respect to the anticanonical embedding. Then, there are at most 27 families of tropical lines meeting the interior of $\mathcal{T}X$. Each family is indexed by a given extremal curve in X, and all its members have the same 5 boundary points. Furthermore, these points are the tropicalization of the 5 nodes associated to the link of the indexing curve in the Schläfli graph.*

Proof. We write our potential tropical line as $\mathcal{T}L$ for some L meeting the dense torus of $\mathbb{P}^{44}$. Our first key observation is that the combinatorics of the boundary of $\mathcal{T}X$, encoded in the Schläfli graph by Lemma 5.4, make it impossible to simultaneously satisfy conditions (i) and (ii). A careful case by case analysis, which is described by Lemmas 8.3, 8.4 and 8.5, shows that $\mathcal{T}L$ meets every boundary tropical line of $\mathcal{T}X$ in at most one point. Furthermore, the point must be a node.

By Remark 5.3 we know that each node lies in exactly 9 boundary hyperplanes. Since all boundary points of $\mathcal{T}L$ must have disjoint sets of ∞-entries, we conclude that $\mathcal{T}L$ has at most 5 boundary points. By the action of $\mathrm{W}(\mathrm{E}_6)$ we may assume that one of them equals $\mathcal{T}E_1 \cap \mathcal{T}F_{12}$. Lemma 8.6 implies that the remaining 4 points in the boundary of $\mathcal{T}L$ are $\mathcal{T}E_i \cap \mathcal{T}F_{2i}$ for $i \neq 1, 2$. They come from the intersections of the 5 pairs of classical lines meeting G_2, so we use G_2 as an index for $\mathcal{T}L$. Notice that the 5 boundary points are complete determined by this label but the tropical line $\mathcal{T}L$ need not be unique. There are 27 such labels and they are all $\mathrm{W}(\mathrm{E}_6)$-conjugate. This concludes our proof. □

Lemma 8.3. *$\mathcal{T}L$ cannot meet 3 tropical boundary lines at non-nodal points.*

Proof. We argue by contradiction. Our hypothesis ensures that the three boundary lines are pairwise disjoint since, otherwise, their intersection point with $\mathcal{T}L$ would be a node. Without loss of generality, by the action of the Weyl group $\mathrm{W}(\mathrm{E}_6)$ we may assume they are $\mathcal{T}E_1$, $\mathcal{T}E_2$ and $\mathcal{T}E_3$.

We consider the anticanonical triangles X_{14}, X_{25} and X_{36}. Our hypothesis and (ii) ensures that $\mathcal{T}L$ cannot meet any of the three lines $\mathcal{T}F_{14}$, $\mathcal{T}F_{25}$ and $\mathcal{T}F_{36}$. We conclude that $\mathcal{T}L$ does not intersect the boundary hyperplane corresponding to Y_{123456}, thus violating condition (i). □

Lemma 8.4. *$\mathcal{T}L$ cannot meet 2 boundary tropical lines at non-nodal points.*

Proof. We argue by contradiction. As with Lemma 8.3, we know that the two boundary lines in the statement cannot intersect and we may assume they are $\mathcal{T}E_1$ and $\mathcal{T}E_2$. Since their intersection with $\mathcal{T}L$ is not a node and $\mathcal{T}L$ meets the hyperplanes indexed by X_{1*} and X_{2*}, condition (ii) implies that $\mathcal{T}L$ cannot intersection any of the boundary tropical lines $\mathcal{T}F_{1j}$ nor $\mathcal{T}F_{2l}$ where $j \neq 1$ and $l \geq 3$. Since it also intersects all the hyperplanes Y_{1k2l**}, we conclude that $\mathcal{T}L$ must intersect all tropical lines F_{ij} with $i, j \notin \{1, 2\}$.

An analogous argument shows that $\mathcal{T}L$ does not intersect any $\mathcal{T}G_k$ for $k = 1, \ldots, 6$. Looking at the triangle Y_{123456}, we conclude that $\mathcal{T}L$ meets the associated hyperplane at the node $\mathcal{T}F_{34} \cap \mathcal{T}F_{56}$. In turn, the intersection between $\mathcal{T}L$ and each of the three hyperplanes X_{32} and X_{34}, allows us to conclude not only that $\mathcal{T}L$ intersects E_3, but that the intersection point must be in $\mathcal{T}E_3 \cap \mathcal{T}F_{34}$ and $\mathcal{T}E_3 \cap \mathcal{T}F_{35}$. This cannot happen since $\mathcal{T}F_{34} \cap \mathcal{T}F_{56}$ and $\mathcal{T}E_3 \cap \mathcal{T}F_{34}$ are distinct points in $\mathcal{T}L \cap \mathcal{T}F_{34}$ by Lemma 5.4. □

Lemma 8.5. *$\mathcal{T}L$ cannot meet a boundary tropical line at a non-nodal point.*

Proof. We argue by contradiction. Using the action of $\mathrm{W}(\mathrm{E}_6)$ if necessary, we may assume $\mathcal{T}L$ meets $\mathcal{T}E_1$ at a non-nodal point. Our first claim is that $\mathcal{T}L$ must also intersect $\mathcal{T}E_2, \ldots, \mathcal{T}E_6$. Otherwise, by the action of $\mathfrak{S}_6$, we may assume $\mathcal{T}L$ does not meet $\mathcal{T}E_6$. By considering the hyperplanes X_{1i} for $i \neq 1$, we conclude from condition (ii) that

$$\mathcal{T}L \cap \mathcal{T}F_{1i} = \emptyset \qquad \text{and} \qquad \mathcal{T}L \cap \mathcal{T}G_i = \emptyset \quad \text{for all } i \neq 1. \tag{8.1}$$

On the contrary, since $\mathcal{T}L \cap \mathcal{T}E_6 = \emptyset$, the non-empty intersection between $\mathcal{T}L$ and the hyperplane X_{61} together with (8.1) guarantees that $\mathcal{T}L \cap \mathcal{T}G_1 \neq \emptyset$. Furthermore, Lemma 8.4 forces this intersection to be a nodal point of $\mathcal{T}G_1$. Again, (8.1) and the combinatorics of the anticanonical triangles restrict the nature of this node: it must lie in some $\mathcal{T}E_j$, which we may take as $\mathcal{T}E_2$.

Considering the hyperplanes X_{2i} for $i \neq 2$, conditions (i) and (ii) and the absence of Eckardt points implies that $\mathcal{T}L$ does not intersect any F_{2i} for $i \neq 2$. A similar argument using the hyperplane X_{31}, the properties on $\mathcal{T}G_1$ discussed above and (8.1) ensures that $\mathcal{T}L$ does not meet $\mathcal{T}E_3$.

Since $\mathcal{T}L \cap \mathcal{T}E_3 = \mathcal{T}L \cap \mathcal{T}G_2 = \mathcal{T}L \cap \mathcal{T}F_{23} = \emptyset$, we conclude that $\mathcal{T}L$ does not meet the hyperplane X_{32}, in disagreement with condition (i). Therefore,

$$\mathcal{T}L \cap \mathcal{T}E_i \neq \emptyset \quad \text{for all } i = 1, \ldots, 6. \tag{8.2}$$

The analysis for both hyperplanes X_{ij} and X_{ji} for all pairs $i \neq j$ together with (8.2) implies that $\mathcal{T}L \cap \mathcal{T}F_{ij} = \emptyset$. For if this were not the case, the intersection would necessary contain the nodal points $\mathcal{T}F_{ij} \cap \mathcal{T}E_i$ and $\mathcal{T}F_{ij} \cap \mathcal{T}E_j$, contradicting (ii).

Finally, our last claim implies that $\mathcal{T}L$ does not meet the hyperplane defined by Y_{123456}, violating condition (i). This concludes our proof. □

Lemma 8.6. *Let X be a smooth cubic del Pezzo without Eckardt points and let $\mathcal{T}L$ be a tropical line in $\mathbb{TP}^{44}$ meeting the interior of $\mathcal{T}X$. Assume $\mathcal{T}L$ meets the line $\mathcal{T}E_1$ at the node $\mathcal{T}E_1 \cap \mathcal{T}F_{1j}$ with $j \neq 1$. Then, $\mathcal{T}L$ has exactly 5 points in its boundary, namely $\mathcal{T}E_i \cap \mathcal{T}F_{ij}$ with $i \neq j$.*

Proof. By the action of $\mathfrak{S}_6$, we may assume that $j = 2$. Since $\mathcal{T}L$ intersects $\mathcal{T}F_{12}$ at $\mathcal{T}E_1 \cap \mathcal{T}F_{12}$, conditions (i) and (ii) ensure that $\mathcal{T}L$ cannot intersect $\mathcal{T}E_2$, $\mathcal{T}G_1$, $\mathcal{T}G_2$ nor any $\mathcal{T}F_{kl}$ with $k, l \notin \{1, 2\}$. Likewise, since $\mathcal{T}L$ intersects $\mathcal{T}E_1$ at the same node, we conclude that $\mathcal{T}L$ misses $\mathcal{T}F_{1k}$ and $\mathcal{T}G_k$ for all $k \neq 1, 2$.

There are 8 remaining boundary tropical lines to consider, namely $\mathcal{T}E_i$ and $\mathcal{T}F_{2i}$ for $i = 3, \ldots, 6$. Notice that they give rise to 4 nodal points, so $\mathcal{T}L$ itself has at most 5 boundary points. Since $\mathcal{T}L$ avoids both $\mathcal{T}E_2$ and $\mathcal{T}G_i$ for $i = 3, \ldots, 6$, but meets the boundary hyperplane X_{2i} for each $i = 3 \ldots, 6$, we conclude that $\mathcal{T}L$ meets all $\mathcal{T}F_{2i}$ for $i \neq 1, 2$. Similarly, analyzing the intersection between $\mathcal{T}L$ and the boundary hyperplane X_{ij} for a fix $i = 3, \ldots, 6$ and $j \notin \{1, 2, i\}$, it follows that $\mathcal{T}L$ meets $\mathcal{T}E_i$ for all $i \neq 1, 2$.

Finally, the nontrivial intersection between $\mathcal{T}L$ and each of the boundary hyperplanes X_{i2} for $i \neq 2$ ensures that $\mathcal{T}L$ meets both $\mathcal{T}E_i$ and $\mathcal{T}F_{2i}$ at their intersection point $\mathcal{T}E_i \cap \mathcal{T}F_{2i}$. Therefore, $\mathcal{T}L$ has exactly 5 boundary points and they have the desired description. □

Proposition 8.2 provides strong combinatorial conditions on potential tropical lines meeting the interior of $\mathcal{T}X$, by characterizing 27 possible five tuples of boundary points as the tropicalization of the five points in $\mathbb{P}^{44}$ associated to the link of a vertex in the Schläfli graph. Such tuples of points have coordinates in the function field associated to the 40 Yoshida functions from Subsection 2.3. The next result describes their coordinates, and its proof provides an algorithm for expressing them. An implementation in `Python`and `Sage` is available in the Supplementary material.

Lemma 8.7. *Let X be a smooth cubic del Pezzo surface with no Eckardt points anticanonically embedded in $\mathbb{TP}^{44}$. Then, each of the 135 nodes of X obtained as pairwise intersections between its 27 extremal curves has exactly 9 zero coordinates. The remaning ones are Laurent monomials in the Yoshida and Cross functions. Furthermore, each nonzero coordinate has at least one Cross function factor.*

Proof. By the action of $\mathrm{W}(\mathrm{E}_6)$, it is enough to show the validity of the statement for the 5 points associated to the link of G_2, namely $E_1 \cap F_{12}$ and $E_i \cap F_{2i}$ for $i \neq 1, 2$. By Remark 5.3, these points are characterized by the vanishing of precisely 9 anticanonical coordinates in the linear span of X obtained as the solution to the 270 linear equations (3.10).

Using `Sage` we compute a basis $\{v_0, \ldots, v_3\}$ of the 4-dimensional solution set to this linear system over the function field associated to the 40 Yoshida functions (see the Supplementary material). We encode it as a 4×45 matrix M all of whose entries are Laurent monomials in both the Yoshidas and Cross functions. Since these coordinates are not algebraically indepedent, we choose to work instead with the matrix M', whose entries are the rational functions in the parameters $d_1, \ldots, d_6$ expressing the corresponding entries in M. The new matrix is obtained from (2.3) and (2.4).

We obtain the precise coordinates for the classical node associated to a giving boundary point of $\mathcal{T}L$ by finding a generator of the 1-dimensional left-kernel of the corresponding 4×9-submatrix of M': this generators provides the scalars in the linear combination of the vectors v_i giving the node. For example, the point $E_1 \cap F_{12}$ coincides with the basis element v_3, but in general, the scalars will be rational functions in parameters $d_1, \ldots, d_6$. A `Python` script allows us to re-express these coordinates as rational functions in the Yoshidas functions.

By factorizing the 36 nonzero coordinates of $E_1 \cap F_{12}$, we certify the claim in the statement (see the Supplementary material). The factors are monomials and binomials in the Yoshidas, and these binomial expressions yield Cross functions. By acting via the transpositions $(1\,i)$ in $\mathfrak{S}_6$ we see that the remaining 4 nodes associated to the link of G_2 have the desired factorization. The action of $\mathrm{W}(\mathrm{E}_6)$ proves the statement for all vertices in the Schläfli graph. $\square$

Remark 8.8. The parameterization of the 135 nodes described in Lemma 8.7 in terms of Yoshida and Cross functions is solely obtained from the coordinates of the $\mathbb{P}^3$ linearly spanned by X, and makes no use of the binomial cubic equation cutting out the X in $\mathbb{P}^3$. Explicit coordinates for the 27 families of 5 nodes are available in the Supplementary material.

Lemma 7.4 provides a powerful tool to check when a finite set of points in $\mathbb{TP}^{n-1}$ is not collinear: it suffices to find a tropically non-singular 3×3-minor in the associated $r \times n$-matrix with entries in $\overline{\mathbb{R}}$. Our strategy to rule out any tropical line on $\mathcal{T}X$ beyond the tropicalization of the 27 classical lines in X is to find precise coordinates for the 5 boundary points on each potential tropical lines in the interior of $\mathcal{T}X$ and search for tropically non-singular 3×3-minors that show these five points are not collinear in $\mathbb{TP}^{44}$. This method discards any potential tropical line meeting the interior of $\mathcal{T}X$ whenever some Yoshida function has non-trivial valuation. A separate analysis will be required for the apex of the Naruki fan. We do this in Section 9.

As discussed in Section 4, the Naruki fan $\mathcal{N}$ in $\mathbb{R}^{40}$ encodes the combinatorial types of tropical smooth del Pezzo cubics with no Eckardt points in $\mathbb{TP}^{44}$ by means of the valuations of the 40 Yoshida functions. Proposition 8.2 expresses the 5 boundary points of the potential line $\mathcal{T}L$ as the tropicalizations of nodes in the boundary of the surface X. By Lemma 8.7, the coordinates of these nodes come in two flavors: 9 of them are zero, and the remaining 36 are Laurent monomials in the Yoshidas and Cross functions. Furthermore, Cross functions do appear in all nonzero coordinates.

The Cross functions, expressed as linear binomials in the 40 Yoshida functions yields the *Cross tropical hyperplane arrangement* in $\mathbb{R}^{40}$ generated by the single hyperplane $\mathcal{Y}_3 = \mathcal{Y}_{37}$ (associated to the Cross_{116} function) and its 134 $\mathrm{W}(\mathrm{E}_6)$-conjugates. The Cross arrangement is compatible with the fan structure of $\mathcal{N}$ discussed in Section 4. The valuation of each Cross function cannot be completely determined from a given point in $\mathcal{N}$ if the point lies in the tropical hyperplane determined by the Cross. Indeed, in the presence of ties between the summands, the valuation of the binomial expression can be higher than expected. Our previous observation allows us to check this condition uniformly on each cone of $\mathcal{N}$ using any point in its relative interior. We choose the sum of the rays spanning each cone as our witness point.

The uncertainty of the valuations of these 135 Cross functions makes it a priori impossible to determine all coordinates of the 5 boundary points of $\mathcal{T}L$ from the coordinates of the associated 5 classical nodes. To take this uncertainty into account, given a cone σ in $\mathcal{N}$ and an extremal curve C in X indexing a potential tropical line $\mathcal{T}L$, we compute two 5×45-matrices describing the 5 boundary points of $\mathcal{T}L$:

(1) a matrix M_{exp} with the expected valuations of each of the 5 classical nodes in $\mathbb{P}^{44}$ (assuming the valuation of a Cross factors is the minimum of the valuation of the constituent 2 Yoshida functions). The entries are linear functions in the valuations of the Yoshida functions. For a given point in rel int(σ) it gives an element in $\overline{\mathbb{R}}$;
(2) a matrix M_{true} whose (i,j) entries come in two types. Given σ, we conside the collection $\mathcal{C}$ indexing all Cross hyperplanes intersecting its relative interior. If the jth. coordinate of the ith. node in $\mathbb{P}^{44}$ does not contain any Cross function in $\mathcal{C}$, then the valuation of this coordinate is the expected one, so $M_{\mathrm{true}}[i,j] = M_{\mathrm{exp}}[i,j]$. Otherwise, its valuation cannot be determined. We record the entry of M_{true} as a '`None`'.

The values of these 27 pairs of matrices is constant along the relative interior of each cone in $\mathcal{N}$. Our implementation stores the values of these two matrices at the center of mass of each cone σ, so the entries of our matrices take values in $\mathbb{R} \cup \{\infty\} \cup \{\texttt{None}\}$. For each cone σ in $\mathcal{N}$, we collect the 27 pairs of (evaluated) matrices into a family

$$\mathcal{F}_\sigma := \{(M_{\exp}[C,\sigma], M_{\mathrm{true}}[C,\sigma]) \colon C \text{ extremal curve in } X\}. \tag{8.3}$$

We use the families $\mathcal{F}_\sigma$ to rule out all the potential extra tropical lines described in Proposition 8.2 for all cones in $\mathcal{N}$ except the apex. The method is described in algorithm 1 and its implementation in `Python` can be found in the Supplementary material. Most of the cones of $\mathcal{N}$ are covered by Proposition 8.9. The cone σ of type (a) requires a different strategy for 15 of the extremal curves. Lemma 8.10 discusses these special cases. The apex of $\mathcal{N}$ is analyzed in Section 9.

Proposition 8.9. *Assume some Yoshida function on X has non-trivial valuation. Given a vertex C of the Schläfli graph, we consider the 5 nodes on X associated to the link of C in this graph. Then, their tropicalizations in $\mathbb{TP}^{44}$ are not tropically collinear.*

Proof. Since not all 40 Yoshida functions on X have trivial valuation, we know the associated point p in $\mathcal{N}$ describing the combinatorial type of $\mathcal{T}X \subset \mathbb{TP}^{44}$ is not the apex. We let σ be the smallest cone containing the point p, and consider the family $\mathcal{F}_\sigma$ in (8.3) encoding the expected and true coordinatewise valuations of the 27 families of 5 nodes from Lemma 8.7.

Given an extremal curve C, algorithm 1 uses the pair of matrices $(M_{\exp}[C,\sigma], M_{\mathrm{true}}[C,\sigma])$ and a fixed 3-element set J of $\{0,\dots,5\}$ and searches for a 3-element set J' of $\{0,\dots,44\}$ giving a tropically non-singular minor of $M_{\mathrm{true}}[C,\sigma]$ with rows J and columns J' that agrees with the associated minor of $M_{\exp}[C,\sigma]$. We test this by checking the minor of $M_{\mathrm{true}}[C,\sigma]$ has no '`None`' entries. The existence of such a tropically non-singular minor together with Lemma 7.4 would prove that the tropicalization of the 5 nodes associated to C gives 5 not tropically collinear points in $\mathbb{TP}^{44}$. Whenever the output is the list `NonSingMinors` for some choice of J, our `Python` script checks the next set J in the lexicographic order and attempts to find 3 suitable columns. We repeat this process until all triples of rows have been tested.

For each non-apex cone σ in $\mathcal{N}$ and each extremal curve C in X, Table A.4 gives all instances where algorithm 1 succeeds. We record the information of rows and columns giving a tropically non-singular 3×3-minor of $M_{\mathrm{true}}[C,\sigma]$. With this method we find a tropically non-singular minor for all non-apex cones in $\mathcal{N}$ and all curves C with one exception: the combination of the (a) cone and the 15 extremal curves

$$\mathscr{F} := \{E_3, E_4, E_5, E_6, F_{12}, F_{34}, F_{35}, F_{36}, F_{45}, F_{46}, F_{56}, G_3, G_4, G_5, G_6\}. \tag{8.4}$$

In all these cases, Lemma 8.10 and algorithm 2 provides a tropically non-singular minor for the corresponding element in $\mathcal{F}_{(a)}$. This concludes our proof. ∎

The failure of algorithm 1 in providing a certifying minor with entries in $\overline{\mathbb{R}}$ for a pair in $\mathcal{F}_\sigma$ need not imply that the corresponding 5 points in $\mathbb{TP}^{44}$ are tropically collinear. An analysis of the entry patterns of tropically non-singular 3×3-minors of $M_{\exp}[C,(a)]$ (computed in the body of algorithm 1) gives many instances where the tropical permanents consist of two non-infinite terms but when evaluated in $M_{\mathrm{true}}[C,(a)]$ both terms involve '`None`' entries. Furthermore, up to permutations of rows and columns, the associated minors in $M_{\mathrm{true}}[C,(a)]$ have the form

$$\begin{pmatrix} * & \infty & \infty \\ \infty & * & \texttt{None} \\ * & * & \texttt{None} \end{pmatrix}, \tag{8.5}$$

where $*$ indicates an entry in $\mathbb{R}$. This shape indicates the nature of the 3 coordinates of 3 classical nodes in X. In particular, the third entries of the last 2 nodes involve Cross functions whose associated tropical hyperplanes contain the cone (a). The following lemma shows that, nonetheless, we can completely determine the valuation of their ratio. Therefore, these two '`None`' entries will

Algorithm 1: Ruling out a potential tropical line in $\mathbb{TP}^{44}$ meeting the interior of $\mathcal{T}X$.

Input: An ordered list J in $\{0,\dots,4\}$ of size 3 and a pair of 5×45 matrices $(M_{\text{exp}}, M_{\text{true}})$ in $\mathcal{F}_\sigma$ of expected and certain coordinates of 5 points in the boundary of $\mathcal{T}X \subset \mathbb{TP}^{44}$, for a fixed cone σ in the Naruki fan $\mathcal{N}$.

Assumption: X is a smooth del Pezzo cubic with no Eckardt points, embedding in $\mathbb{P}^{44}$ via the Yoshida-adapted anticanonical coordinates, where the valuation of the 40 Yoshida functions yield a point in the cone σ.

Output: A 3-element *subset* in $\{0,\dots,44\}$ giving the columns of a tropically non-singular 3×3 minor with rows in J for each pair $(M_{\text{exp}}, M_{\text{true}})$ in $\mathcal{F}_\sigma$, or a *list* of all 3-element subsets of $\{0,\dots,44\}$ giving tropically non-singular minors of M_{exp} if no tropically non-singular minor without '`None`' entries can be detected.

```
NonSingMinors ← [ ]
for J′ in Subsets([44], 3) do
    S_exp(J, J′) ← 3 × 3-submatrix of M_exp with rows J and columns J′;
    if S_exp(J, J′) is tropically non-singular then
        NonSingMinors ← NonSingMinors + [J′]
        S_true(J, J′) ← 3 × 3-submatrix of M_true with rows J and columns J′;
        if S_true(J, J′) = S_exp(J, J′) then
            return J′
return NonSingMinors.
```

not change the number of terms realizing the minimum in the tropical permanent, showing that the 3×3-minor of M_{true} is tropically non-singular.

Lemma 8.10. *Let C be a curve in the collection $\mathscr{F}$ from (8.4) and let N be the 5×45 matrix giving the 5 nodes associated to the link of C in the Schläfli graph. Then, there exists a pair of unordered 3-element sets J in $\{0,\dots,5\}$ and J' in $\{0,\dots,44\}$ satisfying the following three conditions:*

(i) the submatrix of $M_{\text{exp}}[C,(a)]$ with rows J and columns J' is tropically non-singular,
(ii) the submatrix of $M_{\text{true}}[C,(a)]$ with rows J and columns J' has the shape (8.5),
(iii) the expression $(\frac{N[J[1],J'[2]]}{N[J[2],J'[2]]})^3$ is a Laurent monomial in the Yoshida functions.

Proof. The discussion preceeding the statement shows why it is conceivable to find a pair (J,J') satisfying (i) and (ii). It gives rise to algorithm 2, which enables us to prove (iii) by explicit computations. A direct factorization of the expression in (iii) is not possible since the Yoshida functions are not algebraically independent for dimensional reasons. We are forced to express all relevant functions in terms of the parameters $d_1,\dots,d_6$.

From the shape (8.5) we deduce that the only entries in the submatrix of N with rows J and columns J' with undetermined valuations are $N[J[1],J'[2]]$ and $N[J[2],J'[2]]$. This means that both entries contain Cross functions in their factorization whose valuations cannot be determined from the cone (a). We let A (respectively B) be the product of all the Cross factors in $N[J[1],J'[2]]$ (resp. $N[J[2],J'[2]]$). A simple calculation shows that the ratio A/B is a Laurent monomial in the 36 positive roots of E_6.

By Remark 4.1 we know that the exponent vectors of the 40 Yoshida functions span a rank 16 sublattice of $\mathbb{Z}^{36}$ of index 3. A basis of this lattice is provided by the Yoshida functions indexed by the set $\mathcal{B} = \{5\}\cup\{17,\dots,31\}$ (see the Supplementary material). In particular, even though the ratio A/B need not be a Laurent monomial in the Yoshida functions indexed by $\mathcal{B}$, its cube will be. We certify this last step as follows. We factor $(A/B)^3$ as a Laurent monomial in the positive roots of E_6 and let v be its exponent vector in $\mathbb{Z}^{36}$. We solve the linear system of equations $M^t x = v^t$, where M is the 16×36-submatrix of the Yoshida matrix with rows in $\mathcal{B}$. If it exists, its unique solution will lie in $\mathbb{Z}^{16}$ and it will be the exponent vector giving the desired factorization of $(A/B)^3$. From this it follows that the cubed ratio in (iii) has the same desired property.

For each choice of J and C, we run our `Python` implementation of algorithm 2 (available in the Supplementary material). As the input set $\mathscr{S}$ we use the output `NonSingMinors` from algorithm 1 obtained from the input J and C. Table 8.1 shows the choice of rows J for each curve C in $\mathscr{F}$ from (8.4) that failed to give the desired output in algorithm 1 and the non-empty output of algorithm 2 in each case. This concludes our proof. □

Algorithm 2: Finding tropically non-singular 3×3-minors involving unknown valuations.

Input: A 3-element set $J \subset \{0, \dots, 5\}$, an extremal curve C in X, a 5×45 matrix N giving the 5 nodes associated to the link of C in the Schläfli graph, a list $\mathscr{S}$ of 3-element sets in $\{0, \dots, 44\}$ encoding all tropically non-singular 3×3-minors in $M_{\exp}[C, \sigma]$ for a fixed cone σ in the Naruki fan $\mathcal{N}$, and a list $\mathcal{B}$ of row indices giving a $\mathbb{Z}$-basis of the row space of the Yoshida matrix

Assumptions: X is a smooth del Pezzo cubic with no Eckardt points, embedding in $\mathbb{P}^{44}$ via the Yoshida-adapted anticanonical coordinates. The valuation of its 40 Yoshida functions is a point in the relative interior of σ. The output of algorithm 1 for C and J is the empty list.

Output: The empty list or a pair consisting of (1) an element J' in $\mathscr{S}$ giving the 3 columns of a tropically non-singular minor with rows J for the pair $(M_{\exp}[C, \sigma], M_{\text{true}}[C, \sigma])$, and (2) the expression of the third power of the ratio of the two relevant entries with unknown valuations in N as a Laurent monomial in the Yoshidas indexed by $\mathcal{B}$.

for J' *in* $\mathscr{S}$ **do**
- $S_{\exp}(J, J') \leftarrow 3 \times 3$-submatrix of $M_{\exp}[C, \sigma]$ with rows J and columns J';
- **if** $S_{\exp}(J, J')$ *is tropically non-singular* **then**
  - $S_{\text{true}}(J, J') \leftarrow 3 \times 3$-submatrix of $M_{\text{true}}[C, \sigma]$ with rows J and columns J';
  - **for** $(\tau_1, \tau_2) \in \mathfrak{S}_3 \times \mathfrak{S}_3$ **do**
    - $(N^{\text{new}}, S^{\text{new}}_{\text{true}}) \leftarrow ((\tau_1, \tau_2) \cdot N, (\tau_1, \tau_2) \cdot S_{\text{true}}(J, J'))$;
    - **if** $S^{\text{new}}_{\text{true}}$ *has the entries-pattern of* (8.5) *(where $*$ indicates a real number)* **then**
      - $(A, B) \leftarrow$ product of all Cross factors in each $(N^{\text{new}}[2, 3], N^{\text{new}}[3, 3])$, converted into a rational function in $d_1, \dots, d_6$;
      - **if** $(A/B)^3$ *is a monomial in the Yoshida functions indexed by* $\mathcal{B}$ **then**
        - **return** $(J', (A/B)^3)$.

return [ ].

---

9. Tropical lines in $\mathcal{T}X$ in the trivial valuation case

In Section 8 we discussed the combinatorics of potential tropical lines meeting the interior of $\mathcal{T}X$. In particular, Proposition 8.2 showed that any tropical line on $\mathcal{T}X$ meeting its interior intersects the boundary of $\mathbb{TP}^{44}$ at precisely 5 points. These points are the tropicalization of the intersections of pairs of (-1)-curves meeting a fixed (-1)-curve, which we use to label the potential tropical line. There are exactly 27 families of such potential tropical lines and they are all conjugated by $\mathrm{W}(\mathrm{E}_6)$. The quintuple of boundary points of each potential line gives 5 disjoint sets $B_1, \dots, B_5$ associated to the nine ∞ coordinates of each boundary point. These sets determine the 5 rays $e_{B_1}, \dots, e_{B_5}$ in the recession fan of $\mathcal{T}L$.

When some of the Yoshida functions parameterizing X has non-trivial valuation, these potential lines are ruled out after checking that their five boundary points are not collinear in $\mathbb{TP}^{44}$. When

Curve	Rows	Columns	Minor Shape	$(A/B)^3$ as a Laurent monomial in the Yoshidas
E_3	$[0,1,2]$	$[8,9,24]$	$\begin{pmatrix} * & \infty & \infty \\ \infty & * & \texttt{None} \\ * & * & \texttt{None} \end{pmatrix}$	-1
E_4	$[0,1,2]$	$[8,9,13]$	$\begin{pmatrix} * & \infty & \texttt{None} \\ \infty & * & \infty \\ * & * & \texttt{None} \end{pmatrix}$	$Y_5\,Y_{22}\,Y_{26}^2\,Y_{30}/(Y_{18}^2\,Y_{20}\,Y_{21}\,Y_{27})$
E_5	$[0,1,2]$	$[7,32,33]$	$\begin{pmatrix} * & \texttt{None} & \infty \\ \infty & \infty & * \\ * & \texttt{None} & * \end{pmatrix}$	$(-1)\,Y_{17}^3\,Y_{18}^2\,Y_{20}\,Y_{26}\,Y_{27}\,Y_{30}^2/(Y_5\,Y_{19}^3\,Y_{21}^2\,Y_{22}\,Y_{29}^3)$
E_6	$[0,1,2]$	$[7,8,12]$	$\begin{pmatrix} \infty & * & \infty \\ * & \infty & \texttt{None} \\ * & * & \texttt{None} \end{pmatrix}$	1
F_{12}	$[0,2,4]$	$[17,26,44]$	$\begin{pmatrix} \texttt{None} & \infty & * \\ \infty & * & \infty \\ \texttt{None} & * & * \end{pmatrix}$	-1
F_{34}	$[0,2,4]$	$[5,7,23]$	$\begin{pmatrix} * & \infty & \infty \\ \infty & * & \texttt{None} \\ * & * & \texttt{None} \end{pmatrix}$	1
F_{35}	$[0,1,2]$	$[6,15,23]$	$\begin{pmatrix} * & * & \texttt{None} \\ \infty & * & \infty \\ * & \infty & \texttt{None} \end{pmatrix}$	$Y_5^2\,Y_{21}\,Y_{25}^3\,Y_{26}\,Y_{27}\,Y_{29}^3/(Y_{18}\,Y_{20}^2\,Y_{22}\,Y_{24}^3\,Y_{28}^3\,Y_{30})$
F_{36}	$[0,1,2]$	$[6,9,19]$	$\begin{pmatrix} * & \infty & \infty \\ \infty & * & \texttt{None} \\ * & * & \texttt{None} \end{pmatrix}$	-1
F_{45}	$[0,1,2]$	$[7,10,12]$	$\begin{pmatrix} * & * & \texttt{None} \\ * & \infty & \infty \\ \infty & * & \texttt{None} \end{pmatrix}$	$Y_5\,Y_{18}\,Y_{26}^2\,Y_{31}^3/(Y_{20}\,Y_{21}\,Y_{22}^2\,Y_{27}\,Y_{30}^2)$
F_{46}	$[0,1,2]$	$[8,9,13]$	$\begin{pmatrix} * & \infty & \texttt{None} \\ \infty & * & \infty \\ * & * & \texttt{None} \end{pmatrix}$	$Y_{18}\,Y_{20}^2\,Y_{22}\,Y_{24}^3\,Y_{28}^3\,Y_{30}/(Y_5^2\,Y_{21}\,Y_{25}^3\,Y_{26}\,Y_{27}\,Y_{29}^3)$
F_{56}	$[0,1,2]$	$[8,9,13]$	$\begin{pmatrix} * & \infty & \texttt{None} \\ \infty & * & \infty \\ * & * & \texttt{None} \end{pmatrix}$	$Y_5^2\,Y_{20}\,Y_{22}^2\,Y_{26}\,Y_{28}^3\,Y_{31}^3/(Y_{17}^3\,Y_{18}\,Y_{21}^2\,Y_{25}^3\,Y_{27}^2\,Y_{30})$
G_3	$[0,2,4]$	$[30,35,44]$	$\begin{pmatrix} \texttt{None} & \infty & * \\ \infty & * & \infty \\ \texttt{None} & * & * \end{pmatrix}$	1
G_4	$[0,3,4]$	$[20,32,38]$	$\begin{pmatrix} * & \texttt{None} & \infty \\ \infty & \infty & * \\ * & \texttt{None} & * \end{pmatrix}$	$(-1)\,Y_{18}\,Y_{20}^2\,Y_{22}\,Y_{30}/(Y_5^2\,Y_{21}\,Y_{26}\,Y_{27})$
G_5	$[0,2,4]$	$[32,34,42]$	$\begin{pmatrix} \texttt{None} & * & * \\ \infty & * & \infty \\ \texttt{None} & * & * \end{pmatrix}$	1
G_6	$[0,1,3]$	$[11,32,42]$	$\begin{pmatrix} * & \texttt{None} & \infty \\ \infty & \infty & * \\ * & \texttt{None} & * \end{pmatrix}$	1

TABLE 8.1. Ruling out the remaining 15 potential interior tropical lines for the combinatorial type induced by the cone (**a**) in $\mathcal{N}$ that are not covered by Table A.4. The entries A and B correspond to products of all Cross factors in the coordinate of the 2 classical nodes responsible for the two '`None`' entries in the (J, J')-minor of M_{true}, read from top to bottom.

all 40 Yoshida functions have valuation zero, the methods from Section 8 fail. Maria: Explain combinatorial type refers to stable surfaces.

Our main result in this section says that when the valuation on $\mathbb{K}$ is trivial, these tuples of 5 points are tropically collinear and, furthermore, the tropical line through them lies in the surface $\mathcal{T}X$. Surprisingly, no tropical cycle supported on such curves can be lifted to a curve in X. Thus, the challenging relative lifting problem for such cycles on $\mathcal{T}X$ has a negative answer for the anticanonical embedding. The monomial map (4.1) will yield the same answer for tropicalizations with respect to the Cox embedding of X.

Theorem 9.1. *Assume that $\mathbb{K}$ has trivial valuation. Then, there are exactly 27 extra (non-generic) tropical lines in $\mathcal{T}X$ meeting its interior. Furthermore, no tropical cycle supported on each such tropical line can be lifted to an effective curve on the del Pezzo cubic X.*

Proof. The group $\mathrm{W}(\mathrm{E}_6)$ acts transitively on all 27 potential tropical lines meeting the interior of $\mathcal{T}X$. Therefore, the count gives either no such lines or exactly 27 of them. Without loss of generality, we assume our tropical line is indexed by the extremal curve E_1.

We prove both assertions in the statement for this particular curve. By Proposition 8.2, the five points associated to the link of E_1 in the Schläfli graph are the tropicalization of the nodes $F_{1j}\cap G_j$ for $j=2,\ldots,6$. By Remark 5.3, each one of these nodes has exacly 9 coordinates equal to 0. These coordinates become ∞ under tropicalization. The corresponding five sets of ∞-coordinates are $B_j=\{X_{ij}\colon i\neq j\}\cup\{Y_{1jklmn}\colon k,l,m,n\}$ $(j=2,\ldots,6)$. They partition the set $\{0,\ldots,44\}$.

Since $\mathbb{K}$ has trivial valuation, the remaining 36 coordinates on the tropicalization of each node have value 0. Therefore, the fan Σ_{E_1} with rays e_{B_j} $(j=2,\ldots,6)$ is a non-generic tropical line passing through these five points at infinity. It is the only tropical line containing these five points. Since $\mathcal{T}X$ is also a fan, Σ_{E_1} lies in $\mathcal{T}X$.

A tropical cycle supported on Σ_{E_1} has an integer multiplicity on each ray and it must satisfy the balancing condition at the origin: the five primitive vectors for each ray scaled by their multiplicity should add up to a multiple of the all-ones vector. The disjoint support property forces these five multiplicities to agree, hence the cycle can be written as $m\cdot\Sigma_{E_1}$ for some integer m. Note that the tropical cycle will be effective whenever $m\geq 1$. We claim any such cycle (effective or not) cannot be lifted to an effective curve $\mathcal{C}$ on X. We argue by contradiction.

By construction, all boundary points on the curve $\mathcal{C}$ tropicalize to one of the five boundary points on Σ_{E_1}. In particular, $\mathcal{C}$ contains five boundary points $p_2,\ldots,p_6$, where $\mathrm{trop}(p_j)$ is the leaf associated to the ray e_{B_j} for $j=2,\ldots,6$. Each point p_j lies in the 9 hyperplanes indexed by B_j. We claim that $p_j=F_{ij}\cap G_j$ for $j=2,\ldots,6$. For this, it suffices to show that

$$\mathcal{C}\cap G_1=\emptyset\,,\qquad \mathcal{C}\cap E_k \text{ for all } k\neq 1 \qquad\text{and}\qquad \mathcal{C}\cap F_{ik}=\emptyset \text{ for all } i,k\neq 1. \tag{9.1}$$

The assertion follows by tropicalization. For example, if $\mathcal{C}\cap G_1\neq\emptyset$, then Σ_{E_1} contain a point in its boundary with all coordinates $X_{21}=X_{31}=\ldots=X_{61}=\infty$. This contradicts the definition of Σ_{E_1}. The other two assertions follow similarly by the combinatorics of the anticanonical triangles. Since each p_j has vanishing coordinates Y_{1jklmn} and X_{ij} with $i\neq j$, the conditions (9.1) imply that $p_j=F_{ij}\cap G_j$ for each $j=2,\ldots,6$.

The 27 lines $E_1,\ldots,G_6$ generate the effective cone, thus we can write the class of $\mathcal{C}$ as

$$[\mathcal{C}]=\sum_i a_i[E_i]+\sum_{i<j} b_{ij}[F_{ij}]+\sum_i c_i[G_i]\quad \text{ for some } a_i,b_{ij},c_i\geq 0.$$

Conditions (9.1) translate this identity to the following system of 16 linear equations:

$$\begin{cases} 0=[\mathcal{C}]\cdot[G_1]=\displaystyle\sum_{i\neq 1}a_i+\sum_{j>1}b_{1j}-c_1, \\ 0=[\mathcal{C}]\cdot[E_k]=-a_k+\displaystyle\sum_{i<k}b_{ik}+\sum_{i>k}b_{ki}+\sum_{i\neq k}c_i \quad \text{for } k\neq 1, \\ 0=[\mathcal{C}]\cdot[F_{ik}]=a_i+a_k-b_{ik}+\displaystyle\sum_{\{p,q\}\cap\{i,k\}=\emptyset} b_{pq}+c_i+c_k \quad \text{for } i,k\neq 1. \end{cases}$$

A simple calculation with `Sage`, available in the supplementary material, confirms that the system has rank 6 and no solutions in the positive orthant other than the trivial one. Therefore, no multiple of the tropical line Σ_{E_1} lifts to an effective curve in X. This concludes our proof. ∎

Maria: We can use tropical intersection theory on Gubler models to get rid of the effectiveness assumption on $\mathcal{C}$.

Roots r_0 to r_5	Roots r_6 to r_{11}	Roots r_{12} to r_{17}	Roots r_{18} to r_{23}	Roots r_{24}-r_{29}	Roots r_{30} to r_{35}
$-d_1 + d_2$	$-d_2 + d_4$	$-d_1 + d_4$	$-d_2 + d_6$	$-d_1 + d_6$	$d_2 + d_4 + d_6$
$d_1 + d_2 + d_3$	$d_1 + d_2 + d_4$	$d_1 + d_2 + d_5$	$-d_1 + d_5$	$d_2 + d_3 + d_6$	$d_2 + d_5 + d_6$
$-d_2 + d_3$	$-d_4 + d_6$	$-d_3 + d_6$	$d_1 + d_3 + d_5$	$d_2 + d_4 + d_5$	$d_3 + d_4 + d_6$
$-d_3 + d_4$	$-d_1 + d_3$	$d_1 + d_3 + d_4$	$d_1 + d_4 + d_5$	$d_1 + d_4 + d_6$	$d_3 + d_5 + d_6$
$-d_4 + d_5$	$-d_3 + d_5$	$d_2 + d_3 + d_4$	$d_2 + d_3 + d_5$	$d_1 + d_5 + d_6$	$d_4 + d_5 + d_6$
$-d_5 + d_6$	$-d_2 + d_5$	$d_1 + d_2 + d_6$	$d_1 + d_3 + d_6$	$d_3 + d_4 + d_5$	$d_1 + d_2 + d_3 + d_4 + d_5 + d_6$

TABLE A.1. 36 positive roots of E_6

$k = 0$ to 9	10 to 19	20 to 29	30 to 39
$r_{10}r_{11}r_{12}r_2r_{22}r_{24}r_{27}r_{35}r_8$	$r_{13}r_{16}r_{19}r_2r_{20}r_{25}r_{27}r_{34}r_8$	$r_0r_{10}r_{12}r_{14}r_{33}r_{35}r_5r_6r_7$	$r_0r_{13}r_{14}r_{23}r_{25}r_{29}r_{34}r_4r_7$
$r_1r_{18}r_{21}r_{23}r_{29}r_{30}r_{31}r_4r_9$	$r_1r_{12}r_{16}r_2r_{21}r_{27}r_{31}r_{33}r_5$	$r_1r_{15}r_{26}r_{28}r_{30}r_{33}r_5r_6r_9$	$r_{11}r_{14}r_{23}r_{24}r_{26}r_{35}r_4r_6r_9$
$r_{15}r_{17}r_{18}r_{20}r_{25}r_{26}r_{34}r_4r_9$	$r_{11}r_{13}r_{15}r_{22}r_{23}r_{30}r_{34}r_8r_9$	$r_0r_1r_2r_{34}r_{35}r_4r_5r_8r_9$	$r_{10}r_{12}r_{15}r_{17}r_{18}r_{21}r_{22}r_{30}r_{33}$
$r_{16}r_{19}r_2r_{24}r_{28}r_3r_{35}r_5r_6$	$r_{14}r_{19}r_{20}r_{25}r_{26}r_{28}r_{32}r_6r_7$	$r_0r_{10}r_{17}r_{20}r_{22}r_{32}r_{34}r_7r_8$	$r_{11}r_{15}r_{17}r_{22}r_{24}r_{26}r_{28}r_3r_{32}$
$r_1r_{19}r_2r_{21}r_{22}r_{28}r_{30}r_{32}r_8$	$r_{11}r_{12}r_{13}r_{14}r_{15}r_{25}r_{26}r_{27}r_{33}$	$r_{11}r_{12}r_{14}r_{21}r_{22}r_{23}r_{31}r_{32}r_7$	$r_{10}r_{16}r_{17}r_{20}r_{24}r_{26}r_{27}r_{33}r_6$
$r_{12}r_{15}r_2r_{22}r_{25}r_{28}r_{34}r_5r_7$	$r_{13}r_{14}r_{16}r_{19}r_{21}r_{23}r_{30}r_{33}r_6$	$r_{12}r_{14}r_{18}r_{19}r_2r_{21}r_{25}r_{35}r_4$	$r_0r_{10}r_{17}r_{18}r_{24}r_{29}r_3r_{35}r_4$
$r_0r_{13}r_{15}r_{16}r_{17}r_3r_{33}r_{34}r_5$	$r_{16}r_{17}r_2r_{21}r_{22}r_{23}r_{24}r_{34}r_4$	$r_0r_1r_{10}r_{13}r_{27}r_{29}r_{30}r_{33}r_8$	$r_0r_{11}r_{13}r_{14}r_{19}r_3r_{32}r_{35}r_8$
$r_{10}r_{18}r_{19}r_{20}r_{30}r_{35}r_6r_8r_9$	$r_{11}r_{12}r_{15}r_{18}r_3r_{31}r_{35}r_5r_9$	$r_{10}r_{22}r_{23}r_{24}r_{28}r_{29}r_{30}r_6r_7$	$r_0r_1r_{28}r_{29}r_3r_{31}r_{32}r_5r_7$
$r_{16}r_{20}r_{23}r_{31}r_{34}r_5r_6r_7r_9$	$r_1r_2r_{24}r_{25}r_{26}r_{27}r_{28}r_{29}r_4$	$r_{13}r_{15}r_{18}r_{19}r_{25}r_{28}r_{29}r_3r_{30}$	$r_0r_1r_{14}r_{17}r_{21}r_{26}r_{32}r_{33}r_4$

TABLE A.2. Yoshida functions $\mathcal{Y}_k$ for $0 \leq k \leq 39$ expressed as products of roots

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Appendix A. Computation

A.1. **Coble covariants.** Table A.1, Table A.2, and Table A.3 give our choices of positive roots, Yoshida functions, and Cross functions, respectively.

A.2. $\mathrm{W(E_6)}$ **action.**

$k=0$ to 14	15 to 29	30 to 44	45 to 59	60 to 74	75 to 89	90 to 104	105 to 119	120 to 134
$-\mathcal{Y}_{12}+\mathcal{Y}_{36}$	$-\mathcal{Y}_{14}+\mathcal{Y}_{5}$	$-\mathcal{Y}_{10}+\mathcal{Y}_{4}$	$-\mathcal{Y}_{1}+\mathcal{Y}_{38}$	$-\mathcal{Y}_{0}-\mathcal{Y}_{35}$	$\mathcal{Y}_{28}-\mathcal{Y}_{32}$	$\mathcal{Y}_{1}+\mathcal{Y}_{9}$	$\mathcal{Y}_{18}-\mathcal{Y}_{35}$	$\mathcal{Y}_{31}-\mathcal{Y}_{36}$
$-\mathcal{Y}_{2}+\mathcal{Y}_{35}$	$-\mathcal{Y}_{27}-\mathcal{Y}_{31}$	$-\mathcal{Y}_{34}-\mathcal{Y}_{38}$	$-\mathcal{Y}_{34}+\mathcal{Y}_{7}$	$-\mathcal{Y}_{10}-\mathcal{Y}_{30}$	$-\mathcal{Y}_{23}+\mathcal{Y}_{37}$	$-\mathcal{Y}_{31}+\mathcal{Y}_{34}$	$-\mathcal{Y}_{36}+\mathcal{Y}_{4}$	$\mathcal{Y}_{1}-\mathcal{Y}_{26}$
$\mathcal{Y}_{20}-\mathcal{Y}_{8}$	$-\mathcal{Y}_{39}+\mathcal{Y}_{4}$	$-\mathcal{Y}_{19}+\mathcal{Y}_{37}$	$\mathcal{Y}_{17}-\mathcal{Y}_{33}$	$\mathcal{Y}_{37}-\mathcal{Y}_{39}$	$-\mathcal{Y}_{10}+\mathcal{Y}_{11}$	$-\mathcal{Y}_{22}+\mathcal{Y}_{23}$	$-\mathcal{Y}_{17}+\mathcal{Y}_{5}$	$\mathcal{Y}_{34}-\mathcal{Y}_{9}$
$\mathcal{Y}_{30}-\mathcal{Y}_{36}$	$-\mathcal{Y}_{12}-\mathcal{Y}_{29}$	$\mathcal{Y}_{2}-\mathcal{Y}_{30}$	$\mathcal{Y}_{18}-\mathcal{Y}_{19}$	$\mathcal{Y}_{12}-\mathcal{Y}_{8}$	$\mathcal{Y}_{2}-\mathcal{Y}_{25}$	$-\mathcal{Y}_{3}+\mathcal{Y}_{33}$	$-\mathcal{Y}_{2}+\mathcal{Y}_{31}$	$-\mathcal{Y}_{31}-\mathcal{Y}_{39}$
$\mathcal{Y}_{1}-\mathcal{Y}_{17}$	$-\mathcal{Y}_{21}+\mathcal{Y}_{37}$	$\mathcal{Y}_{15}-\mathcal{Y}_{21}$	$-\mathcal{Y}_{24}+\mathcal{Y}_{4}$	$\mathcal{Y}_{28}+\mathcal{Y}_{3}$	$-\mathcal{Y}_{0}+\mathcal{Y}_{12}$	$\mathcal{Y}_{14}+\mathcal{Y}_{20}$	$-\mathcal{Y}_{23}-\mathcal{Y}_{27}$	$-\mathcal{Y}_{25}-\mathcal{Y}_{4}$
$\mathcal{Y}_{27}-\mathcal{Y}_{33}$	$-\mathcal{Y}_{19}-\mathcal{Y}_{7}$	$\mathcal{Y}_{24}-\mathcal{Y}_{39}$	$\mathcal{Y}_{21}-\mathcal{Y}_{8}$	$-\mathcal{Y}_{11}+\mathcal{Y}_{14}$	$-\mathcal{Y}_{3}+\mathcal{Y}_{8}$	$\mathcal{Y}_{39}+\mathcal{Y}_{7}$	$\mathcal{Y}_{10}-\mathcal{Y}_{15}$	$\mathcal{Y}_{20}-\mathcal{Y}_{23}$
$-\mathcal{Y}_{21}-\mathcal{Y}_{32}$	$\mathcal{Y}_{27}+\mathcal{Y}_{35}$	$-\mathcal{Y}_{26}+\mathcal{Y}_{37}$	$\mathcal{Y}_{7}-\mathcal{Y}_{8}$	$\mathcal{Y}_{23}-\mathcal{Y}_{38}$	$-\mathcal{Y}_{28}-\mathcal{Y}_{5}$	$\mathcal{Y}_{33}-\mathcal{Y}_{5}$	$-\mathcal{Y}_{1}+\mathcal{Y}_{19}$	$-\mathcal{Y}_{0}-\mathcal{Y}_{27}$
$-\mathcal{Y}_{39}-\mathcal{Y}_{9}$	$\mathcal{Y}_{4}+\mathcal{Y}_{9}$	$-\mathcal{Y}_{10}-\mathcal{Y}_{9}$	$-\mathcal{Y}_{26}+\mathcal{Y}_{30}$	$\mathcal{Y}_{31}+\mathcal{Y}_{8}$	$\mathcal{Y}_{1}-\mathcal{Y}_{24}$	$-\mathcal{Y}_{22}+\mathcal{Y}_{35}$	$\mathcal{Y}_{15}-\mathcal{Y}_{20}$	$-\mathcal{Y}_{15}+\mathcal{Y}_{36}$
$-\mathcal{Y}_{16}+\mathcal{Y}_{18}$	$\mathcal{Y}_{5}+\mathcal{Y}_{6}$	$-\mathcal{Y}_{12}-\mathcal{Y}_{33}$	$-\mathcal{Y}_{11}+\mathcal{Y}_{25}$	$\mathcal{Y}_{14}-\mathcal{Y}_{19}$	$-\mathcal{Y}_{23}+\mathcal{Y}_{26}$	$-\mathcal{Y}_{8}+\mathcal{Y}_{9}$	$\mathcal{Y}_{16}+\mathcal{Y}_{5}$	$\mathcal{Y}_{1}-\mathcal{Y}_{39}$
$-\mathcal{Y}_{29}+\mathcal{Y}_{9}$	$-\mathcal{Y}_{24}+\mathcal{Y}_{30}$	$-\mathcal{Y}_{18}-\mathcal{Y}_{26}$	$\mathcal{Y}_{18}-\mathcal{Y}_{39}$	$-\mathcal{Y}_{17}+\mathcal{Y}_{36}$	$-\mathcal{Y}_{16}-\mathcal{Y}_{35}$	$\mathcal{Y}_{29}+\mathcal{Y}_{39}$	$\mathcal{Y}_{15}-\mathcal{Y}_{24}$	$\mathcal{Y}_{12}-\mathcal{Y}_{15}$
$\mathcal{Y}_{12}-\mathcal{Y}_{4}$	$\mathcal{Y}_{26}-\mathcal{Y}_{39}$	$-\mathcal{Y}_{16}-\mathcal{Y}_{23}$	$-\mathcal{Y}_{3}-\mathcal{Y}_{36}$	$-\mathcal{Y}_{11}+\mathcal{Y}_{38}$	$-\mathcal{Y}_{4}-\mathcal{Y}_{7}$	$\mathcal{Y}_{23}-\mathcal{Y}_{6}$	$\mathcal{Y}_{0}-\mathcal{Y}_{26}$	$\mathcal{Y}_{22}-\mathcal{Y}_{3}$
$\mathcal{Y}_{25}+\mathcal{Y}_{30}$	$-\mathcal{Y}_{17}-\mathcal{Y}_{28}$	$\mathcal{Y}_{13}-\mathcal{Y}_{15}$	$-\mathcal{Y}_{12}+\mathcal{Y}_{16}$	$-\mathcal{Y}_{13}+\mathcal{Y}_{19}$	$\mathcal{Y}_{19}-\mathcal{Y}_{24}$	$\mathcal{Y}_{13}-\mathcal{Y}_{3}$	$\mathcal{Y}_{3}-\mathcal{Y}_{37}$	$-\mathcal{Y}_{13}+\mathcal{Y}_{39}$
$\mathcal{Y}_{35}-\mathcal{Y}_{7}$	$\mathcal{Y}_{12}-\mathcal{Y}_{24}$	$\mathcal{Y}_{14}-\mathcal{Y}_{31}$	$-\mathcal{Y}_{12}-\mathcal{Y}_{27}$	$\mathcal{Y}_{13}-\mathcal{Y}_{28}$	$-\mathcal{Y}_{13}-\mathcal{Y}_{30}$	$\mathcal{Y}_{14}-\mathcal{Y}_{39}$	$-\mathcal{Y}_{15}+\mathcal{Y}_{6}$	$\mathcal{Y}_{22}-\mathcal{Y}_{7}$
$-\mathcal{Y}_{37}-\mathcal{Y}_{4}$	$-\mathcal{Y}_{19}+\mathcal{Y}_{2}$	$\mathcal{Y}_{10}-\mathcal{Y}_{16}$	$-\mathcal{Y}_{14}-\mathcal{Y}_{18}$	$\mathcal{Y}_{1}+\mathcal{Y}_{8}$	$\mathcal{Y}_{22}-\mathcal{Y}_{39}$	$\mathcal{Y}_{22}+\mathcal{Y}_{6}$	$-\mathcal{Y}_{13}+\mathcal{Y}_{37}$	$-\mathcal{Y}_{34}-\mathcal{Y}_{6}$
$-\mathcal{Y}_{10}-\mathcal{Y}_{6}$	$\mathcal{Y}_{11}-\mathcal{Y}_{21}$	$\mathcal{Y}_{1}+\mathcal{Y}_{28}$	$-\mathcal{Y}_{17}+\mathcal{Y}_{3}$	$\mathcal{Y}_{38}+\mathcal{Y}_{6}$	$-\mathcal{Y}_{29}-\mathcal{Y}_{3}$	$\mathcal{Y}_{15}-\mathcal{Y}_{4}$	$-\mathcal{Y}_{25}-\mathcal{Y}_{35}$	$-\mathcal{Y}_{13}+\mathcal{Y}_{5}$

TABLE A.3. Cross functions Cross_k for $0 \leq k \leq 134$ in terms of Yoshida functions. The table shows only one of the four ways of writing a Cross function in this way.

Cone	Rows	E_1	E_2	E_3	E_4	E_5	E_6	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{23}	F_{24}	F_{25}	F_{26}	F_{34}	F_{35}	F_{36}	F_{45}	F_{46}	F_{56}
(aa2a3a4)	012	0,1,2	0,2,12	0,1,2	0,1,4	0,1,4	0,1,15	0,8,36	0,2,5	0,1,6	0,1,5	0,1,5	0,1,2	0,1,2	0,1,3	0,1,2	0,2,12	0,1,2	0,1,3	0,1,7	0,2,3	0,2,3
(aa2a3b)	012	0,2,9	0,2,18	1,2,7	0,1,7	0,1,7	0,2,7	0,2,7	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,2	0,1,10	0,2,18	0,1,2	0,1,2	0,1,7	0,2,4	0,2,4
(aa2a3)	012	0,2,12	0,2,18	1,2,5	0,1,7	0,1,7	0,1,15	0,8,36	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,3	0,1,2	0,2,18	0,1,2	0,1,4	0,1,7	0,2,4	0,2,4
(aa2a4)	012	0,1,2	0,2,12	0,2,4	0,1,4	0,1,4	0,1,15	0,8,36	0,2,5	0,1,6	0,1,5	0,1,5	0,1,2	0,1,2	0,1,3	0,1,2	0,2,12	0,1,2	0,1,3	0,1,7	0,2,3	0,2,3
(aa3a4)	012	0,1,2	0,2,12	0,1,2	0,1,4	0,1,4	0,1,15	0,8,36	0,2,5	0,1,6	0,1,5	0,1,5	0,1,2	0,1,2	0,1,3	0,1,2	0,2,12	0,1,2	0,1,3	0,1,20	0,2,3	0,2,3
(a2a3a4)	012	0,2,9	0,2,12	0,2,4	0,2,4	0,2,4	0,2,12	0,8,36	0,2,5	0,2,8	0,2,5	0,2,5	0,2,3	0,2,4	0,2,3	0,2,5	0,2,12	0,2,5	0,2,3	0,2,7	0,2,3	0,2,3
(aa2b)	012	0,2,13	0,2,23	1,3,7	0,2,8	0,2,7	0,2,7	0,2,7	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,2	0,1,10	0,2,25	0,1,2	0,1,2	0,1,7	0,2,7	0,2,8
(aa3b)	012	0,2,9	0,2,18	1,2,7	0,1,7	0,1,7	0,2,7	0,2,7	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,2	0,1,10	0,2,18	0,1,2	0,1,2	0,1,33	0,2,4	0,2,4
(a2a3b)	012	0,2,9	0,2,18	2,3,7	0,2,4	0,2,4	0,2,7	0,2,7	0,5,12	0,2,8	0,5,10	0,2,7	0,2,7	0,2,4	0,2,7	0,2,4	0,2,18	0,2,3	0,2,10	0,2,7	0,2,4	0,2,4
(aa2)	012	1,8,13	0,2,23	1,3,5	0,6,8	0,6,7	0,6,7	2,18,33	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,3	0,1,2	0,2,25	0,1,2	0,1,4	0,1,7	0,2,7	0,3,8
(aa3)	012	0,2,12	0,2,18	1,2,5	0,1,7	0,1,7	0,2,13	0,8,36	0,5,11	0,1,6	0,5,10	0,1,7	0,1,2	0,1,2	0,1,3	0,1,2	0,2,18	0,1,2	0,1,4	0,1,33	0,2,4	0,2,4
(aa4)	012	0,2,9	0,2,12	0,2,4	0,2,4	0,2,4	0,2,15	0,8,36	0,2,5	0,2,8	0,2,5	0,2,5	0,2,3	0,2,4	0,2,3	0,2,7	0,2,12	0,2,5	0,2,3	0,2,7	0,2,3	0,2,3
(a2a3)	012	0,2,12	0,2,19	2,3,5	0,2,4	0,2,4	0,2,13	0,8,36	0,5,14	0,2,8	0,5,10	0,2,7	0,2,7	0,2,10	0,2,7	0,2,7	0,2,18	0,2,5	0,2,16	0,2,7	0,2,4	0,2,4
(a2a4)	012	0,2,9	0,2,12	0,2,4	0,2,4	0,2,4	0,2,12	0,8,36	0,2,5	0,2,8	0,2,5	0,2,5	0,2,3	0,2,4	0,2,3	0,2,5	0,2,12	0,2,5	0,2,3	0,2,7	0,2,3	0,2,3
(a3a4)	012	0,2,9	0,2,12	0,2,4	0,2,4	0,2,4	0,2,12	0,8,36	0,2,5	0,2,8	0,2,5	0,2,5	0,2,3	0,2,4	0,2,3	0,2,5	0,2,12	0,2,5	0,2,3	0,2,7	0,2,3	0,2,3
(ab)	012	0,2,20	—	1,3,12	1,2,8	1,2,7	1,2,7	0,2,7	0,5,11	0,5,16	0,5,11	0,5,10	0,1,2	0,1,2	0,2,7	0,1,10	—	0,1,2	0,1,3	0,1,37	0,2,7	0,2,8
	013		2,3,20														0,2,3					
(a2b)	012	0,2,13	0,2,23	2,3,7	0,2,8	0,2,7	0,2,7	0,2,7	0,5,12	0,2,8	0,5,10	0,2,7	0,2,7	0,2,10	0,2,7	0,2,4	0,2,25	0,2,3	0,2,10	0,2,7	0,2,7	0,2,8
(a3b)	012	0,2,9	0,2,18	2,3,7	0,2,4	0,2,4	0,2,7	0,2,7	0,5,12	0,2,8	0,5,10	0,2,7	0,2,7	0,2,4	0,2,9	0,2,4	0,2,18	0,2,3	0,2,10	0,2,7	0,2,4	0,2,4
(a)	012	2,8,20	—	—	—	—	—	—	0,5,11	0,5,16	0,5,11	0,5,10	0,1,10	0,1,6	0,2,7	0,1,10	—	—	—	—	—	—
	013		3,4,20	—	—	—	—	—									—	—	—	—	—	—
	rest			—	—	—	—	—									—	—	—	—	—	—
(a2)	012	2,8,13	0,2,33	2,3,5	0,8,13	0,7,12	0,7,13	2,18,36	0,5,14	0,3,8	0,5,10	0,2,7	0,2,7	0,2,10	0,2,7	0,2,7	0,2,25	0,2,15	0,2,26	0,2,7	0,2,7	0,3,8
(a3)	012	0,2,12	0,2,19	2,3,7	0,2,4	0,2,4	0,2,13	0,8,36	0,5,14	0,2,8	0,5,10	0,2,7	0,2,7	0,2,10	0,3,9	0,2,7	0,2,18	0,2,5	0,2,16	0,2,7	0,2,4	0,2,4
(a4)	012	0,2,9	0,2,12	0,2,4	0,2,4	0,2,4	0,2,15	0,8,36	0,2,5	0,2,8	0,2,5	0,2,5	0,2,3	0,2,4	0,2,3	0,2,7	0,2,12	0,2,5	0,2,3	0,2,7	0,2,3	0,2,3
(b)	012	2,8,21	—	—	2,7,8	2,7,8	3,7,8	2,4,7	2,4,7	—	2,15,16	—	2,3,7	—	2,15,16	—	—	2,15,16	—	2,3,8	2,7,8	2,7,8
	013		2,3,20	2,3,20						2,16,21		2,8,15		2,3,15		2,3,16	2,3,15		2,3,16			

Cone	Rows	G_1	G_2	G_3	G_4	G_5	G_6
(aa2a3a4)	012	0,1,2	0,2,7	0,2,5	0,2,5	0,1,5	0,2,5
(aa2a3b)	012	0,1,2	0,2,7	0,5,7	0,2,5	0,5,10	0,1,5
(aa2a3)	012	0,1,2	0,2,7	0,5,7	0,2,5	0,12,14	0,2,5
(aa2a4)	012	0,1,2	0,2,7	0,5,20	0,2,5	0,1,5	0,2,5
(aa3a4)	012	0,1,2	0,2,7	0,2,5	0,2,5	0,1,5	0,2,5
(a2a3a4)	012	0,2,7	0,2,7	0,2,5	0,2,5	0,2,5	0,2,5
(aa2b)	012	0,1,2	0,2,7	0,5,7	0,2,5	0,5,10	0,1,5
(aa3b)	012	0,1,2	0,2,7	0,5,7	0,2,5	0,5,10	0,1,5
(a2a3b)	012	0,10,12	0,2,7	0,5,7	0,2,5	0,5,10	0,2,5
(aa2)	012	0,1,2	0,2,7	0,5,21	0,2,5	0,12,14	0,2,5
(aa3)	012	0,1,2	0,2,7	0,5,7	0,2,5	0,12,14	0,2,5
(aa4)	012	0,2,7	0,2,7	0,5,20	0,2,5	0,2,5	0,2,5
(a2a3)	012	0,10,22	0,2,7	0,5,7	0,2,5	0,12,14	0,2,5
(a2a4)	012	0,2,7	0,2,7	0,5,20	0,2,5	0,2,5	0,2,5
(a3a4)	012	0,2,7	0,2,7	0,2,5	0,2,5	0,2,5	0,2,5
(ab)	012	0,1,2	0,2,7	1,2,7	1,2,6	1,5,7	0,1,7
(a2b)	012	0,10,12	0,2,7	0,5,7	0,2,5	0,5,10	0,2,5
(a3b)	012	0,10,12	0,2,7	0,5,7	0,2,5	0,5,10	0,2,5
(a)	012	0,1,11	0,2,7	—	—	—	—
	013			—	—	—	—
	rest			—	—	—	—
(a2)	012	0,10,12	0,2,7	0,5,21	0,2,5	0,12,14	0,10,36
(a3)	012	0,11,12	0,2,7	0,5,7	0,2,5	0,12,14	0,2,5
(a4)	012	0,2,7	0,2,7	0,5,20	0,2,5	0,2,5	0,2,5
(b)	012	2,3,7	2,4,7	2,4,7	—	7,12,17	—
	013				3,8,20		2,7,20

TABLE A.4. Ruling out all 27 potential non-boundary tropical lines for all non-apex cells in the Naruki fan $\mathcal{N}$. Each entry gives the 3 columns of a tropically non-singular 3×3-minor of the pair of matrices $(M_{\exp}[C,\sigma], M_{\text{true}}[C,\sigma])$ in the family $\mathcal{F}_\sigma$ from (8.3) for each cone σ.

An absence of a triple for an extremal ray is indicated by '—' and it should be interpreted as algorithm 1 failing to find a singular minor with the prescribed rows. Whenever a choice of 3 rows does not rule out all extremal curves, we move on to the next choice of rows (in the lexicographic order) and only check the remaining extremal curves.

All 27 extremal curves are covered by a suitable choice of rows with the exception of the cell (**a**), for which the method only rules out 12 potential lines. The remaining 15 cases are treated in Table 8.1.