

Alternate Compactifications of Hurwitz Spaces

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by

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## Abstract

We construct several modular compactifications of the Hurwitz space  $H_{g/h}^d$  of genus  $g$  curves expressed as  $d$ -sheeted, simply branched covers of genus  $h$  curves. They are obtained by allowing the branch points of the cover to collide to a variable extent, generalizing the spaces of twisted admissible covers of Abramovich, Corti, and Vistoli [2]. The resulting spaces are very well-behaved if  $d$  is small or if relatively few collisions are allowed. In particular, for  $d = 2$  and  $3$ , they are always well-behaved. For  $d = 2$ , we recover the spaces of hyperelliptic curves of Fedorchuk [9]. For  $d = 3$ , we obtain new birational models of the space of triple covers.

We describe in detail the birational geometry of the spaces of triple covers of  $\mathbf{P}^1$  with a marked fiber. In this case, we obtain a sequence of birational models that begins with the space of marked (twisted) admissible covers and proceeds through the following transformations:

- (1) sequential contractions of the boundary divisors,
- (2) contraction of the hyperelliptic divisor,
- (3) sequential flips of the higher Maroni loci,
- (4) contraction of the Maroni divisor (for even  $g$ ).

The sequence culminates in a Fano variety in the case of even  $g$ , which we describe explicitly, and a variety fibered over  $\mathbf{P}^1$  with Fano fibers in the case of odd  $g$ .

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## Introduction

The most significant development in the modern theory of algebraic curves is the construction of their moduli space  $M_g$ . It is difficult to get a grasp on the geometry of  $M_g$ , largely because it is difficult to get a grasp on a curve in the abstract. We often approach  $M_g$  through moduli spaces of particular realizations of curves, which are more accessible. One such class of moduli spaces is the *Hurwitz spaces*.

The Hurwitz space  $H_g^d$  is the moduli space of genus  $g$  curves realized as  $d$ -sheeted, simply branched covers of  $\mathbf{P}^1$ . It admits a natural morphism to  $M_g$  that only remembers the abstract moduli of the curve. The images of  $H_g^d$  in  $M_g$  are some of the most important subvarieties of  $M_g$ . For  $d = 2$ , this is the locus of hyperelliptic curves; for  $d = 3$ , this is the locus of trigonal curves, and so on. For odd  $g$ , the case of  $d = (g + 1)/2$  plays a crucial role in the celebrated theorem of Harris and Mumford [15] that  $M_g$  is of general type for large  $g$ . For sufficiently large  $d$ , the map  $H_g^d \rightarrow M_g$  is surjective. This is a basis for the oldest proof of the irreducibility of  $M_g$ , due to Hurwitz [21] in characteristic zero and due to Fulton [12] in sufficiently high characteristic.

While its relation to  $M_g$  makes  $H_g^d$  especially interesting, its relation to a simpler moduli space  $M_{0;b}$  makes it accessible to study. The space  $M_{0;b}$  is the moduli space of  $b$  distinct unordered points<sup>1</sup> on  $\mathbf{P}^1$ . There is a map  $\text{br} : H_g^d \rightarrow M_{0;b}$  that assigns to a simply branched cover the location of its branch points, where  $b = 2g + 2d - 2$  by the Riemann–Hurwitz formula. The map  $\text{br}$  expresses  $H_g^d$  as a finite covering space of  $M_{0;b}$ :

$$(0.0.1) \quad \begin{array}{c} H_g^d \longrightarrow M_g . \\ \text{br} \downarrow \\ M_{0;b} \end{array}$$

The three quasi-projective varieties in (0.0.1) admit well-known modular compactifications: the space  $\overline{M}_g$  of Deligne–Mumford stable curves, the space  $\overline{M}_{0;b}$  of Mumford–Knutsen stable marked curves and the space  $\overline{H}_g^d$  of Harris–Mumford admissible covers.

---

<sup>1</sup>Ordered branch points are customary; however, we focus almost exclusively on the unordered case.

A particularly exciting development in the study of these moduli spaces is the construction of *alternate* modular compactifications. Let us illustrate with the example of  $M_g$ . The story begins in 1969, when Deligne and Mumford discovered that  $M_g$  could be compactified by adding to it the moduli of curves with the simplest singularities: nodes. A remarkable development came in 1991, when Schubert [36] discovered that this is not the only way to compactify  $M_g$ . He constructed a compactification that allows cuspidal curves! A couple of decades later, Hassett and Hyeon [19] constructed a still different compactification that allows tacnodal curves! The story for  $M_{0;b}$  is similar—the Mumford–Knutson compactification is one choice among the many discovered by Hassett [18].

The alternate compactifications are fascinating for several reasons. Firstly, they provide tools to systematically study different kinds of degenerations of nice geometric objects. For example, in certain cases, it is beneficial to consider a cuspidal degeneration of smooth curves rather than a nodal degeneration. In this case, Schubert’s compactification of  $M_g$  is more pertinent than the standard compactification.

Secondly, the several alternate compactifications provide an unprecedented opportunity to study explicitly the birational geometry of extremely interesting higher dimensional varieties. As a result, the discoveries of Schubert, Hassett and Hyeon have spurred a major research program in algebraic geometry to understand the birational geometry of moduli spaces by constructing different birational models as alternate modular compactifications. This program lies at the confluence of several important areas, such as the classical geometry of the objects being parametrized, the birational geometry of higher dimensional varieties, especially the Minimal Model Program, Geometric Invariant Theory (GIT), the study of algebraic stacks, and so on.

There have been fascinating developments in such a program for  $M_g$  and  $M_{0;b}$ . Our goal is to explore it for  $H_g^d$ . Towards that goal, we systematically construct a number of compactifications of  $H_g^d$ . The idea is to include degenerate covers where the branch points are allowed to coincide to a variable extent. The resulting spaces are very well-behaved if  $d$  is small or if relatively few collisions are allowed. In particular, for  $d = 2$  and  $3$ , they are always well-behaved. For  $d = 2$ , we recover the spaces of hyperelliptic curves of Fedorchuk [9]. For  $d = 3$ , we obtain new birational models of the space of trigonal curves. These spaces describe a fascinating picture of the birational geometry of a slight variant of the spaces of trigonal curves, namely the space of trigonal curves with a marked fiber.



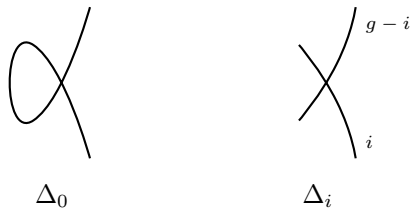


FIGURE 1. Generic points of the boundary components of  $\overline{M}_g$ .

In what follows, we describe the context and the statements of our results in more detail. We begin by giving a brief overview of some of the known alternate compactifications of  $M_g$  and  $M_{0;b}$ , respectively. We then describe alternate compactifications of  $H_g^d$ . Finally, we describe the particularly interesting case of trigonal curves.

### Compactifications of $M_g$

The standard compactification of  $M_g$  is the Deligne–Mumford compactification  $\overline{M}_g$ , parametrizing stable curves of arithmetic genus  $g$ . Recall that a curve  $C$  is *Deligne–Mumford stable* if

- (1)  $C$  has at worst nodes ( $y^2 - x^2$ ) as singularities, and
- (2) the dualizing sheaf  $\omega_C$  is ample.

$\overline{M}_g$  has only quotient singularities and a normal crossings boundary divisor  $\overline{M}_g \setminus M_g$ . The curves corresponding to the generic points of the boundary components  $\Delta_i$  (for  $i = 0, \dots, \lfloor g/2 \rfloor$ ) are displayed in Figure 1.

Schubert’s compactification  $\overline{M}_g^\psi$  is the space of pseudo-stable curves. A curve  $C$  is *pseudo-stable* if

- (1)  $C$  has at worst nodes ( $y^2 - x^2$ ) or cusps ( $y^2 - x^3$ ) as singularities,
- (2) if  $E \subset C$  is a connected sub-curve of arithmetic genus one, then it meets  $\overline{C} \setminus E$  in at least two points (no “elliptic tails”),
- (3)  $\omega_C$  is ample.

$\overline{M}_g^\psi$  is related to  $\overline{M}_g$  by a divisorial contraction—there is a morphism  $\overline{M}_g \rightarrow \overline{M}_g^\psi$  that contracts the boundary divisor  $\Delta_1$  to the locus of cuspidal curves.

Hassett and Hyeon’s compactification  $\overline{M}_g^{\text{tn}}$  allows curves with at worst tacnodal singularities ( $y^2 - x^4$ ) while disallowing elliptic tails and elliptic bridges (a connected, genus one sub-curve meeting the rest of the curve in two points).  $\overline{M}_g^{\text{tn}}$  is related to  $\overline{M}_g^\psi$  by a flip  $\overline{M}_g^\psi \dashrightarrow \overline{M}_g^{\text{tn}}$ .

Motivated by the above examples, Hassett and Keel have pioneered a systematic program to study the birational geometry of  $\overline{M}_g$ , which we now recall. Before we proceed, let us quickly recall

the notion of Mori chambers and their relation to the birational geometry of a variety. Let  $X$  be a suitably nice (normal,  $\mathbf{Q}$ -factorial) projective variety. A *birational contraction* of  $X$  is a birational map  $\beta: X \dashrightarrow Y$ , where  $Y$  is likewise nice and the exceptional locus of  $\beta^{-1}$  has codimension at least two. The *Mori chamber*  $\text{Mor}(\beta)$  is the cone in  $\text{Pic}_{\mathbf{Q}}(X)$  spanned by the pullback of the ample cone of  $Y$  and the exceptional divisors of  $\beta$ . The map  $\beta$  can be recovered from its Mori cone as

$$X \dashrightarrow \text{Proj} \bigoplus_{m \geq 0} H^0(X, mD)$$

for any  $D \in \text{Mor}(\beta)$ . Thus, understanding the Mori chambers of  $X$  is equivalent to understanding all the birational contractions of  $X$ . In nice cases (for example, when  $X$  is toric or Fano), the Mori chambers form a finite polyhedral partition of a piece of  $\text{Pic}_{\mathbf{Q}}(X)$ .

The aim of the Hassett–Keel program is twofold:

- (1) Describe the Mori chamber decomposition of the  $\langle \lambda, \delta \rangle$  plane in  $\text{Pic}_{\mathbf{Q}}(\overline{M}_g)$ , where  $\lambda$  is the class of the Hodge bundle and  $\delta$  the class of the boundary.
- (2) Give a modular description, if possible, of the spaces corresponding to each Mori chamber.

One reason to restrict to the  $\langle \lambda, \delta \rangle$  plane is that it contains the canonical divisor ( $K = 13\lambda - 2\delta$ ) and ample divisors ( $a\lambda - \delta$  for  $a > 11$ ). Hence, knowing the Mori chamber decomposition for this plane would give a sequence of birational models of  $\overline{M}_g$  beginning with  $\overline{M}_g$  itself and culminating in the canonical model, realizing the Minimal Model Program for  $\overline{M}_g$ . The intermediate spaces can be interpreted as *log canonical models*

$$\text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_g, m(K + \alpha\delta)).$$

The spaces  $\overline{M}_g$ ,  $\overline{M}_g^{\psi}$  and  $\overline{M}_g^{\text{tn}}$  are the first steps of the Hassett–Keel program. They correspond to the following chambers (also shown in Figure 2)

$$\overline{M}_g : \quad a\lambda - b\delta \text{ for } a > 0 \text{ and } a/b > 11,$$

$$\overline{M}_g^{\psi} : \quad a\lambda - b\delta \text{ for } a > 0 \text{ and } 11 \geq a/b > 10,$$

$$\overline{M}_g^{\text{tn}} : \quad a\lambda - b\delta \text{ for } a > 0 \text{ and } 10 > a/b > 10 - \epsilon.$$

The precise value of  $\epsilon > 0$  in the last case is not known. The above is the extent of our knowledge for arbitrary  $g$ . We also know that the hyperelliptic locus must be in the base locus of the birational

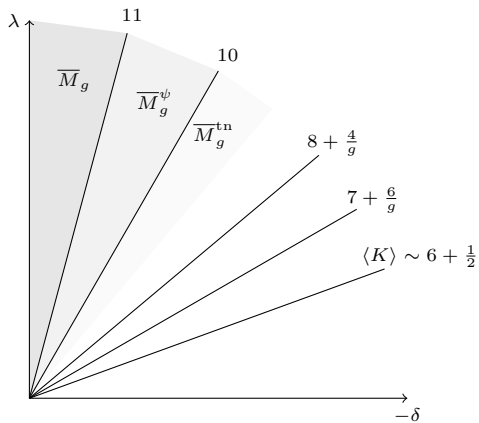


FIGURE 2. Mori chambers of the first three birational models of  $\overline{M}_g$  along with some important rays in the  $\langle \lambda, \delta \rangle$  plane. The numbers represent slopes.

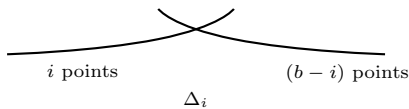


FIGURE 3. Generic points of the boundary components of  $\overline{M}_{0,b}$ .

contraction for the rays  $\langle a\lambda - \delta \rangle$  for  $a < 8 + 4/g$  and likewise for the trigonal locus for  $a < 7 + 6/g$ . There are heuristics about what class of singular curves should replace these curves, assuming that the model has an interpretation as the moduli of some class of curves. We refer the reader to the article by Fedorchuk and Smyth [10] for a comprehensive survey.

### Compactifications of $M_{0,b}$

The standard compactification of  $M_{0,b}$  is the Mumford–Knutsen compactification  $\overline{M}_{0,b}$ , parametrizing stable marked rational curves. Recall that a curve  $P$  along with a marked divisor  $\Sigma$  is a *stable marked curve* if

- (1)  $P$  has at worst nodes as singularities,
- (2)  $\Sigma$  lies in the smooth locus of  $P$ ,
- (3)  $\Sigma$  is reduced,
- (4)  $\omega_P \otimes \mathcal{O}_P(\Sigma)$  is ample.

$\overline{M}_{0,b}$  has only quotient singularities and a normal crossings boundary divisor  $\overline{M}_{0,b} \setminus M_{0,b}$ . The marked curves corresponding to the generic points of the boundary components  $\Delta_i$  (for  $i = 3, \dots, \lfloor b/2 \rfloor$ ) are displayed in Figure 3. The philosophy behind the compactification is to force the points of  $\Sigma$  to remain distinct at the cost of allowing  $P$  to degenerate into a reducible curve.

A sequence of alternate compactifications of  $\overline{M}_{0,b}$  is provided by the spaces of *weighted marked rational curves* of Hassett [18]. In these spaces, collisions of marked points are allowed to a certain extent, specified by a rational number  $\epsilon$ . The space  $\overline{M}_{0,b}(\epsilon)$  parametrizes  $\epsilon$ -stable marked rational curves. A curve  $P$  with a marked divisor  $\Sigma$  is  $\epsilon$ -stable if

- (1)  $P$  has at worst nodes as singularities,
- (2)  $\Sigma$  lies in the smooth locus of  $P$ ,
- (3)  $\epsilon \cdot \text{mult}_p \Sigma \leq 1$  for all  $p \in P$ ,
- (4)  $\omega_P \otimes O_P(\epsilon \Sigma)$  is ample.

For  $\epsilon = 1$ , we recover the standard compactification  $\overline{M}_{0,b}$ . For  $\epsilon \geq \epsilon'$ , we have a morphism  $\overline{M}_{0,b}(\epsilon) \rightarrow \overline{M}_{0,b}(\epsilon')$  that sends  $(P, \Sigma)$  to  $(P', \Sigma')$ , where  $P'$  is obtained from  $P$  by contracting the components on which  $\omega_P(\epsilon \Sigma)$  is not ample. The morphism  $\overline{M}_{0,b}(\epsilon) \rightarrow \overline{M}_{0,b}(\epsilon')$  is a divisorial contraction; it contracts a component of the boundary  $\overline{M}_{0,b}(\epsilon) \setminus \overline{M}_{0,b}$  if its generic point parametrizes a marked curve with at most  $1/\epsilon'$  points on one of the components (see Figure 4).

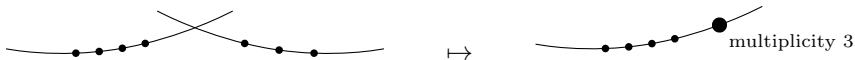


FIGURE 4. The divisorial contraction  $\overline{M}_{0,b}(1/2) \rightarrow \overline{M}_{0,b}(1/3)$

The spaces  $\overline{M}_{0,b}(\epsilon)$  can be interpreted as log canonical models of  $\overline{M}_{0,b}$ . They provide a partial Mori chamber decomposition of the rational Picard group.

### Compactifications of $H_g^d$

We now come to the focus of our work. We first describe the standard compactification of  $H_g^d$  due to Harris and Mumford [15]. The challenge in compactifying  $H_g^d$  is handling degenerations of covers when the branch points come together. Harris and Mumford ingeniously circumvent this issue by forcing the branch points to remain distinct, following the idea behind the Mumford–Knutsen compactification  $\overline{M}_{0,b}$ . Their compactification  $\overline{H}_g^d$  parametrizes admissible covers—these are  $d$ -sheeted covers  $\phi: C \rightarrow P$  satisfying the following conditions:

- (1)  $(P, \text{br } \phi)$  is a stable marked curve,

- (2)  $\phi$  is admissible over the nodes of  $P$  in the following sense: the local picture of  $C \rightarrow P$  near a node of  $C$  is of the following form, for some  $m \geq 1$ :

$$\mathrm{Spec} k[u, v]/uv \rightarrow \mathrm{Spec} k[x, y]/xy$$

$$x, y \mapsto u^m, v^m.$$

Strictly speaking, by  $\mathrm{br} \phi$ , we mean the branch divisor of  $\phi: C^{\mathrm{sm}} \rightarrow P^{\mathrm{sm}}$ . The admissibility condition is better handled using the idea of Abramovich, Corti, and Vistoli [2] of considering covers of orbifold modifications of  $P$  étale over the (orbi)nodes instead of covers of  $P$  itself. This results in a nicer moduli space called the space of *twisted admissible covers*. For the moment, we suppress this subtlety.

The central theme of the present work is to explore what happens when we let the branch points collide. We construct spaces of *weighted admissible covers*, following the compactification  $\overline{M}_{0;b}(\epsilon)$ , and thus allowing the branch points to collide to a certain extent.

**THEOREM (Theorem 1.0.1).** *Let  $\overline{H}_g^d(\epsilon)$  be the space of  $\epsilon$ -admissible covers, which is, roughly speaking, the moduli of  $\phi: C \rightarrow P$ , where  $C$  is a curve of genus  $g$ ,  $P$  a curve of genus 0 and  $\phi$  a map of degree  $d$  satisfying two conditions*

- (1)  $(P, \mathrm{br} \phi)$  is  $\epsilon$ -stable,
- (2)  $\phi$  is admissible over the nodes of  $P$ .

*Then  $\overline{H}_g^d(\epsilon)$  is a projective coarse moduli space. It contains  $H_g^d$  as an open subspace and admits a branch morphism  $\mathrm{br}: \overline{H}_g^d(\epsilon) \rightarrow \overline{M}_{0;b}(\epsilon)$ .*

In the actual definition, the admissibility is handled using orbifold nodes.

For  $\epsilon = 1$ , we recover the Harris–Mumford compactification. As we take smaller and smaller values of  $\epsilon$ , more and more branch points are allowed to coincide, leading to progressively nastier singularities on  $C$ . These singularities are not necessarily Gorenstein. For example, for  $\epsilon \leq 1/4$ , and  $d \geq 3$ , the curve  $C$  can have a spatial triple point singularity (Example 1.7.7). The branch morphism  $\mathrm{br}: \overline{H}_g^d(\epsilon) \rightarrow \overline{M}_{0;b}(\epsilon)$  is not necessarily finite; it has positive dimensional fibers for  $\epsilon \leq 1/6$  (Example 1.7.8).

For  $d = 2$ , the only singularities  $C$  can have are  $A_n$  singularities, namely singularities of the form  $y^2 - x^{n+1}$  (Example 1.7.6). We thus recover the spaces of hyperelliptic curves first constructed by Fedorchuk [9].

In general, the local structure of  $\overline{H}_g^d(\epsilon)$  is horrible. In fact, if  $d$  is large enough and  $\epsilon$  small enough, then  $\overline{H}_g^d(\epsilon)$  is reducible (Example 1.7.9). Nonetheless, for  $d = 2$  and 3, it is irreducible and has at worst quotient singularities; the associated moduli stack is a smooth Deligne–Mumford stack (Theorem 1.5.5).

### Geometry of spaces of trigonal curves

We describe in detail the case of  $d = 3$ . In this case, we get a fascinating picture of the birational geometry of a slight variant of the space of trigonal curves. The spaces of weighted admissible covers form half the story in this picture; the other half is a sequence of *yet* more compactifications, special to the case of  $d = 3$ , that features an interplay of the global geometry of triple covers and the local geometry of triple point singularities.

Let us introduce some notation to describe this picture. Denote by  $T_{g;1}$  the moduli space of  $(\phi: C \rightarrow P, \sigma)$ , where  $P \cong \mathbf{P}^1$ ,  $C$  is a smooth curve of genus  $g$ ,  $\phi$  a simply branched triple cover and  $\sigma \in P$  an additional marked point over which  $\phi$  is unramified. Then  $T_{g;1}$  is a unirational variety of dimension  $2g + 2$ . A standard compactification  $\overline{T}_{g;1}$  is provided by the space of (appropriately marked) admissible covers. In addition, the spaces of weighted admissible covers  $\overline{T}_{g;1}(\epsilon)$  provide a sequence of alternate compactifications. In these spaces, at most  $1/\epsilon$  points of  $\text{br } \phi$  are allowed to coincide; the additional marked point  $\sigma$  is always away from  $\text{br } \phi$ . Note that the only relevant values of  $\epsilon$  are  $\epsilon = 1/j$  for  $j = 1, 2, \dots$ . Furthermore, for  $\omega_P(\epsilon\Sigma + \sigma)$  to be ample, we must have  $\epsilon > 1/b$ , where  $b = 2g + 4$  is the degree of  $\Sigma$ . We thus get the following sequence of birational modifications:

$$(0.0.2) \quad \overline{T}_{g;1} \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}(1/j) \dashrightarrow \overline{T}_{g;1}(1/(j+1)) \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}(1/(b-1)).$$

The rational maps in the above sequence are divisorial contractions; they contract the boundary divisors to loci of higher codimension. The final model  $\overline{T}_{g;1}(1/(b-1))$  parametrizes  $(\phi: C \rightarrow P, \sigma)$  where  $P \cong \mathbf{P}^1$  and the only condition on  $\phi$  is that not all of its branch points be coincident.

Since  $\overline{T}_{g;1}$  is a uniruled variety, there is no canonical model to look for. The goal in this case, according to the Minimal Model Program, is a Fano-fibration. The final space  $\overline{T}_{g;1}(1/(b-1))$  is not such a model; we must search further. A natural idea to proceed is to consider spaces where all the branch points are allowed to coincide. However, it is not clear how to do this. After all, there is no Hassett space that allows all the points of  $\Sigma$  to be coincident.

It turns out that there is not one, but a sequence of compactifications where all the branch points are allowed to coincide. Clearly, if we want to allow such covers, we must disallow some previously allowed covers to have a separated moduli problem. The precise conditions to determine which covers to include and which to exclude are captured by the notion of an  $l$ -balanced cover, depending on an integer  $l$  satisfying  $0 \leq l \leq g$  and  $l \equiv g \pmod{2}$ . The condition of being  $l$ -balanced depends on two invariants: the Maroni invariant and the  $\mu$  invariant.

The Maroni invariant is global in nature. For a connected triple cover  $\phi: C \rightarrow \mathbf{P}^1$  of genus  $g$ , the sheaf  $\phi_* O_C/O_{\mathbf{P}^1}$  is isomorphic to  $O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n)$  for some positive integers  $m, n$  satisfying  $m + n = g + 2$ . The Maroni invariant is the difference  $|m - n|$ .

The  $\mu$  invariant is local in nature and pertains only to those covers whose branch divisor is concentrated at one point. For a connected triple cover  $\phi: C \rightarrow \mathbf{P}^1$  of genus  $g$  with  $\text{br}(\phi) = b \cdot p$  for some  $p \in \mathbf{P}^1$ , denote by  $\tilde{C}$  the normalization of  $C$ . The sheaf  $\phi_*(O_{\tilde{C}}/O_C)$  is isomorphic to  $k[t]/t^m \oplus k[t]/t^n$ , for some positive integers  $m, n$  satisfying  $m + n = g + 2$ . The  $\mu$  invariant is the difference  $|m - n|$ .

We say that a cover  $\phi: C \rightarrow \mathbf{P}^1$  is  $l$ -balanced if

- (1) the Maroni invariant of  $\phi$  is at most  $l$ ,
- (2) if  $\text{br} \phi$  is supported at one point, then the  $\mu$  invariant of  $\phi$  is greater than  $l$ .

**THEOREM** (Theorem 3.3.4, Theorem 4.0.5(1)). *Let  $\overline{T}_{g,1}^l$  be the moduli space of  $(\phi: C \rightarrow P, \sigma)$ , where  $P \cong \mathbf{P}^1$ ,  $\sigma \notin \text{br} \phi$  and  $\phi$  is  $l$ -balanced. Then  $\overline{T}_{g,1}^l$  is a projective coarse moduli space birational to  $T_{g,1}$ .*

It is easy to see that  $\overline{T}_{g,1}^g = \overline{T}_{g,1}(1/(b-1))$ . We thus get a sequence of yet more compactifications that extends (0.0.2):

$$(0.0.3) \quad \overline{T}_{g,1}^g \dashrightarrow \overline{T}_{g,1}^{g-2} \dashrightarrow \dots \dashrightarrow \overline{T}_{g,1}^l \dashrightarrow \overline{T}_{g,1}^{l-2} \dashrightarrow \dots \dashrightarrow \overline{T}_{g,1}^{0 \text{ or } 1}.$$

Most of these steps are flips. The divisorial contractions in (0.0.2) and the flips in (0.0.3) exhibit a sequence of models of  $\overline{T}_{g,1}$  that culminates in a Fano-fibration. Furthermore, we can explicitly describe the Mori chamber decomposition corresponding to the final sequence of flips. The following theorem summarizes the geometry of (0.0.3).

**THEOREM 0.0.1.** *(Theorem 4.0.5) Let  $l$  be an integer with  $0 \leq l \leq g$  and  $l \equiv g \pmod{2}$ .*

(1) The rational map

$$\beta_g: \overline{T}_{g;1}^g \dashrightarrow \overline{T}_{g;1}^{g-2}$$

extends to a morphism, which contracts the “hyperelliptic divisor” to a point.

(2) If  $g$  is even, then the rational map

$$\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$$

extends to a morphism, which contracts the “Maroni divisor” to a  $\mathbf{P}^1$ .

(3) Except in the two cases mentioned above, the rational maps

$$\beta_l: \overline{T}_{g;1}^l \dashrightarrow \overline{T}_{g;1}^{l-2}$$

are isomorphisms away from codimension two. In these cases,  $\text{Exc}(\beta_l)$  is covered by  $K$ -negative curves and  $\text{Exc}(\beta_l^{-1})$  by  $K$ -positive curves, where  $K$  is the canonical divisor.

(4) For even  $g$ , the final model  $\overline{T}_{g;1}^0$  is the quotient of a weighted projective space by an action of  $\mathbf{S}_3$ . In particular, it is Fano of Picard rank one.

(5) For odd  $g$ , the final model  $\overline{T}_{g;1}^1$  admits a morphism to  $\mathbf{P}^1$  whose fibers are Fano of Picard rank one.

(6) For  $0 < l < g$ , the rational Picard group of  $\overline{T}_{g;1}^l$  has rank two. For  $g \neq 3$  it is generated by  $\lambda$  and  $\delta$ . The canonical divisor is given by

$$K = \frac{2}{(g+2)(g-3)} (3(2g+3)(g-1)\lambda - (g^2-3)\delta).$$

(7) There are elements  $D_l$  in the rational Picard group, given in the case of  $g \neq 3$  by

$$\left(\frac{g-3}{2}\right) D_l = \{(7g+6)\lambda - g\delta\} + \frac{l^2}{g+2} \cdot \{9\lambda - \delta\},$$

such that the following hold. For  $l > 0$ , the interior of the cone  $\langle D_l, D_{l+2} \rangle$  is the Mori chamber associated to the model  $\overline{T}_{g;1}^l$ . For even  $g$ , the cone  $\langle D_0, D_2 \rangle$  is the Mori chamber associated to the model  $\overline{T}_{g;1}^0$ . For even (resp. odd)  $g$ , the ray  $\langle D_0 \rangle$  (resp.  $\langle D_1 \rangle$ ) is an edge of the effective cone.

Figure 1 shows a sketch of the Mori chamber decomposition along with an approximate location of the ray  $\langle K \rangle$ .



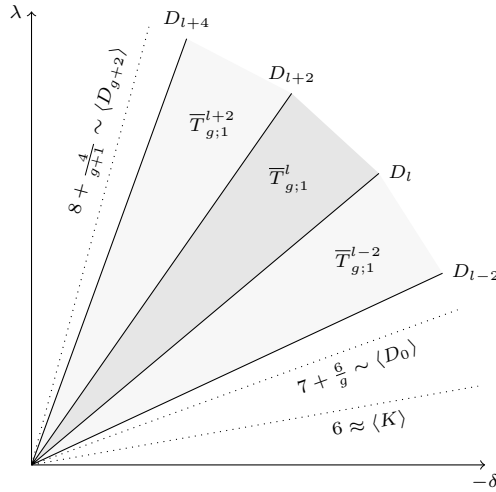


FIGURE 5. The Mori chamber decomposition of  $\text{Pic}_{\mathbf{Q}}$  given by the models  $\overline{T}_{g;1}^l$  and an approximate location of the ray spanned by the canonical class  $K$ .

### 0.1. Conventions

We work over a field  $\mathbf{K}$  of characteristic zero. All schemes are understood to be locally Noetherian schemes over  $\mathbf{K}$ . We reserve the letter  $k$  for (variable) algebraically closed  $\mathbf{K}$ -fields. While working over an algebraically closed field  $k$ , “point” means “ $k$ -point,” unless specified otherwise.

If  $X$  is an algebraic space, and  $x \rightarrow X$  a geometric point then  $O_{X,x}$  denotes the stalk of  $O_X$  at  $x$  in the étale topology; if  $X$  is a scheme, then this is the strict henselization of the local ring of  $X$  at  $x$ . We set  $X_x = \text{Spec } O_{X,x}$ . The reader unfamiliar with these algebraic notions need not worry: nothing is lost by imagining  $O_{X,x}$  to be the ring of convergent power series around  $x$  and  $X_x$  to be a small simply-connected analytic neighborhood of  $x$  in  $X$ . For a local ring  $R$ , the symbol  $R^{\text{sh}}$  denotes its strict henselization and  $\widehat{R}$  its completion.

Stacks are usually denoted by curly symbols and their coarse spaces by the roman equivalents. A category fibered in groupoids is often described by specifying the objects and keeping the morphisms implicit; the morphisms are given by standard commutative squares.

Given morphisms  $\phi: X \rightarrow Y$  and  $Z \rightarrow Y$ , we set  $X_Z = X \times_Y Z$  and  $\phi_Z = \phi \times_Y Z$ . If the  $Y$  is clear from context, we omit it from the notation and simply write  $X \times Z$ .

We use *vector bundle* and *locally free sheaf* interchangeably. The projectivization of a vector bundle  $E$  is denoted by  $\mathbf{P}E$ ; this is the space of one-dimensional *quotients* of  $E$ . The space of one-dimensional sub-bundles of  $E$  is denoted by  $\mathbf{P}_{\text{sub}}E$ . A morphism  $X \rightarrow Y$  is *projective* if it factors as a closed embedding  $X \hookrightarrow \mathbf{P}E$  followed by  $\mathbf{P}E \rightarrow Y$  for some vector bundle  $E$  on  $Y$ .

A *curve* over a scheme  $S$  is a flat, proper morphism whose geometric fibers are purely one-dimensional. The source of this morphism could be a scheme, an algebraic space or a Deligne–Mumford stack; in the last case it is usually denoted by a curly letter. A curve over  $S$  is connected if its geometric fibers are connected. *Genus* always means arithmetic genus. By the genus of a stacky curve, we mean the genus of its coarse space. A *cover* is a representable, flat, surjective morphism.

The symbol  $\mu_n$  denotes the group of  $n$ th roots of unity; its elements are usually denoted by  $\zeta$ .

The methods we use are purely functorial—there is no use of GIT. We construct a moduli space by first constructing the Deligne–Mumford stack, prove that it is proper using the valuative criterion, deduce the existence of a coarse (algebraic) space by the theorem of Keel and Mori [22], and finally prove that it is a projective scheme by exhibiting ample line bundles on it. Throughout, an *algebraic stack* or an *algebraic space* is understood to be in the sense of Laumon and Moret-Bailly [26].

## The big Hurwitz stack

Let  $d$  be a positive integer. Denote by  $H_{g/h}^d$  the classical small Hurwitz space with unordered branch points. It is the coarse space of the small Hurwitz stack  $\mathcal{H}_{g/h}^d$  that parametrizes

$$(1.0.1) \quad \phi: C \rightarrow P,$$

where  $P$  is a smooth curve of genus  $h$ ,  $C$  a smooth curve of genus  $g$ , and  $\phi$  a cover of degree  $d$  with simple branching.

In this chapter, we lay the groundwork for constructing a number of compactifications of  $\mathcal{H}_{g/h}^d$  and its variants. Before we dive into the technical details, let us roughly describe the kinds of compactifications we get as a direct consequence of the main theorem of this chapter. Fix non-negative integers  $g$ ,  $h$  and  $b$ , related by

$$2g - 2 = d(2h - 2) + b.$$

**THEOREM 1.0.1.** *Let  $\epsilon > 0$  be a rational number such that  $\epsilon \cdot b + 2h - 2 > 0$ . Denote by  $\overline{\mathcal{M}}_{h;b}(\epsilon)$  the stack of  $\epsilon$ -stable  $b$ -pointed genus  $h$  curves (as in [18]). Let  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  be the stack of  $\epsilon$ -admissible covers. Roughly speaking, this is the moduli of  $\phi: C \rightarrow P$ , where  $C$  is a curve of genus  $g$ ,  $P$  a curve of genus  $h$  and  $\phi$  a map of degree  $d$  satisfying the following conditions:*

- (1)  $(P, \text{br } \phi)$  is  $\epsilon$ -stable,
- (2)  $\phi$  is admissible over the nodes of  $P$ .

*Then  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  is a proper Deligne–Mumford stack. It contains the classical Hurwitz stack  $\mathcal{H}_{g/h}^d$  as an open substack and admits a projective coarse space  $\overline{H}_{g/h}^d(\epsilon)$ .*

**PROOF.** See Section 1.7 for a precise definition of  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$ . The proof of Theorem 1.0.1 is subsumed by Corollary 1.7.4. □

We follow Abramovich, Corti, and Vistoli [2] to take care of the admissibility condition over the nodes, by considering covers of orbifold modifications of  $P$  instead of  $P$  itself.

Theorem 1.0.1 is a consequence of a more general observation. Let  $\mathcal{M}_{h;b}$  be the stack of  $(P; \Sigma)$  where  $P$  is a nodal curve of arithmetic genus  $g$  and  $\Sigma \subset P$  a divisor of degree  $b$  supported in the smooth locus. Then  $\mathcal{M}_{h;b}$  is an unscrupulous (read non-separated) enlargement of  $\mathcal{M}_{h;b}$ , and the various compactifications  $\overline{\mathcal{M}}_{h;b}(\epsilon)$  are open substacks of  $\mathcal{M}_{h;b}$ , carefully chosen so that they are proper over the base field. In this chapter, we construct an analogous unscrupulous enlargement  $\mathcal{H}_{g/h}^d$  of  $\mathcal{H}_{g/h}^d$ . It admits a morphism  $\text{br} : \mathcal{H}_{g/h}^d \rightarrow \mathcal{M}_{h;b}$  that assigns to a cover its branch divisor. The main theorem of the chapter is the following.

THEOREM 1.0.2. *The morphism  $\text{br} : \mathcal{H}_{g/h}^d \rightarrow \mathcal{M}_{h;b}$  defined by*

$$(\phi : C \rightarrow P) \mapsto (P; \text{br } \phi)$$

*is proper.*

PROOF. Theorem 1.0.2 is subsumed by Theorem 1.3.8. □

$\mathcal{M}_{h;b}$  should be seen as the moduli space of all possible branching data of our covers;  $\mathcal{H}_{g/h}^d$  should be seen as the moduli space of all possible branched covers; and  $\text{br}$  should be seen as the map that assigns to a branched cover the branching data. Theorem 1.0.2 says that any compactification of the moduli of the branching data gives a corresponding compactification of the moduli of branched covers by simply taking the preimage under the branch morphism. The spaces of  $\epsilon$ -admissible covers are obtained in this way using the compactifications  $\overline{\mathcal{M}}_{h;b}(\epsilon)$ .

The following idea motivates our construction of  $\mathcal{H}_{g/h}^d$ . A finite flat cover of degree  $d$ , say  $\phi : C \rightarrow P$ , can be viewed as a family of length  $d$  schemes parametrized by  $P$ . Now, we recall that there exists a moduli stack of length  $d$  schemes; it is an Artin stack  $\mathcal{A}_d$  obtained as the quotient of an affine scheme by a linear group. Said differently, a finite flat cover  $\phi : C \rightarrow P$  of degree  $d$  is simply a map  $\chi : P \rightarrow \mathcal{A}_d$ . In this way, we can interpret the small Hurwitz space  $\mathcal{H}_{g/h}^d$  as the space of maps from smooth curves of genus  $h$  into  $\mathcal{A}_d$ , satisfying certain (deformation open) conditions. The merit of this re-interpretation is that it allows us to use techniques from the well-studied topic of compactifications of spaces of maps into stacks to construct compactifications of  $\mathcal{H}_{g/h}^d$ . The stack  $\mathcal{H}_{g/h}^d$  is the fruit of this approach.

In [1], Abramovich and Vistoli develop techniques to compactify spaces of maps from smooth curves into Deligne–Mumford stacks. Their insightful observation is that to have proper moduli,

the source curve must be allowed to degenerate into not just a nodal curve but an *orbinodal curve*—this is a nodal curve with a particular type of orbifold structure at the node. Our construction of  $\mathcal{H}_{g/h}^d$  follows the same theme. Although we use the central idea of [1], we cannot directly use the results of [1] since  $\mathcal{A}_d$  is not Deligne–Mumford. Nevertheless, the fact that  $\mathcal{A}_d$  is the quotient of an affine scheme by the general linear group allows us to extend the essential arguments without much trouble.

The main case of interest for us is the Hurwitz stack  $\mathcal{H}_g^d$  of covers of  $\mathbf{P}^1$ . However, it is no more challenging to treat covers of curves of arbitrary genus. It is also useful to have the flexibility to fix the ramification type of some fibers of the cover. Therefore, we work in the general context of covers with arbitrary branching over a divisor and prescribed branching over distinct marked points on the base. Furthermore, it is notationally easier and conceptually no harder to refrain from fixing any numerical invariants as far as we can. Therefore, instead of working with  $\mathcal{H}_{g/h}^d$  and  $\mathcal{M}_{h;b}$  we simply have  $\mathcal{H}^d$  and  $\mathcal{M}$ .

The chapter is organized as follows. In Section 1.1, we introduce  $\mathcal{A}_d$ , following Poonen [34]. In Section 1.2, we recall the notion of orbinodal curves. In Section 1.3, we define  $\mathcal{H}^d$  and state the main theorem (Theorem 1.3.8). In Section 1.4, we prove the main theorem. In Section 1.5, we study the local structure of  $\mathcal{H}^d$ . In Section 1.6, we prove that certain open substacks of  $\mathcal{H}^d$  admit projective coarse spaces. In Section 1.7, we describe how to get various compactifications of the small Hurwitz space using  $\mathcal{H}^d$ . Section 1.4 is by far the most technical. The reader who is put off by technicalities may wish to skip most of it, except possibly the proofs of the valuative criteria (Subsection 1.4.5), which contain the main geometric ideas.

### 1.1. The classifying stack of length $d$ schemes

The object of study in this section is the classifying stack of schemes of length  $d$ . We at once begin with the desired functorial description. Consider the category  $\mathcal{A}_d$  fibered over **Schemes** whose objects over a scheme  $S$  are  $(\phi: X \rightarrow S)$ , where  $\phi$  is a finite flat morphism of degree  $d$ .

To prove that  $\mathcal{A}_d$  is indeed an algebraic stack, we consider a more rigidified version. The data of a finite flat morphism  $\phi: X \rightarrow S$  is equivalent to the data of an  $O_S$  algebra  $A$  which is locally free of rank  $d$  as an  $O_S$  module. In the rigidified version of  $\mathcal{A}_d$ , we consider such algebras along with a marked  $O_S$  basis. Namely, we consider the contravariant functor  $\mathcal{B}_d: \mathbf{Schemes} \rightarrow \mathbf{Sets}$  defined by

$$\mathcal{B}_d: S \mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of } (A, \tau), \text{ where } A \text{ is an } O_S \text{ algebra and } \tau: A \rightarrow \\ O_S^{\oplus d} \text{ an isomorphism of } O_S \text{ modules.} \end{array} \right\}.$$

PROPOSITION 1.1.1. ([34, Proposition 1.1]) *The functor  $\mathcal{B}_d$  is represented by an affine scheme  $B_d$  of finite type.*

PROOF. Let  $e_1, \dots, e_d$  be the standard basis of  $O_S^{\oplus d}$ . Then the data  $(A, \tau)$  is equivalent to an  $O_S$  algebra structure on  $O_S^{\oplus d}$ . An  $O_S$  algebra structure is specified by maps of  $O_S$  modules

$$i: O_S \rightarrow O_S^{\oplus d}, \text{ say } 1 \mapsto \sum d_i e_i$$

and

$$m: O_S^{\oplus d} \otimes_S O_S^{\oplus d} \rightarrow O_S^{\oplus d}, \text{ say } e_i \otimes e_j \mapsto \sum c_{ij}^k e_k.$$

These maps make  $O_S^{\oplus d}$  an  $O_S$  algebra with identity  $i(1)$  and multiplication  $m$  if and only if the  $c_{ij}^k$  and the  $d_i$  satisfy certain polynomial conditions. Thus  $\mathcal{B}_d$  is represented by a closed subscheme of  $\mathbf{A}^{d^3+d} = \mathbf{A}\langle c_{ij}^k, d_i \rangle$ .  $\square$

The scheme  $B_d$  admits a natural  $\mathrm{Gl}_d$  action, which is most easily described on the functor of points. A matrix  $M \in \mathrm{Gl}_d(S)$  acts on  $B_d(S)$  by

$$(1.1.1) \quad M: (A, \tau) \mapsto (A, M \circ \tau).$$

PROPOSITION 1.1.2.  *$\mathcal{A}_d$  is equivalent to the quotient  $[B_d/\mathrm{Gl}_d]$ .*

PROOF. The proof is straightforward. There is a morphism from  $\mathcal{A}_d$  to  $[B_d/\mathrm{Gl}_d]$  defined as follows. Consider an object  $\phi: X \rightarrow S$  in  $\mathcal{A}_d(S)$ . Let  $A = \phi_* O_X$ . Then  $A$  is an  $O_S$  algebra which is locally free of rank  $d$  as an  $O_S$  module. Set  $P = \mathrm{Isom}_{O_S\text{-mod}}(A, O_S^{\oplus d})$ . Then  $\pi: P \rightarrow S$  is a principal  $\mathrm{Gl}_d$  bundle. We have a tautological isomorphism

$$\tau: \pi^* A \xrightarrow{\sim} O_P^{\oplus d}.$$

The data  $(\pi^* A, \tau)$  gives a map  $P \rightarrow B_d$ , which is visibly  $\mathrm{Gl}_d$  equivariant. The assignment

$$(\phi: X \rightarrow S) \mapsto (\pi: P \rightarrow S, P \rightarrow B_d)$$

defines a morphism  $\mathcal{A}_d \rightarrow [B_d/\mathrm{Gl}_d]$  which is easily seen to be an isomorphism.  $\square$

Let  $\phi: \mathcal{X}_d \rightarrow \mathcal{A}_d$  be the universal object; set  $\mathcal{A} = \phi_* \mathcal{O}_{\mathcal{X}_d}$  and  $\mathcal{L} = \det \mathcal{A}^\vee$ . We have the trace map

$$\mathrm{tr}: \mathcal{A} \rightarrow \mathcal{O}_{\mathcal{A}_d},$$

which pre-composed with the multiplication  $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  yields a map

$$\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{O}_{\mathcal{A}_d},$$

or equivalently a map

$$\mathcal{A} \rightarrow \mathcal{A}^\vee.$$

Taking determinants and dualizing once more, we obtain a map

$$(1.1.2) \quad \delta: \mathcal{O}_{\mathcal{A}_d} \rightarrow \mathcal{L}^{\otimes 2}.$$

This is the familiar discriminant construction.

**PROPOSITION 1.1.3.** *Let  $\mathcal{E}_d \subset \mathcal{A}_d$  be the maximal open substack over which  $\phi$  is étale. Then*

- (1)  $\mathcal{E}_d \subset \mathcal{A}_d$  is the locus where  $\delta$  is invertible;
- (2)  $\mathcal{E}_d$  is equivalent to  $B\mathbf{S}_d$ , where  $\mathbf{S}_d$  is the symmetric group on  $d$  letters.

**PROOF.** The first assertion is a standard fact in commutative algebra.

For the second, we exhibit an isomorphism from  $\mathcal{E}_d$  to  $B\mathbf{S}_d$ . Let  $\phi: X \rightarrow S$  be an element of  $\mathcal{E}_d(S)$ . Then  $\phi: X \rightarrow S$  is a finite étale morphism of degree  $d$ . Set  $P = \mathrm{Isom}_S(X, \{1, \dots, d\} \times S)$ . Then  $\pi: P \rightarrow S$  is a principal  $\mathbf{S}_d$  bundle. The assignment

$$(\phi: X \rightarrow S) \mapsto (\pi: P \rightarrow S),$$

defines a morphism  $\mathcal{E}_d \rightarrow B\mathbf{S}_d$ , which can be easily checked to be an isomorphism.  $\square$

We denote the zero locus of  $\delta$  in  $\mathcal{A}_d$  by  $\Sigma_d$  and call it the *universal branch locus*. We call the ideal of  $\Sigma_d \subset \mathcal{A}_d$  the *universal discriminant*. Given a map  $\chi: S \rightarrow \mathcal{A}_d$ , given by a cover  $\phi: X \rightarrow S$ , we denote by  $\mathrm{br} \phi$  the pullback  $\chi^* \Sigma_d$  and call it the *branch locus*.

## 1.2. Orbinodal curves

In this section, we recall the notion of an orbinodal curve as introduced by Abramovich and Vistoli [1]. Our brief exposition is based on the work of Olsson [33]. Orbinodal curves are called “balanced twisted curves” in [1] and “twisted curves” in [33].

An orbinodal curve is a stacky modification of a nodal curve at the node points. A nodal curve  $C$  is of the form  $\text{Spec } k[x, y]/xy$  at a node, in the étale topology. An orbinodal curve  $\mathcal{C}$  has the form

$$[\text{Spec } k[u, v]/uv]/\mu_n,$$

where  $\mu_n$  acts by  $u \mapsto \zeta x$ ,  $v \mapsto \zeta^{-1}y$ . Thus, the coarse space of an orbinodal curve is a nodal curve, as seen by computing the ring of invariants

$$(k[u, v]/uv)^{\mu_n} = k[x, y]/xy, \text{ where } x = u^n, y = v^n.$$

A pointed orbinodal curve is an orbinodal curve  $\mathcal{C}$  along with marked points on its coarse space  $C$  such that over a marked point,  $\mathcal{C} \rightarrow C$  has the form

$$[\text{Spec } k[u]/\mu_n] \rightarrow \text{Spec } k[x], \quad x \mapsto u^n,$$

in the étale topology.

Here is the formal definition.

DEFINITION 1.2.1. Let  $S$  be a scheme. We say that

$$(\mathcal{C} \rightarrow C \rightarrow S; p_1, \dots, p_n : S \rightarrow C)$$

is a *pointed orbinodal curve* if

- (1)  $C \rightarrow S$  is a nodal curve and  $p_i : S \rightarrow C$  pairwise disjoint sections.
- (2)  $\mathcal{C} \rightarrow C$  is the coarse space that is required to be an isomorphism over the open set  $C^{\text{gen}} \subset C$  which is the complement of the images of  $p_i$  and the singular locus of  $C \rightarrow S$ :

$$\mathcal{C} \times_C C^{\text{gen}} \xrightarrow{\sim} C^{\text{gen}}.$$

- (3) Let  $c \rightarrow C$  be a geometric point lying over  $s \rightarrow S$ . If  $c$  is a node of  $C_s$ , then there is an étale neighborhood  $U \rightarrow C$  of  $c$ , an open set  $T \subset S$  containing  $s$ , some  $t \in O_T$ , and  $n \geq 1$  for which we have the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} \times_C U & \longrightarrow & U \\ \text{étale} \downarrow & \square & \downarrow \text{étale} \\ [\text{Spec } O_T[u, v]/(uv - t)/\mu_n] & \longrightarrow & \text{Spec } O_T[x, y]/(xy - t^n) \end{array},$$



Here,  $\mu_n$  acts by  $u \mapsto \zeta u$  and  $v \mapsto \zeta^{-1}v$ , and the map on the bottom is given by  $x \mapsto u^n$  and  $y \mapsto v^n$ .

- (4) Let  $s \rightarrow S$  be a geometric point and set  $c = p_i(s)$ . Then there is an étale neighborhood  $U \rightarrow C$  of  $c$  and  $n \geq 1$  for which we have the Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} \times_C U & \longrightarrow & U \\ \text{étale} \downarrow & \square & \downarrow \text{étale} \\ [\text{Spec } O_S[u]/\mu_n] & \longrightarrow & \text{Spec } O_S[x] \end{array},$$

Here,  $\mu_n$  acts by  $u \mapsto \zeta u$ , and the map on the bottom is given by  $x \mapsto u^n$ .

We abbreviate  $(\mathcal{C} \rightarrow C \rightarrow S; p_1, \dots, p_n : S \rightarrow C)$  by  $(\mathcal{C} \rightarrow C; p)$ .

A *morphism* between two pointed orbinodal curves  $(\mathcal{C}_1 \rightarrow C_1; p_{1j})$  and  $(\mathcal{C}_2 \rightarrow C_2; p_{2j})$  is a 1-morphism  $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  such that the induced map  $F: C_1 \rightarrow C_2$  takes  $p_{1j}$  to  $p_{2j}$ .

Although the structure of  $\mathcal{C}$  is specified for *some* étale neighborhood, it holds for any sufficiently small neighborhood. The precise statement from [33] follows.

PROPOSITION 1.2.2. *Let  $(\mathcal{C} \rightarrow C; p)$  be a pointed orbinodal curve over  $S$ . For a geometric point  $c \rightarrow C$ , set*

$$\mathcal{C}^{\text{sh}} = \mathcal{C} \times_C \text{Spec } O_{C,c}.$$

Let  $s \rightarrow S$  be the image of  $c \rightarrow C$ .

- (1) *Suppose  $c$  is a node of  $C_s$  and  $t \in O_{S,s}$  and  $x, y \in O_{C,c}$  are such that  $O_{C,c}$  is isomorphic to the strict henselization of  $O_{S,s}[x, y]/(xy - t^n)$  at the origin. Then, for some  $n \geq 1$ , we have*

$$\mathcal{C}^{\text{sh}} \cong [\text{Spec } O_{C,c}[u, v]/(uv - t, u^n - x, v^n - y)/\mu_n],$$

where  $\mu_n$  acts by  $x \mapsto \zeta x$ ,  $y \mapsto \zeta^{-1}y$ .

- (2) *Suppose  $c = p_i(s)$  and  $x \in O_{C,c}$  is such that  $O_{C,c}$  is isomorphic to the strict henselization of  $O_{S,s}[x]$  at the origin. Then, for some  $n \geq 1$ , we have*

$$\mathcal{C}^{\text{sh}} \cong [\text{Spec } O_{C,c}[u]/(u^n - x)/\mu_n],$$

where  $\mu_n$  acts by  $u \mapsto \zeta u$ .

PROOF. See [33, Proposition 2.2, Definition 2.3]. □

### 1.3. The big Hurwitz stack $\mathcal{H}^d$

Fix a positive integer  $d$ . The goal of this section is to define the big Hurwitz stack  $\mathcal{H}^d$  as a moduli stack of  $d$  sheeted covers of pointed orbifold curves. We begin by defining the stack of divisorially marked, pointed nodal curves. This should be interpreted as the moduli space of the branching data of our branched covers.

DEFINITION 1.3.1. Define the stack  $\mathcal{M}$  of divisorially marked, pointed nodal curves as the category fibered over  $\mathbf{Schemes}_{\mathbf{K}}$  whose objects over  $S$  are

$$\mathcal{M}(S) = \{(P \rightarrow S; \Sigma; \sigma_1, \dots, \sigma_n)\},$$

where

- (1)  $P$  is an algebraic space and  $P \rightarrow S$  a connected nodal curve;
- (2)  $\Sigma \subset P$  is a Cartier divisor, flat over  $S$ , that lies in the smooth locus of  $P \rightarrow S$ ;
- (3)  $\sigma_j: S \rightarrow P$  are pairwise disjoint sections lying in the smooth locus of  $P \rightarrow S$  and away from  $\Sigma$ .

PROPOSITION 1.3.2.  $\mathcal{M}$  is a smooth algebraic stack, locally of finite type.

PROOF. Postponed to Section 1.4. □

We now define  $\mathcal{H}^d$ . Recall our notation from Section 1.1:

$\mathcal{A}_d$  is the classifying stack of schemes of length  $d$ ;

$\mathcal{X}_d \rightarrow \mathcal{A}_d$  is the universal scheme of length  $d$ ;

$\Sigma_d \subset \mathcal{A}_d$  is the universal branch locus;

$\mathcal{E}_d = \mathcal{A}_d \setminus \Sigma_d$  is the locus of étale covers.

DEFINITION 1.3.3. Define the big Hurwitz stack  $\mathcal{H}^d$  as the category fibered over  $\mathbf{Schemes}_{\mathbf{K}}$  whose objects over  $S$  are

$$(1.3.1) \quad \mathcal{H}^d(S) = \{(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \chi: \mathcal{P} \rightarrow \mathcal{A}_d)\},$$

where

- (1)  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n)$  is a pointed orbinodal curve;
- (2)  $\chi: \mathcal{P} \rightarrow \mathcal{A}_d$  is a representable morphism that maps the following to  $\mathcal{E}_d$ : the generic points of the components of  $\mathcal{P}_s$ , the nodes of  $\mathcal{P}_s$ , and the preimages of the marked points in  $\mathcal{P}_s$ , for every fiber  $\mathcal{P}_s$  of  $\mathcal{P} \rightarrow S$ .

A morphism between two objects  $(\mathcal{P}_1 \rightarrow P_1 \rightarrow S_1; \{\sigma_{1j}\}; \chi_1: \mathcal{P}_1 \rightarrow \mathcal{A}_d)$  and  $(\mathcal{P}_2 \rightarrow P_2 \rightarrow S_2; \{\sigma_{2j}\}; \chi_2: \mathcal{P}_2 \rightarrow \mathcal{A}_d)$  over a morphism  $S_1 \rightarrow S_2$  consists of two pieces of data:  $(F, \alpha)$ , where

- (1)  $F$  is a morphism of pointed orbinodal curves:  $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ , and
- (2)  $\alpha$  is a 2-morphism:  $\alpha: \chi_1 \rightarrow \chi_2 \circ F$ ,

such that  $(F, \alpha)$  fits in a Cartesian diagram

$$(1.3.2) \quad \begin{array}{ccc} & & \mathcal{A}_d \\ & \chi_1 \curvearrowright & \nearrow \alpha \\ & & \chi_2 \\ \mathcal{P}_1 & \xrightarrow{F} & \mathcal{P}_2 \\ \downarrow & \square & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

We abbreviate  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \chi: \mathcal{P} \rightarrow \mathcal{A}_d)$  by  $(\mathcal{P} \rightarrow P; \sigma; \chi)$ .

REMARK 1.3.4. The careful reader may wonder what happened to the 2-morphisms between the 1-morphisms from  $\mathcal{P}_1$  to  $\mathcal{P}_2$ . After all, the objects of  $\mathcal{H}^d$  involve stacks, which makes it, *a priori*, a 2-category. However, by [1, Lemma 4.2.3], the 2-automorphism group of any 1-morphism  $\mathcal{P}_1 \rightarrow \mathcal{P}_2$  is trivial. Thus,  $\mathcal{H}^d$  is equivalent to a 1-category [1, Proposition 4.2.2]. What this means explicitly is that we treat two morphisms given by  $(F, \alpha)$  and  $(F', \alpha')$  as *the same* if they are related by a 2-morphism between  $F$  and  $F'$ .

REMARK 1.3.5. Let us explain the condition of representability of  $\chi$  (Definition 1.3.3 (2)). A morphism between two Deligne–Mumford stacks  $F: \mathcal{X} \rightarrow \mathcal{Y}$  is representable if and only if for every geometric point  $x \rightarrow \mathcal{X}$ , the induced map of automorphism groups  $\text{Aut}_x(\mathcal{X}) \rightarrow \text{Aut}_{F(x)}(\mathcal{Y})$  is injective [1, Lemma 4.4.3]. Thus the representability of  $\chi$  means that the stack structure on  $\mathcal{P}$  is the minimal one that affords a morphism to  $\mathcal{A}_d$ .

REMARK 1.3.6. Let us explain the role played by the orbinodes. Consider a local piece of an orbinodal curve near a node; say  $\mathcal{U} = [\text{Spec}(k[x, y]/xy)/\mu_n]$  and an étale cover  $\mathcal{C} \rightarrow \mathcal{U}$ . Observe that the induced map on the coarse spaces  $C \rightarrow U$  is precisely an admissible cover in the sense of

Harris and Mumford [15]. In this way, the orbifoldes provide a way to deal with the admissibility condition.

REMARK 1.3.7. Let us explain the role played by the marked points. Consider a local piece of an orbifold curve near a marked point; say  $\mathcal{U} = [\mathrm{Spec} k[u]/\mu_n]$ . The morphism  $\chi$  maps such a piece into  $\mathcal{E}_d \cong B\mathbf{S}_d$ , corresponding to an étale cover  $\mathcal{C} \rightarrow \mathcal{U}$ . Note that in contrast to the fundamental group of a small piece of a schematic curve, the fundamental group of the stacky curve  $\mathcal{U}$  is not trivial; it is precisely  $\mu_n$ . Thus,  $\mathcal{C} \rightarrow \mathcal{U}$  may be a non-trivial étale cover, specified by the monodromy

$$\mathrm{Aut}_0(\mathcal{U}) = \mu_n \rightarrow \mathrm{Aut}_0(B\mathbf{S}_d) = \mathbf{S}_d.$$

The condition of representability implies that this monodromy map is injective. On the level of coarse spaces, we thus get a cover  $C \rightarrow U$  with monodromy around 0 given by an element of order  $n$  in  $\mathbf{S}_d$ . By taking the open and closed substack of  $\mathcal{H}^d$  where  $\mathrm{Aut}_{\sigma_i}(\mathcal{P})$  has order  $n$ , we in effect impose the condition that the monodromy of  $\mathcal{C} \rightarrow \mathcal{P}$  around  $\sigma_i$  is a permutation  $\pi \in \mathbf{S}_d$  of order exactly  $n$ . By further restricting to the open and closed substack where  $\pi$  has a specific cycle structure, we can fully prescribe the monodromy. In this way, we can get moduli spaces of covers with prescribed ramification type over distinct marked points on the base.

It is useful to have a formulation of  $\mathcal{H}^d$  purely in terms of finite covers. Since a map to  $\mathcal{A}_d$  is nothing but a finite cover of degree  $d$ , we see that  $\mathcal{H}^d$  may be equivalently described as the category whose objects over a scheme  $S$  are

$$(1.3.3) \quad \{(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \phi: \mathcal{C} \rightarrow \mathcal{P})\},$$

where

- (1)  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n)$  is a pointed orbifold curve;
- (2)  $\phi$  is a finite cover of degree  $d$ , étale over the following: the generic points of the components of  $\mathcal{P}_s$ , the nodes of  $\mathcal{P}_s$ , and the preimages of the marked points in  $\mathcal{P}_s$ , for every fiber  $\mathcal{P}_s$  of  $\mathcal{P} \rightarrow S$ .
- (3) Furthermore, the following condition is satisfied: for every open subset  $\mathcal{U} \subset \mathcal{P} \setminus \mathrm{br} \phi$ , the morphism  $\mathcal{U} \rightarrow B\mathbf{S}_d$  corresponding to the étale cover  $\mathcal{C}|_{\mathcal{U}} \rightarrow \mathcal{U}$  is representable.

In this formulation, a morphism between  $(\mathcal{P}_1 \rightarrow P_1 \rightarrow S_1; \sigma_{1j}; \phi_1: \mathcal{C}_1 \rightarrow \mathcal{P}_1)$  and  $(\mathcal{P}_2 \rightarrow P_2 \rightarrow S_2; \sigma_{2j}; \phi_2: \mathcal{C}_2 \rightarrow \mathcal{P}_2)$  is given by  $(F, G)$  where  $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$  is a morphism of pointed orbifold curves and  $G: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  a morphism over  $F$  such that we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{G} & \mathcal{C}_2 \\ \downarrow & \square & \downarrow \\ \mathcal{P}_1 & \xrightarrow{F} & \mathcal{P}_2 \\ \downarrow & \square & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array} .$$

We abbreviate  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \phi: \mathcal{C} \rightarrow \mathcal{P})$  by  $(\mathcal{P} \rightarrow P; \sigma; \phi)$ . We use the formulation of  $\mathcal{H}^d$  in terms of maps to  $\mathcal{A}_d$  or in terms of finite covers depending on whichever is convenient.

$\mathcal{H}^d$  is related to  $\mathcal{M}$  via the branch morphism, which we now define. Consider an object  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \phi: \mathcal{C} \rightarrow \mathcal{P})$  in  $\mathcal{H}^d(S)$ . Identify  $\text{br } \phi$  with its image in  $P$  (note that  $\text{br } \phi$  is anyway disjoint from the stacky points of  $\mathcal{P}$ ). Then  $\text{br } \phi \subset P$  is an  $S$ -flat Cartier divisor. The branch morphism  $\text{br}: \mathcal{H}^d \rightarrow \mathcal{M}$  is defined by

$$\text{br}: (\mathcal{P} \rightarrow P \rightarrow S; \sigma_1, \dots, \sigma_n; \phi: \mathcal{C} \rightarrow \mathcal{P}) \mapsto (P \rightarrow S; \text{br } \phi; \sigma_1, \dots, \sigma_n).$$

**THEOREM 1.3.8 (Main).**  *$\mathcal{H}^d$  is an algebraic stack, locally of finite type. The morphism*

$$\text{br}: \mathcal{H}^d \rightarrow \mathcal{M}$$

*is represented by proper Deligne–Mumford stacks.*

Theorem 1.3.8 is motivated by the treatment of Hurwitz spaces as spaces of maps into a suitable stack by Abramovich, Corti, and Vistoli [2], building on the work of Abramovich and Vistoli [1]. The proof of the main theorem in [1] is quite involved. However, thanks to the advancement of technology related to stacks, we can give a fairly short and conceptual proof of Theorem 1.3.8. We rely most notably on the careful study of orbifold curves by Olsson [33] and the construction of Quot schemes by Olsson and Starr [32]. There is a very general result for the existence of Hom stacks due to Aoki [3], but it is not suitable for our purpose because it does not yield the required finiteness properties.

We prove Theorem 1.3.8 in Section 1.4.

REMARK 1.3.9. The key properties of  $\mathcal{A}_d$  used in the proof are that it is the quotient of an affine scheme by an action of the general linear group and it contains a proper Deligne–Mumford stack  $\mathcal{E}_d$  as an open (to which the generic points, the marked points and the nodes are required to map). It certainly seems possible to prove a generalization of Theorem 1.3.8 where  $\mathcal{A}_d$  is replaced by a suitable such global quotient  $[U/G]$ , generalizing the construction by Ciocan-Fontanine, Kim, and Maulik [5]. However, such a generalization is beyond the scope of the present work.

#### 1.4. Proof of the main theorem

This section is devoted to proving Theorem 1.3.8. The proof is broken down into parts.

**1.4.1. That  $\mathcal{M}$  is a smooth algebraic stack, locally of finite type.** This result is essentially [33, Lemma 5.1]. We sketch a proof for completeness, following Olsson [33] and Hall [14].

LEMMA 1.4.1. [14, Proposition 2.1] *Let  $\pi: P \rightarrow S$  be a nodal curve, where  $P$  is an algebraic space and  $S$  a scheme. Let  $s \rightarrow S$  be a geometric point. Then there is an étale neighborhood  $T \rightarrow S$  of  $s$  such that  $\pi_T: P_T \rightarrow T$  is projective.*

PROOF. Pick points  $x_1, \dots, x_n \in P_s$  in the smooth locus such that the Cartier divisor  $x_1 + \dots + x_n$  is ample on  $P_s$ . Since  $P \rightarrow S$  is smooth along the chosen points  $x_i$ , there is an étale neighborhood  $T \rightarrow S$  of  $s$  such that each  $x_i$  extends to a section  $\sigma_i$  of  $\pi_T: P_T \rightarrow T$ . By passing to a Zariski open, if necessary, assume that the sections map to the smooth locus of  $\pi_T$ . Then the divisor  $\sigma_1(T) + \dots + \sigma_n(T)$  is a Cartier divisor on  $P_T$  which is ample on the fiber over  $s$ . Again, by passing to a Zariski open, if necessary, we get a  $\pi_T$ -ample divisor. Hence  $\pi_T$  is projective.  $\square$

We are ready to prove Proposition 1.3.2, which we recall for convenience.

PROPOSITION 1.3.2.  *$\mathcal{M}$  is a smooth algebraic stack, locally of finite type.*

PROOF. Let  $\mathcal{M}^{b,n} \subset \mathcal{M}$  be the subcategory where the degree of the marked divisor is  $b$  and the number of marked points is  $n$ . It suffices to prove that  $\mathcal{M}^{b,n}$  is an algebraic stack, locally of finite type. For brevity, set  $\mathcal{U} = \mathcal{M}^{0,0}$ .

Clearly, the obvious forgetful morphism  $\mathcal{M}^{b,n} \rightarrow \mathcal{U}$  is representable by smooth algebraic spaces of finite type. Hence, it suffices to prove that  $\mathcal{U}$  is an algebraic stack, locally of finite type.

That  $\mathcal{U}$  is a stack over **Schemes $_{\mathbf{K}}$**  follows from standard descent theory; it will not be reproduced here. Note, however, that it is important to allow algebraic spaces (and not merely schemes)  $P \rightarrow S$  in the definition of  $\mathcal{M}$ .

We now prove that  $\mathcal{U}$  is algebraic. Recall that this means the following two conditions:

- (1) the diagonal  $\mathcal{U} \rightarrow \mathcal{U} \times \mathcal{U}$  is representable by separated algebraic spaces of finite type;
- (2)  $\mathcal{U}$  admits a smooth, surjective morphism from a scheme, locally of finite type.

For (1), we must check that given two objects  $P_i \rightarrow S$  of  $\mathcal{U}$ , for  $i = 1, 2$ , the sheaf  $\text{Isom}_S(P_1, P_2)$  on  $S$  is representable by a separated algebraic space of finite type. It suffices to check this étale locally on  $S$ . Also, this is well-known if  $P_i \rightarrow S$  are projective. Thanks to Lemma 1.4.1, the general case follows.

For (2), it suffices to exhibit a smooth, surjective map to  $\mathcal{U}$  from an algebraic stack, which is itself locally of finite type. Denote by  $\mathcal{M}_{g,k}^{\text{DM}}$  the stack of Deligne–Mumford stable curves of genus  $g$  and  $k$  marked points. This is an algebraic stack of finite type. The forgetful morphism  $\mathcal{M}_{g,k}^{\text{DM}} \rightarrow \mathcal{U}$  is easily seen to be smooth, and the morphism from the disjoint union

$$\bigsqcup_{k \geq 0, g \geq 0} \mathcal{M}_{g,k}^{\text{DM}} \rightarrow \mathcal{U}$$

is surjective.

Finally, the smoothness of  $\mathcal{U}$  follows from the smoothness of  $\mathcal{M}_{g,k}^{\text{DM}}$ . □

**1.4.2. That  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$  is an algebraic stack, locally of finite type.** The overall strategy is to work our way up from  $\mathcal{M}$  to  $\mathcal{H}^d$  via a series of intermediate algebraic stacks. We first introduce some convenient notation. Denote by  $\mathcal{M}^{b,*}$  (resp.  $\mathcal{M}^{*,n}$ ,  $\mathcal{M}^{b,n}$ ) the open substack of  $\mathcal{M}$  where the marked divisor has degree  $b$  (resp. there are  $n$  marked points, degree  $b$  and  $n$  marked points).

The first intermediate step is the stack of pointed orbifold curves. Let  $\mathcal{M}^{\text{orb}}$  be the category over **Schemes** $_{\mathbf{K}}$  whose objects over  $S$  are pointed orbifold curves  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma)$ . Denote by  $\mathcal{M}^{\text{orb} \leq N}$  the subcategory of  $\mathcal{M}^{\text{orb}}$  where the order of the automorphism groups at the points of the orbifold curve is bounded above by  $N$ . There is a morphism  $\mathcal{M}^{\text{orb}} \rightarrow \mathcal{M}^{0,*}$  given by

$$(\mathcal{P} \rightarrow P \rightarrow S; \sigma) \rightarrow (P \rightarrow S; \sigma).$$

We quote, without proof, a theorem of Olsson [33].

**THEOREM 1.4.2.** *[33, Theorem 1.9, Corollary 1.11]  $\mathcal{M}^{\text{orb}}$  and  $\mathcal{M}^{\text{orb} \leq N}$  are smooth algebraic stacks, locally of finite type.  $\mathcal{M}^{\text{orb} \leq N}$  is an open substack of  $\mathcal{M}^{\text{orb}}$ . The morphism  $\mathcal{M}^{\text{orb} \leq N} \rightarrow \mathcal{M}^{0,*}$  is representable by Deligne–Mumford stacks of finite type.*

We have a morphism  $\mathcal{H}^d \rightarrow \mathcal{M}^{\text{orb}}$  given by

$$(\mathcal{P} \rightarrow P; \sigma; \mathcal{C} \rightarrow \mathcal{P}) \mapsto (\mathcal{P} \rightarrow P; \sigma).$$

Define categories  $\mathcal{F}inCov^d$  and  $\mathcal{V}ect^d$  fibered over  $\mathbf{Schemes}_{\mathbf{K}}$  as follows

$$\mathcal{F}inCov^d(S) = \{(\mathcal{P} \rightarrow P \rightarrow S; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P}), \text{ where } \phi \text{ is finite, flat of degree } d\},$$

$$\mathcal{V}ect^d(S) = \{(\mathcal{P} \rightarrow P \rightarrow S; \sigma; \mathcal{F}), \text{ where } \mathcal{F} \text{ is locally free of rank } d \text{ on } \mathcal{P}\}.$$

In both definitions,  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma)$  is a pointed orbinodal curve. We have morphisms

$$(1.4.1) \quad \mathcal{H}^d \rightarrow \mathcal{F}inCov^d \rightarrow \mathcal{V}ect^d \rightarrow \mathcal{M}^{\text{orb}}.$$

Indeed, the first is obvious; the second is given by

$$(\mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P}) \mapsto (\mathcal{P} \rightarrow P; \sigma; \phi_* \mathcal{O}_{\mathcal{C}});$$

and the last by

$$(\mathcal{P} \rightarrow P; \sigma; \mathcal{F}) \mapsto (\mathcal{P} \rightarrow P; \sigma).$$

We analyze each morphism in (1.4.1) one by one.

Before we proceed, we need some results on the structure of orbinodal curves. We first recall the notion of a *generating sheaf* on a Deligne–Mumford stack from [25, § 5.2]. Let  $\mathcal{X}$  be a Deligne–Mumford stack with coarse space  $\rho: \mathcal{X} \rightarrow X$ . A locally free sheaf  $\mathcal{E}$  on  $\mathcal{X}$  is a *generating sheaf* if for every quasi coherent sheaf  $\mathcal{F}$ , the morphism

$$\rho^* \rho_*(\mathcal{H}om_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}) \rightarrow \mathcal{F}$$

is surjective. Equivalently,  $\mathcal{E}$  is a generating sheaf if and only if for every point  $x$  of  $\mathcal{X}$ , the representation of  $\text{Aut}_x(\mathcal{X})$  on the fiber of  $\mathcal{E}$  at  $x$  contains every irreducible representation of  $\text{Aut}_x(\mathcal{X})$ .

**PROPOSITION 1.4.3.** *Let  $S$  be a scheme and  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma)$  a pointed orbinodal curve. There is a scheme  $T$  and a surjective étale morphism  $T \rightarrow S$  such that*

- (1)  $\mathcal{P}_T$  admits a finite, flat morphism from a projective scheme  $Z$ ;
- (2)  $\mathcal{P}_T$  is the quotient of a quasi projective scheme by a linear algebraic group;
- (3)  $\mathcal{P}_T$  admits a generating sheaf.



PROOF. The first statement is due to Olsson [33, Theorem 1.13]. The existence of a finite flat cover  $Z \rightarrow \mathcal{P}_T$  implies that  $\mathcal{P}_T$  is the quotient of an algebraic space  $Y$  by the action of a linear algebraic group by [8, Theorem 2.14]. We may assume that  $T$  is affine and, by Lemma 1.4.1, that  $P_T$  is projective over  $T$ . Then  $P_T$  is quasi-projective. In this case,  $Y$  can be proved to be quasi-projective [25, Remark 4.3]. Finally, since  $\mathcal{P}$  is a quotient stack with a quasi projective coarse space, the third statement follows directly from [25, Theorem 5.3].  $\square$

PROPOSITION 1.4.4.  $\mathcal{V}ect^d \rightarrow \mathcal{M}^{\text{orb}}$  is an algebraic stack, locally of finite type.

PROOF. Let  $S$  be a scheme and  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma)$  an object of  $\mathcal{M}^{\text{orb}}$ . We must prove that the category of vector bundles of rank  $d$  on  $\mathcal{P}$  is an algebraic stack, locally of finite type. It suffices to prove this after passing to an étale cover of  $S$ . By Proposition 1.4.3, we can assume that  $\mathcal{P} \rightarrow S$  admits a generating sheaf and by Lemma 1.4.1, that  $P \rightarrow S$  is projective. Now it can be shown that the stack  $\mathcal{C}oh_{\mathcal{P}/S}$  of coherent sheaves on  $\mathcal{P}$ , flat over  $S$ , is an algebraic stack, locally of finite type. A smooth atlas is given by the Quot schemes of Olsson and Starr [32]. We omit the details; see the pre-print by Nironi [31, § 2.1] for a complete proof. Clearly, the stack of vector bundles of rank  $d$  on  $\mathcal{P}$  is an open substack of  $\mathcal{C}oh_{\mathcal{P}/S}$ .  $\square$

PROPOSITION 1.4.5.  $\mathcal{F}inCov^d \rightarrow \mathcal{V}ect^d$  is representable by algebraic spaces of finite type.

For the proof, we need two easy lemmas.

LEMMA 1.4.6. Let  $S$  be an affine scheme and  $\mathcal{X} \rightarrow S$  be a proper Deligne–Mumford stack with coarse space  $\rho: \mathcal{X} \rightarrow X$ , where  $X$  is a scheme. Let  $\mathcal{F}$  be a coherent sheaf on  $\mathcal{X}$ , flat over  $S$ . Then, there is a finite complex  $M_\bullet$  of locally free sheaves on  $S$ :

$$M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$$

such that for every  $f: T \rightarrow S$ , we have natural isomorphisms

$$H^i(f^* M_\bullet) \xrightarrow{\sim} H^i(\mathcal{X}_T, \mathcal{F}_T);$$

PROOF. Let  $F = \rho_* \mathcal{F}$ . Then  $F$  is a coherent sheaf on  $X$ , flat over  $S$ . Since  $X$  is a proper scheme over  $S$ , the standard theorem on cohomology and base change for schemes [30, §II.5], gives a finite complex of locally free sheaves  $M_\bullet$  with natural isomorphisms

$$(1.4.2) \quad H^i(f^* M_\bullet) \xrightarrow{\sim} H^i(X_T, F_T).$$

Now, the map  $\rho_T: \mathcal{X}_T \rightarrow X_T$  is the map to the coarse space. Since maps to the coarse spaces are cohomologically trivial for quasi-coherent sheaves, we have  $\rho_{T*}(\mathcal{F}_T) = F_T$  and a natural identification

$$(1.4.3) \quad H^i(X_T, F_T) = H^i(\mathcal{X}_T, \mathcal{F}_T).$$

Combining (1.4.2) and (1.4.3), we obtain the result.  $\square$

LEMMA 1.4.7. *Let  $\mathcal{X} \rightarrow S$  and  $\mathcal{F}$  be as in Lemma 1.4.6. Then the contravariant functor from **Schemes** <sub>$S$</sub>  to **Sets** defined by*

$$(f: T \rightarrow S) \mapsto H^0(\mathcal{X}_T, \mathcal{F}_T)$$

*is represented by an affine scheme  $Sect_{\mathcal{F}/S}$  over  $S$ .*

When no confusion is likely, we denote  $Sect_{\mathcal{F}/S}$  by  $Sect_{\mathcal{F}}$ .

PROOF. Let  $M_{\bullet}$  be as in Lemma 1.4.6. Let  $T_i = \text{Spec}_S(\text{Sym}^*(M_i^{\vee}))$  be the total spaces of the vector bundles  $M_i$  (we only care about  $i = 0, 1$ ). Then  $T_i$  are vector bundles over  $S$  and we have a morphism  $T_0 \rightarrow T_1$ . Let  $Sect_{\mathcal{F}} \subset T_0$  be the scheme theoretic preimage of the zero section of  $T_1$ . From the natural isomorphism

$$H^0(f^*M_{\bullet}) \xrightarrow{\sim} H^0(\mathcal{X}_T, \mathcal{F}_T),$$

it is easy to see that  $Sect_{\mathcal{F}}$  represents the desired functor.  $\square$

We now have the tools to prove Proposition 1.4.5.

PROOF OF PROPOSITION 1.4.5. Let  $S$  be a scheme and  $S \rightarrow \mathcal{V}ect^d$  a morphism given by the object  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma; \mathcal{F})$  of  $\mathcal{V}ect^d(S)$ . We must prove that  $\mathcal{F}inCov^d \times_{\mathcal{V}ect^d} S$  is an algebraic space of finite type. It suffices to prove this after passing to an étale cover of  $S$ . So, assume that  $S$  is affine and  $P$  is projective over  $S$ . By an  $O_{\mathcal{P}}$ -algebra structure on  $\mathcal{F}$ , we mean a pair  $(i, m)$ , where  $i: O_{\mathcal{P}} \rightarrow \mathcal{F}$  and  $m: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$  are morphisms of  $O_{\mathcal{P}}$  modules that make  $\mathcal{F}$  a sheaf of  $O_{\mathcal{P}}$ -algebras. Let  $\mathcal{A}lg_{\mathcal{F}}$  be the stack of  $O_{\mathcal{P}}$ -algebra structures on  $\mathcal{F}$ . The operation of taking the spectrum gives an equivalence

$$\mathcal{A}lg_{\mathcal{F}} \xrightarrow{\sim} \mathcal{F}inCov^d \times_{\mathcal{V}ect^d} S.$$

Now, an algebra structure on  $\mathcal{F}$  is determined by a global section (corresponding to  $i$ ) of  $\mathcal{F}$  and one (corresponding to  $m$ ) of  $\mathcal{H}om(\mathcal{F} \otimes \mathcal{F}, \mathcal{F})$  subject to the conditions

$$m \circ (i \otimes \text{id}) = m \circ (\text{id} \otimes 1) = \text{id} \quad (\text{multiplicative identity})$$

$$m \circ \text{sw} = m \quad (\text{symmetry})$$

$$m \circ (\text{id} \otimes m) = m \circ (m \otimes \text{id}) \quad (\text{associativity}),$$

where  $\text{sw}: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$  is the switch  $x \otimes y \mapsto y \otimes x$ . Each of these equations can be interpreted as the vanishing (agreeing with the zero section) of a morphism from  $\text{Sect}_{\mathcal{F}} \times_S \text{Sect}_{\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})}$  to a suitable  $\text{Sect}$  space. For example, the equality

$$m \circ (\text{id} \otimes 1) = \text{id}$$

can be phrased as the vanishing of the morphism

$$\text{Sect}_{\mathcal{F}} \times_S \text{Sect}_{\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})} \rightarrow \text{Sect}_{\mathcal{H}om(\mathcal{F}, \mathcal{F})}$$

defined by

$$(i, m) \mapsto m \circ (i \otimes \text{id}) - \text{id}.$$

Thus,  $\mathcal{A}lg_{\mathcal{F}}$  is represented by the closed subscheme of  $\text{Sect}_{\mathcal{F}} \times_S \text{Sect}_{\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{F})}$  defined by vanishing of the equations given by the conditions above.  $\square$

We finish the final piece of (1.4.1).

PROPOSITION 1.4.8.  $\mathcal{H}^d \rightarrow \mathcal{F}inCov^d$  is an open immersion.

PROOF. Let  $S$  be a scheme and  $S \rightarrow \mathcal{F}inCov^d$  a morphism corresponding to an object  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$  of  $\mathcal{F}inCov^d(S)$ . Let  $\pi: \mathcal{P} \rightarrow S$  be the projection. Denote by  $\Sigma \subset P$  the image in  $P$  of the branch divisor of  $\phi$ . Clearly, the locus  $S_1 \subset S$  over which  $\Sigma$  is disjoint from the singular locus of  $P \rightarrow S$  and the sections  $\sigma_i$  is an open subscheme. Over  $S_1$ , the Cartier divisor  $\Sigma \subset P$  does not contain any components of the fibers and hence it is  $S_1$ -flat.

Let  $\chi: \mathcal{P} \rightarrow \mathcal{A}_d$  be the morphism corresponding to the degree  $d$  cover  $\mathcal{C} \rightarrow \mathcal{P}$ . Let  $\mathcal{I}_{\chi} \rightarrow \mathcal{P}$  be the inertia stack of  $\chi$ . Then  $\mathcal{I}_{\chi} \rightarrow \mathcal{P}$  is a representable finite morphism. The set  $Z \subset \mathcal{P}$  over which  $\mathcal{I}_{\chi}$  has a fiber of cardinality higher than one is a closed subset and its complement is exactly the locus where  $\chi$  is representable. Let  $S_2 = S_1 \cap (S \setminus \pi(Z))$ .

Then, by definition,  $\mathcal{H}^d \times_{\mathcal{F}inCov^d} S = S_2$ , which is an open subscheme of  $S$ .  $\square$

We have finished the first part of the proof of Theorem 1.3.8.

PROPOSITION 1.4.9. *The morphism  $br : \mathcal{H}^d \rightarrow \mathcal{M}$  is an algebraic stack, locally of finite type.*

PROOF. The forgetful morphism  $\mathcal{M} \rightarrow \mathcal{M}^{0,*}$  is represented by algebraic spaces of finite type. Hence, it suffices to show that  $\mathcal{H}^d \rightarrow \mathcal{M}^{0,*}$  is an algebraic stack, locally of finite type. We have the sequence of morphisms

$$(1.4.4) \quad \mathcal{H}^d \rightarrow \mathcal{F}inCov^d \rightarrow \mathcal{V}ect^d \rightarrow \mathcal{M}^{orb} \rightarrow \mathcal{M}^{0,*}.$$

Starting from the right, Theorem 1.4.2, Proposition 1.4.4, Proposition 1.4.5 and Proposition 1.4.8 imply that each of the morphisms above is an algebraic stack, locally of finite type. Hence, so is their composite.  $\square$

**1.4.3. That  $br : \mathcal{H}^d \rightarrow \mathcal{M}$  is of finite type.** The strategy in this section is to study (1.4.4) more carefully and trim down the intermediate stacks so that they are of finite type.

PROPOSITION 1.4.10. *The morphism  $\mathcal{H}^d \rightarrow \mathcal{M}^{orb}$  factors through the open substack  $\mathcal{M}^{orb \leq N}$  for any  $N \geq d!$ .*

As the proof shows, one can do better than  $d!$ , but the actual number is not very important.

PROOF. Take an object  $(\mathcal{P} \rightarrow P \rightarrow S; \sigma; \chi)$  of  $\mathcal{H}^d$ . Let  $p$  be a point of  $\mathcal{P}$  which is either a node or a marked point in its fiber. Then  $\chi$  maps a neighborhood of  $p$  into  $\mathcal{E}_d \cong BS_d$ . Since  $\chi$  is required to be representable, we have

$$\text{Aut}_p(\mathcal{P}) \hookrightarrow \text{Aut}_{\chi(p)}(BS_d) = \mathbf{S}_d.$$

In particular, the size of  $\text{Aut}_p(\mathcal{P})$  is at most  $d!$ .  $\square$

Set

$$\mathcal{H}_b^d = \mathcal{M}^{b,*} \times_{\mathcal{M}} \mathcal{H}^d,$$

and denote by  $\mathcal{V}ect_{l,N}^d$  the open substack of  $\mathcal{V}ect^d$  parametrizing vector bundles of fiberwise degree  $l$  and at most  $N$  dimensional space of global sections.

PROPOSITION 1.4.11. *The morphism  $\mathcal{H}_b^d \rightarrow \mathcal{V}ect^d$  factors through the open substack  $\mathcal{V}ect_{l,N}^d$  for  $l = -b/2$  and any  $N \geq d$ .*

PROOF. Consider a geometric point  $(\mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$  of  $\mathcal{H}_b^d$ . Then, the branch divisor of  $\phi$ , which is a section of  $\det \phi_* O_{\mathcal{C}}^{\otimes(-2)}$ , has degree  $b$ . Hence  $\phi_* O_{\mathcal{C}}$  has degree  $l$ . Furthermore, since  $\mathcal{C}$  is a reduced curve which is a degree  $d$  cover of the connected curve  $\mathcal{P}$ , we must have  $h^0(\phi_* O_{\mathcal{C}}) \leq d$ .  $\square$

PROPOSITION 1.4.12. *The morphism  $\mathcal{V}ect_{l,N}^d \rightarrow \mathcal{M}^{\text{orb}}$  is of finite type.*

For the proof, we need some results about the boundedness of families of sheaves on Deligne–Mumford stacks. Let  $S$  be an affine scheme and  $\mathcal{X} \rightarrow S$  a Deligne–Mumford stack with coarse space  $\rho: \mathcal{X} \rightarrow X$ , and a generating sheaf  $\mathcal{E}$ . Let  $O_X(1)$  be an  $S$ -relatively ample line bundle on  $X$ . Let  $U$  be an  $S$ -scheme, not necessarily of finite type, and  $\mathcal{F}$  a sheaf on  $\mathcal{X}_U$ . We say that the family of sheaves  $(\mathcal{X}_U, \mathcal{F})$  is *bounded* if there is an  $S$ -scheme  $T$  of finite type and a sheaf  $\mathcal{G}$  on  $\mathcal{X}_T$  such that every geometric fiber  $(\mathcal{X}_u, \mathcal{F}_u)$  appearing in  $(\mathcal{X}_U, \mathcal{F})$  over  $U$  appears in  $(\mathcal{X}_T, \mathcal{G})$  over  $T$ . In this case, we say that  $(\mathcal{X}_T, \mathcal{G})$  *bounds*  $(\mathcal{X}_U, \mathcal{F})$ .

Set

$$F_{\mathcal{E}}(-) = \rho_* \mathcal{H}om_{\mathcal{X}}(\mathcal{E}, -).$$

Then  $F_{\mathcal{E}}$  takes exact sequences of quasi-coherent sheaves on  $\mathcal{X}$  to exact sequences on of quasi-coherent sheaves on  $X$ , because  $\rho_*$  is cohomologically trivial.

LEMMA 1.4.13. *In the above setup, if the family  $(X_U, F_{\mathcal{E}}(\mathcal{F}))$  is bounded, then the family  $(\mathcal{X}_U, \mathcal{F})$  is also bounded.*

PROOF. Since  $F_{\mathcal{E}}(\mathcal{F})$  is bounded, we have a surjection

$$O_X(-M)^{\oplus N} \otimes_S O_U \twoheadrightarrow F_{\mathcal{E}}(\mathcal{F})$$

for large enough  $M$  and  $N$ . Since  $\mathcal{E}$  is a generating sheaf, this gives a surjection

$$(1.4.5) \quad \mathcal{E} \otimes_{\mathcal{X}} O_X(-M)^{\oplus N} \otimes_S O_U \twoheadrightarrow \mathcal{F}.$$

Let  $\mathcal{K}$  be the kernel. Then,  $(X_U, F_{\mathcal{E}}(\mathcal{K}))$  is also bounded, and by the same reasoning as above, we have a surjection

$$(1.4.6) \quad \mathcal{E} \otimes_{\mathcal{X}} O_X(-M')^{\oplus N'} \otimes_S O_U \twoheadrightarrow \mathcal{K}$$

for large enough  $M'$  and  $N'$ . Combining (1.4.5) and (1.4.6),  $\mathcal{F}$  can be expressed as the cokernel

$$\mathcal{E} \otimes_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}(-M')^{\oplus M'} \otimes_S \mathcal{O}_U \rightarrow \mathcal{E} \otimes_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}(-M)^{\oplus M} \otimes_S \mathcal{O}_U \rightarrow \mathcal{F}.$$

Set

$$\mathcal{H} = \mathcal{H}om_{\mathcal{X}} \left( \mathcal{E} \otimes_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}(-M')^{\oplus N'}, \mathcal{E} \otimes_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}(-M)^{\oplus N} \right),$$

and  $T = Sect_{\mathcal{H}/S}$ . By Lemma 1.4.7,  $T \rightarrow S$  is of finite type. Letting  $\mathcal{G}$  be the cokernel of the universal homomorphism on  $\mathcal{X}_T$ , we see that  $(\mathcal{X}_T, \mathcal{G})$  bounds  $(\mathcal{X}_U, \mathcal{F})$ .  $\square$

REMARK 1.4.14. In the case of a curve  $\mathcal{X} \rightarrow S$ , the family  $(X_U, F_{\mathcal{E}}(\mathcal{F}))$  is bounded if the degree, rank and  $h^0$  of  $F_{\mathcal{E}}(\mathcal{F})_u$  are bounded for  $u \in U$ .

We now have the tools to prove Proposition 1.4.12.

PROOF OF PROPOSITION 1.4.12. Let  $S$  be a connected affine scheme and  $S \rightarrow \mathcal{M}^{\text{orb}}$  a morphism given by the pointed orbinodal curve  $(\mathcal{P} \xrightarrow{\rho} P \rightarrow S; \sigma)$ . We must prove that  $\mathcal{V}ect_{i,N}^d \times_{\mathcal{M}^{\text{orb}}} S \rightarrow S$  is of finite type. After passing to an étale cover of  $S$  if necessary, assume that

- (1)  $P \rightarrow S$  is projective with relatively ample line bundle  $\mathcal{O}_P(1)$  (this is possible by Lemma 1.4.1),
- (2) We have a generating sheaf  $\mathcal{E}$  on  $\mathcal{P}$  (this is possible by Proposition 1.4.3).

Set  $E = \rho_* \mathcal{E}$ . Since  $\mathcal{E} \otimes \rho^* \mathcal{O}_P(-1)$  is also a generating sheaf, by twisting  $\mathcal{E}$  by  $\rho^* \mathcal{O}_P(-1)$  enough times, assume that we have a surjection  $\mathcal{O}_{\mathcal{P}}^{\oplus M} \rightarrow \mathcal{E}$  for some  $M$ .

Let  $U \rightarrow \mathcal{V}ect_{i,N}^d \times_{\mathcal{M}^{\text{orb}}} S$  be a surjective map from a scheme (not necessarily of finite type), given by the family  $(\mathcal{P}_U \rightarrow P_U \rightarrow U; \sigma; \mathcal{F})$ .

CLAIM.  $(\mathcal{P}_U, \mathcal{F})$  is a bounded family of sheaves.

PROOF. Set

$$F = F_{\mathcal{E}}(\mathcal{F}) = \rho_* \mathcal{H}om_{\mathcal{P}}(\mathcal{E}, \mathcal{F}).$$

By Remark 1.4.14, it suffices to show that the degree, rank and  $h^0$  of  $F_u$  are bounded. The rank of  $F_u$  is constant; the degree of  $\mathcal{H}om(\mathcal{E}, \mathcal{F})_u$  is constant. It is easy to see that the degree of  $\mathcal{H}om(\mathcal{E}, \mathcal{F})_u$  and the degree of  $F_u$  differ by a bounded amount, depending only on  $\mathcal{P} \rightarrow P$  and  $\mathcal{E}$ . Hence the degree of  $F_u$  is bounded. Likewise, it is easy to see that  $h^0(F_u)$  and  $h^0(\mathcal{H}om(\rho_* \mathcal{E}, \rho_* \mathcal{F})_u)$

differ by a bounded amount, depending only on  $\mathcal{P} \rightarrow P$  and  $\mathcal{E}$ . On the other hand,

$$\begin{aligned} H^0(\mathcal{H}om(\rho_*\mathcal{E}, \rho_*\mathcal{F})_u) &= \text{Hom}(E_u, \rho_*\mathcal{F}_u) \\ &\subset \text{Hom}(O_{P_u}^{\oplus M}, \rho_*\mathcal{F}_u) \\ &= H^0(\mathcal{F}_u)^{\oplus M}. \end{aligned}$$

By hypothesis, the final vector space has dimension at most  $MN$ . It follows that  $h^0(F_u)$  is bounded.  $\square$

Let  $T \rightarrow S$  be of finite type and  $(\mathcal{P}_T, \mathcal{G})$  a family that bounds  $(\mathcal{P}_U, \mathcal{F})$ . By shrinking  $T$  if necessary,  $\mathcal{G}$  is a vector bundle of rank  $d$  with fiberwise degree  $l$  and dimension of global sections at most  $N$ . Then we have a surjective map  $T \rightarrow \mathcal{V}ect_{l,N}^d \times_{\mathcal{M}^{\text{orb}}} S$ . It follows that the latter is of finite type.  $\square$

We have now finished the second part of the proof of Theorem 1.3.8.

PROPOSITION 1.4.15.  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$  is of finite type.

PROOF. Since the open substacks  $\mathcal{M}^{b,*}$  cover  $\mathcal{M}$  for  $b = 0, 1, 2, \dots$ , it suffices to show that  $\text{br} : \mathcal{H}_b^d = \mathcal{H}^d \times_{\mathcal{M}} \mathcal{M}^{b,*} \rightarrow \mathcal{M}^{b,*}$  is of finite type. With  $l = -b/2$ , and  $N$  large enough, we have the following diagram,

$$\begin{array}{ccccc} & \mathcal{H}_b^d & \xrightarrow{0} & \mathcal{F}inCov^d & \\ & \searrow & & \downarrow 1 & \\ & & \mathcal{V}ect_{l,N}^d & \longrightarrow & \mathcal{V}ect^d \\ & \searrow & \searrow 3 & & \downarrow \\ \mathcal{M}^{b,*} & \longrightarrow & \mathcal{M}^{\text{orb} \leq N} & \longrightarrow & \mathcal{M}^{\text{orb}} \\ & \uparrow 4 & \longleftarrow 2 & & \\ & \mathcal{H}_b^d & & & \end{array}$$

The thick arrows in the diagram are known to be of finite type: (0) is an open immersion, (1) is of finite type by Proposition 1.4.5, (2) by Theorem 1.4.2, and (3) by Proposition 1.4.12. Recall that for algebraic stacks  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ , all locally of finite type, and morphisms  $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ , we have the following:

- (1) If  $\mathcal{X} \rightarrow \mathcal{Y}$  and  $\mathcal{Y} \rightarrow \mathcal{Z}$  are of finite type, then  $\mathcal{X} \rightarrow \mathcal{Z}$  is also of finite type;
- (2) If  $\mathcal{X} \rightarrow \mathcal{Z}$  is of finite type, then  $\mathcal{X} \rightarrow \mathcal{Y}$  is also of finite type.

Using the two repeatedly reveals that (4) is also of finite type.  $\square$

**1.4.4. That  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$  is Deligne–Mumford.**

PROPOSITION 1.4.16.  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$  is represented by Deligne–Mumford stacks.

PROOF. The proof is straightforward. By Theorem 1.4.2, it suffices to check that  $\mathcal{H}^d \rightarrow \mathcal{M}^{\text{orb}}$  is represented by Deligne–Mumford stacks. In other words, we want this morphism to have unramified inertia. This can be checked on points. Let  $(\mathcal{P} \rightarrow P \rightarrow \text{Spec } k; \sigma; \mathcal{C} \rightarrow \mathcal{P})$  be a geometric point of  $\mathcal{H}^d$ . We must show that  $\mathcal{C}$  has no infinitesimal automorphisms over the identity of  $\mathcal{P}$ . As  $\mathcal{C} \rightarrow \mathcal{P}$  is a finite cover, these automorphisms are classified by  $\text{Hom}_{\mathcal{C}}(\Omega_{\mathcal{C}/\mathcal{P}}, O_{\mathcal{C}})$ . Since  $\mathcal{C} \rightarrow \mathcal{P}$  is unramified on the generic points of the components,  $\Omega_{\mathcal{C},\mathcal{P}}$  is supported on a zero dimensional locus. Since  $\mathcal{C}$  is reduced, it follows that  $\text{Hom}_{\mathcal{C}}(\Omega_{\mathcal{C}/\mathcal{P}}, O_{\mathcal{C}}) = 0$ .  $\square$

**1.4.5. That  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$  is proper.** We check that  $\text{br}$  is proper by verifying the valuative criterion. Two pieces of notation will be helpful. If  $S$  is the spectrum of a local ring, denote by  $S^\circ$  the punctured spectrum

$$S^\circ = S \setminus \{\text{closed point of } S\}.$$

For a Deligne–Mumford stack  $\mathcal{X}$  with coarse space  $\mathcal{X} \rightarrow X$  and a geometric point  $x \rightarrow X$ , set

$$\mathcal{X}_x = \mathcal{X} \times_X \text{Spec } O_{X,x}.$$

It will be convenient to work with the spectrum of a *henselian* DVR. The reader unfamiliar with this notion should imagine it to be a small (in particular, simply-connected) complex disk.

We begin with a simple lemma about the following setup. Let  $r$  be a positive integer and  $G$  a finite group. Let  $R$  be a henselian DVR with residue field  $k$  and uniformizer  $t$ . Let  $O_S$  the henselization of  $R[x, y]/(xy - t^r)$  at the point corresponding to  $(t, x, y)$ . For a positive integer  $a$  dividing  $r$ , define a finite extension  $S_a \rightarrow S$  by

$$O_{S_a} = O_S[u, v]/(u^a - x, v^a - y, uv - t^{r/a}).$$

We have an action of  $\mu_a$  on  $S_a$  over the identity of  $S$  by  $u \mapsto \zeta u$  and  $v \mapsto \zeta^{-1}v$ .

LEMMA 1.4.17. *Let  $\chi : S^\circ \rightarrow BG$  be a morphism given by a  $G$  torsor  $E \rightarrow S^\circ$ . Then  $\chi$  extends to a morphism  $[S_r/\mu_r] \rightarrow BG$ . More generally,  $\chi$  extends to a morphism  $[S_a/\mu_a] \rightarrow BG$  if and only if the pullback of  $E$  to  $S_a^\circ$  is trivial. Furthermore, in this case the extension of  $\chi$  is representable if and only if  $a$  is the smallest with the above property.*



PROOF. To extend  $\chi$ , we may work étale locally on the source. We use the étale cover  $S_a \rightarrow [S_a/\mu_a]$ . Note that  $S_a$  is simply connected (it is henselian). Hence the pullback of  $E$  to  $S_a^\circ$  extends to  $S_a$  if and only if this pullback is trivial. Being trivial over  $S_r^\circ$  is automatic, since  $S_r^\circ$  is simply connected.

Note that  $S_r^\circ \rightarrow S^\circ$  is the universal covering space—it is a  $\mu_r$ -torsor where the source  $S_r^\circ$  is simply connected. The  $G$  torsor  $E \rightarrow S^\circ$  corresponds to a homomorphism  $\mu_r \rightarrow G$ . By the theory of covering spaces, the pullback of  $E$  along  $S_a^\circ \rightarrow S^\circ$  is trivial if and only if  $\mu_r \rightarrow G$  factors as

$$(1.4.7) \quad \mu_r \rightarrow \mu_a \rightarrow G,$$

where  $\mu_r \rightarrow \mu_a$  is the map  $\zeta \mapsto \zeta^{r/a}$ . As we saw, in this case, we get a morphism  $\chi: [S_a/\mu_a] \rightarrow BG$ . Let  $s \rightarrow [S_a/\mu_a]$  be the stacky point. Observe that the map on automorphism groups  $\text{Aut}_s([S_a/\mu_a]) \rightarrow \text{Aut}_s(BG)$  is exactly the map  $\mu_a \rightarrow G$  in (1.4.7). Since  $\chi$  is representable precisely when  $\text{Aut}_s([S_a/\mu_a]) \rightarrow \text{Aut}_s(BG)$  is injective, the result follows.  $\square$

PROPOSITION 1.4.18.  $\text{br}: \mathcal{H}^d \rightarrow \mathcal{M}$  is separated.

PROOF. As  $\text{br}$  is of finite type, we may use the valuative criterion. Let  $R$  be a henselian DVR with residue field  $k$ , fraction field  $K$  and uniformizer  $t$ . Set  $\Delta = \text{Spec } R$ . Denote the special, the general and a geometric general point of  $\Delta$  by  $0, \eta$  and  $\bar{\eta}$  respectively. Let  $(\mathcal{P}_i \rightarrow P_i \rightarrow \Delta; \sigma; \chi_i: \mathcal{P}_i \rightarrow \mathcal{A}_d)$ , for  $i = 1, 2$ , be two objects of  $\mathcal{H}^d(\Delta)$  over an object  $(P; \Sigma; \sigma)$  of  $\mathcal{M}(\Delta)$ . Let  $\phi_i: \mathcal{C}_i \rightarrow \mathcal{P}_i$  be the corresponding degree  $d$  covers and let

$$\psi: (\mathcal{C}_1 \rightarrow \mathcal{P}_1)|_\eta \rightarrow (\mathcal{C}_2 \rightarrow \mathcal{P}_2)|_\eta$$

be an isomorphism over the identity of  $P$ . We must show that  $\psi$  extends to an isomorphism of the orbifold curves  $\mathcal{P}_1 \rightarrow \mathcal{P}_2$  and the covers  $\mathcal{C}_1 \rightarrow \mathcal{C}_2$  over all of  $\Delta$ . Recall that  $P^{\text{gen}}$  is the complement of the markings  $\sigma_j$  in the smooth locus of  $\mathcal{P} \rightarrow \Delta$ .

**Step 1: Extending  $\psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  over  $P^{\text{gen}}$ :** Since  $\mathcal{C}_i \rightarrow \mathcal{P}$  is étale over the generic points of the components of  $\mathcal{P}|_0$ , the map  $\psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  extends, except possibly at finitely many points on the central fiber. As a result, on  $P^{\text{gen}}$  we get an isomorphism of vector bundles

$$\psi^\# : \phi_{2*} \mathcal{O}_{\mathcal{C}_2}|_{P^{\text{gen}}} \rightarrow \phi_{1*} \mathcal{O}_{\mathcal{C}_1}|_{P^{\text{gen}}}$$

away from a locus of codimension two. Since  $P^{\text{gen}}$  is smooth, by Hartog's theorem, this isomorphism extends over all of  $P^{\text{gen}}$ . The extension must also be an isomorphism of algebras by continuity.

**Step 2: Extending  $\psi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$  at the non-generic nodes:** Let  $p \rightarrow P|_0$  be a node not in the closure of  $P|_\eta^{\text{sing}}$ . It suffices to extend  $\psi$  étale locally around  $p$ . The local ring  $O_{P,p}$  must be the strict henselization of the ring  $R[x, y]/(xy - t^r)$  at the point corresponding to  $(t, x, y)$  for some positive integer  $r$ . Recall that the  $\chi_i$  are required to map the nodes to the substack  $\mathcal{E}_d \cong BS_d$  corresponding to étale covers. By the first step, the two maps  $\chi_i: \text{Spec } O_{P,p}^\circ \rightarrow BS_d$  are isomorphic. Since both  $\chi_i$  are representable, the structure of orbifoldal curves (Proposition 1.2.2) and Lemma 1.4.17 imply that

$$(\mathcal{P}_1)_p \cong (\mathcal{P}_2)_p \cong \text{Spec}[O_{P,p}[u, v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a],$$

for some divisor  $a$  of  $r$ . Thus, we can get an extension  $\psi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ .

**Step 3: Extending  $\psi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$  at the marked points:** Let  $p \rightarrow P|_0$  be one of the marked points  $\sigma_j(0)$ . Then  $O_{P,p}$  is the henselization of  $R[x]$  at  $(t, x)$ . Let  $\bar{\sigma}_j$  be a geometric generic point of  $P$  over  $\sigma_j: \eta \rightarrow P|_\eta$ . By the structure of orbifoldal curves (Proposition 1.2.2) for  $p$  and  $\bar{\sigma}_j$ , we have the picture for  $i = 1, 2$ :

$$\begin{array}{ccc} \mathcal{P}_{i,p} & \longrightarrow & \text{Spec } O_{P,p} \\ \parallel & & \parallel \\ [\text{Spec } R[v]^{\text{sh}}/\mu_{r_i}] & \longrightarrow & \text{Spec } R[x]^{\text{sh}} \\ \uparrow & & \uparrow \\ [\text{Spec } K[v]^{\text{sh}}/\mu_{r_i}] & \longrightarrow & \text{Spec } K[x]^{\text{sh}} \\ \parallel & & \parallel \\ \mathcal{P}_{i,\bar{\sigma}_j} & \longrightarrow & \text{Spec } O_{P,\bar{\sigma}_j} \end{array},$$

where  $\mu_{r_i}$  acts by  $v \mapsto \zeta v$ . The isomorphism  $\mathcal{P}_1|_\eta \rightarrow \mathcal{P}_2|_\eta$  gives an isomorphism  $\mathcal{P}_{1,\bar{\sigma}_j} \rightarrow \mathcal{P}_{2,\bar{\sigma}_j}$ . In particular, we get  $r_1 = r_2 = r$ . Furthermore, it is easy to see that an isomorphism  $[\text{Spec } K[v]^{\text{sh}}/\mu_r] \rightarrow [\text{Spec } K[v]^{\text{sh}}/\mu_r]$  over the identity of the coarse spaces  $\text{Spec } K[x]^{\text{sh}} \rightarrow \text{Spec } K[x]^{\text{sh}}$  must be of the form  $v \mapsto \zeta v$  for some  $r$ th root of unity  $\zeta$ . Clearly, such an isomorphism can be extended to an isomorphism  $[\text{Spec } R[v]^{\text{sh}}/\mu_r] \rightarrow [\text{Spec } R[v]^{\text{sh}}/\mu_r]$ .

**Step 4: Extending  $\psi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$  at the generic nodes:** This step mirrors the previous step; only the orbifoldal structures are a little different. We give the details for completeness.

Let  $p \rightarrow P|_0$  be a node in the closure of  $P|_\eta^{\text{sing}}$ . Then  $O_{P,p}$  is the henselization of  $R[x, y]/xy$  at  $(t, x, y)$ . Since  $\Delta$  is henselian, we have a section  $\sigma: \Delta \rightarrow P^{\text{sing}}$  with  $\sigma(0) = p$ . Let  $\bar{\sigma}$  be a geometric

generic point of  $P|_\eta$  over  $\sigma: \eta \rightarrow P|_\eta$ . By the structure of orbifold curves (Proposition 1.2.2) for  $p$  and  $\bar{\sigma}$ , we have the picture for  $i = 1, 2$ :

$$\begin{array}{ccc}
\mathcal{P}_{i,p} & \longrightarrow & \text{Spec } O_{P,p} \\
\parallel & & \parallel \\
[\text{Spec } R[u_i, v_i]^{\text{sh}} / (u_i v_i, u_i - x^{r_i}, v_i - y^{r_i}) / \mu_{r_i}] & \longrightarrow & \text{Spec } (R[x, y] / xy)^{\text{sh}} \\
\uparrow & & \uparrow \\
[\text{Spec } K[u_i, v_i]^{\text{sh}} / (u_i v_i, u_i - x^{r_i}, v_i - y^{r_i}) / \mu_{r_i}] & \longrightarrow & \text{Spec } (K[x, y] / xy)^{\text{sh}} \\
\parallel & & \parallel \\
\mathcal{P}_{i,\bar{\sigma}} & \longrightarrow & \text{Spec } O_{P,\bar{\sigma}}
\end{array}$$

The isomorphism  $\psi: \mathcal{P}_1|_\eta \rightarrow \mathcal{P}_2|_\eta$  gives an isomorphism  $\mathcal{P}_{1,\bar{\sigma}} \rightarrow \mathcal{P}_{2,\bar{\sigma}}$ . In particular, we get  $r_1 = r_2 = r$ . Furthermore, see that an isomorphism  $\psi: \mathcal{P}_{1,\bar{\sigma}} \rightarrow \mathcal{P}_{2,\bar{\sigma}}$  over the identity of the coarse spaces  $P_{1,\bar{\sigma}} \rightarrow P_{1,\bar{\sigma}}$  must be of the form  $u_1 \mapsto \zeta_1 u_2$  and  $v_1 \mapsto \zeta_2 v_2$  for some  $r$ th roots of unity  $\zeta_1$  and  $\zeta_2$ . Such an isomorphism can be extended to an isomorphism  $\mathcal{P}_{1,p} \rightarrow \mathcal{P}_{2,p}$ .

**Step 5: Extending  $\psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ :** By Step 2, Step 3 and Step 4, we have an isomorphism  $\psi: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ . By Step 1, we also have an isomorphism  $\psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  except over the node points and the marked points of  $\mathcal{P}_i|_0$ . However,  $\mathcal{C}_i \rightarrow \mathcal{P}_i$  is étale over these points; hence  $\psi$  must extend to an isomorphism  $\psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ . □

The crucial ingredient for properness is the following theorem of Horrocks [20].

**PROPOSITION 1.4.19.** [20, Corollary 4.1.1] *Let  $S$  be the spectrum of a regular local ring. If  $\dim S = 2$ , then every vector bundle on the punctured spectrum  $S^\circ$  is trivial.*

**PROOF.** We only describe the main idea. See [20] for the full details.

Denote by  $i: S^\circ \rightarrow S$  the inclusion map. Let  $E$  be a vector bundle on  $S^\circ$ . If  $\dim S \geq 2$ , then  $i_* E$  can be shown to be a coherent sheaf on  $S$  with depth at least 2. If  $\dim S = 2$ , by the Auslander–Buchsbaum formula, we conclude that  $i_* E$  is free. As every vector bundle on  $S$  is trivial, we conclude that  $E$  is trivial. □

**PROPOSITION 1.4.20.**  $\text{br}: \mathcal{H}^d \rightarrow \mathcal{M}$  is proper.

PROOF. A large chunk of the proof is identical to the proof in the paper of Abramovich and Vistoli [1, Proposition 6.0.4]. The final step is new; it uses Proposition 1.4.19 and the expression of  $\mathcal{A}_d$  as the quotient of an affine scheme by  $\mathrm{Gl}_d$ .

As  $\mathrm{br}$  is of finite type, we may use the valuative criterion. As before, let  $R$  be a henselian DVR with residue field  $k$ , fraction field  $K$  and uniformizer  $t$ . Set  $\Delta = \mathrm{Spec} R$ . Denote the special, the general and a geometric general point of  $\Delta$  by  $0$ ,  $\eta$  and  $\bar{\eta}$  respectively. Let  $(P \rightarrow \Delta; \Sigma; \sigma)$  be an object of  $\mathcal{M}(\Delta)$  and  $(\mathcal{P}|_\eta \rightarrow P|_\eta; \sigma; \chi)$  an object of  $\mathcal{H}^d(\eta)$ . We want to extend it to an object over all of  $\Delta$ , possibly after a base change.

**Step 1. Extending  $\chi$  at the generic points of the components:** This step follows Step 2 in [1, Proposition 6.0.4].

We work étale locally. Let  $\zeta$  be a geometric generic point of a component of  $P|_0$ . Then the local ring  $O_{P,\zeta}$  is also a DVR. Since the branch divisor  $\Sigma$  does not contain any component of  $P|_0$ , the morphism  $\chi$  sends the punctured spectrum  $P_\zeta^\circ$  to  $\mathcal{E}_d$ . We must extend it to a morphism  $\chi: P_\zeta \rightarrow \mathcal{E}_d$ . Since  $\mathcal{E}_d \cong \mathbf{BS}_d$  is a proper Deligne–Mumford stack, such an extension is possible after passing to a finite cover  $\tilde{P}_\zeta \rightarrow P_\zeta$ . By Abhyankar’s lemma, there is an  $n$  such that  $\tilde{P}_\zeta \rightarrow P_\zeta$  is isomorphic to  $P_\zeta \times_{\mathrm{Spec} R} \mathrm{Spec} R[\sqrt[n]{t}] \rightarrow P_\zeta$ . Thus, by passing to a sufficiently big cover  $\mathrm{Spec} R[\sqrt[n]{t}] \rightarrow \mathrm{Spec} R = \Delta$ , we can extend  $\chi$  along the generic points of all the components of  $P|_0$ . Henceforth, replace  $R$  by  $R[\sqrt[n]{t}]$ .

At this point, we have a morphism  $\chi: P \rightarrow \mathcal{A}_d$  defined away from finitely many points on  $P|_0$ .

**Step 2. Extending  $\chi$  at the non-generic nodes:** This step follows Step 3 in [1, Proposition 6.0.4].

Let  $p \rightarrow P|_0$  be a node not in the closure of  $P_\eta|_{\mathrm{sing}}$ . We must describe an orbifold structure at  $p$  and a representable extension of  $\chi$ . It suffices to do both things in the étale topology. The stalk  $O_{P,p}$  is isomorphic to  $R[x, y]^{\mathrm{sh}}/(xy - t^r)$  for some  $r \geq 1$ . Since  $\Sigma$  is supported away from the nodes, the morphism  $\chi$  sends the punctured spectrum  $P_p^\circ$  to  $\mathcal{E}_d \cong \mathbf{BS}_d$ . As in Lemma 1.4.17, let  $a$  be the smallest integer dividing  $r$  such that  $\chi$  extends to a morphism

$$\chi: [\mathrm{Spec} O_{P,p}[u, v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a] \rightarrow \mathcal{E}_d \cong \mathbf{BS}_d,$$

where  $\mu_a$ , as usual, acts by  $u \mapsto \zeta$  and  $v \mapsto \zeta^{-1}v$ . Construct  $\mathcal{P}$  over  $P$  such that

$$\mathcal{P}_p = [\mathrm{Spec} O_{P,p}[u, v]/(u^a - x, v^a - y, uv - t^{r/a})/\mu_a].$$

By Lemma 1.4.17, we have a representable extension  $\chi: \mathcal{P}_p \rightarrow \mathcal{E}_d \cong \mathbf{BS}_d$ .

**Step 3: Extending  $\chi$  at the generic nodes and marked points:** This step follows Step 4 in [1, Proposition 6.0.4].

Let  $p \rightarrow P|_0$  be in the closure of  $P|_\eta^{\text{sing}}$ . First, we extend the orbinode structure  $\mathcal{P}|_\eta$  over  $p$ . Note that  $O_{P,p}$  is isomorphic to the henselization of  $R[x, y]/xy$  at  $(t, x, y)$ . Since  $\Delta$  is henselian, we have a section  $\sigma: \Delta \rightarrow P^{\text{sing}}$  with  $\sigma(0) = p$ . Letting  $\bar{\sigma}$  be a geometric generic point of this section, we get by Proposition 1.2.2

$$\mathcal{P}_{\bar{\sigma}} \cong [\text{Spec } K[u, v]^{\text{sh}} / (uv, u^a - x, v^a - y) / \mu_a],$$

for some positive integer  $a$ . We extend  $\mathcal{P}$  over  $P_p$  by the same formula

$$\mathcal{P}_p \cong [\text{Spec } R[u, v]^{\text{sh}} / (uv, u^a - x, v^a - y) / \mu_a].$$

Having defined the orbinodal structure, we must extend  $\chi$ . Again, note that  $\chi$  sends a neighborhood of  $p$  to the étale locus  $\mathcal{E}_d \cong \mathbf{BS}_d$ . We must find an extension  $\chi: \mathcal{P}_p \rightarrow \mathcal{E}_d$ . We may work étale locally on the source, in particular, on the étale cover  $\text{Spec } O_{P,p}[u, v] / (uv, u^a - x, v^a - y) \rightarrow \mathcal{P}_p$ . We already have  $\chi$  on the punctured spectrum  $(\text{Spec } O_{P,p}[u, v] / (uv, u^a - x, v^a - y))^\circ$ . Since this punctured spectrum is simply connected,  $\chi$  extends to a map  $\chi: O_{P,p}[u, v] / (uv, u^a - x, v^a - y) \rightarrow \mathcal{E}_d$ .

The case of marked points  $p = \sigma_j(0)$  is entirely analogous, if not easier.

**Step 4. Extending  $\chi$  over all of  $\mathcal{P}$ :** By the previous steps, we have a pointed orbinodal structure  $\mathcal{P} \rightarrow P$  and an extension of  $\chi$  on  $\mathcal{P}$  away from finitely many smooth, unmarked (i.e. different from  $\sigma_j$ ) points of  $P|_0$ . Let  $p \rightarrow P|_0$  be a smooth, unmarked point. Recall that  $\mathcal{A}_d \cong [B_d / \text{Gl}_d]$ , where  $B_d$  is an affine scheme (Proposition 1.1.2). The morphism  $\chi: P_p^\circ \rightarrow \mathcal{A}_d$  is equivalent to a  $\text{Gl}_d$  torsor  $E^* \rightarrow P_p^\circ$  and a  $\text{Gl}_d$  equivariant morphism  $E^* \rightarrow B_d$ . However, by Proposition 1.4.19, there are no nontrivial  $\text{Gl}_d$  torsors on  $P_p^\circ$ . In particular,  $E^*$  extends to a  $\text{Gl}_d$  torsor  $E \rightarrow P_p$ . Next,  $E^* \subset E$  is the complement of the codimension two locus  $E|_p$ . Since  $E$  is smooth and  $B_d$  affine, we have an extension  $E \rightarrow B_d$  by Hartog's theorem. The extension is  $\text{Gl}_d$  equivariant by continuity. Thus, we get an extension  $\chi: P_p \rightarrow \mathcal{A}_d$ .

Finally, note that the two divisors  $\chi^* \Sigma_d$  and  $\Sigma$  are supported in the general locus  $P^{\text{gen}}$  and are equal, by construction, on the complement of a codimension two set. Hence, they must be equal.

□

REMARK 1.4.21. It may be helpful to recast Step 4 in terms of finite covers. Let  $p \rightarrow P|_0$  be a smooth, unmarked point. Assume that we have a finite cover  $\phi: C \rightarrow U \setminus \{p\}$ , where  $U$  is a neighborhood of  $p$ . We wish to extend it to a cover over all of  $U$ . By Proposition 1.4.19, the vector bundle  $\phi_*O_C$  extends to a vector bundle over  $U$ . Next, we must extend the  $O_P$  algebra structure of  $\phi_*O_C$ . The algebra structure is specified by maps of vector bundles, which all extend over  $p$  by Hartog's theorem. The extensions continue to satisfy the identities to be an algebra by continuity. The result is an extension of  $\phi$  over all of  $U$ .

The proof of the main theorem is now complete. We recall the statement and collect the pieces of the proof.

THEOREM 1.3.8 (Main).  $\mathcal{H}^d$  is an algebraic stack, locally of finite type. The morphism

$$\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$$

is represented by proper Deligne–Mumford stacks.

PROOF. That  $\text{br}$  is an algebraic stack, locally of finite type is the content of Subsection 1.4.2, culminating in Proposition 1.4.9. That  $\text{br}$  is of finite type is done in Subsection 1.4.3, culminating in Proposition 1.4.15. That  $\text{br}$  is Deligne–Mumford is Proposition 1.4.16. Finally, the properness is checked in Subsection 1.4.5 in Proposition 1.4.18 and Proposition 1.4.20.  $\square$

### 1.5. The local structure of $\mathcal{H}^d$

In this section, we analyze the local structure of  $\mathcal{H}^d$ . The main consequence of our analysis is that  $\mathcal{H}^d$  is smooth for  $d = 2$  and  $3$  (Theorem 1.5.5). Throughout the section, we use the formulation of  $\mathcal{H}^d$  in terms of finite covers instead of in terms of maps to  $\mathcal{A}_d$ .

We recall the standard setup of deformation theory. Let  $k$  be an algebraically closed field over  $\mathbf{K}$ . Denote by  $\mathbf{Art}_k$  the category of local Artin rings with residue field  $k$ . For any object  $(A, m)$  of  $\mathbf{Art}_k$ , denote by  $0$  the special point of  $\text{Spec } A$ . Let  $(A, m)$  and  $(A', m')$  be two object of  $\mathbf{Art}_k$  related by an exact sequence

$$0 \rightarrow J \rightarrow A' \rightarrow A \rightarrow 0.$$

Say that  $A'$  is a *small extension* of  $A$  by  $J$  if  $m' \cdot J = 0$ .

We denote by  $\text{Def}_X$  the standard functor on  $\mathbf{Art}_k$  classifying deformations of  $X$ :

$$\text{Def}_X(A) = \{(X_A \rightarrow \text{Spec } A, i)\},$$

where  $X_A \rightarrow \text{Spec } A$  is a flat morphism and  $i: X_A|_0 \rightarrow X$  an isomorphism. We often shorten  $(X_A \rightarrow \text{Spec } A, i)$  to just  $X_A$ , and call it a *deformation of  $X$  over  $A$* .

Likewise, for a morphism  $\phi: X \rightarrow Y$ , we denote by  $\text{Def}_\phi$  the functor classifying deformations of  $\phi$  (allowing both  $X$  and  $Y$  to vary):

$$\text{Def}_\phi(A) = \{(X_A \rightarrow \text{Spec } A, Y_A \rightarrow \text{Spec } A, \phi_A: X_A \rightarrow Y_A, i_X, i_Y)\},$$

where  $X_A \rightarrow \text{Spec } A$  and  $Y_A \rightarrow \text{Spec } A$  are flat morphisms and  $i_X: X_A|_0 \rightarrow X$  and  $i_Y: Y_A|_0 \rightarrow Y$  are isomorphisms making the obvious commutative diagram

$$(1.5.1) \quad \begin{array}{ccc} X_A|_0 & \xrightarrow{\phi_A|_0} & Y_A|_0 \\ \downarrow i_X & & \downarrow i_Y \\ X & \xrightarrow{\phi} & Y \end{array} .$$

We often shorten the unwieldy  $(X_A \rightarrow \text{Spec } A, Y_A \rightarrow \text{Spec } A, \phi_A: X_A \rightarrow Y_A, i_X, i_Y)$  to just  $(\phi_A: X_A \rightarrow Y_A)$  and call it a *deformation of  $\phi$  over  $A$* .

Let  $\xi = (\mathcal{P} \rightarrow P; \sigma_1, \dots, \sigma_n; \phi: \mathcal{C} \rightarrow \mathcal{P})$  be such that  $(\mathcal{P} \rightarrow P; \sigma_1, \dots, \sigma_n)$  is a (not necessarily proper) pointed orbifold curve over  $k$  and  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  a finite cover, étale over the nodes and the marked points of  $P$ . Denote by  $\text{Def}_\xi$  the functor classifying deformations of  $\xi$ :

$$\text{Def}_\xi(A) = \{(\mathcal{P}_A \rightarrow P_A \rightarrow \text{Spec } A; \sigma_{i,A}; \phi_A: \mathcal{C}_A \rightarrow \mathcal{P}_A, i_C, i_P)\},$$

where  $(\mathcal{P}_A \rightarrow P_A \rightarrow \text{Spec } A; \sigma_{i,A})$  is a (not necessarily proper) pointed orbifold curve,  $\phi: \mathcal{C}_A \rightarrow \mathcal{P}_A$  a finite cover, and  $i_P: \mathcal{P}_A|_0 \rightarrow \mathcal{P}$  and  $i_C: \mathcal{C}_A|_0 \rightarrow \mathcal{C}$  isomorphisms commuting with  $\phi_A$  and  $\phi$  as in (1.5.1).

If  $\xi$  corresponds to a point of  $\mathcal{H}^d$ , then we have a formally smooth morphism

$$\text{Def}_\xi \rightarrow \mathcal{H}^d.$$

Our goal is to understand  $\text{Def}_\xi$ .

Following Fedorchuk [9, § 4.1], we first simplify the task of studying the deformations of  $\xi$  into the study of its deformations on Zariski local pieces. Following his terminology from [9, § 4.1], let  $\{U_i\}$  be an adapted affine open cover of  $P$ . This means that each  $U_i$  contains exactly one from the

following: a node, a marked point or a point of  $\text{supp}(\text{br } \phi)$ . Set

$$\begin{aligned}\mathcal{U}_i &= U_i \times_P \mathcal{P} \\ \mathcal{V}_i &= \mathcal{C} \times_{\mathcal{P}} \mathcal{U}_i \\ \phi_i &= \phi|_{\mathcal{V}_i}: \mathcal{V}_i \rightarrow \mathcal{U}_i \\ \xi_i &= (\mathcal{U}_i \rightarrow U_i; \sigma_i; \phi_i: \mathcal{V}_i \rightarrow \mathcal{U}_i).\end{aligned}$$

In the last equation,  $\sigma_i$  is ignored if  $U_i$  does not contain any marked point. Set  $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ ,  $\mathcal{V}_{ij} = \mathcal{V}_i \cap \mathcal{V}_j$ ,  $\mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ , and so on. Observe that  $\mathcal{U}_{ij}$  does not contain orbinodes, marked points or branch points. To emphasize that the multiple intersections are schemes, we denote them by roman letters  $U_{ij}$ ,  $V_{ij}$ ,  $U_{ijk}$ , and so on.

We have restriction maps  $\text{Def}_{\xi} \rightarrow \text{Def}_{\xi_i}$ .

PROPOSITION 1.5.1. *With the above notation, the map  $\text{Def}_{\xi} \rightarrow \prod_i \text{Def}_{\xi_i}$  is formally smooth.*

PROOF. Let  $0 \rightarrow k \rightarrow A' \rightarrow A \rightarrow 0$  be a small extension. Assume that we are given a deformation  $\xi_A$  of  $\xi$  on  $A$ . Denote the restriction of  $\xi_A$  over  $\mathcal{U}_i$  by  $\xi_{i,A}$ ; it is a deformation of  $\xi_i$ . Suppose, furthermore, that we are given extensions  $\xi_{i,A'}$  of  $\xi_{i,A}$ . We must prove that the  $\xi_{i,A'}$  can be glued to get a global extension  $\xi_{A'}$  of  $\xi_A$ .

Note that, by construction,  $U_{ij}$  is a nonsingular affine scheme. Therefore, its deformations are trivial. Let  $p_{ij}: \mathcal{O}_{\mathcal{U}_{i,A'}}|_{U_{ij}} \rightarrow \mathcal{O}_{\mathcal{U}_{j,A'}}|_{U_{ij}}$  be an isomorphism over the identity

$$\mathcal{O}_{\mathcal{P}_A}|_{U_{ij}} = \mathcal{O}_{\mathcal{U}_{i,A}}|_{U_{ij}} \rightarrow \mathcal{O}_{\mathcal{U}_{j,A}}|_{U_{ij}} = \mathcal{O}_{\mathcal{P}_A}|_{U_{ij}}.$$

The choice of  $p_{ij}$  is given by an element of  $\text{Hom}(\Omega_{U_{ij}}, \mathcal{O}_{U_{ij}})$ . The isomorphisms  $p_{ij}$  may not be compatible on the triple overlaps  $U_{ijk}$ . However, since

$$H^2(\mathcal{H}om(\Omega_{\mathcal{P}}, \mathcal{O}_{\mathcal{P}})) = 0,$$

the two co-cycle defined by  $p_{ij} + p_{jk} - p_{ik}$  on  $U_{ijk}$  is in fact a co-boundary. As a result, by changing the choice of the  $p_{ij}$ , we can assure that they are compatible on triple overlaps. Thus, we obtain an orbinodal curve  $(\mathcal{P}_{A'} \rightarrow P_{A'}; \sigma_{A'})$  over  $A'$  extending  $(\mathcal{P}_A \rightarrow P_A; \sigma_A)$  over  $A$ . This takes care of one piece of an extension  $\xi_{A'}$  of  $\xi_A$ .



Having constructed  $\mathcal{P}_{A'}$ , we construct  $\mathcal{C}_{A'}$  similarly by choosing isomorphisms

$$c_{ij}: \mathcal{O}_{\mathcal{V}_{i,A'}}|_{V_{ij}} \rightarrow \mathcal{O}_{\mathcal{V}_{j,A'}}|_{V_{ij}}.$$

Since  $\phi: V_{ij} \rightarrow U_{ij}$  is étale, we have an equality  $\phi^*\Omega_{U_{ij}} = \Omega_{V_{ij}}$ . Observe that if we wish to extend  $\phi_A: \mathcal{C}_A \rightarrow \mathcal{P}_A$  to  $\phi_{A'}: \mathcal{C}_{A'} \rightarrow \mathcal{P}_{A'}$ , where  $\mathcal{P}_{A'}$  is glued by the  $p_{ij}$  and  $\mathcal{C}_{A'}$  by the  $c_{ij}$ , then  $c_{ij} \in \text{Hom}(\Omega_{V_{ij}}, \mathcal{O}_{V_{ij}})$  must be the pullback of  $p_{ij} \in \text{Hom}(\Omega_{U_{ij}}, \mathcal{O}_{U_{ij}})$ . By choosing the  $c_{ij}$  in this way, we obtain the desired extension  $\mathcal{C}_{A'}$  of  $\mathcal{C}_A$  along with an extension  $\phi_{A'}: \mathcal{C}_{A'} \rightarrow \mathcal{P}_{A'}$  of  $\phi_A: \mathcal{C}_A \rightarrow \mathcal{P}_A$ , completing the second piece of the extension  $\xi_{A'}$  of  $\xi_A$ .  $\square$

Next, we analyze  $\text{Def}_{\xi_i}$ . We use the forgetful morphisms  $\text{Def}_{\xi_i} \rightarrow \text{Def}_{\mathcal{U}_i}$  and  $\text{Def}_{\xi_i} \rightarrow \text{Def}_{\mathcal{V}_i}$ .

PROPOSITION 1.5.2. *Retain the notation of Proposition 1.5.1.*

- (1) *If  $\mathcal{U}_i$  does not contain a point of  $\text{br } \phi$ , then  $\text{Def}_{\xi_i}$  is formally smooth.*
- (2) *If  $\mathcal{U}_i$  contains a point of  $\text{br } \phi$ , then  $\text{Def}_{\xi_i} \rightarrow \text{Def}_{\mathcal{V}_i}$  is formally smooth.*

REMARK 1.5.3. In the second case,  $\mathcal{U}_i$  does not contain any orbifold or marked point. Hence, it is a nonsingular scheme and  $\text{Def}_{\xi_i}$  is simply  $\text{Def}_{\phi_i}$ .

PROOF. In the first case, the map  $\phi_i: \mathcal{V}_i \rightarrow \mathcal{U}_i$  is étale. Therefore, the forgetful map  $\text{Def}_{\xi_i} \rightarrow \text{Def}_{(\mathcal{U}_i; \sigma_i)}$  is an isomorphism. We are thus reduced to showing that the deformations of the pointed orbifold curve  $(\mathcal{U}_i; \sigma_i)$  are unobstructed. This is shown in [2, § 3]. We briefly recall the argument. The obstructions to the deformations lie in  $\mathcal{E}xt^2(\Omega_{\mathcal{U}_i}, \mathcal{O}_{\mathcal{U}_i})$ . Étale locally,  $\mathcal{U}_i$  is at worst a nodal curve; hence  $\mathcal{E}xt^2(\Omega_{\mathcal{U}_i}, \mathcal{O}_{\mathcal{U}_i}) = 0$ .

In the second case,  $\mathcal{U}_i = U_i$  is a nonsingular affine scheme; its deformations are trivial. To verify the smoothness of  $\text{Def}_{\phi_i} \rightarrow \text{Def}_{\mathcal{V}_i}$ , take an extension of rings in  $\mathbf{Art}_k$ , say  $A' \rightarrow A \rightarrow 0$ , a deformation  $\phi_{i,A}: V_{i,A} \rightarrow U_i \times \text{Spec } A$  of  $\phi_i$  over  $A$  and an extension  $V_{i,A'} \rightarrow \text{Spec } A'$  of  $V_{i,A}$ . We must construct an extension  $\phi_{i,A'}: \mathcal{V}_{i,A'} \rightarrow U_i \times \text{Spec } A'$  of  $\phi_{i,A}$ . By the infinitesimal lifting property for  $U_i$ , the map  $V_{i,A} \rightarrow U_i$  extends to a map  $V_{i,A'} \rightarrow U_i$ , yielding such an extension  $\phi_{i,A'}: \mathcal{V}_{i,A'} \rightarrow U_i \times \text{Spec } A'$ .  $\square$

Recall that a scheme (stack) is *smoothable* if it is the flat limit of non-singular schemes (stacks). Let  $\mathcal{H}^d \subset \mathcal{H}^d$  be the open locus consisting of

$$(\mathcal{P} \rightarrow \mathcal{P}; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P}),$$

where  $\mathcal{C}$  and  $\mathcal{P}$  are smooth and  $\phi$  is simply branched.

PROPOSITION 1.5.4. *Retain the notation of Proposition 1.5.1. Let  $S$  be the set of indices  $i$  for which  $U_i$  contains a point of  $\text{br } \phi$ .*

(1)  *$\text{Def}_\xi$  is smooth if and only if  $\text{Def}_{V_i}$  is smooth for all  $i \in S$ .*

(2) *The point of  $\mathcal{H}^d$  given by  $\xi$  is in the closure of  $\mathcal{H}^d$  if and only if  $V_i$  is smoothable for all  $i \in S$ .*

PROOF. Proposition 1.5.1 and Proposition 1.5.2 together give a smooth morphism  $\text{Def}_\xi \rightarrow \prod_{i \in S} \text{Def}_{V_i}$ , proving the first assertion.

For the second assertion, consider the smooth morphism

$$(1.5.2) \quad \text{Def}_\xi \rightarrow \prod_{i \notin S} \text{Def}_{U_i} \times \prod_{i \in S} \text{Def}_{V_i}.$$

For  $i \notin S$ , the  $U_i$  is either a smooth curve or an orbifold curve. In either case, it is smoothable. By the smoothness of (1.5.2), if all the  $V_i$  are smoothable for  $i \in S$  then  $\xi$  is in the closure of the locus of

$$(\mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P}),$$

with smooth  $\mathcal{C}$  and  $\mathcal{P}$ . It is not hard to see that this locus is in the closure of  $\mathcal{H}^d$ , where the only additional constraint is that  $\phi$  be simply branched.  $\square$

We record two important special cases.

THEOREM 1.5.5. *For  $d = 2$  and  $3$ , the stack  $\mathcal{H}^d$  is smooth and contains  $\mathcal{H}^d$  as a dense open substack.*

PROOF. We begin with a general observation. For a finite cover  $\phi: X \rightarrow Y$  of degree  $d$ , we have an exact sequence

$$0 \rightarrow O_Y \rightarrow \phi_* O_X \rightarrow F \rightarrow 0,$$

split by  $1/d$  times the trace map  $\text{tr}: \phi_* O_X \rightarrow O_Y$ . Therefore, the vector bundle  $F$  admits a map  $F \rightarrow \phi_* O_X$ . Since  $\phi_* O_X$  is a sheaf of  $O_Y$  algebras, we get a map  $\text{Sym}^*(F) \rightarrow \phi_* O_X$ , which is clearly surjective. In other words,  $\phi: X \rightarrow Y$  naturally factors as an embedding

$$(1.5.3) \quad \iota: X \hookrightarrow \text{Spec}_Y \text{Sym}^*(F)$$

followed by the projection  $\text{Spec}_Y \text{Sym}^*(F) \rightarrow Y$ .

We return to the proof of Theorem 1.5.5. By Proposition 1.5.4, it suffices to prove that  $\text{Def}_{V_i}$  is smooth and  $V_i$  is smoothable for all  $i$  for which  $\phi_i: V_i \rightarrow U_i$  is ramified. In the case of  $d = 2$ , the embedding  $\iota$  in (1.5.3) exhibits  $V_i$  as a divisor in a nonsingular affine surface. It is now well known that  $\text{Def}_{V_i}$  is smooth. In the case of  $d = 3$ , the embedding  $\iota$  exhibits  $V_i$  as a subscheme of a nonsingular affine threefold. Since  $V_i$  is a reduced curve, it is Cohen–Macaulay. Thus  $V_i$  is a Cohen–Macaulay subscheme of codimension two in a nonsingular affine variety. This lets us conclude that  $\text{Def}_{V_i}$  is smooth (see the book by Hartshorne [17, § 2.8] for a discussion of deformations of Cohen–Macaulay subschemes in codimension two).

By the embedding  $\iota$ , we see that the  $V_i$  have singularities with embedding dimension at most three. Such singularities are known to be smoothable by the work of Schaps [35, Theorem 2].  $\square$

## 1.6. Projectivity

In this section, we prove that the branch morphism is projective on coarse spaces by showing that the Hodge line bundle is relatively anti-ample. We begin by defining the Hodge bundle.

Let  $(\mathcal{P} \rightarrow \mathcal{P}; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$  be the universal object over  $\mathcal{H}^d$ . Let  $\pi_{\mathcal{P}}: \mathcal{P} \rightarrow \mathcal{H}^d$  and  $\pi_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{H}^d$  be the projections. When no confusion is likely, we denote both projections by  $\pi$ . Define the *Hodge bundle*  $\Lambda$  on  $\mathcal{H}^d$  by

$$\Lambda = (R^1\pi_*\mathcal{O}_{\mathcal{C}})^\vee.$$

Then  $\Lambda$  is a locally free sheaf on  $\mathcal{H}^d$ . Define the line bundle  $\lambda$  by

$$\lambda = \det \Lambda.$$

We use additive notation for  $\lambda$ . So,  $-\lambda$  denotes the dual of  $\lambda$ .

Throughout the section, we use without explicit reference that separated Deligne–Mumford stacks have coarse spaces [22, Corollary 1.3]. We also repeatedly use that Deligne–Mumford stacks admit a finite surjective map from a scheme [41, Proposition 2.6]. This is typically used in the following guise: if we have a map from  $X$  to the coarse space  $Y$  of a Deligne–Mumford stack  $\mathcal{Y}$ , then there is  $\tilde{X} \rightarrow X$ , finite and surjective, such that  $\tilde{X} \rightarrow Y$  lifts to  $\tilde{X} \rightarrow \mathcal{Y}$ .

**THEOREM 1.6.1.** *Let  $\mathcal{M}$  be a Deligne–Mumford stack separated over  $\mathbf{K}$  and let  $\mathcal{M} \rightarrow \mathcal{M}$  be a morphism. Set  $\mathcal{H} = \mathcal{M} \times_{\mathcal{M}} \mathcal{H}^d$ . Denote by  $H$  and  $M$  the coarse spaces of  $\mathcal{H}$  and  $\mathcal{M}$  respectively. Then the induced morphism*

$$\text{br}: H \rightarrow M$$

is projective. In particular, if  $M$  is projective, so is  $H$ .

The essential ingredient in the proof is the following lemma.

LEMMA 1.6.2. *Let  $s: \text{Spec } k \rightarrow \mathcal{M}$  be a geometric point, and  $X$  a scheme with a quasi-finite morphism  $X \rightarrow s \times_{\mathcal{M}} \mathcal{H}^d$ . Then the pullback of  $-\lambda$  to  $X$  is ample.*

PROOF. Without loss of generality,  $X$  is reduced and connected. By replacing  $X$  by its normalization  $X^\nu \rightarrow X$  if necessary, assume further that  $X$  is normal.

Let  $(P; \Sigma; \sigma)$  be the marked nodal curve over  $k$  corresponding to the point  $s$  and  $(\mathcal{P} \rightarrow P \times X; \sigma \times X; \phi: \mathcal{C} \rightarrow \mathcal{P})$  the family over  $X$  giving the map to  $s \times_{\mathcal{M}} \mathcal{H}^d$ .

Construct  $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$  by normalizing  $\mathcal{C}$  over  $\mathcal{P}^{\text{sm}}$ . Explicitly,  $\tilde{\mathcal{C}}$  is such that we have

$$\begin{aligned} \tilde{\mathcal{C}} \times_P (P \setminus \Sigma) &= \mathcal{C} \times_P (P \setminus \Sigma), \text{ and} \\ \tilde{\mathcal{C}} \times_P \mathcal{P}^{\text{sm}} &= (\mathcal{C} \times_P \mathcal{P}^{\text{sm}})^\nu. \end{aligned}$$

By the result of Teissier [39], we conclude that the fibers of  $\tilde{\mathcal{C}} \times_P \mathcal{P}^{\text{sm}} \rightarrow X$  are the normalizations of the corresponding fibers of  $\mathcal{C} \times_P \mathcal{P}^{\text{sm}} \rightarrow X$ .

Consider the family of finite covers  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \mathcal{P}$  over  $X$ . Let  $t \rightarrow X$  be a  $k$ -point. Then  $\tilde{\mathcal{C}}_t$  is smooth except over the nodes of  $\mathcal{P}_t$ . It is easy to see that there are only finitely many isomorphism types for the cover  $\mathcal{C}_t \rightarrow \mathcal{P}$ . Since  $X$  is connected, the fibers over  $X$  of  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \mathcal{P}$  must all be isomorphic as finite covers. By replacing  $X$  by a finite cover if necessary, we can make  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \mathcal{P}$  a constant family. In other words, we get  $\tilde{\phi}_0: \tilde{\mathcal{C}}_0 \rightarrow \mathcal{P}_0$  over  $k$  such that

$$\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_0 \times X, \quad \mathcal{P} = \mathcal{P}_0 \times X, \text{ and } \tilde{\phi} = \tilde{\phi}_0 \times X.$$

In the rest of the proof, we treat  $O_{\mathcal{C}}$  and  $O_{\tilde{\mathcal{C}}}$  as bundles on  $\mathcal{P}$ , omitting  $\phi_*$  and  $\tilde{\phi}_*$  to lighten notation. Denote by  $I_\Sigma$  the ideal of  $\Sigma$  in  $\mathcal{P}$ . The inclusion  $O_{\mathcal{C}} \subset O_{\tilde{\mathcal{C}}}$  is an isomorphism except over  $\Sigma \times X$ . Hence, the quotient  $O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}}$  is annihilated by  $I_{\Sigma_0 \times X}^N$  for  $N$  large enough. In other words, for every point  $t$  of  $X$ , we have

$$(1.6.1) \quad I_\Sigma^N \cdot O_{\tilde{\mathcal{C}}_t} \subset O_{\mathcal{C}_t}.$$

As a result,  $O_{\mathcal{C}_t}$  is determined by the subspace  $H^0(O_{\mathcal{C}_t}/I_\Sigma^N \cdot O_{\tilde{\mathcal{C}}_t})$  of  $H^0(O_{\tilde{\mathcal{C}}_t}/I_\Sigma^N \cdot O_{\tilde{\mathcal{C}}_t})$ .

Consider the following sequence on  $\mathcal{P}$ :

$$0 \rightarrow O_{\mathcal{C}}/(I_{\Sigma \times X}^N \cdot O_{\tilde{\mathcal{C}}}) \rightarrow O_{\tilde{\mathcal{C}}}/(I_{\Sigma \times X}^N \cdot O_{\tilde{\mathcal{C}}}) \rightarrow O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}} \rightarrow 0.$$

Applying  $\pi_*$ , we obtain a sequence of vector bundles on  $X$ :

$$(1.6.2) \quad 0 \rightarrow \pi_* (O_{\mathcal{C}}/(I_{\Sigma \times X}^N \cdot O_{\tilde{\mathcal{C}}})) \rightarrow \pi_* (O_{\tilde{\mathcal{C}}}/(I_{\Sigma \times X}^N \cdot O_{\tilde{\mathcal{C}}})) \rightarrow \pi_* (O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}}) \rightarrow 0.$$

Since  $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_0 \times X$ , the middle vector bundle is in fact trivial:

$$\pi_* (O_{\tilde{\mathcal{C}}}/(I_{\Sigma_0 \times X}^N \cdot O_{\tilde{\mathcal{C}}})) = V \otimes O_X, \text{ where } V = H^0(O_{\tilde{\mathcal{C}}_0}/(I_{\Sigma_0}^N \cdot O_{\tilde{\mathcal{C}}_0})).$$

The sequence (1.6.2) gives us a morphism  $\mu: X \rightarrow \mathbf{G}$ , where  $\mathbf{G}$  is the Grassmannian of quotients of  $V$  of the appropriate dimension. Moreover, by our discussion above, for every point  $t$  of  $X$ , the fiber  $\phi_t: \mathcal{C}_t \rightarrow \mathcal{P}_t$  is determined by  $\mu(t)$ . Since  $X \rightarrow s \times_{\mathcal{M}} \mathcal{H}^d$  is quasi-finite,  $\mu$  must also be quasi-finite. We conclude that the pullback to  $X$  of the Plücker line bundle on  $\mathbf{G}$  is ample. By (1.6.2), this pullback is simply  $\det \pi_* (O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}})$ . On the other hand, applying  $\pi_*$  to the exact sequence

$$0 \rightarrow O_{\mathcal{C}} \rightarrow O_{\tilde{\mathcal{C}}} \rightarrow O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}} \rightarrow 0,$$

and keeping in mind that  $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_0 \times X$  is a constant family, we get

$$\det \pi_* (O_{\tilde{\mathcal{C}}}/O_{\mathcal{C}}) \cong \det R^1 \pi_* O_{\mathcal{C}}.$$

We deduce that the right hand side, which is the pullback of  $-\lambda$  to  $X$ , is ample.  $\square$

**PROOF OF THEOREM 1.6.1.** We want to show that  $\text{br}: H \rightarrow M$  is projective. Denote also by  $\lambda$  the pullback to  $\mathcal{H}$  of  $\lambda$  on  $\mathcal{H}^d$ . Since  $\text{Pic}(\mathcal{H}) \otimes \mathbf{Q} = \text{Pic}(H) \otimes \mathbf{Q}$ , we may treat  $\lambda$  as a  $\mathbf{Q}$  line bundle on  $H$ . We claim that  $-\lambda$  is  $\text{br}$ -ample. It suffices to check this on the fibers of  $\text{br}: H \rightarrow M$ . Let  $s \rightarrow M$  be a  $k$ -point and set  $H_s = \text{br}^{-1}(s)$ . Choose a lift  $\bar{s} \rightarrow \mathcal{M}$  of  $s \rightarrow M$ . Then  $H_s$  is the coarse space of  $\bar{s} \times_{\mathcal{M}} \mathcal{H}$ . There is a scheme  $X$  and a finite surjective map  $X \rightarrow \bar{s} \times_{\mathcal{M}} \mathcal{H}$ . Lemma 1.6.2 implies that  $-\lambda$  is ample on  $X$ . Since  $X \rightarrow H_s$  is finite and surjective, we deduce that  $-\lambda$  is ample on  $H_s$ .  $\square$

### 1.7. Spaces of weighted admissible covers

The proper morphism  $\mathcal{H} \rightarrow \mathcal{M}$  lets us construct several compactifications of different variants of the Hurwitz spaces. Some of these have appeared in literature in different guises. In this section, we describe a particularly interesting sequence of new projective compactifications.

Let  $g$ ,  $h$ , and  $b$  be non-negative integers related by

$$2g - 2 = d(2h - 2) + b.$$

Let  $\mathcal{M}_{h;b} \subset \mathcal{M}$  be the open and closed substack whose  $k$  points correspond to  $(P; \Sigma)$ , where  $P$  is a connected curve of arithmetic genus  $h$  and  $\Sigma \subset P$  a divisor of degree  $b$ . Let  $\mathcal{M}_{h;b} \subset \mathcal{M}_{h;b}$  be the open substack where  $P$  is smooth and  $\Sigma$  is reduced. Then  $\mathcal{M}_{h;b}$  is a smooth stack and it contains  $\mathcal{M}_{h;b}$  as a dense open substack.

Let  $\mathcal{H}_{g/h}^d$  be the open and closed substack of  $\mathcal{M}_{h;b} \times_{\mathcal{M}} \mathcal{H}^d$  whose  $k$  points correspond to  $(\mathcal{P} \rightarrow P; \phi: \mathcal{C} \rightarrow \mathcal{P})$  where  $\mathcal{C}$  is connected. By the Riemann–Hurwitz formula,  $\mathcal{C}$  has arithmetic genus  $g$ . Observe that the small Hurwitz stack  $\mathcal{H}_{g/h}^d$  is simply the open substack defined by

$$\mathcal{H}_{g/h}^d = \mathcal{M}_{h;b} \times_{\mathcal{M}_{h;b}} \mathcal{H}_{g/h}^d.$$

We recall a sequence of open substacks of  $\mathcal{M}_{h;b}$  that contain  $\mathcal{M}_{h;b}$  and are proper over the base field. These are the spaces of *weighted pointed stable curves* constructed by Hassett [18].

**DEFINITION 1.7.1.** Let  $\epsilon$  be a rational number. Let  $P$  be a nodal curve over  $k$  and  $\Sigma \subset P$  a divisor supported in the smooth locus. We say that  $(P, \Sigma)$  is  $\epsilon$ -stable if

- (1) for every point  $p$  of  $P$ , we have

$$\epsilon \cdot \text{mult}_p(\Sigma) \leq 1;$$

- (2) the  $\mathbf{Q}$  line bundle  $\omega_P \otimes O_P(\epsilon\Sigma)$  is ample, where  $\omega_P$  is the dualizing line bundle of  $P$ .

Denote by  $\overline{\mathcal{M}}_{h;b}(\epsilon) \subset \mathcal{M}_{h;b}$  the open substack parametrizing  $\epsilon$ -stable marked curves.

Recall the main theorem from [18].

**THEOREM 1.7.2.** [18, Theorem 2.1, Variation 2.1.3]  $\overline{\mathcal{M}}_{h;b}(\epsilon)$  is a Deligne–Mumford stack, proper over  $\mathbf{K}$ . It admits a projective coarse space  $\overline{\mathcal{M}}_{h;b}(\epsilon)$ .

If  $\deg(\omega_{\mathcal{P}}(\epsilon\Sigma)) = \epsilon \cdot b + 2h - 2 \leq 0$ , then  $\overline{\mathcal{M}}_{h;b}(\epsilon)$  is empty. Otherwise, it contains  $\mathcal{M}_{h;b}$  as a dense open substack.

DEFINITION 1.7.3. Define the stack  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  of  $\epsilon$ -admissible covers by the formula

$$\overline{\mathcal{H}}_{g/h}^d(\epsilon) = \overline{\mathcal{M}}_{h;b}(\epsilon) \times_{\mathcal{M}_{h;b}} \mathcal{H}_{g/h}^d.$$

We sometimes call  $\epsilon$ -admissible covers *weighted admissible covers*.

COROLLARY 1.7.4.  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  is a Deligne–Mumford stack, proper over  $\mathbf{K}$ . It admits a projective coarse space  $\overline{H}_{g/h}^d(\epsilon)$  and a morphism

$$\text{br} : \overline{\mathcal{H}}_{g/h}^d(\epsilon) \rightarrow \overline{\mathcal{M}}_{h;b}(\epsilon).$$

PROOF. Follows directly from Theorem 1.3.8 and Theorem 1.6.1.  $\square$

As before, if  $\epsilon \cdot b + 2h - 2 \leq 0$ , then  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  is empty. Otherwise, it contains  $\mathcal{H}_{g/h}^d$  as an open substack (but it may not be dense; see Example 1.7.9).

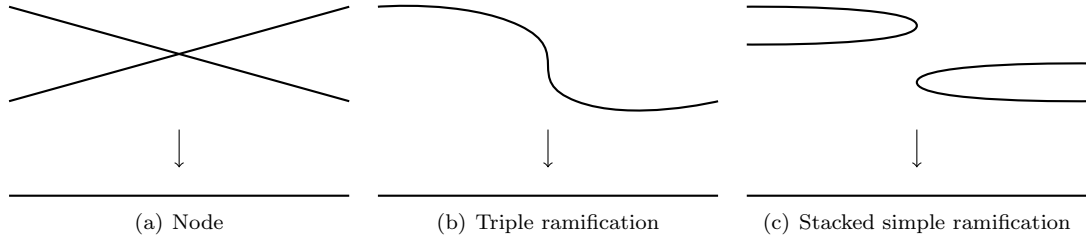
**1.7.1. Examples.** We describe the geometry of the spaces of weighted admissible covers by some illustrative examples.

These spaces generalize some known compactifications of Hurwitz spaces, mentioned in the following two examples.

EXAMPLE 1.7.5 (Twisted admissible covers). Consider the case  $\epsilon = 1$  and the resulting stack of 1-admissible covers  $\overline{\mathcal{H}}_{g/h}^d(1)$ . It parametrizes  $(\mathcal{P} \rightarrow P; \phi: \mathcal{C} \rightarrow \mathcal{P})$ , where  $\text{br } \phi \subset P$  is étale over the base. The induced morphism on coarse spaces  $C \rightarrow P$  is an admissible cover in the sense of Harris and Mumford [15] (but with unordered branch points).

By Proposition 1.5.4, the stack  $\overline{\mathcal{H}}_{g/h}^d(1)$  is smooth and contains the small Hurwitz stack  $\mathcal{H}_{g/h}$  as a dense open substack. In fact,  $\overline{\mathcal{H}}_{g/h}^d(1)$  is essentially the stack of *twisted admissible covers* of Abramovich, Corti, and Vistoli [2]; the only difference is that in [2], the branch points are ordered, whereas in  $\overline{\mathcal{H}}_{g/h}^d(1)$ , they are unordered.

EXAMPLE 1.7.6 (Spaces of hyperelliptic curves). Consider the case  $h = 0$  and  $d = 2$ , and the resulting stacks  $\overline{\mathcal{H}}_g^2(\epsilon)$  of  $\epsilon$ -admissible covers. Consider a  $k$ -point of  $\overline{\mathcal{H}}_g^2(\epsilon)$ , given by a cover  $(\mathcal{P} \rightarrow P; \phi: \mathcal{C} \rightarrow \mathcal{P})$ . Say  $[1/\epsilon] = n$ . Away from over the nodes of  $\mathcal{P}$ , the singularities of  $\mathcal{C}$  are

FIGURE 1. Possible local pictures of  $\phi$  for  $1/3 < \epsilon \leq 1/2$ 

(étale) locally of the form

$$y^2 - x^m,$$

for  $m \leq n$ . Thus, the spaces  $\overline{H}_g^2(\epsilon)$  are just the spaces of hyperelliptic curves with  $A_n$  singularities constructed by Fedorchuk [9].

The singularities of  $\mathcal{C}$  get much more interesting for higher degrees, as illustrated in the next example.

EXAMPLE 1.7.7 (Singularities of  $\mathcal{C}$ ). Let  $(\mathcal{P} \rightarrow P; \phi: \mathcal{C} \rightarrow \mathcal{P})$  be a  $k$ -point of  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$ . Notice that we do not explicitly restrict the singularities of  $\mathcal{C}$ ; the restrictions are imposed indirectly by the allowed multiplicity of the branch divisor. We list some examples of the singularities that appear on  $\mathcal{C}$  for small values of  $1/\epsilon$  and  $d \geq 3$ .

- (1)  $1/2 < \epsilon \leq 1$

In this case,  $\mathcal{C}$  is smooth (except, of course, over the nodes of  $\mathcal{P}$ ) and simply branched over  $\mathcal{P}$ .

- (2)  $1/3 < \epsilon \leq 1/2$

In this case,  $\mathcal{C}$  can have only nodal singularities. Also, the branches of the nodes must be individually unramified over  $\mathcal{P}$  as in Figure 1(a). This case also allows certain kinds of multiple ramification in  $\phi$ : it can be triply ramified as in Figure 1(b) or it can have two simple ramification points lying over the same point of  $\mathcal{P}$  as in Figure 1(c).

- (3)  $1/4 < \epsilon \leq 1/3$

In this case,  $\mathcal{C}$  can have nodal and cuspidal (formally  $k[[x, y]]/(y^2 - x^3)$ ) singularities as in Figure 2(a). This case also allows even more multiple ramification in  $\phi$ ; for example, it is possible to have ramification types (4), or (3, 2) or (2, 2, 2) in a fiber of  $\phi$ .

Another interesting possibility is a *ramified node* (Figure 2(b))—it is a combination of multiple ramification and the development of a singularity. This is a node on  $\mathcal{C}$ , one of



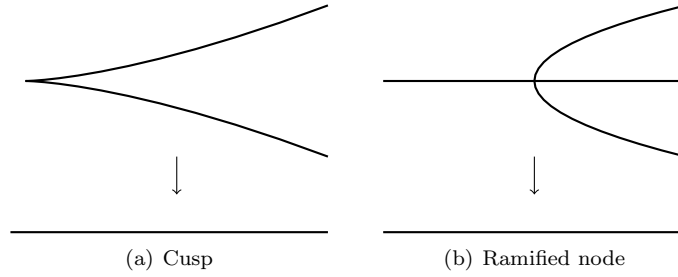


FIGURE 2. Some of the possible local pictures of  $\phi$  for  $1/4 < \epsilon \leq 1/3$

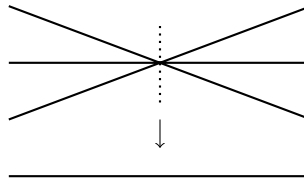


FIGURE 3. Planar triple points are allowed for  $\epsilon \leq 1/6$

whose branches is simply ramified over  $\mathcal{P}$ , formally expressed by

$$k[[t]] \rightarrow k[[t, x]]/x(x^2 - t).$$

(4)  $\epsilon \leq 1/4$

In this case,  $\mathcal{C}$  can have non-Gorenstein singularities. Indeed, the spatial triple point (formally the union of the coordinate axes in  $\mathbf{A}^3$ ) is a branched cover of a line with branch divisor of multiplicity four. Since multiplicity four is allowed in the branch divisor for  $\epsilon \leq 1/4$ , the cover  $\mathcal{C} \rightarrow \mathcal{P}$  can have formal local picture of a spatial triple point:

$$k[[t]] \rightarrow k[[t, x, y]]/(xy, y(x - t), x(y - t)).$$

In the case of admissible covers ( $\epsilon = 1$ ) and in the case of hyperelliptic curves ( $d = 2$ ), the morphism

$$\text{br} : \overline{\mathcal{H}}_{g/h}^d(\epsilon) \rightarrow \overline{\mathcal{M}}_{h;b}(\epsilon)$$

is finite. This is no longer the case if  $d \geq 3$  and  $\epsilon$  is sufficiently small. In fact, as soon as  $\epsilon \leq 1/6$ , we have positive dimensional fibers, as illustrated in the next example.

EXAMPLE 1.7.8 (Non-finiteness of the branch morphism). For every  $c \in k$ , consider the the planar triple point expressed as a triple cover of a smooth curve (Figure 3) by the formal description:

$$(1.7.1) \quad k[[t]] \rightarrow k[[t, x]]/x(x-t)(x-ct).$$

The discriminant is the ideal  $\langle t^6 \rangle$ . Although the rings  $k[[t, x]]/x(x-t)(x-ct)$  are isomorphic for different choices of  $c$ , they are *not* necessarily isomorphic as  $k[[t]]$  algebras. Said differently, although the singularities  $\text{Spec } k[[t, x]]/x(x-t)(x-ct)$  are isomorphic abstractly, they are *not* necessarily isomorphic as triple covers of  $\text{Spec } k[[t]]$ . One way to see this is the following. Consider the tangent space to  $\text{Spec } k[[t, x]]/x(x-t)(x-ct)$  at  $(0, 0)$ . In this two dimensional vector space, there are four distinguished one dimensional subspaces: the three tangent spaces of the branches and the kernel of the projection to the tangent space of  $\text{Spec } k[[t]]$ . The moduli of the configuration of these four subspaces depends on  $c$ . Up to a finite ambiguity, different choices of  $c$  give non-isomorphic triple covers.

For  $d \geq 3$ ,  $\epsilon \leq 1/6$  and  $h, b$  large enough to allow  $\epsilon \cdot b + 2h - 2 \geq 0$ , the formal descriptions in (1.7.1) are realizable in covers of a fixed genus  $h$  curve with a fixed branch divisor. We thus get infinitely many points in a fiber of  $\text{br} : \overline{\mathcal{H}}_{g/h}^d(\epsilon) \rightarrow \overline{\mathcal{M}}_{h;b}(\epsilon)$ .

In the case of admissible covers ( $\epsilon = 1$ ), the small Hurwitz space  $\mathcal{H}_{g/h}^d$  is dense in  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$ . By Theorem 1.5.5, this remains the case for arbitrary  $\epsilon$  if  $d \leq 3$ . However, this is not true in general, as illustrated by the following example.

EXAMPLE 1.7.9 (Extraneous components in  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$ ). For a sufficiently large  $d$  and a sufficiently small  $\epsilon$ , we exhibit a point in  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  that is not in the closure of  $\mathcal{H}_{g/h}^d$ . For simplicity, take  $h = 0$ ; the phenomenon is local, so the case of  $h = 0$  can be used to construct examples for any  $h$ .

Let  $C$  be a reduced, connected curve that is not a flat limit of smooth curves (see the article by Mumford [29] for the existence of such curves). For sufficiently large  $d$ , we have a finite map  $\phi: C \rightarrow \mathbf{P}^1$  of degree  $d$ . Let  $\epsilon$  be so small that  $\epsilon \cdot \text{mult}_p(\text{br } \phi) \leq 1$  for all  $p \in \mathbf{P}^1$ . Then  $(\mathbf{P}^1; \phi: C \rightarrow \mathbf{P}^1)$  is a point in  $\overline{\mathcal{H}}_g^d(\epsilon)$  which, by construction, is not in the closure of  $\mathcal{H}_g^d$ . Hence  $\overline{\mathcal{H}}_g^d(\epsilon)$  has *extraneous components*—components other than the closure of  $\mathcal{H}_g^d$ .

Thanks to Theorem 1.5.5, there are no extraneous components for  $d = 2$  or  $3$ . By Proposition 1.5.4, unsmoothable singularities are the only reason for extraneous components.

We end the chapter with a question prompted by Example 1.7.9.

QUESTION 1.7.10. *For which  $d, g$  and  $h$  is  $\mathcal{H}_{g/h}^d$  irreducible? More precisely, for which  $d, g, h$  and  $\epsilon$  is  $\overline{\mathcal{H}}_{g/h}^d(\epsilon)$  irreducible?*

## Moduli of $d$ -gonal singularities and crimping

In Chapter 1, we constructed the stack  $\mathcal{M}$  of divisorially marked, pointed nodal curves and the stack  $\mathcal{H}^d$  of degree  $d$  covers of pointed orbifold curves. They are related by the branch divisor morphism

$$\mathrm{br} : \mathcal{H}^d \rightarrow \mathcal{M}.$$

The goal of this short chapter is to understand the fibers of this morphism.

Consider a point  $s : \mathrm{Spec} k \rightarrow \mathcal{M}$ . For simplicity, assume that it corresponds to a smooth curve  $P$  with a marked divisor  $\Sigma$ . The fiber of  $\mathrm{br}$  over  $s$  consists precisely of degree  $d$  covers  $\phi : C \rightarrow P$  with  $\mathrm{br} \phi = \Sigma$ . Let  $\tilde{C} \rightarrow C$  be the normalization. Since  $\tilde{C}$  is smooth, the cover  $\tilde{C} \rightarrow P$  is determined by its restriction  $\tilde{C}|_{P \setminus \Sigma} \rightarrow P \setminus \Sigma$ , which is étale. Since there are only finitely many étale covers of degree  $d$  of a smooth curve, there are only finitely many possibilities for  $\tilde{\phi} : \tilde{C} \rightarrow P$ . The fiber of  $\mathrm{br}$  over  $s$  thus decomposes into finitely many (open and closed) components corresponding to the choice of  $\tilde{\phi} : \tilde{C} \rightarrow P$ . Within each component,  $C \rightarrow P$  is obtained by *crimping* a fixed  $\tilde{C} \rightarrow P$  over the points of  $\Sigma$ . The crimping can be described formally locally around the points of  $\Sigma$  in  $P$ . In this way, the description of the fibers of  $\mathrm{br}$  includes the discrete global data of the normalization and the continuous local data of the crimping.

The chapter is organized as follows. In Section 2.1, we define the functor of crimps of a finite cover and reduce its study to the study of the functor of crimps over a formal disk. In Section 2.2, we prove that the functor of crimps over a formal disk is represented by a projective scheme. In Section 2.3 and Section 2.4, we describe the space of crimps of double and triple covers of a disk, respectively. In addition to providing explicit examples, the study of crimps of triple covers will be relevant in the later chapters about the Mori theory of spaces of trigonal curves.

Moduli of singular curves and the phenomenon of crimping have been studied extensively by van der Wyck [40]. Our study of crimping in the context of finite covers, however, is much more elementary.

### 2.1. The space of crimps of a finite cover

Let  $\mathcal{Y}$  be a reduced, purely one dimensional Deligne–Mumford stack over  $k$  and  $\Sigma \subset \mathcal{Y}$  a Cartier divisor. Let  $\tilde{\phi}: \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$  a finite cover of degree  $d$ , étale over  $\mathcal{Y} \setminus \Sigma$ . In all the cases we consider,  $\mathcal{Y}$  is either a (pointed) orbifold curve or the spectrum of a DVR.

Define the functor  $\text{Crimp}_{\tilde{\phi}, \Sigma}: \mathbf{Schemes}_k \rightarrow \mathbf{Sets}$  of *crimps of  $\tilde{\phi}$  over  $\Sigma$*  by

$$\text{Crimp}_{\tilde{\phi}, \Sigma}(T) = \{(\tilde{\mathcal{X}} \times T \rightarrow \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T)\} / \text{Isomorphism},$$

where  $\phi: \mathcal{X} \rightarrow \mathcal{Y} \times T$  is a finite cover of degree  $d$  with  $\text{br}(\phi) = \Sigma \times T$ . Two such crimps  $\tilde{\mathcal{X}} \times T \rightarrow \mathcal{X}_i \rightarrow \mathcal{Y} \times T$ , for  $i = 1, 2$ , are isomorphic if there is an isomorphism  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  that commutes with the relevant maps

$$\begin{array}{ccccc} \tilde{\mathcal{X}} \times T & \longrightarrow & \mathcal{X}_1 & \longrightarrow & \mathcal{Y} \times T \\ \parallel & & \downarrow & & \parallel \\ \tilde{\mathcal{X}} \times T & \longrightarrow & \mathcal{X}_2 & \longrightarrow & \mathcal{Y} \times T \end{array} .$$

We sometimes write  $\text{Crimp}(\tilde{\phi}, \Sigma)$  instead of  $\text{Crimp}_{\tilde{\phi}, \Sigma}$  for better readability.

If  $\mathcal{Z} \rightarrow \mathcal{Y}$  is a morphism such that  $\Sigma_{\mathcal{Z}} \subset \mathcal{Z}$  is also a divisor, then we have a natural transformation

$$\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow \text{Crimp}(\tilde{\phi}_{\mathcal{Z}}, \Sigma_{\mathcal{Z}})$$

defined by

$$(\tilde{\mathcal{X}} \times T \rightarrow \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T) \mapsto (\tilde{\mathcal{X}}_{\mathcal{Z}} \times T \rightarrow \mathcal{X}_{\mathcal{Z}} \xrightarrow{\phi_{\mathcal{Z}}} \mathcal{Y}_{\mathcal{Z}} \times T).$$

Let  $G = \text{Aut}(\tilde{\phi})$  be the group of automorphisms of  $\tilde{\mathcal{X}}$  over the identity of  $\mathcal{Y}$ . This is a finite group, which acts on  $\text{Crimp}(\tilde{\phi}, \Sigma)$  as follows.

$$G \ni \alpha: (\tilde{\mathcal{X}} \times T \xrightarrow{\nu} \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T) \mapsto (\tilde{\mathcal{X}} \times T \xrightarrow{\nu \circ \alpha^{-1}} \mathcal{X} \xrightarrow{\phi} \mathcal{Y} \times T).$$

**REMARK 2.1.1.** A crimp may be equivalently thought of as a suitable subalgebra  $\phi_* O_{\mathcal{X}}$  of the algebra  $\tilde{\phi}_* O_{\tilde{\mathcal{X}} \times T}$  on  $\mathcal{Y} \times T$ . Then isomorphism of crimps simply becomes equality of subalgebras. The action of  $G$  is induced by the action of  $G$  on  $\tilde{\phi}_* O_{\tilde{\mathcal{X}}}$ .

Throughout, we view  $O_{\tilde{\mathcal{X}} \times T}$  and  $O_{\mathcal{X}}$  as sheaves of algebras on  $\mathcal{Y} \times T$ , omitting  $\tilde{\phi}_*$  and  $\phi_*$  to lighten notation. Observe that the quotient  $O_{\tilde{\mathcal{X}} \times T} / O_{\mathcal{X}}$  is an  $O_{\mathcal{Y} \times T}$  module supported entirely on  $\text{supp}(\Sigma) \times T$ . In other words,  $\tilde{\mathcal{X}} \times T \rightarrow \mathcal{X}$  is an isomorphism away from  $\Sigma \times T$ .

Having defined  $\text{Crimp}(\tilde{\phi}, \Sigma)$  in wide generality, we immediately turn to the case of interest. Let  $(\mathcal{P} \rightarrow P; \sigma_1, \dots, \sigma_n)$  be a pointed orbifold curve and  $\Sigma \subset \mathcal{P}$  a divisor supported in the general locus  $P^{\text{gen}} = \mathcal{P}^{\text{sm}} \setminus \sigma_1, \dots, \sigma_n$ . Let  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \mathcal{P}$  be a finite cover, étale over  $\mathcal{P} \setminus \Sigma$ . We begin by making precise our remark that crimps can be described formally locally around the points of  $\Sigma$ .

PROPOSITION 2.1.2. *Let  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \mathcal{P}$  and  $\Sigma$  be as above.*

(1) *Let  $U \subset \mathcal{P}$  be an open set containing  $\Sigma$ . Then the transformation*

$$\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow \text{Crimp}(\tilde{\phi}_U, \Sigma)$$

*is an isomorphism.*

(2) *The transformation*

$$\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow \prod_{s \in \text{supp}(\Sigma)} \text{Crimp}(\tilde{\phi} \times_{\mathcal{P}} \text{Spec } O_{P,s}, \Sigma \times_{\mathcal{P}} \text{Spec } O_{P,s})$$

*is an isomorphism.*

(3) *The transformation*

$$\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow \prod_{s \in \text{supp}(\Sigma)} \text{Crimp}(\tilde{\phi} \times_{\mathcal{P}} \text{Spec } \widehat{O}_{P,s}, \Sigma \times_{\mathcal{P}} \text{Spec } \widehat{O}_{P,s})$$

*is an isomorphism.*

PROOF. The last assertion is the strongest, so we prove that. Following Remark 2.1.1, we treat crimps as subalgebras. For brevity, we set

$$\widehat{P}_s = \text{Spec } \widehat{O}_{P,s}, \quad \Sigma_s = \Sigma \times_P \widehat{P}_s, \quad \text{and } \tilde{C}_s = \tilde{\mathcal{C}} \times_P \widehat{P}_s,$$

Given crimps  $\tilde{C}_s \times T \rightarrow C_s \rightarrow \widehat{P}_s \times T$  for  $s \in \text{supp}(\Sigma)$ , construct a subalgebra  $O_C$  of  $O_{\tilde{C} \times T}$  as the fiber product of algebras

$$\begin{array}{ccc} O_C & \longrightarrow & O_{\tilde{C} \times T} \\ \downarrow & & \downarrow \\ \prod_s O_{C_s} & \longrightarrow & \prod_s O_{\tilde{C}_s \times T} \end{array}$$

We thus get a natural transformation

$$\prod_{s \in \text{supp}(\Sigma)} \text{Crimp}(\tilde{\phi} \times_{\mathcal{P}} \text{Spec } \widehat{O}_{P,s}, \Sigma \times_{\mathcal{P}} \text{Spec } \widehat{O}_{P,s}) \rightarrow \text{Crimp}(\tilde{\phi}, \Sigma).$$

It is easy to check that it is inverse to the transformation in (3).  $\square$

## 2.2. Crimps over a disk

Thanks to Proposition 2.1.2, we now focus on the crimps of covers of the formal disk. Set  $R = k[[t]]$  and  $\Delta = \text{Spec } R$ . Let  $\Delta^*$  be the punctured disk  $\Delta \setminus \{0\}$ . Fix a finite cover  $\tilde{\phi}: \tilde{C} \rightarrow \Delta$  of degree  $d$ , étale over  $\Delta^*$ , with  $\text{br}(\tilde{\phi})$  given by  $\langle t^a \rangle$ . Fix a divisor  $\Sigma \subset \Delta$  given by  $\langle t^b \rangle$  and set  $\delta = (b - a)/2$ .

**PROPOSITION 2.2.1.** *Let  $\tilde{C} \times T \rightarrow C \xrightarrow{\phi} \Delta \times T$  be a crimp with  $\text{br}(\phi) = \Sigma \times T$ . Set  $Q = O_{\tilde{C} \times T}/O_C$ . Then  $Q$  is a  $T$ -flat sheaf on  $\Delta \times T$  annihilated by  $t^b$ . When restricted to the fibers of  $\Delta \times T \rightarrow T$ , the sheaf  $Q$  has length  $\delta$ .*

**PROOF.** In the proof, all the linear-algebraic operations are over  $O_{\Delta \times T}$ .

First,  $Q$  is  $T$ -flat simply because the inclusion  $i: O_C \hookrightarrow O_{\tilde{C} \times T}$  remains an inclusion when restricted to the fibers of  $\Delta \times T \rightarrow T$ . For the rest, consider the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & O_{\Delta \times T} & \xrightarrow{\tilde{\delta}} & (\det O_{\tilde{C} \times T}^\vee)^{\otimes 2} & \rightarrow & \tilde{B} \rightarrow 0 \\ & & \parallel & & \downarrow \det(i^\vee)^2 & & \downarrow \\ 0 & \rightarrow & O_{\Delta \times T} & \xrightarrow{\delta} & (\det O_C^\vee)^{\otimes 2} & \longrightarrow & B \rightarrow 0 \end{array} .$$

The horizontal maps  $\tilde{\delta}$  and  $\delta$  define the respective branch divisors as in Section 1.1. In particular,  $B$  is annihilated by  $\langle t^b \rangle$ . The snake lemma yields the sequence

$$(2.2.1) \quad 0 \rightarrow \tilde{B} \rightarrow B \rightarrow \text{cok}(\det(i^\vee)^2) \rightarrow 0.$$

Since  $t^b$  annihilates  $B$ , it annihilates  $\text{cok}(\det(i^\vee)^2)$ , hence  $\text{cok}(\det(i^\vee))$ , hence  $\text{cok}(i^\vee)$  and hence  $\text{cok } i = Q$ .

To compute the length of  $Q$  on the fibers, replace  $T$  by a field. By (2.2.1), we get

$$\begin{aligned} 2 \text{ length } Q &= \text{length}(\text{cok}(\det(i^\vee)^2)) \\ &= \text{length } B - \text{length } \tilde{B} \\ &= b - a = 2\delta. \end{aligned}$$

$\square$

REMARK 2.2.2. By Proposition 2.2.1, if  $\tilde{C}$  is smooth, then  $\delta$  is indeed the  $\delta$  invariant of  $C$ .

We now exhibit the space of crimps over a disk explicitly as a projective variety. Set  $F = O_{\tilde{C}}/t^b O_{\tilde{C}}$  and denote by  $\text{Quot} = \text{Quot}(F, \delta)$  the Quot scheme of length  $\delta$  quotients of the  $\Delta$  module  $F$ . Since  $\text{supp } F$  is projective (it is finite!),  $\text{Quot}$  is a projective scheme. The idea is to identify quotients which arise as  $O_{\tilde{C}}/O_C$ . For this to be true, the quotient must satisfy the following two properties:

- (1) The kernel must be closed under multiplication, to get a subalgebra  $O_C$  of  $O_{\tilde{C}}$ ;
- (2) The resulting  $C \rightarrow \Delta$  must have the right branch divisor.

We now formalize both conditions. Let  $\pi: \Delta \times \text{Quot} \rightarrow \Delta$  be the projection. On  $\Delta \times \text{Quot}$  we have the universal sequence

$$0 \rightarrow S \rightarrow F \otimes_k O_{\text{Quot}} \rightarrow Q \rightarrow 0.$$

The multiplication  $F \otimes_{\Delta} F \rightarrow F$  induces maps

$$S \otimes_{\Delta \times \text{Quot}} S \rightarrow (F \otimes_{\Delta} F) \otimes_k O_{\text{Quot}} \rightarrow F \otimes_k O_{\text{Quot}} \rightarrow Q.$$

Define the closed subscheme  $X \subset \text{Quot}$  as the annihilator of the composite map  $\pi_*(S \otimes_{\Delta \times \text{Quot}} S) \rightarrow \pi_* Q$  on  $\text{Quot}$ . This takes care of (1).

On  $\Delta \times X$ , the sheaf  $S$  inherits the structure of an  $O_{\Delta \times X}$  algebra. Form the subalgebra  $O_C$  of  $O_{\tilde{C} \times X}$  as the fiber product

$$\begin{array}{ccc} O_C & \longrightarrow & O_{\tilde{C} \times X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & F \otimes_k O_X \end{array},$$

and set  $C = \text{Spec } O_C$ .

CLAIM. *In the above setup,  $C \rightarrow \Delta \times X$  is flat.*

PROOF. By the definition of  $O_C$ , we have the sequence

$$0 \rightarrow O_C \rightarrow O_{\tilde{C} \times X} \rightarrow Q \rightarrow 0.$$

Since  $Q$  is  $X$ -flat, we conclude that  $O_C$  is  $X$ -flat and  $O_C \rightarrow O_{\tilde{C} \times X}$  remains an inclusion when restricted to the fibers of  $\Delta \times X \rightarrow X$ . For every point  $x \in X$ , the sheaf  $O_{C_x}$  is a subsheaf of the free sheaf  $O_{\tilde{C}}$  and hence is free. It follows that  $O_C$  is a locally free  $\Delta \times X$  module.  $\square$



At this point, we have  $\tilde{C} \times X \rightarrow C \xrightarrow{\phi} \Delta \times X$ , where  $\tilde{C} \rightarrow C$  is an isomorphism over  $\Delta^* \times X$  and  $C \rightarrow \Delta \times X$  is finite and flat. We now enforce (2). Define  $B$  by

$$0 \rightarrow O_{\Delta \times X} \xrightarrow{\delta} (\det O_C^\vee)^{\otimes 2} \rightarrow B \rightarrow 0,$$

where the linear algebraic operations are over  $O_{\Delta \times X}$ , and  $\delta$  is the usual discriminant as in Section 1.1. Observe that  $\delta$  remains an injection when restricted to the fibers of  $\pi: \Delta \times X \rightarrow X$ , and hence  $B$  is  $X$ -flat. By Proposition 2.2.1 applied to a fiber, we conclude that  $B$  has fiberwise length  $b$ . Define the closed subscheme  $Y \subset X$  as the annihilator of

$$\pi_* B \xrightarrow{t^b} \pi_* B.$$

This condition would be superfluous if  $X$  were reduced. However, it appropriately restricts the non-reduced structure on  $X$ , taking care of (2).

By construction, we have a crimp  $\tilde{C} \times Y \rightarrow C \xrightarrow{\phi} \Delta \times Y$  with  $\text{br } \phi = \Sigma \times Y$ . We thus get a morphism

$$(2.2.2) \quad Y \rightarrow \text{Crimp}(\tilde{\phi}, \Sigma).$$

**PROPOSITION 2.2.3.** *The morphism  $Y \rightarrow \text{Crimp}(\tilde{\phi}, \Sigma)$  in (2.2.2) is an isomorphism. In particular,  $\text{Crimp}(\tilde{\phi}, \Sigma)$  is a projective scheme.*

**PROOF.** We construct a transformation  $\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow Y$ , inverse to (2.2.2). Let  $T$  be a scheme and  $\tilde{C} \times T \rightarrow C \xrightarrow{\phi} \Delta \times T$  a crimp with branch divisor  $\Sigma \times T$ . Define the quotient  $Q = O_{\tilde{C} \times T} / O_C$ . By Proposition 2.2.1,  $Q$  is a  $T$ -flat quotient of  $O_{\tilde{C} \times T} / t^b O_{\tilde{C} \times T} = F \otimes_k O_T$ , fiberwise of length  $\delta$ . This gives a map  $T \rightarrow \text{Quot}(F, \delta)$ . Since the kernel of  $F \otimes_k O_T \rightarrow Q$  is the image of  $O_C$ , it is closed under multiplication. Hence  $T \rightarrow \text{Quot}$  factors through  $T \rightarrow X$ . Since  $\text{br}(\phi) = \Sigma \times T$ , the cokernel of

$$O_{\Delta \times T} \xrightarrow{\delta} (\det O_C^\vee)^{\otimes 2}$$

is annihilated by  $t^b$ . Therefore, the map  $T \rightarrow X$  factors through  $T \rightarrow Y$ . In this way, we get a morphism  $\text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow Y$ , which is clearly inverse to (2.2.2).  $\square$

**COROLLARY 2.2.4.** *Let  $\tilde{C} \rightarrow \mathcal{P}$  be a finite cover of an orbifold curve and  $\Sigma \subset P^{\text{gen}}$  a divisor. Then  $\text{Crimp}(\tilde{C} \rightarrow \mathcal{P}, \Sigma)$  is represented by a projective scheme.*

**PROOF.** Follows immediately from Proposition 2.1.2 and Proposition 2.2.3.  $\square$

Finally, we relate the spaces of crimps with the fibers of  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$ . Let  $p : \text{Spec } k \rightarrow \mathcal{M}$  be a point corresponding to a divisorially marked, pointed curve  $(P; \Sigma; \sigma_1, \dots, \sigma_n)$ . As usual, we abbreviate  $\sigma_1, \dots, \sigma_n$  by  $\sigma$ . Let  $\Gamma$  be the set of  $(\mathcal{P} \rightarrow P; \sigma; \tilde{\phi} : \tilde{\mathcal{C}} \rightarrow \mathcal{P})$ , where  $(\mathcal{P} \rightarrow P; \sigma)$  is a pointed orbifold curve and  $\tilde{\phi}$  a finite cover of degree  $d$  such that

- (1)  $\tilde{\mathcal{C}} \times_{\mathcal{P}} \mathcal{P}^{\text{sm}}$  is smooth;
- (2)  $\tilde{\phi}$  is étale over  $\mathcal{P} \setminus \Sigma$ ; and
- (3)  $\tilde{\phi}$  corresponds to a representable classifying map  $\mathcal{P} \rightarrow \mathcal{A}_d$ .

Assume that no two elements of  $\Gamma$  are isomorphic over the identity of  $P$ . Then  $\Gamma$  is a finite set. We have a morphism

$$(2.2.3) \quad \bigsqcup_{\Gamma} \text{Crimp}(\tilde{\phi}, \Sigma) \rightarrow p \times_{\mathcal{M}} \mathcal{H}^d.$$

given by

$$(\tilde{\mathcal{C}} \times T \rightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{P} \times T) \mapsto (\mathcal{P} \times T \rightarrow P \times T; \sigma \times T; \mathcal{C} \xrightarrow{\phi} \mathcal{P} \times T).$$

Recall that we have an action of  $\text{Aut}(\tilde{\phi})$  on  $\text{Crimp}(\tilde{\phi}, \Sigma)$ . The morphism above clearly descends to a morphism

$$(2.2.4) \quad \bigsqcup_{\Gamma} [\text{Crimp}(\tilde{\phi}, \Sigma) / \text{Aut}(\tilde{\phi})] \rightarrow p \times_{\mathcal{M}} \mathcal{H}^d.$$

**PROPOSITION 2.2.5.** *The morphism in (2.2.3) is finite and surjective. The morphism in (2.2.4) is representable and a bijection on  $k$ -points.*

**PROOF.** The statement is true almost by design. The details are straightforward.  $\square$

Proposition 2.2.5 is as close as we can come to explicitly identifying the fibers of  $\text{br} : \mathcal{H}^d \rightarrow \mathcal{M}$ . However, this is good enough for determining many crude properties like the dimension.

### 2.3. Crimps of double covers

Let  $\tilde{\phi} : \tilde{\mathcal{C}} \rightarrow \Delta$  be a double cover with  $\tilde{\mathcal{C}}$  smooth. Let  $\Sigma \subset \Delta$  be the divisor given by  $\langle t^b \rangle$ . In this section, we describe  $\text{Crimp}(\tilde{\phi}, \Sigma)$ . Let the branch divisor of  $\tilde{\mathcal{C}} \rightarrow \Delta$  be given by  $\langle t^a \rangle$ . Observe that

$$a = \begin{cases} 0 & \text{if } \tilde{\phi} \text{ is étale} \\ 1 & \text{if } \tilde{\phi} \text{ is ramified.} \end{cases}$$

PROPOSITION 2.3.1. *With the above notation,*

$$\text{Crimp}(\tilde{\phi}, \Sigma) = \begin{cases} \text{Point} & \text{if } b \equiv a \pmod{2} \\ \emptyset & \text{otherwise.} \end{cases}$$

*In the first case, the point corresponds to the cover*

$$R \rightarrow R[x]/(x^2 - t^{b-a}).$$

PROOF. The restriction on  $b - a$  modulo 2 comes from Proposition 2.2.1. The rest is straightforward.  $\square$

## 2.4. Crimps of triple covers

Let  $\tilde{\phi}: \tilde{C} \rightarrow \Delta$  be a triple cover with  $\tilde{C}$  smooth. Let  $\Sigma \subset \Delta$  be the divisor given by  $\langle t^b \rangle$ . Observe that there are three possibilities for  $\tilde{\phi}$ :

- (1) étale:  $\tilde{C} = \Delta \sqcup \Delta \sqcup \Delta \rightarrow \Delta$ ,
- (2) totally ramified:  $\tilde{C} = \text{Spec } R[x]/(x^3 - t) \rightarrow \Delta$ .
- (3) simply ramified:  $\tilde{C} = \Delta \sqcup \text{Spec } R[x]/(x^2 - t) \rightarrow \Delta$ ,

In this section, we describe  $\text{Crimp}(\tilde{\phi}, \Sigma)$  for each of the three cases. The description is much more involved than the case of double covers. Instead of describing  $\text{Crimp}(\tilde{\phi}, \Sigma)$  completely, we describe a stratification given by a numerical invariant called the  $\mu$  invariant.

**2.4.1. The  $\mu$ -invariant.** Let  $\phi: C = \text{Spec } S \rightarrow \Delta$  be a cover of degree three and  $\tilde{C} = \text{Spec } \tilde{S} \rightarrow C$  the normalization of  $C$ . We first treat the case where  $\tilde{C} \rightarrow \Delta$  is étale. Set

$$Q = O_{\tilde{C}}/O_C = (\tilde{S}/R)/(S/R).$$

Then  $Q$  is an  $R$ -module of finite length and is a quotient of the free  $R$ -module  $\tilde{S}/R$  of rank two. It follows that

$$Q \cong k[t]/t^m \oplus k[t]/t^n,$$

for some  $m, n \geq 0$ . We denote by  $\mu$  the difference  $|m - n|$

$$\mu(\phi) = |m - n|.$$

We now treat the case where  $\tilde{C} \rightarrow \Delta$  is not necessarily étale. In this case, let  $\Delta' \rightarrow \Delta$  be a finite cover such that  $\Delta'$  is smooth and the normalization  $\tilde{C}'$  of  $C' = C \times_{\Delta} \Delta'$  is étale over  $\Delta'$ . Define

$$\mu(\phi) = \mu(C' \rightarrow \Delta') / \deg(\Delta' \rightarrow \Delta).$$

In general, the  $\mu$ -invariant is only a rational number. It is easy to check that  $\mu(\phi)$  does not depend on the cover  $\Delta' \rightarrow \Delta$ , as long as  $\tilde{C}' \rightarrow \Delta'$  is étale. A canonical choice is simply  $\Delta' = \tilde{C}$ .

**PROPOSITION 2.4.1.**  *$\mu$  is a lower semicontinuous function on  $\text{Crimp}(\tilde{\phi}, \Sigma)$ .*

**PROOF.** It suffices to treat the case where  $\tilde{\phi}: \tilde{C} \rightarrow \Delta$  is étale. The remaining two cases follow after replacing  $\Delta$  by an appropriate  $\Delta'$ .

Let  $\tilde{C} \rightarrow C \xrightarrow{\phi} \Delta$  be a crimp. Let  $Q = O_{\tilde{C}}/O_C$ . By Proposition 2.2.1, we see that  $\text{length}(Q) = b/2$ . Let

$$Q \cong k[t]/t^m \oplus k[t]/t^n,$$

where  $n + m = b/2$  and  $n - m = \mu(\phi)$ . Then  $\mu(\phi) \leq l$  if and only if  $t^{\lceil b/4 + l/2 \rceil}$  annihilates  $Q$ . This is clearly a closed condition.  $\square$

Throughout the rest of the section, fix a non-negative rational number  $l \geq 0$  and a divisor  $\Sigma \subset \Delta$  given by  $\langle t^b \rangle$ . Denote by  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  the locally closed subset of  $\text{Crimp}(\tilde{\phi}, \Sigma)$  consisting of crimps with  $\mu$  invariant  $l$ . Although we can put a natural scheme structure on  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$ , we only describe the underlying variety. Clearly, the action of  $\text{Aut}(\tilde{\phi})$  on  $\text{Crimp}(\tilde{\phi}, \Sigma)$  preserves the  $\mu$ -invariant and hence induces an action of  $\text{Aut}(\tilde{\phi})$  on  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$ .

**2.4.2.  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  for  $\tilde{\phi}$  étale.** Fix  $\tilde{\phi}: \tilde{C} = \text{Spec } \tilde{S} \rightarrow \Delta$ , a triple cover with  $\tilde{\phi}$  étale. Fix an isomorphism of  $R$ -algebras

$$\tilde{S} \cong R \oplus R \oplus R.$$

Then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  can be thought of as the parameter space of certain  $R$ -subalgebras of  $\tilde{S}$ .

**PROPOSITION 2.4.2.** *Let  $S \subset \tilde{S}$  be an  $R$ -module such that*

$$(2.4.1) \quad R \subset S \text{ and } \tilde{S}/S \cong k[t]/t^m \oplus k[t]/t^n,$$

with  $m \leq n$ . Then

- (1)  $S$  contains  $t^n \tilde{S}$ .

(2) The quotient  $S/\langle R, t^n \tilde{S} \rangle$  is an  $R$ -submodule of  $\tilde{S}/\langle R, t^n \tilde{S} \rangle$  generated by the image of one element  $t^m f$ , for some  $f \in \tilde{S}$  nonzero modulo  $\langle R, t \rangle$ .

(3)  $S$  is an  $R$ -subalgebra if and only if  $t^{2m} f^2 \in S$ .

In particular, if  $2m \geq n$ , then every  $R$ -submodule  $S$  of  $\tilde{S}$  satisfying (2.4.1) is an  $R$ -subalgebra.

PROOF. From (2.4.1), it follows that  $S$  is generated as an  $R$ -module by  $1$ ,  $t^m f$  and  $t^n g$  for some  $f$  and  $g$  in  $\tilde{S}$  such that  $1$ ,  $f$  and  $g$  are linearly independent modulo  $t$ . The first two assertions follow from this observation. For the third, see that  $S$  is closed under multiplication if and only if the pairwise products of the generators lie in  $S$ . By (1), this is automatic for all products except  $t^{2m} f^2$ . Finally, if  $2m \geq n$  then the condition (3) is vacuous by (1).  $\square$

Using Proposition 2.4.2, we can readily describe  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$ .

PROPOSITION 2.4.3. *Retain the setup introduced at the beginning of Subsection 2.4.2. Let  $m, n$  be such that*

$$n + m = b/2 \text{ and } n - m = l.$$

*First,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is non-empty only if  $m$  and  $n$  are non-negative integers. If this numerical condition is satisfied, then we have the following two cases:*

(1) *If  $2m \geq n$ , then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $l$ . Its  $k$ -points correspond to  $R$ -subalgebras of  $\tilde{S}$  generated as an  $R$ -module by  $1$ ,  $t^n \tilde{S}$  and  $t^m f$  for some  $f \in \tilde{S}$  nonzero modulo  $\langle R, t \rangle$ .*

(2) *If  $n > 2m$ , then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is a disjoint union of three irreducible components of dimension  $m$ , conjugate under the  $\text{Aut}(\tilde{\phi}) = \mathbf{S}_3$  action. Its  $k$ -points correspond to  $R$ -subalgebras of  $\tilde{S}$  generated as an  $R$ -module by  $1$ ,  $t^n \tilde{S}$  and  $t^m f$ , where  $f$  has the form*

$$f = (1, h, -h) \text{ or } (h, 1, -h) \text{ or } (h, -h, 1),$$

*with  $h \equiv 0 \pmod{t^{n-2m}}$ .*

PROOF. Let  $\tilde{C} \rightarrow C = \text{Spec } S \rightarrow \Delta$  be a crimp with  $\mu$  invariant  $l$  and branch divisor  $\Sigma$ . Then

$$\tilde{S}/S \cong k[t]/t^n \oplus k[t]/t^m.$$

In particular,  $m$  and  $n$  must be integers.

The space  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  may be identified with the space of  $R$ -modules  $S$  satisfying

$$(2.4.2) \quad R \subset S \subset \tilde{S} \text{ and } \tilde{S}/S \cong k[t]/t^n \oplus k[t]/t^m,$$

satisfying the additional condition that  $S$  be closed under multiplication.

Set  $F = \tilde{S}/R$ . By Proposition 2.4.2 (2), the space of  $S$  as in (2.4.2) is simply the space of submodules of the  $(R/t^{n-m}R)$ -module  $t^m F/t^n F$  generated by one element  $t^m f$ , where  $f$  is nonzero modulo  $t$ . To specify such a submodule, it suffices to specify the image  $\bar{f}$  of  $f$  in  $F/t^{n-m}F$ , such that it is nonzero modulo  $t$ . Two such  $\bar{f}$  define the same submodule if and only if they are related by multiplication by a unit of  $R/t^{n-m}R$ .

In the case  $2m \geq n$ , the condition of being closed under multiplication is superfluous. Thus,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  may be identified with the quotient

$$(F/t^{n-m}F)^*/(R/t^{n-m}R)^* \cong ((k[t]/t^{n-m})^{\oplus 2})^*/(k[t]/t^{n-m})^*,$$

where the superscript  $*$  denotes elements nonzero modulo  $t$ . This quotient is simply the jet-scheme of order  $(n-m-1)$  jets of  $\mathbf{P}^1 = \mathbf{P}_{\text{sub}}(F/tF)$ . In particular, it is irreducible of dimension  $(n-m) = l$ .

In the case  $n > 2m$ , we must check when  $S$  is closed under multiplication. It is not too hard to check that after multiplying by a unit of  $R/t^{n-m}R$ , the element  $\bar{f}$  in  $F/t^{n-m}F$  can be represented as the image in  $F/t^{n-m}F$  of

$$(2.4.3) \quad (1, h, -h) \text{ or } (h, 1, -h), \text{ or } (h, -h, 1), \text{ for some } h \in R/t^{n-m}R.$$

It is easy to check that  $t^{2m}f^2$  lies in the  $R$ -submodule  $S$  of  $\tilde{S}$  generated by 1,  $t^n \tilde{S}$  and  $t^m f$  if and only if  $t^{2m} \bar{f}^2 \in R \langle t^m \bar{f} \rangle$  in  $t^m F/t^n F$ , or equivalently  $h \equiv 0 \pmod{t^{n-2m}}$ . Hence, the choice of  $\bar{f}$  that gives an  $R$ -subalgebra  $S$  is equivalent to the choice of  $h$  from  $t^{n-2m}R/t^{n-m}R \cong k^m$ . Also, see that different choices of  $h$  give different  $S$ . Hence,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is the disjoint union of three irreducible components of dimension  $m$  corresponding to the three possibilities in (2.4.3). Since the group  $\mathbf{S}_3$  acts by permuting the three entries, these three components are conjugate.  $\square$

Let us illustrate the  $\mu$  stratification of  $\text{Crimp}(\tilde{\phi}, \Sigma)$  and the dichotomy in Proposition 2.4.3 in the example of  $b = 8$  (this is the first non-trivial but manageable case).

**EXAMPLE 2.4.4.** We have three choices for  $(m, n)$ , namely  $(2, 2)$ ,  $(1, 3)$  and  $(0, 4)$ , corresponding to  $\mu$  invariants 0, 2, and 4, respectively.

$\text{Crimp}(\tilde{\phi}, \Sigma, 0)$ : This stratum consists of a single point, corresponding to the  $R$ -subalgebra  $S$  of  $\tilde{S}$  generated as an  $R$ -module by 1 and  $t^2\tilde{S}$ . The curve  $\text{Spec } S$  has a spatial (non-Gorenstein) singularity; it is given parametrically as the union of the three branches

$$t \mapsto (t, 0, t^2), \quad t \mapsto (t, t^2, 0), \quad \text{and } t \mapsto (t, 0, 0).$$

$\text{Crimp}(\tilde{\phi}, \Sigma, 2)$ : This stratum is the disjoint union of three one dimensional components, conjugate under the  $\mathbf{S}_3$  action. The points of one of the components correspond to  $R$ -subalgebras  $S$  of  $\tilde{S}$  generated as an  $R$ -module by

$$1, \quad t(1, at, -at), \quad \text{and } t^3\tilde{S},$$

where  $a \in k$ . For  $a \neq 0$ , the singularity of  $\text{Spec } S$  is planar; it is abstractly isomorphic to the singularity  $(u^2 - v^4)(u - v) = 0$ . For  $a = 0$ , the singularity is spatial (non-Gorenstein).

$\text{Crimp}(\tilde{\phi}, \Sigma, 4)$ : This stratum is the disjoint union of three points, conjugate under the  $\mathbf{S}_3$  action. One of them is the  $R$ -subalgebra  $S$  of  $\tilde{S}$  generated as an  $R$ -module by

$$1, \quad (1, 0, 0) \quad \text{and } t^4\tilde{S}.$$

The curve  $\text{Spec } S$  is disconnected (as the case must be if  $m$  or  $n$  is zero); it is the disjoint union of  $\Delta$  and  $\text{Spec } R[y]/(y^2 - t^8)$ .

**2.4.3.  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  for  $\tilde{\phi}$  totally ramified.** Fix  $\tilde{\phi}: \tilde{C} = \text{Spec } \tilde{S} \rightarrow \Delta$ , a triple cover with  $\tilde{C}$  smooth and  $\tilde{\phi}$  totally ramified. Let  $\Delta' \rightarrow \Delta$  be the triple cover given by  $R \rightarrow R' = R[s]/(s^3 - t)$ . Let  $\tilde{S}'$  be the normalization of  $\tilde{S} \otimes_R R'$  and set  $\tilde{C}' = \text{Spec } \tilde{S}'$ . Fix an isomorphism of  $R$  algebras

$$\tilde{S} \cong R[x]/(x^3 - t),$$

and an isomorphism of  $R'$  algebras

$$\tilde{S}' \cong R' \oplus R' \oplus R',$$

such that the normalization map  $\tilde{S} \otimes_R R' \rightarrow \tilde{S}'$  is given by

$$x \mapsto (s, \zeta s, \zeta^2 s),$$

where  $\zeta$  is a third root of unity. Identify  $R$  with its image in  $\tilde{S}$  and  $R'$  with its image in  $\tilde{S}'$ .

**PROPOSITION 2.4.5.** *Let  $M \subset \tilde{S}$  be an  $R$ -submodule of rank three.*

- (1)  $M$  is spanned by three elements  $f_i(x) \in \tilde{S}$ , for  $i = 1, 2, 3$ , having  $x$ -valuations  $v_i$  that are distinct modulo 3.
- (2) Set  $M' = M \otimes_R R'$  and identify it with its image in  $\tilde{S}'$ . Then

$$\tilde{S}'/M' \cong k[s]/s^{v_1} \oplus k[s]/s^{v_2} \oplus k[s]/s^{v_3}.$$

- (3)  $M$  contains  $R$  and is closed under the multiplication map induced from  $\tilde{S}$  if and only if  $M'$  contains  $R'$  and is closed under the multiplication map induced from  $\tilde{S}'$ .

PROOF. Take an  $R$ -basis  $\langle f_i \rangle$  of  $M$  with  $f_i(x) = x^{v_i} g_i(x)$ , where  $g_i(0) \neq 0$  and the  $v_i$  are distinct. Then  $M' \subset \tilde{S}'$  is spanned by the elements

$$s^{v_i}(g_i(s), \zeta^{v_i} g_i(\zeta s), \zeta^{2v_i} g_i(\zeta^2 s)).$$

Since the  $v_i$  are distinct and the three elements above are  $R'$ -linearly independent, the three vectors  $(g_i(0), \zeta^{v_i} g_i(0), \zeta^{2v_i} g_i(0))$  must be  $k$ -linearly independent. It follows that the  $v_i$  are distinct modulo 3 and

$$\tilde{S}'/M' \cong k[s]/s^{v_1} \oplus k[s]/s^{v_2} \oplus k[s]/s^{v_3}.$$

For the last statement, see that  $M$  contains  $R$  if and only if the map  $M \rightarrow \tilde{S}/R$  is zero;  $M$  is closed under multiplication if and only if the map  $M \times_R M \rightarrow \tilde{S}/M$  is zero. Both conditions can be checked after the extension  $R \rightarrow R'$ .  $\square$

Using Proposition 2.4.5 and our analysis of  $R'$ -subalgebras of the étale extension  $R' \rightarrow \tilde{S}'$  from Subsection 2.4.2, we get a description of  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$ .

PROPOSITION 2.4.6. *Retain the setup introduced at the beginning of Subsection 2.4.3. Let  $m, n$  be such that*

$$n + m = 3b/2 \text{ and } n - m = 3l.$$

*First,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is non-empty only if  $m$  and  $n$  are non-negative integers distinct and nonzero modulo 3 and  $2m \geq n$ . If these numerical conditions are satisfied, then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $[l]$ . Its  $k$ -points correspond to  $R$ -subalgebras of  $\tilde{S}$  generated as an  $R$ -module by 1,  $x^n \tilde{S}$  and  $x^m f \in \tilde{S}$ , with  $f$  of the form*

$$f = 1 + \sum_{\substack{0 < i < n-m \\ i \equiv m \pmod{3}}} a_i x^i,$$



for some  $a_i \in k$ .

PROOF. Let  $\tilde{C} \rightarrow C = \text{Spec } S \rightarrow \Delta$  be a crimp with branch divisor given by  $\langle t^b \rangle$  and  $\mu$ -invariant  $l$ . Then  $\tilde{C}' \rightarrow C \times_{\Delta} \Delta' \rightarrow \Delta'$  is a crimp with branch divisor given by  $\langle s^{3b} \rangle$  and  $\mu$ -invariant  $3l$ . Set  $S' = S \otimes_R R'$ . Then

$$\tilde{S}'/S' \cong k[s]/s^m \oplus k[s]/s^n.$$

In particular,  $m$  and  $n$  must be integers. From Proposition 2.4.5,  $0$ ,  $m$  and  $n$  are distinct modulo  $3$ .

$\text{Crimp}(\tilde{\phi}, \Sigma, l)$  may be identified with the space of  $R$ -modules  $S$  satisfying

$$(2.4.4) \quad R \subset S \subset \tilde{S} \text{ and } \tilde{S}'/S' \cong k[s]/s^m \oplus k[s]/s^n,$$

(where  $S' = S \otimes_R R'$ ) with the additional restriction that  $S$  be closed under multiplication. Let  $S$  be an  $R$ -submodule of  $\tilde{S}$  satisfying (2.4.4). From Proposition 2.4.5,  $S$  is generated by  $1$ ,  $x^m f$  and  $x^n g$  where  $f$  and  $g$  are nonzero modulo  $x$ . Then  $S$  is determined by the image of  $f$  in  $\tilde{S}/x^{n-m}\tilde{S}$ . For  $S$  to be closed under multiplication, the image of  $x^{2m}f^2$  in  $\tilde{S}/x^n\tilde{S}$  must be an  $R$ -linear combination of  $1$  and  $x^m f$ . Since all elements of  $S$  lying in  $R$  have  $x$ -valuation divisible by three, this is impossible unless  $x^{2m}f^2 \equiv 0$  in  $\tilde{S}/x^n\tilde{S}$ ; that is  $2m \geq n$ .

For  $2m \geq n$ , every  $f \in \tilde{S}/x^{n-m}\tilde{S}$  nonzero modulo  $\langle R, x \rangle$  yields an  $S$  satisfying (2.4.4) closed under multiplication. Two choices  $f_1$  and  $f_2$  determine the same  $S$  if and only if they are related by

$$x^m f_1 = ax^m f_2 + b,$$

for  $a, b \in R$  with  $a$  invertible. It is not hard to check that  $f$  can be chosen uniquely of the form

$$f = 1 + \sum_{\substack{0 < i < n-m \\ i \equiv m \pmod{3}}} a_i x^i,$$

for some  $a_i \in k$ . Therefore, the stratum  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $\lfloor (n-m)/3 \rfloor = \lfloor l \rfloor$ .  $\square$

Note that since  $n$  and  $m$  are distinct modulo  $3$ , the rational number  $l \in \frac{1}{3}\mathbf{Z}$  is never an integer.

**2.4.4. Crimp( $\tilde{\phi}, \Sigma, l$ ) for  $\tilde{\phi}$  simply ramified.** Fix  $\tilde{\phi}: \tilde{C} = \text{Spec } \tilde{S} \rightarrow \Delta$ , a triple cover with  $\tilde{C}$  smooth and  $\tilde{\phi}$  simply ramified. Let  $\Delta' \rightarrow \Delta$  be the double cover given by  $R \rightarrow R' = R[s]/(s^2 - t)$ . Let  $\tilde{S}'$  be the normalization of  $\tilde{S} \otimes_R R'$  and set  $\tilde{C}' = \text{Spec } \tilde{S}'$ . Set  $\tilde{S}_1 = R[x]/(x^2 - t)$  and fix an

isomorphism of  $R$ -algebras

$$\tilde{S} \cong \tilde{S}_1 \oplus R,$$

and an isomorphism of  $R'$ -algebras

$$\tilde{S}' \cong R' \oplus R' \oplus R',$$

with the normalization map  $\tilde{S} \otimes_R R' \rightarrow \tilde{S}'$  given by

$$(x, 0) \mapsto (s, -s, 0) \quad (0, r) \mapsto (0, 0, r).$$

Identify  $R$  with its image in  $\tilde{S}$  and  $R'$  with its image in  $\tilde{S}'$ .

PROPOSITION 2.4.7. *Let  $M \subset \tilde{S}$  be an  $R$ -submodule of rank three containing  $R$ .*

- (1)  *$M$  is spanned by three elements:  $1$ ,  $(f_1(x), 0)$  and  $(f_2(x), 0)$ , with the  $f_i(x)$  having  $x$ -valuations  $v_i$  that are distinct modulo 2.*
- (2) *Set  $M' = M \otimes_R R'$  and identify it with its image in  $\tilde{S}'$ . Then*

$$\tilde{S}'/M' \cong k[s]/s^{v_1} \oplus k[s]/s^{v_2}.$$

- (3)  *$M$  is closed under the multiplication map induced from  $\tilde{S}$  if and only if  $M'$  is closed under the multiplication map induced from  $\tilde{S}'$ .*

PROOF. Identify  $\tilde{S}/R$  with  $\tilde{S}_1$ . Then there is an equivalence between submodules of  $\tilde{S}$  containing  $R$  and submodules of  $\tilde{S}_1$ . With this modification, the proof is almost identical to the proof of Proposition 2.4.5, with  $\tilde{S}_1$  playing the role of  $\tilde{S}$ .  $\square$

Using Proposition 2.4.7 and our analysis of  $R'$  subalgebras of the étale extension  $R' \rightarrow S'$  from Subsection 2.4.2, we get a description of  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$ .

PROPOSITION 2.4.8. *Retain the setup introduced at the beginning of Subsection 2.4.4. Let  $m, n$  be such that*

$$n + m = b \text{ and } n - m = 2l.$$

*First,  $\text{Crimp}(\tilde{\phi}, l)$  is non-empty only if  $n$  and  $m$  are non-negative integers distinct modulo 2. If these numerical conditions are satisfied, then we have the following two cases:*

- (1) *If  $2m \geq n$ , then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $[l]$ . Its  $k$ -points correspond to  $R$ -subalgebras of  $\tilde{S}$  generated as an  $R$ -module by  $1$ ,  $x^n \tilde{S}$  and  $(x^m f, 0)$  for  $f \in \tilde{S}_1$  of the*

form

$$f = 1 + \sum_{\substack{0 < i < n-m \\ i \text{ odd}}} a_i x^i,$$

(2) If  $n > 2m$ , then we have two further cases:

(a) If  $m$  is odd, then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is empty.

(b) If  $m$  is even, then  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $m/2$ . Its  $k$ -points correspond to  $R$ -subalgebras of  $\tilde{S}$  generated as an  $R$ -module by  $1$ ,  $x^n \tilde{S}$  and  $(x^m f, 0)$  for  $f \in \tilde{S}_1$  of the form

$$f = 1 + \sum_{\substack{n-2m \leq i < n-m \\ i \text{ odd}}} a_i x^i,$$

for some  $a_i \in k$ .

PROOF. Let  $\tilde{C} \rightarrow C = \text{Spec } S \rightarrow \Delta$  be a crimp with branch divisor given by  $\langle t^b \rangle$  and  $\mu$ -invariant  $l$ . Then  $\tilde{C}' \rightarrow C \times_{\Delta} \Delta' \rightarrow \Delta'$  is a crimp with branch divisor given by  $\langle s^{2b} \rangle$  and  $\mu$ -invariant  $2l$ . Set  $S' = S \otimes_R R'$ . Then

$$\tilde{S}'/S' \cong k[s]/s^m \oplus k[s]/s^n.$$

In particular,  $m$  and  $n$  must be integers. From Proposition 2.4.5,  $m$  and  $n$  are distinct modulo 2.

$\text{Crimp}(\tilde{\phi}, \Sigma, l)$  may be identified with the space of  $R$ -modules  $S$  satisfying

$$(2.4.5) \quad R \subset S \subset \tilde{S} \text{ and } \tilde{S}'/S' \cong k[s]/s^m \oplus k[s]/s^n,$$

(where  $S' = S \otimes_R R'$ ) with the additional condition that  $S$  be closed under multiplication. Let  $S$  be an  $R$ -submodule of  $\tilde{S}$  satisfying (2.4.5). See that  $S$  is determined by the image of  $f$  in  $\tilde{S}_1/x^{n-m}\tilde{S}_1$ . For  $S$  to be closed under multiplication  $x^{2m}f^2 \in \tilde{S}_1/x^n\tilde{S}_1$  must be an  $R$ -multiple of  $x^m f$  in  $S_1/x^n\tilde{S}_1$ .

In the case  $2m \geq n$ , any  $f \in \tilde{S}_1/x^{n-m}\tilde{S}_1$  nonzero modulo  $x$  yields an  $S$  satisfying (2.4.5) closed under multiplication. Two  $f_1$  and  $f_2$  give the same  $S$  if and only if they are related by

$$f_1 = a f_2,$$

for some unit  $a \in R$ . It is not hard to check that  $f$  can be chosen uniquely of the form

$$f = 1 + \sum_{\substack{0 < i < n-m \\ i \text{ odd}}} a_i x^i,$$

for some  $a_i \in k$ . Thus,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $\lfloor (n - m)/2 \rfloor = \lfloor l \rfloor$ .

In the case  $n > 2m$ , the condition for being closed under multiplication is non-vacuous. For this to hold,  $x^{2m}f^2 \in \tilde{S}_1/x^n\tilde{S}_1$  must be an  $R$ -multiple of  $x^m f$ . Since elements of  $R$  have even  $x$ -valuation, we conclude that  $m$  must be even. In this case,  $x^{2m}f^2 \equiv x^m f g \pmod{x^n}$  for some  $g \in R$  implies that the image of  $f$  in  $\tilde{S}_1/x^{n-2m}\tilde{S}_1$  is contained in the image of  $R$ . Thus the unique choice of  $f$  as above must have the form

$$f = 1 + \sum_{\substack{n-2m \leq i < n-m \\ i \text{ odd}}} a_i x^i.$$

Thus,  $\text{Crimp}(\tilde{\phi}, \Sigma, l)$  is irreducible of dimension  $m/2$ . □

## Spaces of trigonal curves with a marked unramified fiber

In Chapter 1, we constructed the stack of  $d$ -gonal covers  $\mathcal{H}^d$  in wide generality and explained how it can be used to construct several compactifications of the classical Hurwitz spaces. We observed that these compactifications are especially well-behaved for  $d = 2$  and  $3$ . In the case of  $d = 2$ , we recovered the spaces of hyperelliptic curves first constructed and studied by Fedorchuk [9].

We now take up a detailed study of various birational models of spaces of trigonal curves—degree 3 covers of  $\mathbf{P}^1$ . A particularly interesting picture emerges for the moduli of trigonal curves along with a marked fiber of the trigonal map to  $\mathbf{P}^1$ . In this and the next chapter, we consider the case where the marked fiber is unramified. The generalization to the case of a fiber with other ramification types is carried out in Chapter 5.

The standard compactification of the space of (marked) trigonal curves is the space of (marked) admissible covers. Using the spaces of weighted admissible covers of Section 1.7, we obtain a sequence of new birational models. In this chapter, we construct a sequence of *yet* more birational models that extends the sequence of the spaces of weighted admissible covers. The extended sequence culminates in a Fano fibration, in accordance with the Minimal Model Program. The construction of the new compactifications features an interplay of the classical global geometry of trigonal curves and the local geometry of triple point singularities studied in Chapter 2. The new compactifications are indexed by a positive integer  $l$ ; they parametrize the so-called  *$l$ -balanced triple covers* (Definition 3.2.4). The condition for being  $l$ -balanced involves a global and a local restriction. For a triple cover  $\phi: C \rightarrow \mathbf{P}^1$ , the global restriction is that the *Maroni invariant* of  $\phi$  be at most  $l$ ; this measures, in some sense, how “balanced” the vector bundle  $\phi_*\mathcal{O}_C$  is. The local restriction only pertains to the  $\phi$  for which  $\text{br}(\phi)$  is supported at one point. It requires that the  $\mu$  invariant of the singularity of  $C$  be greater than  $l$ ; this measures, in some sense, how “balanced” the singularity is. The main result of this chapter (Theorem 3.3.4) is that these two restrictions give a proper moduli problem for the space of triple covers with a marked unramified fiber.

The chapter is organized as follows. In Section 3.1, we give an overview of the weighted admissible cover compactifications of the space of marked trigonal curves and motivate the quest for yet more compactifications. In Subsection 3.2.1, we define  $l$ -balanced covers and state the main theorem (Theorem 3.3.4). In Section 1.4, we prove the main theorem. This chapter is primarily about the construction of the new compactifications. The birational geometry of the resulting spaces is the content of the next chapter. Needless to say, the conventions from Section 0.1 are in force throughout.

### 3.1. Background and motivation

Let  $g$  be a non-negative integer and set  $b = 2g + 4$ . Let

$$\mathcal{M}_{0;b,1} \subset \mathcal{M}$$

be the open and closed substack parametrizing  $(P; \Sigma; \sigma_1)$ , where  $P$  is a connected curve of arithmetic genus zero and  $\Sigma$  a divisor of degree  $b$ . Recall that  $\sigma_1$  is required to be away from  $\Sigma$ . Then  $\mathcal{M}_{0;b,1}$  is an irreducible, smooth algebraic stack, containing as a dense open the stack  $\mathcal{M}_{0;b,1}$  where  $P$  is smooth and  $\Sigma$  is reduced.

Let

$$\mathcal{T}_{g;1} \subset \mathcal{H}^3 \times_{\mathcal{M}} \mathcal{M}_{0;b,1}$$

be the open and closed substack parametrizing covers  $(\mathcal{P} \rightarrow P; \sigma_1; \phi: \mathcal{C} \rightarrow \mathcal{P})$ , where  $\text{Aut}_{\sigma_1}(\mathcal{P})$  is trivial and  $\mathcal{C}$  is a connected curve. By the Riemann–Hurwitz formula,  $\mathcal{C}$  has genus  $g$ . By Proposition 1.5.4,  $\mathcal{T}_{g;1}$  is a smooth algebraic stack, containing as a dense open the stack  $\mathcal{T}_{g;1}$  where  $\mathcal{P}$  and  $\mathcal{C}$  are smooth and  $\phi$  is simply branched. Since  $\mathcal{T}_{g;1}$  is irreducible, so is  $\mathcal{T}_{g;1}$ . The stacks  $\mathcal{T}_{g;1}$  and  $\mathcal{M}_{0;b,1}$  are related by the branch morphism

$$\text{br} : \mathcal{T}_{g;1} \rightarrow \mathcal{M}_{0;b,1}$$

$$(\mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P}) \mapsto (P; \text{br}(\phi); \sigma).$$

One of the main results (Theorem 1.3.8) of Chapter 1 is that the branch morphism is proper.

Let  $T_{g;1}$  (resp.  $M_{0;b,1}$ ) be the coarse space of  $\mathcal{T}_{g;1}$  (resp.  $\mathcal{M}_{0;b,1}$ ); these are quasi-projective varieties. As described in Section 1.7, compactifications of  $M_{0;b,1}$  give corresponding compactifications of  $T_{g;1}$  by taking the preimage under the branch morphism. In particular, the compactifications of

$M_{0;b,1}$  given by Hassett’s spaces of weighted marked rational curves give corresponding compactifications of  $T_{g;1}$  by the spaces of weighted admissible covers.

Let us first focus on the compactifications  $\overline{M}_{0;b,1}(\epsilon)$  of the branching data. Recall that  $\overline{M}_{0;b,1}(\epsilon)$  is the coarse space of the open substack  $\overline{\mathcal{M}}_{0;b,1}(\epsilon) \subset \mathcal{M}_{0;b,1}$  that parametrizes  $\epsilon$ -stable marked curves. Also recall that a marked curve  $(P; \Sigma; \sigma)$  is  $\epsilon$ -stable if  $\epsilon \cdot \text{mult}_p \Sigma \leq 1$  for all  $p \in P$  and  $\omega_P(\sigma + \epsilon\Sigma)$  is ample. For  $\epsilon \geq \epsilon'$ , we have a birational morphism  $\overline{M}_{0;b,1}(\epsilon) \rightarrow \overline{M}_{0;b,1}(\epsilon')$ , which sends a marked curve  $(P; \Sigma; \sigma)$  to  $(P'; \Sigma'; \sigma')$ , where  $P'$  is obtained from  $P$  by contracting the components on which  $\omega_P(\sigma + \epsilon'\Sigma)$  is not ample (see Figure 1). The resulting morphism  $\overline{M}_{0;b,1}(\epsilon) \rightarrow \overline{M}_{0;b,1}(\epsilon')$  is a divisorial contraction. Clearly, the only relevant values of  $\epsilon$  are reciprocals of positive integers. Furthermore, we must have  $b \cdot \epsilon + 1 > 2$  to have a non-empty space. We thus get the following sequence of divisorial contractions<sup>1</sup>:

$$(3.1.1) \quad \overline{M}_{0;b,1}(1) \rightarrow \cdots \rightarrow \overline{M}_{0;b,1}(1/j) \rightarrow \overline{M}_{0;b,1}(1/(j+1)) \rightarrow \cdots \rightarrow \overline{M}_{0;b,1}(1/(b-1)).$$

The first model  $\overline{M}_{0;b,1}(1)$  is simply the Mumford–Knutsen compactification of  $M_{0;b,1}$ ; in this model,

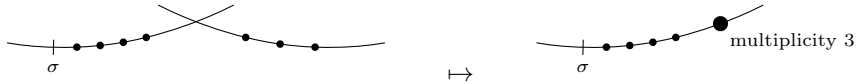


FIGURE 1. The divisorial contraction  $\overline{M}_{0;b,1}(1/3) \rightarrow \overline{M}_{0;b,1}(1/4)$

$\Sigma$  is required to be reduced. The last model  $\overline{M}_{0;b,1}(1/(b-1))$  is a weighted projective space; in this model,  $P \cong \mathbf{P}^1$  and the only restriction on  $\Sigma$  is that it must not be supported at a single point. The picture of the alternate birational models of  $\overline{M}_{0;b,1}(1)$  presented in (3.1.1) agrees perfectly with what is expected by the Minimal Model Program. After all,  $\overline{M}_{0;b,1}$  is a rational (in particular, uniruled) variety. According to the Program, we expect to have a sequence of birational transformations of  $\overline{M}_{0;b,1}$  that culminates in a Fano fibration. The sequence (3.1.1) is indeed such a sequence.

Having described the geometry of the spaces of the branching data, we now turn to the geometry of the spaces of covers. Set

$$\overline{\mathcal{T}}_{g;1}(\epsilon) = \overline{\mathcal{M}}_{0;b,1}(\epsilon) \times_{\mathcal{M}} \mathcal{T}_{g;1}.$$

By Theorem 1.3.8,  $\overline{\mathcal{T}}_{g;1}(\epsilon)$  is a proper Deligne–Mumford stack. By Theorem 1.5.5, it is smooth and irreducible. By Theorem 1.6.1, it has a projective coarse space  $\overline{T}_{g;1}(\epsilon)$ . We thus obtain the following sequence of projective birational models of  $T_{g;1}$ , each lying over the corresponding model of  $M_{0;b,1}$ :

<sup>1</sup>The first map  $\overline{M}_{0;b,1}(1) \rightarrow \overline{M}_{0;b,1}(1/2)$  happens to be an isomorphism.

2

$$(3.1.2) \quad \begin{array}{ccccccc} \overline{T}_{g;1}(1) & \dashrightarrow & \cdots & \dashrightarrow & \overline{T}_{g;1}(1/j) & \dashrightarrow & \overline{T}_{g;1}(1/(j+1)) & \dashrightarrow & \cdots & \dashrightarrow & \overline{T}_{g;1}(1/(b-1)) \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ \overline{M}_{0;b,1}(1) & \rightarrow & \cdots & \rightarrow & \overline{M}_{0;b,1}(1/j) & \rightarrow & \overline{M}_{0;b,1}(1/(j+1)) & \rightarrow & \cdots & \rightarrow & \overline{M}_{0;b,1}(1/(b-1)) \end{array}$$

The first space  $\overline{T}_{g;1}(1)$  is simply the (twisted) admissible cover compactification; in this model, the covers are simply branched. The last space  $\overline{T}_{g;1}(1/(b-1))$  turns out to be a model of Picard rank three; in this model, the base of the covers is  $\mathbf{P}^1$  and the only restriction on the branching is that the branch divisor must not be supported at a single point. Since  $\overline{T}_{g;1}(1/j)$  is normal, the map  $\overline{T}_{g;1}(1/j) \dashrightarrow \overline{T}_{g;1}(1/(j+1))$  is regular away from a locus of codimension two. However, in general, it is not regular everywhere. See that the exceptional locus of its inverse is the locus of covers whose branch divisor contains a  $(j+1)$ -fold point. From our dimension calculation of the spaces of crimps of triple covers (Proposition 2.4.3, Proposition 2.4.6 and Proposition 2.4.8), it follows that this exceptional locus has codimension at least two. In this sense, the maps  $\overline{T}_{g;1}(1/j) \dashrightarrow \overline{T}_{g;1}(1/(j+1))$  are divisorial contractions; they contract certain components of the boundary  $\overline{T}_{g;1}(1/j) \setminus T_{g;1}$  to loci of higher codimension. A component of  $\overline{T}_{g;1}(1/j) \setminus T_{g;1}$  gets contracted precisely if it lies over a component of  $\overline{M}_{0;b,1}(1/j)$  that gets contracted.

The sequence of birational models of  $T_{g;1}$  obtained in this way is incomplete in two respects. First of all, the rational maps  $\overline{T}_{g;1}(1/j) \dashrightarrow \overline{T}_{g;1}(1/(j+1))$  are, in general, not everywhere regular. It would be nice to get an explicit, preferably modular, resolution of these intermediate maps. Secondly, and more importantly, the final model  $\overline{T}_{g;1}(1/(b-1))$  is not what is expected to be an ultimate model according to the Minimal Model Program. It is easy to see that  $T_{g;1}$  is unirational (in particular, uniruled). So we expect to arrive at a Fano-fibration. Therefore, it is natural to ask if (3.1.2) can be extended to reach such a model. The search for the answer to this question motivates the work in this chapter. The spaces of  $l$ -balanced covers constructed in this chapter provide such an extension.

### 3.2. The stack $\overline{\mathcal{T}}_{g;1}^l$ of $l$ -balanced covers

The notion of  $l$ -balanced covers depends on two invariants: the Maroni invariant and the  $\mu$ -invariant.

---

<sup>2</sup>Again, the first map  $\overline{T}_{g;1}(1) \rightarrow \overline{T}_{g;1}(1/2)$  happens to be an isomorphism.



**3.2.1. The Maroni invariant.** Let  $C$  be a reduced curve over  $k$  of arithmetic genus  $g$  and  $\phi: C \rightarrow \mathbf{P}^1$  a triple cover. Set  $F = \phi_* O_C / O_{\mathbf{P}^1}$ . Then  $F$  is a vector bundle on  $\mathbf{P}^1$  of rank two and degree  $-(g+2)$ . We clearly have  $H^0(F(-1)) = 0$ . Therefore, we must have

$$F \cong O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n)$$

for some  $m, n \geq 0$  with  $m+n = g+2$ . In this case, we say that the *splitting type* of  $\phi$  is  $(m, n)$  and its *Maroni invariant* is  $|n-m|$ . We denote the Maroni invariant by  $M(\phi)$ :

$$M(\phi) = |n-m|.$$

We say that a cover with a lower Maroni invariant is *more balanced* than one with a higher Maroni invariant. By the upper semicontinuity of cohomology, the Maroni invariant is upper semicontinuous.

REMARK 3.2.1. The Maroni invariant  $M(\phi)$  satisfies the following numerical conditions:

$$0 \leq M(\phi) \leq g+2 \text{ and } M(\phi) \equiv g \pmod{2}.$$

Furthermore, if  $C$  is connected then  $m, n > 0$  and hence

$$0 \leq M(\phi) \leq g.$$

**3.2.2. The  $\mu$  invariant.** Let  $C$  be a curve over  $k$  of arithmetic genus  $g$  and  $\phi: C \rightarrow \mathbf{P}^1$  a triple cover such that  $\text{supp br}(\phi) = \{p\}$  for some point  $p \in \mathbf{P}^1$ . In this case, we say that  $\phi$  has *concentrated branching at  $p$* . The  $\mu$ -invariant of  $\phi$  is simply the  $\mu$  invariant of the triple cover  $C_p \rightarrow \mathbf{P}_p^1$  as defined in Subsection 3.2.2.

For the convenience of the reader, we recall the definition in the current context. Let  $\tilde{C} \rightarrow C$  be the normalization and  $\tilde{\phi}: \tilde{C} \rightarrow \mathbf{P}^1$  the induced map. Then  $\tilde{C} \cong \mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1$ . Consider the quotient  $Q = \tilde{\phi}_* O_{\tilde{C}} / \phi_* O_C$ . Then  $Q$  is an  $O_{\mathbf{P}^1}$  module of length  $(g+2)$  supported at  $p$ . Since  $Q = (\tilde{\phi}_* O_{\tilde{C}} / O_{\mathbf{P}^1}) / (\phi_* O_C / O_{\mathbf{P}^1})$ , the module  $Q$  is in fact a quotient of the free  $O_{\mathbf{P}^1}$  module  $\tilde{\phi}_* O_{\tilde{C}} / O_{\mathbf{P}^1}$  of rank two. Therefore, we must have

$$Q \cong k[t]/t^m \oplus k[t]/t^n,$$

for some  $m, n \geq 0$  with  $m + n = g + 2$ . In this case, we say that the *splitting type* of the singularity of  $\phi$  over  $p$  is  $(m, n)$  and its  $\mu$ -invariant is  $|n - m|$ . We denote the  $\mu$ -invariant by  $\mu(\phi)$ :

$$\mu(\phi) = |n - m|.$$

By an argument essentially the same as in Proposition 2.4.1, the  $\mu$ -invariant is lower semicontinuous in a family of triple covers with concentrated branching.

REMARK 3.2.2. The  $\mu$  invariant satisfies the same numerical conditions as the Maroni invariant, namely:

$$0 \leq \mu(\phi) \leq g + 2 \text{ and } \mu(\phi) \equiv g \pmod{2}.$$

Furthermore, if  $C$  is connected then  $m, n > 0$  and hence

$$0 \leq \mu(\phi) \leq g.$$

The Maroni invariant and the  $\mu$ -invariant are related by an inequality.

PROPOSITION 3.2.3. *Let  $C$  be a curve over  $k$  of genus  $g$  and  $\phi: C \rightarrow \mathbf{P}^1$  a triple cover with concentrated branching. Then*

$$M(\phi) \leq \mu(\phi).$$

PROOF. Let  $\text{br}(\phi) = b \cdot p$  for some  $p \in \mathbf{P}^1$ , where  $b = 2g + 4$ . Let the splitting type of the singularity over  $p$  be  $(m, n)$ , where  $n \geq m$  and  $n + m = g + 2$ . Set  $F = \phi_* O_C / O_{\mathbf{P}^1}$ . Then  $F \cong O_{\mathbf{P}^1}(-m') \oplus O_{\mathbf{P}^1}(-n')$  for some  $n' \geq m'$  with  $m' + n' = g + 2$ .

Let  $\tilde{C} \rightarrow C$  be the normalization. We have the sequence

$$(3.2.1) \quad 0 \rightarrow F \rightarrow \tilde{\phi}_* O_{\tilde{C}} / O_{\mathbf{P}^1} \rightarrow k[x]/x^m \oplus k[x]/x^n \rightarrow 0.$$

Since  $\tilde{C} \cong \mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1$ , the middle term above is simply  $O_{\mathbf{P}^1}^{\oplus 2}$ . It is easy to see that we have a surjection

$$H^0(O_{\mathbf{P}^1}^{\oplus 2}(n-1)) \twoheadrightarrow H^0(k[x]/x^m \oplus k[x]/x^n).$$

Using the sequence on cohomology of (3.2.1) twisted by  $O_{\mathbf{P}^1}(n-1)$ , we conclude that  $H^1(F(n-1)) = 0$ , or equivalently that  $n' \leq n$ . It follows that

$$M(\phi) = n' - m' = 2n' - (g + 2) \leq 2n - (g + 2) = n - m = \mu(\phi).$$

□

**3.2.3. The stack of  $l$ -balanced covers.**

DEFINITION 3.2.4. Let  $l$  be an integer. A triple cover  $\phi: C \rightarrow \mathbf{P}^1$  is  $l$ -balanced if the following two conditions are satisfied:

- (1) The Maroni invariant of  $\phi$  is at most  $l$

$$M(\phi) \leq l.$$

- (2) If  $\phi$  has concentrated branching, then its  $\mu$ -invariant is greater than  $l$

$$\mu(\phi) > l.$$

DEFINITION 3.2.5. Define  $\overline{\mathcal{T}}_{g;1}^l$  to be the category whose objects over a  $\mathbf{K}$  scheme  $S$  are

$$\overline{\mathcal{T}}_{g;1}^l(S) = \{(P \rightarrow S; \sigma; \phi: C \rightarrow P)\},$$

where

- (1)  $P \rightarrow S$  is a  $\mathbf{P}^1$  bundle and  $\sigma: S \rightarrow P$  a section;
- (2)  $C \rightarrow S$  is a curve with geometrically connected fibers of arithmetic genus  $g$ ;
- (3)  $\phi: C \rightarrow P$  is a triple cover with  $\text{br}(\phi)$  disjoint from  $\sigma(S)$  such that for all geometric points  $s \rightarrow S$ , the cover  $\phi_s: C_s \rightarrow P_s$  is  $l$ -balanced.

We often abbreviate  $(P \rightarrow S; \sigma; \phi: C \rightarrow P)$  to  $(\phi: C \rightarrow P; \sigma)$ .

We have an obvious morphism  $\overline{\mathcal{T}}_{g;1}^l \rightarrow \mathcal{T}_{g;1}$ , given by

$$(P \rightarrow S; \sigma; \phi: C \rightarrow P) \mapsto (P \rightarrow P; \sigma; \phi: C \rightarrow P).$$

PROPOSITION 3.2.6. *The morphism  $\overline{\mathcal{T}}_{g;1}^l \rightarrow \mathcal{T}_{g;1}$  is an open immersion.*

PROOF. Follows easily from the upper semicontinuity of the Maroni invariant and the lower semicontinuity of the  $\mu$  invariant. □

We now state the main theorem.

THEOREM 3.2.7. *Let  $l$  be a non-negative integer. Then  $\overline{\mathcal{T}}_{g;1}^l$  is a Deligne–Mumford stack, smooth and proper over  $\mathbf{K}$ . It is irreducible of dimension  $2g + 2$ .*

The new content in Theorem 3.3.4 is properness; the rest follows from our work in Chapter 1. We prove Theorem 3.3.4 in Section 3.3.

REMARK 3.2.8. For  $l \geq g$ , we have an equality as substacks of  $\mathcal{T}_{g;1}$ :

$$\overline{\mathcal{T}}_{g;1}(1/(b-1)) = \overline{\mathcal{T}}_{g;1}^l.$$

Indeed, since the Maroni and the  $\mu$  invariants lie in the range  $[0, g]$  for connected triple covers, both sides are the open substack of  $\mathcal{T}_{g;1}$  parametrizing  $(P; \sigma; \phi: C \rightarrow P)$  where  $P$  is smooth and  $\phi$  does not have concentrated branching.

COROLLARY 3.2.9.  $\overline{\mathcal{T}}_{g;1}^l$  admits a coarse space  $\overline{T}_{g;1}^l$ , which is an irreducible algebraic space of dimension  $2g+2$  with at worst quotient singularities, proper over  $\mathbf{K}$ . In particular, it is normal and  $\mathbf{Q}$ -factorial.

PROOF. The existence of a coarse space is a theorem of Keel and Mori [22]. The listed properties follow easily from the properties of the corresponding Deligne–Mumford stack.  $\square$

A priori,  $\overline{T}_{g;1}^l$  is only an algebraic space. However, in Chapter 4 we show that it is in fact a projective variety by exhibiting ample line bundles on it (Theorem 4.6.2).

REMARK 3.2.10. By Remark 3.2.1 and Remark 3.2.2, the only pertinent values of  $l$  are the ones satisfying

$$0 \leq l \leq g \text{ and } l \equiv g \pmod{2}.$$

Henceforth, we assume that  $l$  satisfies these conditions.

### 3.3. Proof of the main theorem

This section is devoted to the proof of Theorem 3.3.4. It suffices to prove the theorem after passing to an algebraically closed  $\mathbf{K}$ -field  $k$ . Therefore, we work over  $k$  in the rest of the chapter.

The proof is quite explicit and elementary. The main ingredient is the behavior of vector bundles under pull back and push forward along blow ups of smooth surfaces. Throughout, we use without explicit citation the fact that vector bundles, and hence finite covers, on punctured smooth surfaces admit unique extensions (Proposition 1.4.19).

LEMMA 3.3.1. *Let  $X$  be a smooth surface,  $s \in X$  a point,  $\beta: \text{Bl}_s X \rightarrow X$  the blowup, and  $E \subset \text{Bl}_s X$  the exceptional divisor. Let  $V$  be a locally free sheaf on  $\text{Bl}_s X$ . Denote by  $e$  the natural*

map

$$e: \beta^* \beta_* V \rightarrow V.$$

Then,

- (1)  $\beta_* V$  is torsion free; that is  $\text{Hom}(O_s, \beta_* V) = 0$ .
- (2) If  $H^0(V|_E \otimes O_E(-1)) = 0$ , then  $\beta_* V$  is locally free and  $e$  is injective.
- (3) If  $V|_E$  is globally generated, then  $R^i \beta_* V = 0$  for all  $i > 0$ , and  $e$  is surjective.
- (4) More generally, if  $l \geq 0$  is such that  $V|_E \otimes O_E(l)$  is globally generated, then  $\text{cok}(e)$  is annihilated by  $I_E^l$ , where  $I_E \subset O_{\text{Bl}_s X}$  is the ideal sheaf of  $E$ .

PROOF. Without loss of generality, take  $X$  to be affine.

- (1) We have  $\text{Hom}(O_s, \beta_* V) = \text{Hom}(\beta^* O_s, V) = 0$ , since  $V$  is locally free.
- (2) Set  $X^\circ = X \setminus \{s\} = \text{Bl}_s X \setminus E$  and let  $i: X^\circ \hookrightarrow X$  be the open inclusion. We have a natural map

$$(3.3.1) \quad \beta_* V \rightarrow i_* i^* \beta_* V,$$

which is injective by (1). The target  $i_* i^* \beta_* V$  is locally free—it is the unique locally free extension to  $X$  of  $i^* \beta_* V = V|_{X^\circ}$  on  $X^\circ$  (see, for example, Proposition 1.4.19). We prove that the map (3.3.1) is surjective. Equivalently, we want to prove that every element of  $H^0(X^\circ, V)$  extends to an element of  $H^0(\text{Bl}_s X, V)$ . Take  $f \in H^0(X^\circ, V)$ . Let  $n \geq 0$  be the smallest integer such that  $f$  extends to a section  $\tilde{f}$  in  $H^0(\text{Bl}_s X, V(nE))$ . The minimality of  $n$  means that the restriction of  $\tilde{f}$  to  $E$  is not identically zero. If  $n = 0$ , we are done. If  $n \geq 1$ , then the hypothesis  $H^0(V|_E \otimes O_E(-1)) = 0$  implies that  $\tilde{f}$  restricted to  $E$  is identically zero, contradicting the minimality of  $n$ .

Since  $\beta_* V$  is locally free, so is  $\beta^* \beta_* V$ . The map  $\beta^* \beta_* V \rightarrow V$  is injective away from  $E$ , and hence injective.

- (3) Let  $V|_E$  be globally generated. By the theorem on formal functions, we have

$$(\widehat{R^i \beta_* V})_s = \varprojlim_m H^i(V|_{mE}).$$

To get a handle on  $V|_{mE}$ , we use the exact sequence

$$0 \rightarrow V|_E \otimes I_E^{m-1} \rightarrow V|_{mE} \rightarrow V|_{(m-1)E} \rightarrow 0.$$

Since  $V|_E$  is globally generated, so is  $V|_E \otimes I_E^{m-1} = V|_E \otimes O_E(m-1)$ , for all  $m \geq 1$ . It follows by induction on  $m$  that  $H^i(V|_{mE}) = 0$  for all  $m \geq 1$  and  $i > 0$ . Thus  $R^i \beta_* V = 0$ .

We now prove that  $e$  is surjective. It is an isomorphism away from  $E$ . We need to prove that it is surjective along  $E$ . Since  $V|_E$  is globally generated, it suffices to prove that  $\beta_* V \rightarrow \beta_*(V|_E)$  is surjective. From the exact sequence

$$0 \rightarrow I_E \otimes V \rightarrow V \rightarrow V|_E \rightarrow 0,$$

we get the exact sequence

$$\beta_* V \rightarrow \beta_*(V|_E) \rightarrow R^1 \beta_*(I_E \otimes V).$$

Since  $(I_E \otimes V)|_E = V|_E \otimes O_E(1)$  is globally generated,  $R^1 \beta_*(I_E \otimes V)$  vanishes and we conclude that  $\beta_* V \rightarrow \beta_*(V|_E)$  is surjective.

(4) Consider the diagram

$$\begin{array}{ccccccc} \beta^* \beta_*(I_E^l \otimes V) & \longrightarrow & I_E^l \otimes V & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \beta^* \beta_* V & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & 0 \end{array} .$$

The first row is exact by (3), as  $(I_E^l \otimes V)|_E = V|_E \otimes O_E(l)$  is globally generated. It follows that the multiplication map  $I_E^l \otimes V \rightarrow Q$  is zero. Since  $V \rightarrow Q$  is surjective, the multiplication  $I_E^l \otimes Q \rightarrow Q$  is zero as well.

□

Lemma 3.3.1 allows us to analyze the “blowing down” of trigonal curves. This analysis is the content of the following lemma. Roughly, it says that blowing down a trigonal curve of Maroni invariant  $M$  results in a singularity of  $\mu$  invariant at most  $M$ .

LEMMA 3.3.2. *Let  $X$ ,  $s$ ,  $\beta: \text{Bl}_s X \rightarrow X$ ,  $E$  and  $X^\circ$  be as in Lemma 3.3.1. Let  $F \subset X$  be a smooth curve passing through  $s$  and  $\tilde{F} \subset \text{Bl}_s X$  its proper transform. Let  $\tilde{f}: \tilde{C} \rightarrow \text{Bl}_s X$  be a triple cover, étale over  $\tilde{F}$ . Assume that  $\tilde{C}|_E$  is a reduced curve of genus  $g$  and  $\tilde{\phi}: \tilde{C}|_E \rightarrow E$  has Maroni invariant  $M$ . Denote by  $\phi: C \rightarrow X$  the unique extension to  $X$  of  $\tilde{\phi}: \tilde{C}|_{X^\circ} \rightarrow X^\circ$ . Then  $\phi: C|_F \rightarrow F$  is étale except over  $s$ , and it has a singularity of  $\mu$ -invariant at most  $M$  over  $s$ :*

$$\mu(\phi|_F) \leq M(\tilde{\phi}|_E).$$

The setup is partially described in Figure 2. In this setup, we say that  $C \rightarrow X$  is obtained by

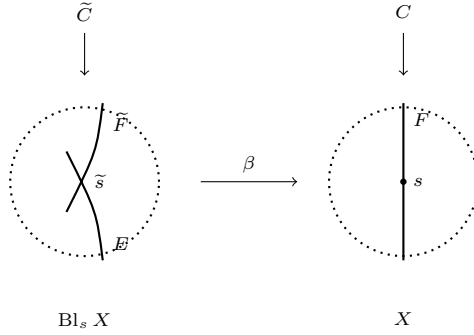


FIGURE 2. The setup of Lemma 3.3.2.

blowing down  $\tilde{C} \rightarrow \text{Bl}_s X$  along  $E$ .

PROOF. To simplify notation, we drop the  $\tilde{\phi}_*$  (resp.  $\phi_*$ ) and simply write  $O_{\tilde{C}}$  (resp.  $O_C$ ), considered as a sheaf of algebras on  $\text{Bl}_s X$  (resp.  $X$ ).

We apply Lemma 3.3.1 to the vector bundle  $O_{\tilde{C}}$  on  $\text{Bl}_s X$ . The condition (2) in Lemma 3.3.1 is clearly satisfied; therefore  $\beta_* O_{\tilde{C}}$  is a locally free sheaf of rank 3. Note that  $\beta_* O_{\tilde{C}}$  is naturally an  $O_X$  algebra which agrees with  $O_C$  on  $X^\circ$ . It follows that

$$O_C = \beta_* O_{\tilde{C}}.$$

Since  $\tilde{C}|_{\tilde{F}} \rightarrow \tilde{F}$  is étale,  $C|_F \rightarrow F$  is étale except possibly over  $s$ . Next, set  $\tilde{s} = \tilde{F} \cap E$ . Consider the map of  $O_{\text{Bl}_s X}$  algebras

$$\nu: \beta^* \beta_* O_{\tilde{C}} = \beta^* O_C \rightarrow O_{\tilde{C}}.$$

This is an isomorphism away from  $E$ , and hence, when restricted to  $\tilde{F}$ , we have a sequence

$$(3.3.2) \quad 0 \rightarrow (\beta^* O_C)|_{\tilde{F}} \xrightarrow{\nu|_{\tilde{F}}} O_{\tilde{C}}|_{\tilde{F}} \rightarrow Q \rightarrow 0,$$

where  $Q$  is supported at  $\tilde{s}$ .

The sequence (3.3.2) exhibits  $O_{\tilde{C}}|_{\tilde{F}}$  as the normalization of  $\beta^* O_C|_{\tilde{F}}$ . Moreover, since  $\beta: \tilde{F} \rightarrow F$  is an isomorphism, the algebra  $\beta^* O_C|_{\tilde{F}}$  on  $\tilde{F}$  can be identified with the algebra  $O_C|_F$  on  $F$  via  $\beta$ . Hence, the splitting type of the singularity of  $C \rightarrow F$  over  $s$  is simply the splitting type of the module  $Q$ .

Let  $x$  be a uniformizer of  $\tilde{F}$  near  $\tilde{s}$ . Suppose

$$Q \cong k[x]/x^m \oplus k[x]/x^n,$$

and

$$O_{\tilde{C}|_E}/O_E \cong O_E(-m') \oplus O_E(-n'),$$

for some positive integers  $m, n, m'$  and  $n'$  with  $m + n = m' + n' = g + 2$ . By Lemma 3.3.1 (4), the ideal  $I_E^{\max\{m', n'\}}$  annihilates the cokernel of  $\nu$ . Restricting to  $\tilde{F}$ , we see that  $x^{\max\{m', n'\}}$  annihilates  $Q$ . In other words,

$$\max\{m, n\} \leq \max\{m', n'\}.$$

Since  $m + n = m' + n'$ , it follows that

$$\mu(\phi|_F) = |m - n| \leq |m' - n'| = M(\tilde{\phi}|_E).$$

□

Next, we prove a precise result about the behavior of rank two bundles under elementary transformations, especially about their splitting type. We first introduce the setup. Let  $R$  be a DVR with uniformizer  $t$ , residue field  $k$ , and fraction field  $K$ . Set  $\Delta = \text{Spec } R$ . Consider  $P = \text{Proj } R[X, Y] = \mathbf{P}_\Delta^1$  with the two disjoint sections  $s_0 \equiv [0 : 1]$  and  $s_\infty \equiv [1 : 0]$ . Denote by  $F$  the central fiber of  $P \rightarrow \Delta$ , and by  $0$  (resp.  $\infty$ ) the point  $F \cap s_0$  (resp.  $F \cap s_\infty$ ). Consider the map

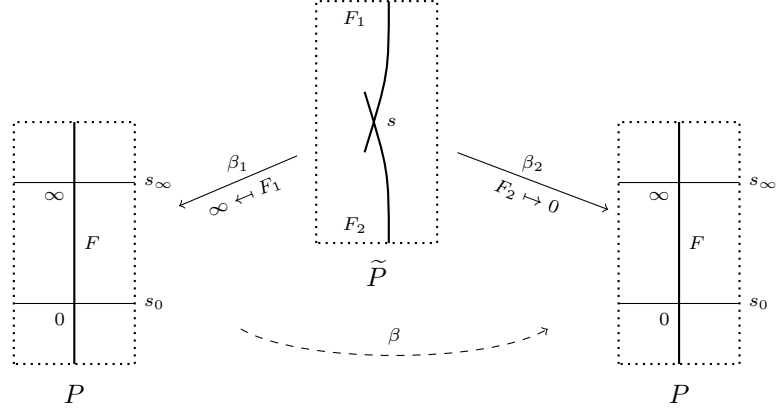
$$\beta: P \setminus \{\infty\} \rightarrow P, \quad [X : Y] \mapsto [tX : Y].$$

Then  $\beta$  has a resolution (see Figure 3)

$$\begin{array}{ccc} & \tilde{P} & \\ \beta_1 \swarrow & & \searrow \beta_2 \\ P & \xrightarrow{\beta} & P \end{array}.$$

Here  $\beta_1: \tilde{P} \rightarrow P$  is the blow up at  $\infty$  and  $\beta_2: \tilde{P} \rightarrow P$  is the blow up at  $0$ . The central fiber of  $\tilde{P} \rightarrow \Delta$  is  $F_1 \cup F_2$ , where  $F_i$  is the exceptional divisor of  $\beta_i$ . The  $F_i$  meet transversely at a point, say  $s$ .



FIGURE 3. Resolution of  $\beta: [X : Y] \mapsto [tX : Y]$ .

LEMMA 3.3.3. Let  $n > m$  be non-negative integers, and  $O(1)$  the dual of the ideal sheaf of  $s_0$ .

Identify

$$\mathrm{Ext}^1(O(-m), O(-n)) = R\langle X^{n-m-2}, \dots, X^i Y^{n-m-2-i}, \dots, Y^{n-m-2} \rangle.$$

Let  $V$  be a vector bundle of rank two on  $P$  given as an extension

$$(3.3.3) \quad 0 \rightarrow O(-n) \rightarrow V \rightarrow O(-m) \rightarrow 0,$$

corresponding to the class  $e(X, Y) \in \mathrm{Ext}^1(O(-m), O(-n))$ . Denote by  $W$  the unique vector bundle on  $P$  obtained by extending  $\beta^*(V)$ . Assume that the class  $t^{m-n+1}e(tX, Y)$ , lying a priori in  $K \otimes_R \mathrm{Ext}^1(O(-m), O(-n))$ , lies in  $\mathrm{Ext}^1(O(-m), O(-n))$ . Then  $W$  can be expressed as an extension

$$0 \rightarrow O(-n) \rightarrow W \rightarrow O(-m) \rightarrow 0,$$

with class  $t^{m-n+1}e(tX, Y)$ . Moreover, in this case, we have an exact sequence

$$(3.3.4) \quad 0 \rightarrow (\beta_1^* W)|_{F_2} \rightarrow (\beta_2^* V)|_{F_2} \rightarrow k[u]/u^m \oplus k[u]/u^n \rightarrow 0,$$

where  $u$  is a uniformizer of  $F_2$  at  $s$ .

We say that a class in  $K \otimes_R \mathrm{Ext}^1(O(-m), O(-n))$  is *integral* if it belongs to  $\mathrm{Ext}^1(O(-m), O(-n))$ . The class  $t^{m-n}e(tX, Y)$  is integral if  $e(X, Y)$  is sufficiently divisible by  $t$ .

PROOF. The proof is by a possibly tedious but straightforward local computation. Write  $P = \mathrm{Spec} R[x] \cup \mathrm{Spec} R[y]$ , where  $x = X/Y$  and  $y = Y/X$ . To ease notation, write  $e(x)$  for  $e(x, 1)$ . Note

that

$$\beta^{-1} \operatorname{Spec} R[x] = \operatorname{Spec} R[x]; \quad \beta^{-1} \operatorname{Spec} R[y] = \operatorname{Spec} K[y].$$

The union  $\beta^{-1} \operatorname{Spec} R[x] \cup \beta^{-1} \operatorname{Spec} K[y]$  is  $P \setminus \{\infty\}$ , as expected.

We can choose local trivializations  $\langle v_x^1, v_x^2 \rangle$  and  $\langle v_y^1, v_y^2 \rangle$  for  $V$  on  $\operatorname{Spec} R[x]$  and  $\operatorname{Spec} R[y]$  respectively, such that they are related on the intersection by

$$(3.3.5) \quad \begin{pmatrix} v_y^1 \\ v_y^2 \end{pmatrix} = \begin{pmatrix} x^{-n} & 0 \\ x^{-n+1}e(x) & x^{-m} \end{pmatrix} \begin{pmatrix} v_x^1 \\ v_x^2 \end{pmatrix}.$$

The bundle  $\beta^*V$  is trivialized by  $\langle \beta^*v_x^1, \beta^*v_x^2 \rangle$  on  $\beta^{-1} \operatorname{Spec} R[x]$  and  $\langle \beta^*v_y^1, \beta^*v_y^2 \rangle$  on  $\beta^{-1} \operatorname{Spec} R[y]$ .

The transition matrix on the intersection is simply the pullback of the matrix in (3.3.5):

$$(3.3.6) \quad \begin{pmatrix} \beta^*v_y^1 \\ \beta^*v_y^2 \end{pmatrix} = \begin{pmatrix} t^{-n}x^{-n} & 0 \\ t^{-n+1}x^{-n+1}e(tx) & t^{-m}x^{-m} \end{pmatrix} \begin{pmatrix} \beta^*v_x^1 \\ \beta^*v_x^2 \end{pmatrix}.$$

Construct  $W$  by gluing trivializations  $\langle w_x^1, w_x^2 \rangle$  on  $\operatorname{Spec} R[x]$  and  $\langle w_y^1, w_y^2 \rangle$  on  $\operatorname{Spec} R[y]$  by

$$(3.3.7) \quad \begin{pmatrix} w_y^1 \\ w_y^2 \end{pmatrix} = \begin{pmatrix} x^{-n} & 0 \\ t^{m-n+1}x^{-n+1}e(tx) & x^{-m} \end{pmatrix} \begin{pmatrix} w_x^1 \\ w_x^2 \end{pmatrix}.$$

Construct an explicit isomorphism  $\psi: \beta^*V \xrightarrow{\sim} W$  on  $P \setminus \{\infty\}$ , as follows:

$$(3.3.8) \quad \begin{aligned} \psi: \begin{pmatrix} \beta^*v_x^1 \\ \beta^*v_x^2 \end{pmatrix} &\mapsto \begin{pmatrix} w_x^1 \\ w_x^2 \end{pmatrix} && \text{on } \beta^{-1} \operatorname{Spec} R[x] = \operatorname{Spec} R[x], \\ \psi: \begin{pmatrix} \beta^*v_y^1 \\ \beta^*v_y^2 \end{pmatrix} &\mapsto \begin{pmatrix} t^{-n}w_y^1 \\ t^{-m}w_y^2 \end{pmatrix} && \text{on } \beta^{-1} \operatorname{Spec} R[y] = \operatorname{Spec} K[y]. \end{aligned}$$

From the transition matrices (3.3.6) and (3.3.7), it is easy to check that this defines a map  $\psi: \beta^*V \rightarrow W$  on  $P \setminus \{\infty\}$ , which is clearly an isomorphism. From (3.3.7), we see that  $W$  is an extension of  $O(-m)$  by  $O(-n)$  corresponding to the class  $t^{m-n+1}e(tX, Y)$ .

Finally, we establish the exact sequence (3.3.4). By Lemma 3.3.1,  $\beta_{1*}\beta_2^*V$  is a vector bundle which is identical to  $\beta^*V$  on  $P \setminus \{\infty\}$ . Therefore, we must have

$$W \cong \beta_{1*}\beta_2^*V.$$

The map  $\beta_1^*W \rightarrow \beta_2^*V$  in (3.3.4) is simply the natural map

$$(3.3.9) \quad \beta_1^*W = \beta_1^*\beta_{1*}(\beta_2^*V) \xrightarrow{\text{ev}} \beta_2^*V.$$

To obtain the cokernel, we express  $\text{ev}$  in local coordinates around  $s$ . Set  $u = \beta_1^*y$ ; this is a function on a neighborhood of  $s$  in  $\tilde{P}$ . A basis for  $\beta_2^*V$  around  $s$  is given by  $\langle \beta_2^*v_x^1, \beta_2^*v_x^2 \rangle$ . A basis for  $\beta_1^*W$  around  $s$  is given by  $\langle \beta_1^*w_y^1, \beta_1^*w_y^2 \rangle$ . From the description of  $\psi$  in (3.3.8), it follows that  $\text{ev}$  is given by

$$\text{ev}: \begin{pmatrix} \beta_1^*w_y^1 \\ \beta_1^*w_y^2 \end{pmatrix} \mapsto \begin{pmatrix} u^n & 0 \\ t^{m-n+1}u^{n-1}e(t/u) & u^m \end{pmatrix} \begin{pmatrix} \beta_2^*v_x^1 \\ \beta_2^*v_x^2 \end{pmatrix}.$$

Note that  $t^{m-n+1}u^{n-1}e(t/u)$  lies in  $R\langle u^{m+1}, \dots, u^{n-1} \rangle$ . Hence, we get

$$\text{cok}(\text{ev}|_{F_2}) \cong k[u]/u^m \oplus k[u]/u^n.$$

Since  $u|_{F_2}$  is a uniformizer for  $F_2$  around  $s$ , the sequence (3.3.4) is established.  $\square$

We now have the tools to prove Theorem 3.3.4, which we restate for the convenience of the reader.

**THEOREM 3.3.4.** *Let  $l$  be a non-negative integer. Then  $\overline{\mathcal{T}}_{g;1}^l$  is a Deligne–Mumford stack, smooth and proper over  $\mathbf{K}$ . It is irreducible of dimension  $2g + 2$ .*

**PROOF OF THEOREM 3.3.4.** Without loss of generality, assume that  $l$  satisfies the conventions in Remark 3.2.10, namely  $0 \leq l \leq g$  and  $l \equiv g \pmod{2}$ . We divide the proof into steps. Recall that  $\mathcal{T}_{g;1}$  is the moduli of  $(\phi: C \rightarrow P; \sigma)$  where  $P \cong \mathbf{P}^1$ ,  $C$  is smooth and connected of genus  $g$ ,  $\phi$  is a simply branched triple cover and  $\sigma \in P \setminus \text{br } \phi$ .

**That  $\overline{\mathcal{T}}_{g;1}^l$  is smooth, of finite type, and irreducible of dimension  $2g + 2$ :** By Proposition 3.2.6,  $\overline{\mathcal{T}}_{g;1}^l$  is an open substack of  $\mathcal{T}_{g;1}$ . By Theorem 1.5.5,  $\mathcal{T}_{g;1}$  is smooth, and hence  $\overline{\mathcal{T}}_{g;1}^l$  is smooth.

Again, by Theorem 1.5.5,  $\mathcal{T}_{g;1}$  contains  $\mathcal{T}_{g;1}$  as a dense open substack. Since  $\mathcal{T}_{g;1}$  is irreducible of dimension  $2g + 2$ , we conclude that  $\overline{\mathcal{T}}_{g;1}^l$  is irreducible of the same dimension.

To see that it is of finite type, denote by  $\mathcal{M}_{0;b,1}^s \subset \mathcal{M}_{0;b,1}$  the open substack parametrizing  $(P; \Sigma; \sigma)$  with  $P$  smooth. It is easy to see that  $\mathcal{M}_{0;b,1}^s$  is of finite type over  $\mathbf{K}$ . By the definition of  $\overline{\mathcal{T}}_{g;1}^l$ , the open immersion  $\overline{\mathcal{T}}_{g;1}^l \hookrightarrow \mathcal{T}_{g;1}$  factors as

$$\overline{\mathcal{T}}_{g;1}^l \hookrightarrow \mathcal{M}_{0;b,1}^s \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1} \subset \mathcal{T}_{g;1}.$$

Since  $\mathcal{T}_{g;1} \rightarrow \mathcal{M}_{0;b,1}$  is of finite type, we conclude that  $\mathcal{M}_{0;b,1}^s \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1}$  and hence  $\overline{\mathcal{T}}_{g;1}^l$  if of finite type over  $\mathbf{K}$ .

**That  $\overline{\mathcal{T}}_{g;1}^l$  is separated:** We use the valuative criterion. Let  $\Delta = \text{Spec } R$  be a the spectrum of a DVR, with special point 0, generic point  $\eta$  and residue field  $k$ . Consider two morphisms  $\Delta \rightarrow \overline{\mathcal{T}}_{g;1}^l$  corresponding to  $(P_i \rightarrow \Delta; \sigma_i; \phi_i: C_i \rightarrow P_i)$  for  $i = 1, 2$ . Let  $\psi_\eta$  be an isomorphism of this data over  $\eta$ , namely isomorphisms  $\psi_\eta^P: P_1|_\eta \rightarrow P_2|_\eta$  and  $\psi_\eta^C: C_1|_\eta \rightarrow C_2|_\eta$  over  $\eta$  that commute with  $\phi_i$  and  $\sigma_i$ . We must show that  $\psi_\eta$  extends to an isomorphism over all of  $\Delta$ .

Suppose that  $\psi_\eta^P$  extends to a morphism  $\psi^P: P_1 \rightarrow P_2$  over  $\Delta$ . Then  $\psi^P$  must be an isomorphism, because the  $P_i \rightarrow \Delta$  are  $\mathbf{P}^1$  bundles and  $\psi_\eta^P$  is an isomorphism. It also follows that  $\psi^P$  must be an isomorphism of marked curves

$$\psi^P: (P_1; \text{br } \phi_1; \sigma_1) \xrightarrow{\sim} (P_2; \text{br } \phi_2; \sigma_2).$$

By the separatedness of  $\mathcal{T}_{g;1} \rightarrow \mathcal{M}_{0;b,1}$ , we conclude that we have an extension  $\psi^C: C_1 \rightarrow C_2$  over  $\psi^P$ .

Therefore, it suffices to show that  $\psi_\eta^P$  extends. Denote by  $\psi^P$  the maximal extension of  $\psi_\eta^P$ . Since  $P_i \rightarrow \Delta$  are  $\mathbf{P}^1$  bundles, the rational map  $\psi^P$  has a resolution of the form

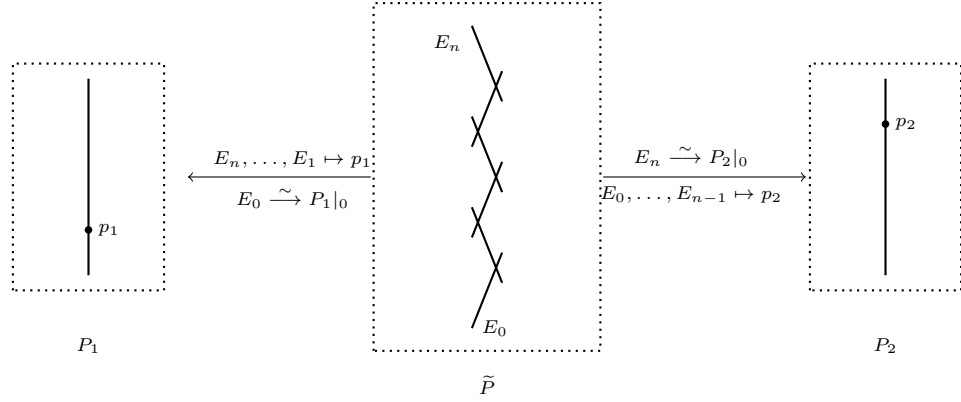
$$(3.3.10) \quad \begin{array}{ccc} & \tilde{P} & \\ & \swarrow \quad \searrow & \\ P_1 & \overset{\psi^P}{\dashrightarrow} & P_2 \end{array},$$

where  $\tilde{P}$  is smooth and its (scheme theoretic) central fiber is a chain of smooth rational curves  $E_0 \cup \dots \cup E_n$ ; the map  $\tilde{P} \rightarrow P_1$  blows down  $E_n, \dots, E_1$  successively to a point  $p_1 \in P_1|_0$ ; and the map  $\tilde{P} \rightarrow P_2$  blows down  $E_0, \dots, E_{n-1}$  successively to a point  $p_2 \in P_2|_0$  (see Figure 4). If  $n = 0$ , then  $\psi^P$  is already a morphism, and we are done. Otherwise, we look for a contradiction.

Since  $\psi_\eta^P$  takes  $\sigma_1(\eta)$  to  $\sigma_2(\eta)$ , either  $p_1 = \sigma_1(0)$  or  $p_2 = \sigma_2(0)$ . By switching 1 and 2 if necessary, say  $p_1 = \sigma_1(0)$ .

Let  $\tilde{C} \rightarrow \tilde{P}$  be the pullback of  $C_1 \rightarrow P_1$ . Since  $C_1 \rightarrow P_1$  is étale over  $\sigma_1(0)$ , the cover  $\tilde{C} \rightarrow \tilde{P}$  is étale over  $E_1, \dots, E_n$ . Then  $C_2 \rightarrow P_2$  is obtained by blowing down  $\tilde{C} \rightarrow \tilde{P}$  successively along  $E_0, \dots, E_{n-1}$ . Thus,  $C_2|_0 \rightarrow P_2|_0$  is has concentrated branching at  $p_2$ ; let  $\mu$  be its  $\mu$ -invariant. On the other hand,  $\tilde{C}|_{E_0} \rightarrow E_0$  is isomorphic to  $C_1|_0 \rightarrow P_1|_0$ ; let  $M$  be its Maroni invariant. Since both  $(C_i \rightarrow P_i)_0$  are  $l$ -balanced, we have

$$\mu > l \geq M.$$

FIGURE 4. The resolution of  $\psi^P: P_1 \rightarrow P_2$ 

However, by repeated application of Lemma 3.3.2 and Proposition 3.2.3, we get

$$\mu \leq M.$$

We have reached a contradiction.

**That  $\overline{\mathcal{T}}_{g;1}^l$  is Deligne–Mumford:** Since we are in characteristic zero, it suffices to prove that a  $k$ -point  $(\phi: C \xrightarrow{\phi} \mathbf{P}^1; \sigma)$  of  $\overline{\mathcal{T}}_{g;1}^l$  has finitely many automorphisms. We have a morphism of algebraic groups

$$\tau: \text{Aut}(\phi: C \rightarrow \mathbf{P}^1, \sigma) \rightarrow \text{Aut}(\mathbf{P}^1).$$

The kernel of  $\tau$  consists of automorphisms of  $\phi$  over the identity of  $\mathbf{P}^1$ . Such an automorphism is determined by its action on a generic fiber of  $\phi$ . Hence  $\ker \tau$  is finite.

Since  $\overline{\mathcal{T}}_{g;1}^l$  is separated,  $\text{Aut}(\phi: C \rightarrow \mathbf{P}^1, \sigma)$  is proper. On the other hand,  $\text{Aut}(\mathbf{P}^1)$  is affine. It follows that  $\text{im } \tau$  is finite. We conclude that  $\text{Aut}(\phi: C \rightarrow \mathbf{P}^1, \sigma)$  is finite.

**That  $\overline{\mathcal{T}}_{g;1}^l$  is proper:** Let  $\Delta = \text{Spec } R$  be as in the proof of separatedness. Denote by  $\bar{\eta}$  a geometric generic point. Let  $(\phi: C_{\bar{\eta}} \xrightarrow{\phi} P_{\bar{\eta}}; \sigma)$  be an object of  $\overline{\mathcal{T}}_{g;1}^l$  over  $\bar{\eta}$ . We need to show that, possibly after a finite base change, it extends to an object of  $\overline{\mathcal{T}}_{g;1}^l$  over  $\Delta$ . Without loss of generality, we may assume that the object over  $\bar{\eta}$  lies in a dense open substack of  $\overline{\mathcal{T}}_{g;1}^l$ . Therefore, we may take  $\phi: C_{\bar{\eta}} \rightarrow P_{\bar{\eta}}$  to not have concentrated branching. Extend  $(P; \text{br } \phi; \sigma)$  to an object  $(P; \Sigma; \sigma)$  of  $\mathcal{M}_{0;b,1}(\Delta)$ . Since  $\overline{\mathcal{T}}_{g;1}^l \rightarrow \mathcal{M}_{0;b,1}$  is proper, we get an extension  $(C \rightarrow P; \sigma)$  of  $(C \rightarrow P; \sigma)_{\bar{\eta}}$  over  $(P; \Sigma; \sigma)$ , possibly after a finite base change. Assume that  $C|_0 \rightarrow P|_0$  satisfies the second condition of Definition 3.2.4. This can be achieved, for instance, by having  $\Sigma|_0$  not supported at a point.

If the Maroni invariant of  $C|_0 \rightarrow P|_0$  is at most  $l$ , we are done. Otherwise, we must modify  $C \rightarrow P$  along the central fiber to make it more balanced. Fix an isomorphism of  $P \rightarrow \Delta$  with  $\text{Proj } R[X, Y] \rightarrow \Delta$  such that the section  $\sigma: \Delta \rightarrow P$  is the zero section  $[0 : 1]$ . Set  $V = \phi_* O_C / O_P$ ; it is a vector bundle of rank 2 on  $P$ . Let

$$V|_{P_0} \cong O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n),$$

where  $m < n$  are positive integers with  $m + n = g + 2$  and  $n - m > l$ . Then we can express  $V$  as an extension

$$(3.3.11) \quad 0 \rightarrow O_P(-n) \rightarrow V \rightarrow O_P(-m) \rightarrow 0.$$

Denote the extension class by

$$e(X, Y) \in \text{Ext}^1(O_P(-m), O_P(-n)) = R\langle X^{n-m-2}, \dots, Y^{n-m-2} \rangle.$$

Since  $C_{\bar{\eta}} \rightarrow P_{\bar{\eta}}$  is  $l$ -balanced but  $n - m > l$ , the class  $e(X, Y)$  is nonzero. However, as the restriction of (3.3.11) to  $P_0$  is split,  $t$  divides  $e(X, Y)$ . By passing to a finite cover  $\tilde{\Delta} \rightarrow \Delta$ , ensure that a sufficiently high power of  $t$  divides  $e(X, Y)$ , so that  $t^{m-n+1}e(tX, Y)$  is integral.

Consider the rational map  $\beta: P \dashrightarrow P$ , sending  $[X : Y]$  to  $[tX : Y]$ . Then  $\beta$  is defined away from  $[1 : 0]$  on the central fiber. Let  $\phi': C' \rightarrow P$  be the unique extension of  $\beta^*C \rightarrow P$ . Then  $C' \rightarrow P$  is isomorphic to  $C \rightarrow P$  on the generic fiber, whereas the central fiber  $C'|_0 \rightarrow P|_0$  is unramified except at  $[1 : 0]$ . The section  $\sigma = [0 : 1]$  of  $P \rightarrow \Delta$  serves as the required marking.

The cover  $C' \rightarrow P$  may be thought of in terms of the resolution of  $\beta$  (as in Figure 3)

$$\begin{array}{ccc} & \tilde{P} & \\ \beta_1 \swarrow & & \searrow \beta_2 \\ P & \xrightarrow{\beta} & P \end{array}$$

Recall that  $\beta_1$  is the blowup at  $[1 : 0]$  and  $\beta_2$  at  $[0 : 1]$  on the central fiber, with exceptional divisors  $F_1$  and  $F_2$  respectively. Set  $\tilde{C} = \beta_2^*C$ . Then  $C' \rightarrow P$  is the blowdown of  $\tilde{C} \rightarrow \tilde{P}$  along  $\beta_1$ . Set  $\tilde{V} = \tilde{\phi}_* O_{\tilde{C}} / O_{\tilde{P}}$  and  $V' = \phi'_* O_{C'} / O_P$ . Then  $\tilde{V} = \beta_2^*V$ , and  $V' = \beta_{1*}\tilde{V}$ .

CLAIM.

- (1)  $V'$  is an extension of  $O_P(-m)$  by  $O_P(-n)$  given by  $e'(X, Y) = t^{m-n+1}e(tX, Y)$ .
- (2) The splitting type of the singularity of  $C'|_0 \rightarrow P|_0$  over  $[1 : 0]$  is  $(m, n)$ .

PROOF. The first claim is directly from Lemma 3.3.3. For the second, consider the natural map

$$\beta_1^* O_{C'} = \beta_1^* \beta_{1*} O_{\tilde{C}} \rightarrow O_{\tilde{C}}.$$

Its restriction  $\beta_1^* O_{C'}|_{F_2} \rightarrow O_{\tilde{C}}|_{F_2}$  expresses  $O_{\tilde{C}}|_{F_2}$  as the normalization of  $\beta_1^* O_{C'}|_{F_2}$ . On the other hand,  $\beta_1$  gives an isomorphism between  $\beta_1^* O_{C'}|_{F_2}$  and  $O_{C'}|_{P_0}$ . Hence, the splitting type of the singularity of  $C'|_0$  over  $[1 : 0]$  is the splitting type of the cokernel of  $\beta_1^* O_{C'}|_{F_2} \rightarrow O_{\tilde{C}}|_{F_2}$ . We can factor out the  $O_{\tilde{P}}|_{F_2}$  summands, by considering the diagram

$$\begin{array}{ccccccc} & & \beta_1^* O_P|_{F_2} & \xlongequal{\quad} & O_{\tilde{P}}|_{F_2} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \beta_1^* O_{C'}|_{F_2} & \longrightarrow & O_{\tilde{C}}|_{F_2} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \beta_1^* V'|_{F_2} & \longrightarrow & \tilde{V}|_{F_2} = \beta_2^* V|_{F_2} & \longrightarrow & Q \longrightarrow 0 \end{array}$$

Thus, the second claim follows from the last exact sequence in Lemma 3.3.3.  $\square$

Returning to the main proof, we see that the operation  $e(X, Y) \mapsto t^{m-n+1}e(tX, Y)$  in coordinates is:

$$X^i Y^{n-m-2-i} \mapsto t^{m-n+1+i} X^i Y^{n-m-2-i}, \text{ for } i = 0, \dots, n-m-2.$$

See that the above operation acts by purely “negative weights”  $t^{m-n+1+i}$ . It follows that after a base change  $\tilde{\Delta} \xrightarrow{t^N \mapsto t} \Delta$  for a sufficiently divisible  $N$  and a sequence of transformations  $[X : Y] \mapsto [tX : Y]$  as above, we can arrange:

- (I) The extension class  $e'(X, Y) \in \text{Ext}^1(O_P(-n), O_P(-m))$  of  $V'$  is nonzero modulo  $t$ .
- (II) The splitting type of the singularity of  $C'|_0 \rightarrow P|_0$  over  $[1 : 0]$  is  $(m, n)$ .

By (I), the new central fiber  $\phi'_0: C'|_0 \rightarrow P|_0$  is more balanced than the original  $C|_0 \rightarrow P|_0$ . Since  $l < n - m$ , the new central fiber also has  $\mu$ -invariant greater than  $l$ . If  $M(\phi'_0) \leq l$ , then the new central fiber is  $l$ -balanced, and we are done. Otherwise, we repeat the entire procedure. After finitely many such iterations, we arrive at a central fiber of Maroni invariant at most  $l$  and  $\mu$ -invariant greater than  $l$ . The proof of properness is thus complete.  $\square$

## The birational geometry of $\overline{T}_{g;1}^l$

In Chapter 3, we constructed a sequence of proper Deligne–Mumford stacks  $\overline{\mathcal{T}}_{g;1}^l$  as compactifications of the stack  $\mathcal{T}_{g;1}$  of trigonal curves with a marked unramified fiber. Their coarse spaces  $\overline{T}_{g;1}^l$  give a sequence of birational models of the space  $T_{g;1}$ :

$$(4.0.12) \quad \overline{T}_{g;1}^g \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}^l \xrightarrow{\beta_l} \overline{T}_{g;1}^{l-2} \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}^{0 \text{ or } 1}.$$

In this chapter, we study these spaces and these birational transformations.

We use  $\text{Exc}$  to denote the locus where a rational map is not an isomorphism. Thus,  $\text{Exc}(\beta_l) \subset \overline{T}_{g;1}^l$  is the locus of covers of Maroni invariant  $l$  and  $\text{Exc}(\beta_l^{-1}) \subset \overline{T}_{g;1}^{l-2}$  the locus of covers with concentrated branching and  $\mu$ -invariant  $l$ .

Here is a summary of the main results of this chapter.

**THEOREM 4.0.5.** *Let  $l$  be an integer with  $0 \leq l \leq g$  and  $l \equiv g \pmod{2}$ .*

(1) *The algebraic spaces  $\overline{T}_{g;1}^l$  are projective schemes.*

(2) *The rational map*

$$\beta_g: \overline{T}_{g;1}^g \dashrightarrow \overline{T}_{g;1}^{g-2}$$

*extends to a morphism, which contracts the “hyperelliptic divisor” to a point.*

(3) *If  $g$  is even, then the rational map*

$$\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$$

*extends to a morphism, which contracts the “Maroni divisor” to a  $\mathbf{P}^1$ .*

(4) *Except in the two cases mentioned above, the rational maps*

$$\beta_l: \overline{T}_{g;1}^l \dashrightarrow \overline{T}_{g;1}^{l-2}$$

*are isomorphisms away from codimension two. In these cases,  $\text{Exc}(\beta_l)$  is covered by  $K$ -negative curves and  $\text{Exc}(\beta_l^{-1})$  by  $K$ -positive curves, where  $K$  is the canonical divisor.*



- (5) For even  $g$ , the final model  $\overline{T}_{g;1}^0$  is the quotient of a weighted projective space by an action of  $\mathbf{S}_3$ . In particular, it is Fano of Picard rank one.
- (6) For odd  $g$ , the final model  $\overline{T}_{g;1}^1$  admits a morphism to  $\mathbf{P}^1$  whose fibers are Fano of Picard rank one.
- (7) For  $0 < l < g$ , the rational Picard group of  $\overline{T}_{g;1}^l$  has rank two. For  $g \neq 3$  it is generated by  $\lambda$  and  $\delta$ . The canonical divisor is given by

$$K = \frac{2}{(g+2)(g-3)} (3(2g+3)(g-1)\lambda - (g^2-3)\delta).$$

- (8) There are elements  $D_l$  in the rational Picard group, given in the case of  $g \neq 3$  by

$$\left(\frac{g-3}{2}\right) D_l = \{(7g+6)\lambda - g\delta\} + \frac{l^2}{g+2} \cdot \{9\lambda - \delta\},$$

such that the following hold. For  $l > 0$ , the interior of the cone  $\langle D_l, D_{l+2} \rangle$  is the Mori chamber associated to the model  $\overline{T}_{g;1}^l$ . For even  $g$ , the cone  $\langle D_0, D_2 \rangle$  is the Mori chamber associated to the model  $\overline{T}_{g;1}^0$ . For even (resp. odd)  $g$ , the ray  $\langle D_0 \rangle$  (resp.  $\langle D_1 \rangle$ ) is an edge of the effective cone.

Figure 1 shows a sketch of the Mori chamber decomposition along with an approximate location of the ray  $\langle K \rangle$ .

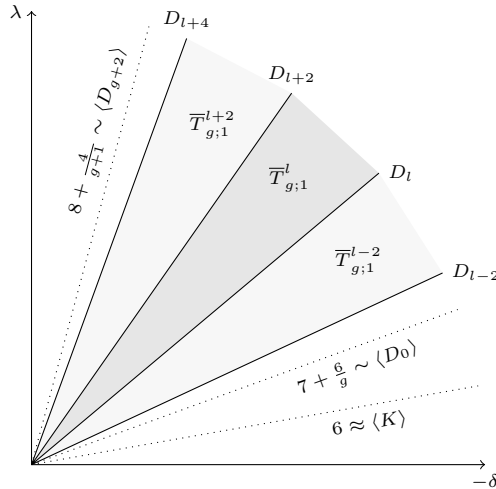


FIGURE 1. The Mori chamber decomposition of  $\text{Pic}_{\mathbf{Q}}$  given by the models  $\overline{T}_{g;1}^l$  and an approximate location of the ray spanned by the canonical class  $K$ .

The statements are not proved in the order in which they are stated. In Section 4.1, we recall some structure theorems for triple covers. In Section 4.2, we use the structure theorems to make

basic dimension counts and prove that most of the  $\beta_l$  are isomorphisms away from codimension two. The assertion about  $K$ -negativity and  $K$ -positivity, however, is proved later in Section 4.6. In Section 4.3, we study the hyperelliptic contraction and in Section 4.4, the Maroni contraction. In Section 4.5, we compute the Picard group and the canonical divisor. In Section 4.6, we compute the ample cones. In Section 4.7, we study the final models.

For the convenience of the reader, we list the statements in the main text corresponding to the statements in the summary above.

- (1) Theorem 4.6.2 for  $0 < l < g$ , Theorem 4.7.2 for  $l = 0$ . The case  $l = g$  follows from the equality  $\overline{T}_{g;1}^g = \overline{T}_{g;1}(1/(b-1))$  (see Remark 3.2.8).
- (2) Theorem 4.3.2,
- (3) Theorem 4.4.3,
- (4) Proposition 4.2.4 and Proposition 4.6.14,
- (5) Theorem 4.7.2,
- (6) Theorem 4.7.4
- (7) Proposition 4.5.1 and Proposition 4.5.4,
- (8) Theorem 4.7.5.

Throughout, we work over an algebraically closed  $\mathbf{K}$  field  $k$ . Recall that  $b = 2g + 4$  is the degree of the branch divisor.

#### 4.1. Structure of triple covers

In this section, we recall some structural results for triple covers. The results of this section hold over  $\mathbf{Z}$ .

**4.1.1. Canonical embedding.** Let  $Y$  be a scheme and  $\phi: X \rightarrow Y$  a triple cover. Assume that the fibers of  $\phi$  are Gorenstein. Define  $F$  by

$$(4.1.1) \quad 0 \rightarrow O_Y \rightarrow \phi_* O_X \rightarrow F \rightarrow 0.$$

Then  $F$  is a vector bundle of rank two on  $Y$ . Setting  $E = F^\vee$  and dualizing, we get a morphism  $E \rightarrow \phi_* \omega_\phi$ , where  $\omega_\phi$  is the dualizing line bundle of  $\phi$ . Equivalently, we have a map

$$\phi^* E \rightarrow \omega_\phi.$$

By explicit verification on the geometric fibers of  $\phi$ , it is easy to check that the above map is surjective and gives an embedding  $X \hookrightarrow \mathbf{P}E$  over  $Y$ . We thus get a sequence

$$(4.1.2) \quad 0 \rightarrow I \rightarrow O_{\mathbf{P}E} \rightarrow O_X \rightarrow 0,$$

where  $I$  has degree  $-3$  on the fibers of  $\pi: \mathbf{P}E \rightarrow Y$ . Twisting (4.1.2) by  $O_{\mathbf{P}E}(1)$  and applying  $\pi_*$  gives

$$0 \rightarrow E \rightarrow \phi_*\omega_\phi \rightarrow R^1\pi_*(I \otimes O_{\mathbf{P}E}(1)) \rightarrow 0.$$

Comparing with the dual of (4.1.1), we get an isomorphism  $R^1\pi_*(I \otimes O_{\mathbf{P}E}(1)) \xrightarrow{\sim} O_Y$ . Using Serre duality for  $\pi_*$ , for which the dualizing line bundle is  $O_{\mathbf{P}E}(-2) \otimes \det E$ , we get an isomorphism

$$\pi_*(I^\vee \otimes O_{\mathbf{P}E}(-3) \otimes \pi^* \det E) \xrightarrow{\sim} O_Y,$$

or equivalently, an isomorphism

$$I \xrightarrow{\sim} O_{\mathbf{P}E}(-3) \otimes \pi^* \det E.$$

Hence (4.1.2) takes the form

$$0 \rightarrow O_{\mathbf{P}E}(-3) \otimes \pi^* \det E \rightarrow O_{\mathbf{P}E} \rightarrow O_X \rightarrow 0.$$

**4.1.2. Structure theorem.** Using the canonical embedding, one can deduce an explicit structure theorem for triple covers. This result is originally due to Miranda [27]. Our exposition is based on the letters of Deligne [6, 7] in response to the work of Gan, Gross, and Savin [13].

Let  $\mathcal{B}$  be the stack over **Schemes** given by  $\mathcal{B}(S) = \{(E, p)\}$ , where  $E$  is a vector bundle of rank two on  $S$  and  $p$  a global section of  $\mathrm{Sym}^3(E) \otimes \det E^\vee$ . Then  $\mathcal{B}$  is an irreducible stack, smooth and of finite type (over  $\mathbf{Z}$ ).

Recall that  $\mathcal{A}_3$  is the classifying stack of triple covers. We define a morphism  $\mathcal{B} \rightarrow \mathcal{A}_3$ . Let  $(\mathcal{E}, p)$  be the universal pair over  $\mathcal{B}$ . Set  $\mathbf{P} = \mathbf{P}\mathcal{E}$  and let  $\pi: \mathbf{P} \rightarrow \mathcal{B}$  be the projection. The section  $p$  gives a map  $i: O_{\mathbf{P}}(-3) \otimes \pi^* \det \mathcal{E} \rightarrow O_{\mathbf{P}}$ . Let  $W \subset \mathcal{B}$  be the locus over which this map is not identically zero. It is easy to see that the complement of  $W \subset \mathcal{B}$  has codimension four. In particular,  $i$  is injective because  $\mathcal{B}$  is smooth and  $i$  is generically injective. Define  $\mathcal{Q}$  as the quotient

$$0 \rightarrow O_{\mathbf{P}}(-3) \otimes \pi^* \det \mathcal{E} \rightarrow O_{\mathbf{P}} \rightarrow \mathcal{Q} \rightarrow 0.$$

Applying  $\pi_*$ , we get

$$0 \rightarrow O_{\mathcal{B}} \rightarrow \pi_* \mathcal{Q} \rightarrow \mathcal{E}^\vee \rightarrow 0.$$

Hence,  $\pi_* \mathcal{Q}$  is an  $O_{\mathcal{B}}$  algebra which is locally free of rank three. We thus get a morphism

$$(4.1.3) \quad f: \mathcal{B} \rightarrow \mathcal{A}_3.$$

**THEOREM 4.1.1.** ([27, Theorem 3.6], [6]) *Let  $\mathcal{B}$  be the stack over **Schemes** described by  $\mathcal{B}(S) = \{(E, p)\}$ , where  $E$  is a vector bundle of rank two on  $S$  and  $p$  a section of  $\mathrm{Sym}^3 E \otimes \det E^\vee$ . Then the morphism  $\mathcal{B} \rightarrow \mathcal{A}_3$  in (4.1.3) is an isomorphism.*

We follow the proof by Deligne [7], which is based on the following observation.

**PROPOSITION 4.1.2.** [34, Proposition 5.1] *The stack  $\mathcal{A}_3$  is smooth. The complement of the Gorenstein locus  $U$  has codimension four.*

**PROOF.** We have a smooth and surjective morphism  $\mathrm{Hilb}_3 \mathbf{A}^2 \rightarrow \mathcal{A}_3$ , where  $\mathrm{Hilb}_3 \mathbf{A}^2$  is the Hilbert scheme of length three subschemes of  $\mathbf{A}^2$ . Since the Hilbert scheme of points on a smooth surface is smooth, we conclude that  $\mathcal{A}_3$  is smooth.

The only non-Gorenstein subschemes of  $\mathbf{A}_k^2$  of length three are  $\mathrm{Spec} k[x, y]/m^2$ , where  $m \subset k[x, y]$  is a maximal ideal. The locus of such has dimension two in the six dimensional space  $\mathrm{Hilb}_3 \mathbf{A}^2$ . It follows that the complement of  $U \subset \mathcal{A}_3$  has codimension four.  $\square$

**PROOF OF THEOREM 4.1.1.** We construct an inverse  $g: \mathcal{A}_3 \rightarrow \mathcal{B}$ . Denote by  $\phi: \mathcal{X} \rightarrow \mathcal{A}_3$  the universal triple cover. Define  $E$  by

$$0 \rightarrow O_{\mathcal{X}} \rightarrow \phi_* O_{\mathcal{X}} \rightarrow E^\vee \rightarrow 0.$$

Then  $E$  is a vector bundle of rank two on  $\mathcal{A}_3$ . We now construct a global section  $p$  of  $\mathrm{Sym}^3 E \otimes \det E^\vee$ . By the procedure of Subsection 4.1.1, over the Gorenstein locus  $U$ , we have an embedding  $\mathcal{X} \hookrightarrow \mathbf{P} = \mathbf{P}E$  giving the sequence

$$0 \rightarrow O_{\mathbf{P}}(-3) \otimes \det E \rightarrow O_{\mathbf{P}} \rightarrow O_{\mathcal{X}} \rightarrow 0.$$

The map  $O_{\mathbf{P}}(-3) \otimes \det E \rightarrow O_{\mathbf{P}}$  gives a section of  $\mathrm{Sym}^3 E \otimes \det E^\vee$  over  $U$ . Since the complement of  $U \subset \mathcal{A}_3$  has codimension at least two and  $\mathcal{A}_3$  is smooth, this section extends to a section  $p$  of  $\mathrm{Sym}^3 E \otimes \det E^\vee$  over all of  $\mathcal{A}_3$ . The pair  $(E, p)$  gives a morphism  $g: \mathcal{A}_3 \rightarrow \mathcal{B}$ .

We must prove that  $f: \mathcal{B} \rightarrow \mathcal{A}_3$  and  $g: \mathcal{A}_3 \rightarrow \mathcal{B}$  are inverses. Here is a sketch. Consider the composite  $f \circ g: \mathcal{A}_3 \rightarrow \mathcal{A}_3$ . It corresponds to a triple cover of  $\mathcal{A}_3$ . To check that  $f \circ g$  is equivalent to the identity, we must check that this triple cover is isomorphic to the universal triple cover. By construction, such an isomorphism exists over the Gorenstein locus  $U$ . Since the complement of  $U$  has codimension higher than two and  $\mathcal{A}_3$  is smooth, the isomorphism extends.

For the other direction, consider the composite  $g \circ f: \mathcal{B} \rightarrow \mathcal{B}$ . It corresponds to a pair  $(\mathcal{E}', p')$  on  $\mathcal{B}$ , where  $\mathcal{E}'$  is a vector bundle of rank two and  $p'$  a section of  $\text{Sym}^3(\mathcal{E}') \otimes \det \mathcal{E}'^\vee$ . To check that  $g \circ f$  is equivalent to the identity, we must check that this pair is isomorphic to the universal pair  $(\mathcal{E}, p)$ . By construction, such an isomorphism exists over  $W$ . Since the complement of  $W$  has codimension higher than two and  $\mathcal{B}$  is smooth, the isomorphism extends. □

#### 4.2. Dimension counts

We return to working over the algebraically closed field  $k$  of characteristic zero.

Let  $0 \leq l \leq g$  be an integer with  $l \equiv g \pmod{2}$ . Denote by  $\mathcal{T}_{g,1}(l) \subset \mathcal{T}_{g,1}$  the locally closed locus consisting of  $(P; \sigma; \phi: C \rightarrow P)$  where  $P \cong \mathbf{P}^1$  and  $\phi$  has Maroni invariant  $l$ . Let  $m \leq n$  be such that

$$n + m = g + 2 \text{ and } n - m = l.$$

Recall that  $\mathcal{T}_{g,1}$  is irreducible of dimension  $b - 2 = 2g + 2$ .

**PROPOSITION 4.2.1.** *Let  $0 \leq l \leq g$  be an integer with  $l \equiv g \pmod{2}$ . Then  $\mathcal{T}_{g,1}(l)$  is irreducible of dimension given by*

$$\dim \mathcal{T}_{g,1}(l) = \begin{cases} 2g + 2 & \text{if } l = 0, \\ 2g + 3 - l & \text{if } 0 < l \leq (g + 2)/3, \\ (3g + l)/2 + 1 & \text{if } (g + 2)/3 < l. \end{cases}$$

*In particular,  $\mathcal{T}_{g,1}(l)$  has codimension one in the following two cases:  $l = g$  and  $l = 2$  (for even  $g$ ). For  $2 < l < g$ , it has codimension at least two.*

PROOF. Let  $E = \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$  and set

$$\begin{aligned} V &= H^0(\mathrm{Sym}^3(E) \otimes \det E^\vee) \\ &= H^0(\mathcal{O}_{\mathbf{P}^1}(2m-n) \oplus \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n) \oplus \mathcal{O}_{\mathbf{P}^1}(2n-m)). \end{aligned}$$

Using  $n+m = g+2$  and  $n-m = l \geq 0$ , we get

$$(4.2.1) \quad \dim V = \begin{cases} 2(g+2) + 4 & \text{if } 2m \geq n, \text{ i.e. } l \leq (g+2)/3 \\ 3(g+l)/2 + 6 & \text{if } 2m < n, \text{ i.e. } l > (g+2)/3 \end{cases}.$$

Using Theorem 4.1.1, a point  $v \in V$  gives a triple cover  $\phi_v: C \rightarrow \mathbf{P}^1$  with  $\phi_{v*} \mathcal{O}_C / \mathcal{O}_{\mathbf{P}^1} = E^\vee$ . Let  $U \subset V \times \mathbf{P}^1$  be the open subset

$$U = \{(v, p) \mid p \notin \mathrm{br}(\phi_v)\}.$$

Then we have a surjective morphism  $U \rightarrow \mathcal{T}_{g,1}(l)$ . Hence  $\mathcal{T}_{g,1}(l)$  is irreducible. The dimension of a general fiber of  $U \rightarrow \mathcal{T}_{g,1}(l)$  is simply

$$\dim \mathrm{Aut} E + \dim \mathrm{Aut}(\mathbf{P}^1) = \dim \mathrm{Aut} E + 3.$$

Hence

$$(4.2.2) \quad \dim \mathcal{T}_{g,1}(l) = \dim U - \dim \mathrm{Aut} E - 3 = \dim V - \dim \mathrm{Aut} E - 2.$$

Observe that

$$(4.2.3) \quad \dim \mathrm{Aut} E = \dim \mathrm{Hom}(E, E) = \begin{cases} 4 & \text{if } l = 0, \text{ i.e. } m = n \\ l + 3 & \text{if } l > 0, \text{ i.e. } m < n \end{cases}.$$

By combining (4.2.1), (4.2.2) and (4.2.3), we get the desired dimension count.  $\square$

Denote by  $\mathcal{T}_{g,1}^\bullet(l) \subset \mathcal{T}_{g,1}$  the locally closed locus consisting of  $(P; \sigma; \phi: C \rightarrow P)$  where  $P \cong \mathbf{P}^1$  and  $\phi$  has concentrated branching with  $\mu$  invariant  $l$ .

PROPOSITION 4.2.2. *Let  $0 \leq l \leq g$  and  $l \equiv g \pmod{2}$ . Then  $\mathcal{T}_{g,1}^\bullet(l)$  is irreducible of dimension given by*

$$\dim \mathcal{T}_{g,1}^\bullet(l) = \begin{cases} l - 1 & \text{if } l \leq (g+2)/3 \\ (g-l)/2 & \text{if } l > (g+2)/3 \end{cases}.$$

In particular,  $\mathcal{T}_{g;1}^\bullet(l) \subset \mathcal{T}_{g;1}$  has codimension at least two.

PROOF. As usual, let  $m, n$  be such that  $n + m = g + 2$  and  $n - m = l$ . Denote by  $\bullet \subset \mathcal{M}_{0;b,1}$  the closed locus consisting of  $(P; \Sigma; \sigma)$ , where  $P \cong \mathbf{P}^1$  and  $\Sigma$  is supported at a point. Then  $\mathcal{T}_{g;1}^\bullet(l)$  is contained in  $\bullet \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1}$ . Note that  $\bullet$  has a unique  $k$ -point, say  $p: \text{Spec } k \rightarrow \mathcal{M}_{0;b,1}$  given by  $(\mathbf{P}^1; b \cdot 0; \infty)$ . However,

$$(4.2.4) \quad \dim(\bullet) = -1,$$

because of the automorphism group  $\text{Aut}(\mathbf{P}^1, 0, \infty) = \mathbf{G}_m$ .

Set  $\tilde{C} = \mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1$  and let  $\tilde{\phi}: \tilde{C} \rightarrow \mathbf{P}^1$  be the projection. Let

$$\text{Crimp}^{\text{conn}}(\tilde{\phi}; b \cdot 0) \subset \text{Crimp}(\tilde{\phi}; b \cdot 0)$$

be the open and closed locus parametrizing crimps  $\tilde{C} \rightarrow C \xrightarrow{\phi} \mathbf{P}^1$  with  $\text{br } \phi = b \cdot 0$  and  $C$  connected.

By Proposition 2.2.5, we get a bijective morphism

$$[\text{Crimp}^{\text{conn}}(\tilde{\phi}; b \cdot 0) / \text{Aut}(\tilde{\phi})] \rightarrow p \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1},$$

Moreover,  $p \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1}^\bullet(l)$  is simply the locus of crimps with  $\mu$ -invariant  $l$ . From Proposition 2.4.3, we get

$$(4.2.5) \quad \dim(p \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1}^\bullet(l)) = \begin{cases} l & \text{if } 2m \leq n, \text{ i.e. } l \leq (g+2)/3, \\ (g+2-l)/2 & \text{if } 2m > n, \text{ i.e. } l > (g+2)/3. \end{cases}$$

Proposition 2.4.3 also implies that the locus in  $[\text{Crimp}^{\text{conn}}(\tilde{\phi}; b \cdot 0) / \text{Aut } \tilde{\phi}]$  of crimps with  $\mu$  invariant  $l$  is irreducible. By combining (4.2.4) and (4.2.5), we get the desired dimension count.  $\square$

Let  $b = b_1 + \dots + b_n$  be a partition of  $b$  with  $b_i \geq 1$  and  $n \geq 2$ . Denote by  $\mathcal{M}_{0;b,1}(\{b_i\}) \subset \mathcal{M}_{0;b,1}$  the locally closed locus consisting of  $k$ -points  $(P; \Sigma; \sigma)$  where  $P \cong \mathbf{P}^1$  and  $\Sigma$  has the form  $\Sigma = \sum_i b_i p_i$  for  $n$  distinct points  $p_1, \dots, p_n \in \mathbf{P}^1$ . Set

$$\mathcal{T}_{g;1}(\{b_i\}) = \mathcal{M}_{0;b,1}(\{b_i\}) \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1}.$$

PROPOSITION 4.2.3. *With the above notation, we have*

$$\dim \mathcal{T}_{g;1}(\{b_i\}) \leq n - 2 + \sum_i \lfloor b_i/6 \rfloor.$$

*In particular,  $\mathcal{T}_{g;1}(\{b_i\})$  has codimension at least two if  $n \leq b - 2$ .*

PROOF. First of all, see that  $\dim \mathcal{M}_{0;b,1}(\{b_i\}) = n - 2$ . Next, we compute the dimensions of the fibers of  $\text{br} : \mathcal{T}_{g;1}(\{b_i\}) \rightarrow \mathcal{M}_{0;b,1}(\{b_i\})$ . Let  $p : \text{Spec } k \rightarrow \mathcal{T}_{g;1}(\{b_i\})$  be a point, given by  $(\mathbf{P}^1; \sigma; \phi : C \rightarrow \mathbf{P}^1)$  with  $\Sigma = \sum_i b_i p_i$ . By Proposition 2.2.5 and Proposition 2.1.2, the dimension of the fiber of  $\text{br}$  containing  $p$  is simply the dimension of  $\prod_i \text{Crimp}(\tilde{\phi}_i, b_i \cdot p_i)$ , where  $\tilde{\phi}$  is the cover of the disk  $\Delta_i$  around  $p_i$  obtained by normalizing  $\phi$ . From the descriptions of  $\text{Crimp}(\tilde{\phi}_i, b_i \cdot p_i)$  in Proposition 2.4.3, Proposition 2.4.6 and Proposition 2.4.8, we see that

$$\dim(\text{Crimp}(\tilde{\phi}_i, b_i \cdot p_i)) \leq \lfloor b_i/6 \rfloor.$$

The result follows. □

A part of Theorem 4.0.5(4) follows immediately from the dimension counts.

PROPOSITION 4.2.4. *For  $2 < l < g$ , the rational map  $\beta_l : \overline{T}_{g;1}^l \dashrightarrow \overline{T}_{g;1}^{l-2}$  is an isomorphism away from codimension two.*

PROOF.  $\text{Exc}(\beta_l)$  is an open subset of the locus of covers of Maroni invariant  $l$ . By Proposition 4.2.1,  $\text{Exc}(\beta_l) \subset \overline{T}_{g;1}^l$  has codimension at least two.

$\text{Exc}(\beta_l^{-1})$  is an open subset of the locus of covers with concentrated branching and  $\mu$  invariant  $l$ . By Proposition 4.2.2,  $\text{Exc}(\beta_l^{-1}) \subset \overline{T}_{g;1}^{l-2}$  has codimension at least two. □

### 4.3. The Hyperelliptic contraction

Let  $g \geq 2$ . In this section, we prove that  $\beta_g : \overline{T}_{g;1}^g \dashrightarrow \overline{T}_{g;1}^{g-2}$  is a divisorial contraction morphism. The idea is to analyze the exceptional loci  $\text{Exc}(\beta_g)$  and  $\text{Exc}(\beta_g^{-1})$ ; the result follows seamlessly from this analysis.

Set  $H = \text{Exc}(\beta_g) \subset \overline{T}_{g;1}^g$ ; this is the locus of curves with Maroni invariant  $g$ . By Proposition 4.2.1,  $H$  is irreducible of dimension  $2g + 1$ . In other words, it is an irreducible divisor. We call  $H$  the *hyperelliptic divisor*. The terminology is justified by the following observation.

PROPOSITION 4.3.1. *A (geometric) generic point of  $H$  corresponds to a marked cover  $(\mathbf{P}^1; \sigma; \phi : C \rightarrow \mathbf{P}^1)$  of the following form (see Figure 2):*



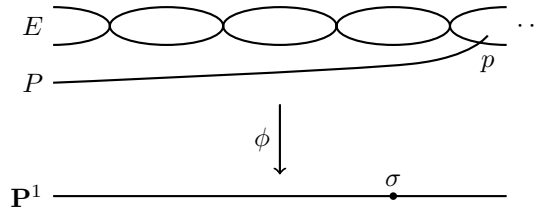


FIGURE 2. A generic point of the hyperelliptic divisor.

- $C = E \cup P$  with  $P \cong \mathbf{P}^1$  and  $E \cap P = \{p\}$ ,
- $E$  is a smooth hyperelliptic curve of genus  $g$  and  $p \in E$  is a non-Weierstrass point,
- $\phi: E \rightarrow \mathbf{P}^1$  has degree two and  $\phi: P \rightarrow \mathbf{P}^1$  has degree one,
- $\sigma \notin \text{br } \phi$ .

PROOF. Let  $(\mathbf{P}^1; \sigma; \phi: C = E \cup P \rightarrow \mathbf{P}^1)$  be such a cover. Then we have

$$h^0(\phi_* O_C \otimes O_{\mathbf{P}^1}(1)) = h^0(\phi^* O_{\mathbf{P}^1}(1)) = 3,$$

which implies that  $\phi_* O_C \cong O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-1) \oplus O_{\mathbf{P}^1}(-g-1)$ . Hence  $\phi$  has Maroni invariant  $g$ . See that the covers  $(\mathbf{P}^1; \sigma; \phi: C = E \cup P \rightarrow \mathbf{P}^1)$  as above form a locus of dimension  $2g+1$ . Since this locus lies in  $H$ , which is irreducible of the same dimension, we conclude that this locus is dense in  $H$ .  $\square$

**THEOREM 4.3.2.** *The birational map  $\beta_g: \overline{T}_{g;1}^g \dashrightarrow \overline{T}_{g;1}^{g-2}$  extends to a morphism. The extension contracts the hyperelliptic divisor  $H$  to a point. The point  $\beta_g(H)$  corresponds to the cover  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$ , where  $\phi$  has concentrated branching and  $C$  has a singularity of type  $D_{2g+2}$ .*

We first prove a lemma (essentially [37, Lemma 4.2]) that gives a simple criterion to check whether an extension defined on  $k$ -points is in fact a morphism.

**LEMMA 4.3.3.** *Let  $X$  and  $Y$  be algebraic spaces over  $k$  with  $X$  normal and  $Y$  proper. Let  $U \subset X$  be a dense open set and  $\phi: U \rightarrow Y$  a morphism. Let  $\phi': X(k) \rightarrow Y(k)$  be a function that agrees with the one given by  $\phi$  on  $U(k)$ . Assume that  $\phi'$  is “continuous in one-parameter families” in the following sense: for every  $\Delta$  which is the spectrum of a DVR with residue field  $k$ , and every morphism  $\gamma: \Delta \rightarrow X$  which sends  $\Delta^*$  to  $U$ , we have*

$$\phi'(\gamma(0)) = (\phi \circ \gamma)(0),$$

where the right hand side is the image of 0 under the unique extension to  $\Delta$  of  $\phi \circ \gamma: \Delta^* \rightarrow Y$ . Then  $\phi: U \rightarrow Y$  extends to a morphism  $\phi: X \rightarrow Y$  that induces  $\phi'$  on  $k$ -points.

PROOF. Let  $\bar{\Phi} \subset X \times Y$  be the closure of the graph  $\Phi$  of  $\phi: U \rightarrow Y$ . Denote by  $\pi_1: \bar{\Phi} \rightarrow X$  and  $\pi_2: \bar{\Phi} \rightarrow Y$  the two projections. Then  $\pi_1: \bar{\Phi} \rightarrow X$  is proper and an isomorphism over  $U$ . We prove that it is an isomorphism. Then the extension of  $\phi$  is obtained by composing  $\pi_1^{-1}: X \rightarrow \bar{\Phi}$  with the projection  $\pi_2: \bar{\Phi} \rightarrow Y$ .

Since  $X$  is normal and  $\pi_1: \bar{\Phi} \rightarrow X$  is proper and dominant, by Zariski's main theorem, it suffices to prove that  $\pi_1$  is an injection on  $k$ -points. Let  $p_1, p_2 \in \bar{\Phi}(k)$  be two points with  $\pi_1(p_i) = x \in X(k)$ . Since  $\Phi$  is dense in  $\bar{\Phi}$ , we can choose maps  $\gamma_i: \Delta \rightarrow \bar{\Phi}$  which send  $\Delta^*$  to  $\Phi$  and 0 to  $p_i$ . Since  $\phi'$  is continuous in one-parameter families, we get

$$\pi_2(p_i) = \pi_2 \circ \gamma_i(0) = (\phi \circ \pi_1 \circ \gamma_i)(0) = \phi'(x).$$

Since  $\pi_1(p_i) = x$ , we get  $p_1 = p_2 = (x, \phi'(x))$ . □

PROOF OF THEOREM 4.3.2. Consider the exceptional locus  $\text{Exc}(\beta_g^{-1}) \subset \bar{T}_{g,1}^{g-2}$ . Let  $p: \text{Spec } k \rightarrow \text{Exc}(\beta_g^{-1})$  be a point, given by a cover  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$ . Then  $M(\phi) \leq g-1$  and  $\phi$  has concentrated branching with  $\mu(\phi) = g$ . Without loss of generality, we may take  $\sigma = \infty$  and  $\text{br } \phi = b \cdot 0$ . The normalization  $\tilde{C}$  of  $C$  is the disjoint union  $\mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1$ . Let  $\text{Spec } k[x] \subset \mathbf{P}^1$  be the standard neighborhood of 0. From Proposition 2.4.3, we see that, up to permuting the three components of  $\tilde{C}$  over  $\mathbf{P}^1$ , the subalgebra  $O_C \subset O_{\tilde{C}}$  is generated locally around 0 as an  $O_{\mathbf{P}^1}$  module by

$$1, \quad x^{g+1}O_{\tilde{C}}, \quad \text{and } (x, ax^g, -ax^g),$$

for some  $a \in k$ . Observe that if  $a = 0$ , then  $M(\phi) = g$ , which is not allowed; hence  $a \neq 0$ . However, two covers given by nonzero  $a, a' \in k$  are isomorphic via the morphism induced by the scaling

$$(\mathbf{P}^1, \infty, 0) \rightarrow (\mathbf{P}^1, \infty, 0), \quad x \mapsto \sqrt[g]{a/a'}x.$$

We conclude that  $\text{Exc}(\beta_g^{-1})$  consists of a single point. Taking  $a = 1$ , we see that the map  $C \rightarrow \mathbf{P}^1$  is given locally by

$$k[x] \rightarrow k[x, y]/(y^2 - x^{2g})(y - x).$$

In particular, the singularity of  $C$  is a  $D_{2g+2}$  singularity.

Next, consider the pointwise extension  $\beta'_g: \overline{T}_{g;1}^g(k) \rightarrow \overline{T}_{g;1}^{g-2}(k)$  which agrees with the one induced by  $\beta_g$  on the complement of  $H = \text{Exc}(\beta_g)$  and sends all the points of  $H$  to the unique point of  $\text{Exc}(\beta_g^{-1})$ . It is clearly continuous in one-parameter families in the sense of Lemma 4.3.3. Since  $\overline{T}_{g;1}^g$  is normal and  $\overline{T}_{g;1}^{g-2}$  proper, we conclude that  $\beta_g$  extends to a morphism that contracts  $H$  to a point.  $\square$

#### 4.4. The Maroni contraction

Let  $g \geq 4$  be even. In this section, we prove that  $\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$  is a divisorial contraction morphism. The idea is the same as in the case of the hyperelliptic contraction; we first define the extension on  $k$ -points and then argue that it is a morphism by checking continuity on one-parameter families. The details are a bit more involved as  $\text{Exc}(\beta_2^{-1})$  is not merely a point. The pointwise extension is obtained by relating the so-called cross-ratio of a marked unbalanced cover on one side and the so-called principal part of an unbalanced crimp on the other side. We begin by defining these two quantities.

For use throughout this section, set  $V = k^{\oplus 3}/k$ , where  $k$  is diagonally embedded and  $\mathbf{P} = \mathbf{P}_{\text{sub}}V/\mathbf{S}_3$ , where  $\mathbf{S}_3$  acts on  $V$  by permuting the three coordinates. The two dimensional vector space  $V$  is to be thought of as the space of functions on  $\{1, 2, 3\} \times \text{Spec } k$  modulo constant functions.

**4.4.1. The cross-ratio of a marked unbalanced cover.** Consider a point  $p: \text{Spec } k \rightarrow \mathcal{T}_{g;1}$  given by  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$ . Set  $F = \phi_*O_C/O_{\mathbf{P}^1}$  and assume that

$$F \cong O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n) \text{ with } 0 < m < n.$$

Define the *cross-ratio of  $\phi$  over  $\sigma$*  as a point of  $\mathbf{P}_{\text{sub}}(F|_{\sigma})$  as the line given by

$$k \cong H^0(F \otimes O_{\mathbf{P}^1}(m)) \hookrightarrow F|_{\sigma} \otimes O_{\mathbf{P}^1}(m) \cong F|_{\sigma}.$$

Since the isomorphisms on both sides are canonical up to the choice of a scalar, this line is well defined. An identification  $C|_{\sigma} \xrightarrow{\sim} \{1, 2, 3\}$  induces an identification  $F|_{\sigma} \xrightarrow{\sim} V$  and lets us treat the cross-ratio as a point of  $\mathbf{P}_{\text{sub}}V$ . Let  $\chi(p)$  be the image of the cross-ratio in  $\mathbf{P} = \mathbf{P}_{\text{sub}}V/\mathbf{S}_3$ . Then  $\chi(p)$  is independent of the identification  $C|_{\sigma} \xrightarrow{\sim} \{1, 2, 3\}$ .

The name ‘‘cross-ratio’’ comes from the following geometric realization of  $\chi(p)$ . For simplicity, assume that  $C$  is Gorenstein. We have the canonical embedding  $C \hookrightarrow \mathbf{F}_M$ , where  $M = n - m$  and

$\mathbf{F}_M = \mathbf{P}F^\vee$  is a Hirzebruch surface. Let  $\tau: \mathbf{P}^1 \rightarrow \mathbf{F}_M$  be the unique section of negative self-intersection and  $P \cong \mathbf{P}^1$  the fiber of  $\mathbf{F}_M \rightarrow \mathbf{P}^1$  over  $\sigma$ . On  $P$ , we have four marked points, namely the three distinct points of  $C|_\sigma$  and the point  $\tau(\sigma)$ . The element  $\chi(p)$  is simply the moduli of  $(P, C|_\sigma, \tau(\sigma))$ . Even if  $C$  is not Gorenstein, it is Gorenstein over an open set  $U \subset \mathbf{P}^1$  containing  $\sigma$ . The geometric description of  $\chi(p)$  goes through if we consider the restricted embedding  $C|_U \hookrightarrow \mathbf{F}_M|_U$ .

**4.4.2. The principal part of an unbalanced crimp.** Analogous to the cross ratio, there is a  $\mathbf{P}$ -valued invariant of a cover with concentrated branching. This is a local invariant, so we consider  $\tilde{\phi}: \tilde{C} \rightarrow \Delta$ , where  $\Delta$  is the disk  $\text{Spec } k[[t]]$  and  $\tilde{\phi}$  an étale triple cover. Let  $\tilde{C} \rightarrow C \xrightarrow{\phi} \Delta$  be a crimp; set  $\tilde{F} = O_{\tilde{C}}/O_\Delta$  and  $F = O_C/O_\Delta$  and  $Q = O_{\tilde{C}}/O_C = \tilde{F}/F$ . Assume that

$$Q \cong k[t]/t^m \oplus k[t]/t^n \text{ with } 0 < m < n.$$

Then the map  $i: F \rightarrow \tilde{F}$  is divisible by  $t^m$  and the rank of the induced map

$$t^{-m}i: F|_0 \rightarrow \tilde{F}|_0$$

is one. Define the *principal part* of the crimp to be the point of  $\mathbf{P}_{\text{sub}}(\tilde{F}|_0)$  given by the image of  $t^{-m}i$ .

More explicitly, from Proposition 2.4.3, we know that  $O_C$  is generated as an  $O_\Delta$  module by 1,  $t^m f$  and  $t^n O_{\tilde{C}}$ , where  $\bar{f} \in \tilde{F}$  is nonzero modulo  $t$ . The principal part is simply the line  $\langle f(0) \rangle \subset \tilde{F}|_0$ . Thus, it partially encodes the moduli of a crimp.

Finally, consider a point  $p: \text{Spec } k \rightarrow \mathcal{T}_{g,1}$  given by  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$ , where  $\phi$  has concentrated branching at 0 with an unbalanced crimp, as above. Identifying  $\tilde{C}|_\sigma \xrightarrow{\sim} \{1, 2, 3\}$  lets us treat the principal part as a point of  $\mathbf{P}_{\text{sub}}V$ . Let  $\rho(p)$  be the image of the principal part in  $\mathbf{P} = \mathbf{P}_{\text{sub}}V/\mathbf{S}_3$ . Then  $\rho(p)$  is independent of the identification  $\tilde{C}|_\sigma \xrightarrow{\sim} \{1, 2, 3\}$ .

The invariants  $\chi(p)$  and  $\rho(p)$  are equal in a particular case.

**PROPOSITION 4.4.1.** *Let  $p = (\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$  be such that  $\phi$  has concentrated branching and  $M(\phi) = \mu(\phi) > 0$ . Then the cross-ratio equals the principal part:*

$$\chi(p) = \rho(p).$$

**PROOF.** Let  $\tilde{C} = \mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1 \rightarrow C$  be the normalization. Set  $F = \phi_* O_C/O_{\mathbf{P}^1}$  and  $\tilde{F} = \tilde{\phi}_* O_{\tilde{C}}/O_{\mathbf{P}^1}$  as usual. Let the splitting type of  $F$  and the splitting type of the singularity be  $(m, n)$

with  $m < n$ . Suppose  $\text{supp br}(\phi) = \{0\}$  and let  $O_{\mathbf{P}^1}(-1) \rightarrow O_{\mathbf{P}^1}$  be the ideal sheaf of  $\{0\}$ . The inclusion  $i: F \rightarrow \tilde{F}$  factors through

$$i': F \otimes O_{\mathbf{P}^1}(m) \rightarrow \tilde{F} \cong O_{\mathbf{P}^1}^{\oplus 2}.$$

Clearly, it is an isomorphism away from 0. Let  $f$  be a nonzero global section of  $F \otimes O_{\mathbf{P}^1}(m)$ . Now,  $\chi(p)$  is defined by the image of  $f$  in  $F \otimes O_{\mathbf{P}^1}(m)|_{\sigma} = \tilde{F}|_{\sigma}$  and  $\rho(p)$  by the image of  $f$  in  $\tilde{F}|_0$ . Since  $\tilde{F}$  is trivial, the two are equal.  $\square$

We now relate the cross-ratio and the principal part in the context of the blowing down of a trigonal curve. Let  $X$  be a smooth surface,  $s \in X$  a point,  $\beta: \text{Bl}_s X \rightarrow X$  the blowup and  $E$  the exceptional divisor. Let  $P \subset X$  be a smooth curve through  $s$ ,  $\tilde{P}$  its proper transform and  $\tilde{s} = E \cap \tilde{P}$ . Let  $\tilde{C} \rightarrow \text{Bl}_s X$  be a triple cover, étale over  $\tilde{P}$ , and set  $\tilde{F} = O_{\tilde{C}}/O_{\text{Bl}_s X}$ . Assume that

$$\tilde{F}|_E \cong O_E(-m) \oplus O_E(-n), \text{ with } 0 < m < n.$$

Let  $C \rightarrow X$  be the cover obtained by blowing down  $\tilde{C} \rightarrow \text{Bl}_s X$  along  $E$ . Then  $C|_P \rightarrow P$  is a crimp over  $s$  with normalization  $\tilde{C}|_{\tilde{P}}$ . Assume that the singularity of  $C|_P \rightarrow P$  over  $s$  also has the splitting type  $(m, n)$ .

**PROPOSITION 4.4.2.** *In the above setup, the cross-ratio of  $(E; \tilde{s}; \tilde{C}|_E \rightarrow E)$  and the principal part of  $(\tilde{C}|_{\tilde{P}} \rightarrow C|_P \rightarrow P)$  are equal.*

**PROOF.** By Lemma 3.3.2, we have  $O_C = \beta_* O_{\tilde{C}}$ . Denote by  $\nu$  the map

$$\nu: \beta^* O_C|_{\tilde{P}} \rightarrow O_{\tilde{C}}|_{\tilde{P}}.$$

This map expresses  $O_{\tilde{C}}|_{\tilde{P}}$  as the normalization of  $\beta^* O_C|_{\tilde{P}} = O_C|_P$ . Set  $F = O_C/O_X$ . As the singularity of  $C|_P \rightarrow P$  over  $s$  has splitting type  $(m, n)$ , there is a nonzero section  $f$  of  $F$  defined around  $s$  such that  $\nu(\beta^* f|_{\tilde{P}})$  has valuation  $m$  with respect to a uniformizer of  $\tilde{P}$  at  $\tilde{s}$ . By shrinking  $X$  if necessary, assume that  $f$  is defined on all of  $X$ . As  $F = \beta_* \tilde{F}$ , we can interpret  $f$  as a section of  $\tilde{F}$  on  $\text{Bl}_s X$ . Since  $\tilde{F}|_E \cong O_E(-m) \oplus O_E(-n)$ , the section  $f$  must in fact be the image of a section  $f'$  under the inclusion

$$\tilde{F} \otimes I_E^m \rightarrow \tilde{F}.$$

Furthermore, since the section  $f|_{\tilde{P}}$  has valuation  $m$  at  $\tilde{s}$ , the section  $f'|_{\tilde{P}}$  has valuation zero at  $\tilde{s}$ . Said differently, the restriction of  $f'$  to  $\tilde{s}$  is nonzero. We see that the cross-ratio of  $(E; \tilde{s}; \tilde{C}|_E \rightarrow E)$

and the principal part of  $(\tilde{C}|_{\tilde{P}} \rightarrow C|_P \rightarrow P)$  are defined by the line spanned by the image of  $f'$  in  $\tilde{F}|_{\tilde{s}}$ .  $\square$

**4.4.3. The Maroni contraction.** We are now ready to tackle  $\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$ . The exceptional locus  $\text{Exc}(\beta_2)$  is the locus in  $\overline{T}_{g;1}^2$  consisting of covers with Maroni invariant two. By Proposition 4.2.1, this locus is an irreducible divisor. We call it the *Maroni divisor*.

**THEOREM 4.4.3.** *Let  $g \geq 4$  be even. The birational map  $\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$  extends to a morphism. The extension contracts the Maroni divisor to  $\mathbf{P} \cong \mathbf{P}^1$ .*

**PROOF.** Set  $g = 2h$ . Consider the exceptional locus  $\text{Exc}(\beta_2^{-1}) \subset \overline{T}_{g;1}^0$ . Let  $p: \text{Spec } k \rightarrow \text{Exc}(\beta_2^{-1})$  be a point, given by  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$ . Then  $M(\phi) = 0$  and  $\phi$  has concentrated branching with  $\mu(\phi) = 2$ . Without loss of generality, we may take  $\sigma = \infty$  and  $\text{br } \phi = b \cdot 0$ . The normalization  $\tilde{C}$  of  $C$  is the disjoint union  $\mathbf{P}^1 \sqcup \mathbf{P}^1 \sqcup \mathbf{P}^1$ . Let  $\text{Spec } k[x] \subset \mathbf{P}^1$  be the standard neighborhood of 0. From Proposition 2.4.3, the subalgebra  $O_C \subset O_{\tilde{C}}$  is generated locally around 0 as an  $O_{\mathbf{P}^1}$  module by

$$1, \quad x^{h+1}O_{\tilde{C}} \text{ and } x^{h-1}f,$$

for some  $f \in O_{\tilde{C}}$  whose image in  $\tilde{F} = O_{\tilde{C}}/O_{\mathbf{P}^1}$  is nonzero modulo  $x$ . Clearly  $O_C$  is determined by  $\bar{f} \in \tilde{F}/x^2\tilde{F}$  and  $\bar{f}$  only matters up to multiplication by a unit in  $k[x]/x^2$ . Let  $\bar{f} = f_1 + xf_2$ , where  $f_i \in \tilde{F}|_0$  with  $f_1 \neq 0$ . By multiplying by units of  $k[x]/x^2$ , we see that  $O_C$  is determined by a line  $\langle f_1 \rangle \subset \tilde{F}|_0$  and an element  $\bar{f}_2$  in the one-dimensional  $k$ -vector space  $\tilde{F}|_0/\langle f_1 \rangle$ . However, if  $\bar{f}_2 = 0$ , then  $M(\phi) = 2$ , which is not allowed. On the other hand, two covers given by  $f_1 + xf_2$  and  $f_1 + axf_2$ , for  $a \in k^*$ , are isomorphic via the map induced by the scaling

$$(\mathbf{P}^1, \infty, 0) \rightarrow (\mathbf{P}^1, \infty, 0), \quad x \mapsto ax.$$

The upshot is that  $p \in \text{Exc}(\beta_2^{-1})$  is determined by the line  $\langle f_1 \rangle \subset \tilde{F}|_0$ , or equivalently by the principal part  $\rho(p) \in \mathbf{P}$ .

We now define an extension  $\beta'_2$  of  $\beta_2: \overline{T}_{g;1}^2 \dashrightarrow \overline{T}_{g;1}^0$  on  $k$ -points. Let  $p \in \text{Exc}(\beta_2)$  be a point corresponding to  $(\mathbf{P}^1; \sigma; \phi: C \rightarrow \mathbf{P}^1)$  with  $M(\phi) = 2$ . Let  $\beta'_2(p)$  be the unique point of  $\text{Exc}(\beta_2^{-1})$  whose principal part equals the cross-ratio  $\chi(p)$  as points of  $\mathbf{P}$ .

By Lemma 4.3.3, it suffices to check that  $\beta'_2$  is continuous in one-parameter families. Let  $\Delta \rightarrow \overline{T}_{g;1}^2$  be a map sending  $\Delta^*$  to the complement of  $\text{Exc}(\beta_2)$  and 0 to a point  $p \in \text{Exc}(\beta_2)$ . By replacing  $\Delta$  by a finite cover, assume that the map lifts to  $\Delta \rightarrow \overline{T}_{g;1}^2$  and is given by the family

$(P; \sigma; \phi: C \rightarrow P)$  over  $\Delta$ . From the proof of the valuative criterion for  $\overline{\mathcal{T}}_{g,1}^l$  (Theorem 3.3.4), we know the procedure to modify  $(P; \sigma; \phi: C \rightarrow P)$  so that the central fiber lies in  $\overline{\mathcal{T}}_{g,1}^0$ . After a sufficiently large base change, it involves blowing up  $\sigma(0)$  and blowing down the proper transform of  $P|_0$ , until the central fiber has Maroni invariant 0. Throughout this process, the central fiber continues to have  $\mu$ -invariant 2. By repeated applications of Proposition 4.4.1 and Proposition 4.4.2, we conclude that the principal part of the resulting limit equals the cross-ratio of the original central fiber. It follows that  $\beta'_2$  is continuous in one-parameter families.  $\square$

#### 4.5. The Picard group

In this section, we compute the rational Picard groups and the canonical divisors of  $\overline{\mathcal{T}}_{g,1}^l$  for  $0 < l < g$ . By Proposition 4.2.4, in this range, all the models  $\overline{\mathcal{T}}_{g,1}^l$  are isomorphic to one another away from codimension two. Hence their Picard groups are identical. Denoting by  $\text{Pic}_{\mathbf{Q}}$  the rational Picard group  $\text{Pic} \otimes \mathbf{Q}$ , we have the equality

$$\text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g,1}^l) = \text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g,1}^l).$$

Moreover, it is clear that the locus of points with nontrivial automorphisms has codimension at least two in  $\overline{\mathcal{T}}_{g,1}^l$ . As a result, the coarse space morphism  $\overline{\mathcal{T}}_{g,1}^l \rightarrow \overline{\mathcal{T}}_{g,1}^l$  is an isomorphism in codimension one. Hence, we may transfer divisors from one to the other without worrying about multiplication factors.

We begin by defining several classes in  $\text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g,1}^l)$ . Let  $(\mathcal{P}; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$  be the universal family over  $\overline{\mathcal{T}}_{g,1}^l$ . Denote by  $\pi_{\mathcal{P}}: \mathcal{P} \rightarrow \overline{\mathcal{T}}_{g,1}^l$  and  $\pi_{\mathcal{C}}: \mathcal{C} \rightarrow \overline{\mathcal{T}}_{g,1}^l$  the projections. When no confusion is likely, we denote both projections by  $\pi$ . Let  $\Sigma \subset \mathcal{P}$  be the branch divisor  $\Sigma = \text{br } \phi$ .

Define the following:

- $\lambda$ : Observe that  $R^1\pi_{\mathcal{C}*}O_{\mathcal{C}}$  is a vector bundle of rank  $g$ . We define  $\lambda$  as the determinant of its dual

$$\lambda = \det(R^1\pi_{\mathcal{C}*}O_{\mathcal{C}})^{\vee}.$$

- $\delta$ : The locus of points in  $\overline{\mathcal{T}}_{g,1}^l$  over which  $\mathcal{C}$  is singular is a divisor. We define  $\delta$  to be its class.

- $T$ : The locus of points in  $\overline{\mathcal{T}}_{g,1}^l$  over which  $\phi$  has a point of triple ramification is a divisor. We define  $T$  to be its class.

- $\text{Br}^2$ : Define  $\text{Br}^2$  as the pushforward

$$\text{Br}^2 = \pi_*(\Sigma^2).$$

$c_1^2$ : Define  $c_1^2$  as the pushforward

$$c_1^2 = \pi_* (c_1(\phi_* O_C)^2[\mathcal{P}]).$$

$c_2$ : Define  $c_2$  as the pushforward

$$c_2 = \pi_* (c_2(\phi_* O_C)[\mathcal{P}]).$$

$\sigma^2$ : Define  $\sigma^2$  as the pushforward

$$\sigma^2 = \pi_* \left( \sigma(\overline{\mathcal{T}}_{g;1}^l)^2 \right).$$

$K$ : Define  $K$  to be the canonical divisor

$$K = K_{\overline{\mathcal{T}}_{g;1}^l} = K_{\overline{\mathcal{T}}_{g;1}^l}.$$

PROPOSITION 4.5.1. *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . Then*

$$\text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g;1}^l) = \mathbf{Q}\langle c_1^2, c_2 \rangle \cong \mathbf{Q}^2.$$

The reason we prefer  $c_1^2$  and  $c_2$  as a basis is that their cohomological nature allows us to compute intersections with curves very easily.

PROOF. We may throw away loci of codimension at least two. Consider the open subset  $V_{g;1}$  (resp.  $U_{g;1}$ ) of  $\overline{\mathcal{T}}_{g;1}^l$  consisting of  $(P; \sigma; \phi: C \rightarrow P)$  where  $\text{br}(\phi)$  has at least  $(b-1)$  (resp.  $b$ ) points in its support. By Proposition 4.2.3, the complement of  $V_{g;1}$  has codimension at least two. Hence

$$\text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g;1}^l) = \text{Pic}_{\mathbf{Q}}(V_{g;1}).$$

On the other hand, it is easy to see that the complement of  $U_{g;1}$  in  $V_{g;1}$  consists of two irreducible divisors, namely  $T$  (the divisor where  $\phi$  has a triple ramification point) and  $\delta$  (the divisor where  $C$  is singular). We thus have an exact sequence

$$\mathbf{Q}\langle T, \delta \rangle \rightarrow \text{Pic}_{\mathbf{Q}}(V_{g;1}) \rightarrow \text{Pic}_{\mathbf{Q}}(U_{g;1}) \rightarrow 0.$$

We claim that  $\text{Pic}_{\mathbf{Q}}(U_{g;1}) = 0$ . Indeed, consider the moduli space  $U_g$  of unmarked trigonal curves

$$U_g = \{(P; \phi: C \rightarrow P)\},$$



where  $P \cong \mathbf{P}^1$ ,  $C$  is a smooth and connected curve of genus  $g$  and  $\phi$  is simply branched. From the work of Stankova-Frenkel [38, Proposition 12.1] or Bolognesi and Vistoli [4, Theorem 1.1], we know that  $\text{Pic}_{\mathbf{Q}}(U_g) = 0$ . Let  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  be the universal object over  $U_g$ . Then  $\mathcal{P} \rightarrow U_g$  is a conic bundle and  $U_{g;1} \subset \mathcal{P}$  is the complement of  $\text{br}(\phi)$ . We conclude that  $\text{Pic}_{\mathbf{Q}}(U_{g;1}) = 0$ .

At this point, we know that the dimension of  $\text{Pic}_{\mathbf{Q}}(\overline{\mathcal{T}}_{g;1}^l)$  is at most two. It is easy to verify (for example, by intersecting with test curves) that  $c_1^2$  and  $c_2$  are linearly independent.  $\square$

We now express all the divisors described above in terms of  $c_1^2$  and  $c_2$ . We are particularly interested in the expressions for  $\lambda$ ,  $\delta$  and  $K$ . As in the proof of Proposition 4.5.1, we may throw away loci of codimension higher than two. Accordingly, let  $W_{0;b,1} \subset \mathcal{M}_{0;b,1}$  be the open locus consisting of points  $(P; \sigma; \Sigma)$ , where  $P \cong \mathbf{P}^1$  and  $\text{supp } \Sigma$  has at least  $(b-1)$  points. As before, let  $V_{g;1}$  be the preimage of  $W_{0;b,1}$  in  $\overline{\mathcal{T}}_{g;1}^l$ . Throughout the rest of this section, we work on  $V_{g;1} \subset \overline{\mathcal{T}}_{g;1}^l$ .

**PROPOSITION 4.5.2.** *Denote by  $D \subset W_{0;b,1}$  the divisor consisting of points where  $\Sigma$  is not reduced. Then*

$$\text{br}^* D = 3T + \delta.$$

**PROOF.** Consider a point  $w$  of  $D$  given by  $(P; \Sigma; \sigma)$ . Then  $\Sigma = 2p + p_3 + \cdots + p_b$  for distinct points  $p, p_3, \dots, p_b \in P$ . Consider a point  $v$  of  $V_{g;1}$  over  $(P; \Sigma; \sigma)$  given by a triple cover  $\phi: C \rightarrow P$ . Then  $\phi$  either has a triple ramification point or a node over  $p$ . Hence,

$$\text{supp } \text{br}^* D = \text{supp } T \cup \text{supp } \delta.$$

We must verify the multiplicities.

We look at the morphism  $\text{Def}_{\phi} \rightarrow \text{Def}_{P;\sigma}$  to compute the multiplicity of  $\text{br}^* D$ . Let  $U \rightarrow \mathbf{P}^1$  be a neighborhood of  $p$  and  $\phi_U: C|_U \rightarrow U$  the restriction of  $\phi$ ; it suffices to look at  $\text{Def}_{\phi_U} \rightarrow \text{Def}_{U,\Sigma|_U}$ . In fact, we may even restrict to an étale neighborhood.

Take the case where  $v \in T$ . Then, étale locally around  $p$ , the cover  $\phi$  has the form  $\phi \equiv \text{Spec } k[x, y]/(x - y^3) \rightarrow \text{Spec } k[x]$ ; the branch divisor is given by  $\Sigma = x^2$ . A versal deformation of the subscheme  $\Sigma \subset \text{Spec } k[x]$  can be given over  $R = k[u, v]$  as the family

$$\text{Spec } R[x]/(x^2 + ux + v).$$

The divisor  $D \subset \text{Spec } R$  corresponding to non-reduced  $\Sigma$  is given by  $\langle u^2 - 4v \rangle$ . A versal deformation of  $\phi$  can be given over  $S = k[a, b]$  as the family

$$\text{Spec } S[x, y]/(x - y^3 - ay - b) \rightarrow \text{Spec } S[x].$$

The divisor  $T \subset \text{Spec } S$  corresponding to covers with a triple ramification point is given by  $\langle a \rangle$ . On the other hand, the branch divisor in  $\text{Spec } S[x]$  is  $\langle 27(x - b)^2 + 4a^3 \rangle$ . Thus, under the induced map  $\text{Spec } S \rightarrow \text{Spec } R$ , the pullback of  $\langle u^2 - 4v \rangle$  is  $\langle a^3 \rangle$ . In other words,  $T$  appears with multiplicity three in  $\text{br}^* D$ .

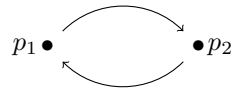
The case where  $v \in \delta$  is similar. Étale locally around  $p$ , the cover  $\phi$  has the form  $\phi \equiv \text{Spec } k[x] \sqcup \text{Spec } k[x, y]/(x^2 - y^2) \rightarrow \text{Spec } k[x]$ . A versal deformation of  $\phi$  can be given over  $S' = k[c]$  as the family

$$\text{Spec } S'[x] \sqcup \text{Spec } S'[x, y]/(x^2 - y^2 - c) \rightarrow \text{Spec } S'[x].$$

The divisor  $\delta \subset \text{Spec } S'$  corresponding to singular covers is given by  $\langle c \rangle$ . On the other hand, the branch divisor in  $\text{Spec } S'[x]$  is  $\langle x^2 - c \rangle$ . Thus, under the induced map  $\text{Spec } S' \rightarrow \text{Spec } R$ , the pullback of  $\langle u^2 - 4v \rangle$  is  $\langle c \rangle$ . In other words,  $\delta$  appears with multiplicity one in  $\text{br}^* D$ .  $\square$

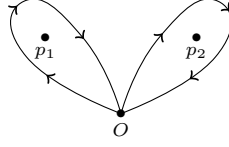
ANOTHER PROOF OF PROPOSITION 4.5.2 OVER  $\mathbf{C}$ . Assume that  $k = \mathbf{C}$ . In this case, we can compute the multiplicities of  $T$  and  $\delta$  in  $\text{br}^* D$  by a beautiful topological argument due to Joe Harris. Take a point  $w \in D$  and let  $v \in V_{g;1}$  be a point over  $w$ . The idea is to compute the multiplicity of the component of  $\text{br}^* D$  containing  $v$  by analyzing the monodromy of  $V_{g;1} \rightarrow W_{0;b,1}$  around  $w$  near  $v$ . Concretely, this can be done by lifting a (real) loop around  $w$ , beginning at a point near  $v$ .

Let  $w \equiv 2p + p_3 + \dots + p_b$ . Take a nearby point  $w' \equiv p_1 + p_2 + p_3 + \dots + p_b$ , where  $p_1 \neq p_2$ . Take a loop  $\gamma$  in  $W_{0;b,1}$  based at  $w'$  that exchanges  $p_1$  and  $p_2$  in the standard way:



Set  $\pi_1 = \pi_1(\mathbf{P}^1 \setminus \{p_1, \dots, p_b\})$ . Take a point  $v'$  over  $w'$  near  $v$ . Then  $v'$  is determined by some monodromy data  $\pi_1 \rightarrow \mathbf{S}_3$ . Choose a basepoint  $O$  in  $\mathbf{P}^1$  and represent the generators of  $\pi_1$  by the

standard circuits around  $p_i$ :



Express the monodromy  $\pi_1 \rightarrow \mathbf{S}_3$  by the  $b$ -tuple  $(\sigma_1, \dots, \sigma_b)$  where  $\sigma_i$  is the image of the standard loop around  $p_i$  described above. Observe that the lift of  $\gamma$  which starts at  $v' \equiv (\sigma_1, \sigma_2, \dots)$  ends at  $v'' \equiv (\sigma_2, \sigma_2^{-1}\sigma_1\sigma_2, \dots)$ . The order of the operation  $v' \mapsto v''$  is the multiplicity of the component of  $\text{br}^* D$  containing  $v$ .

For  $v \in \delta$ , we can take

$$\sigma_1 = \sigma_2 = (12).$$

Then  $v' = v''$ , and hence  $\delta$  appears with multiplicity one in  $\text{br}^* D$ .

For  $v \in T$ , we can take

$$\sigma_1 = (12), \quad \sigma_2 = (23).$$

Then  $v' \mapsto v''$  has order three; the cycle is given by

$$(12), (23) \mapsto (23), (13) \mapsto (13)(12) \mapsto (12), (23).$$

Hence  $T$  appears with multiplicity three in  $\text{br}^* D$ . □

PROPOSITION 4.5.3. *Let  $l$  be such that  $0 < l < g$  and  $l \equiv g \pmod{2}$ . Then the following relations hold in  $\text{Pic}_{\mathbf{Q}}(\overline{T}_{g,1}^l)$ :*

$$\begin{aligned} \text{Br}^2 &= 4c_1^2, \\ \lambda &= \frac{g+1}{2(g+2)}c_1^2 - c_2, \\ T &= 3c_2, \\ \delta &= \frac{4g+6}{(g+2)}c_1^2 - 9c_2, \\ \sigma^2 &= -\frac{1}{(g+2)^2}c_1^2. \end{aligned}$$

PROOF. The relations follow from straightforward Chern class calculations. The first few are scattered throughout [38]. We present the details for completeness.

Take a one-parameter family of triple covers

$$C \xrightarrow{\phi} P \xrightarrow{\pi} B,$$

where  $B$  is a smooth projective curve,  $P \rightarrow B$  a  $\mathbf{P}^1$  bundle and  $C \rightarrow B$  a family of connected curves of genus  $g$ . Let  $\Sigma = \text{br } \phi \subset P$ . Set  $E = (\phi_* O_C / O_P)^\vee$ . Then  $c_1(\phi_* O_C) = -c_1(E)$  and  $c_2(\phi_* O_C) = c_2(E)$ . Choose a (possibly rational) class  $\zeta$  on  $P$  which has degree one on the fibers of  $\pi$  and satisfies  $\zeta^2 = 0$ .

In the calculations that follow, we omit writing pullbacks or push-forwards where they are clear by context. We use  $[\ ]$  to denote the class of a divisor in the Chow ring. Chern classes are understood to be applied to the fundamental class.

Since  $\Sigma = \text{br } \phi$  is the zero locus a section of  $(\det \phi_* O_C^\vee)^{\otimes 2}$ , we have

$$[\Sigma] = 2c_1(E).$$

This gives the first relation.

Using that  $\Sigma$  has relative degree  $b = 2(g+2)$  over  $B$  and  $\zeta^2 = 0$ , we get

$$(4.5.1) \quad [\Sigma] \cdot \zeta = \frac{c_1^2(E)}{g+2} \text{ and } c_1(E) \cdot \zeta = \frac{c_1^2(E)}{2(g+2)}.$$

From Grothendieck–Riemann–Roch applied to  $\pi: P \rightarrow B$ , we get

$$\begin{aligned} \text{ch}(R\pi_* O_C) &= g + \lambda \\ &= \pi_* (\text{ch}(\phi_* O_C) \cdot \text{td}(P/B)) \\ &= \pi_* \left( \left( 1 - c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} \right) \cdot (1 - \zeta) \right). \end{aligned}$$

Combining with (4.5.1), we get the second relation

$$\begin{aligned} \lambda &= \frac{c_1^2(E)}{2} - c_2(E) + \Sigma \cdot \zeta \\ &= \frac{g+1}{2(g+2)} c_1^2 - c_2. \end{aligned}$$

We have an embedding  $C \hookrightarrow \mathbf{P}E$  over  $P$  that exhibits  $C$  as the zero locus of a section  $\alpha$  of the line bundle  $L = \mathcal{O}_{\mathbf{P}E}(3) \otimes \det E^\vee$ . Set  $\xi = c_1(\mathcal{O}_{\mathbf{P}E}(1))$ . Then  $\xi$  satisfies the equation

$$\xi^2 - c_1(E)\xi + c_2(E) = 0.$$

Let  $J_3L$  be the bundle of order three jets of  $L$  along the fibers of  $\pi_E: \mathbf{P}E \rightarrow P$ . More precisely,

$$J_3L = \pi_{1*}(\pi_2^*L \otimes \mathcal{O}_{3\Delta}),$$

where  $\pi_i$  are the two projections  $\mathbf{P}E \times_P \mathbf{P}E \rightarrow P$  and  $\Delta \subset \mathbf{P}E \times_P \mathbf{P}E$  the diagonal divisor. Then the locus  $T \subset B$  of points  $b \in B$  over which  $C|_b \rightarrow P|_b$  has a triple ramification point is simply the image in  $B$  of the zero locus of the section  $J_3\alpha$  of  $J_3L$  induced from the section  $\alpha$  of  $L$ . In particular,

$$\deg T = c_3(J_3L).$$

Since  $J_3L = L + L \otimes \Omega_{\mathbf{P}E/P} + L \otimes \Omega_{\mathbf{P}E/P}^{\otimes 2}$  in the Grothendieck group of sheaves on  $\mathbf{P}E$  and  $c_1(\Omega_{\mathbf{P}E/P}) = -2\xi + c_1(E)$ , we get the third relation

$$\begin{aligned} \deg T &= \pi_{E*}(3\xi - c_1(E)) \cdot \xi \cdot (-\xi + c_1(E)) \\ &= 3c_2(E) = 3c_2. \end{aligned}$$

Our next goal is to compute the divisor  $\text{br}^*D$ . This is simply the branch divisor of  $\Sigma \rightarrow B$ . The ramification divisor of  $\Sigma \rightarrow B$  is given by  $\omega_{\Sigma/B}$ . By adjunction, we have

$$\begin{aligned} \omega_{\Sigma/B} &= (\omega_{P/B} + c_1(\Sigma))|_\Sigma \\ &= (-2\xi + 2c_1(E)) \cdot 2c_1(E) \\ &= \frac{4g+6}{g+2}c_1^2(E), \end{aligned}$$

and hence  $\text{br}^*D = \frac{4g+6}{g+2}c_1^2$ . Using Proposition 4.5.2, we get the third relation

$$\begin{aligned} \delta &= \text{br}^*D - 3T \\ &= \frac{4g+6}{g+2}c_1^2 - 9c_2. \end{aligned}$$

For the last relation, assume furthermore that we have a section  $\sigma: B \rightarrow P$  disjoint from  $\Sigma$ . Abusing notation, also denote by  $\sigma$  the image  $\sigma(B) \subset P$ . Now,  $[\sigma]$  and  $\frac{c_1(E)}{g+2}$  are two divisor classes on  $P$  that have degree one on the fibers and their product is zero. The last relation follows.  $\square$

PROPOSITION 4.5.4. *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . The canonical divisor of  $\overline{T}_{g;1}^l$  (or  $\overline{T}_{g;1}^l$ ) is given by*

$$\begin{aligned} K &= -\frac{2(g+3)(2g+3)}{(g+2)^2}c_1^2 + 6c_2 \\ &= \frac{2}{(g+2)(g-3)}(3(2g+3)(g-1)\lambda - (g^2-3)\delta). \end{aligned}$$

PROOF. Retain the notation introduced just before Proposition 4.5.2. We restrict to the open subset  $V_{g;1}$ . Let  $K_V$  (resp.  $K_W$ ) be the canonical divisor of  $V_{g;1}$  (resp.  $W_{0;b,1}$ ). Since the finite morphism  $V_{g;1} \rightarrow W_{0;b,1}$  is étale except over  $D \subset W_{0;b,1}$  and  $\text{br}^* D = 3T + \delta$ , we have

$$(4.5.2) \quad K_V = \text{br}^* K_W + 2T.$$

We first compute  $K_W$  in terms of  $D$ . Let  $w$  be a point of  $W_{0;b,1}$  corresponding to the data  $(P; \Sigma; \sigma)$ . We have the following canonical identification of the tangent space to  $W_{0;b,1}$  at  $w$ :

$$T_w W_{0;b,1} = \text{Hom}(I_\Sigma/I_\Sigma^2, O_\Sigma) \oplus \text{Hom}(I_\sigma/I_\sigma^2, O_\sigma).$$

With this, the relation between  $D$  and  $K_W$  follows from an easy test-curve calculation. Here are the details. Take a one parameter family  $(\pi: P \rightarrow B; \sigma; \Sigma)$  giving a map  $f: B \rightarrow W_{0;b,1}$ , where  $B$  is a smooth projective curve. Set  $\zeta = \frac{[\sigma]}{2} + \frac{[\Sigma]}{2b}$ . Then  $\zeta$  has degree one on the fibers of  $\pi$  and satisfies  $\zeta^2 = 0$ . Hence  $\omega_{P/B} = -2\zeta$ . By adjunction

$$\deg \omega_{\Sigma/B} = (-2\zeta + [\Sigma]) \cdot [\Sigma] = \frac{b-1}{b} \cdot \Sigma^2.$$

Therefore,

$$\deg D = \deg \omega_{\Sigma/B} = \frac{b-1}{b} \cdot \Sigma^2.$$

On the other hand, we have

$$f^* T_{W_{0;b,1}} = \pi_* \mathcal{H}om_\Sigma(I_\Sigma/I_\Sigma^2, O_\Sigma) \oplus \pi_* \mathcal{H}om_\sigma(I_\sigma/I_\sigma^2, O_\sigma).$$

Observe that

$$\begin{aligned} \deg \pi_* \mathcal{H}om_\Sigma(I_\Sigma/I_\Sigma^2, O_\Sigma) &= \deg(\mathcal{H}om_\Sigma(I_\Sigma/I_\Sigma^2, O_\Sigma)) - \frac{\deg \omega_{\Sigma/B}}{2} \\ &= \Sigma^2 - \frac{\deg D}{2} = \frac{b+1}{2(b-1)} \deg D. \end{aligned}$$

Using  $\sigma^2 = -\Sigma^2/b^2 = -\deg D/(b(b-1))$ , we get

$$\begin{aligned} -\deg K_W &= \deg f^*T_{W_{0;b,1}} \\ &= \deg \pi_* \mathcal{H}om_{\Sigma}(I_{\Sigma}/I_{\Sigma}^2, O_{\Sigma}) + \deg \pi_* \mathcal{H}om_{\sigma}(I_{\sigma}/I_{\sigma}^2, O_{\sigma}) \\ &= \frac{b+1}{2(b-1)} \deg D + \sigma^2 = \frac{b+2}{2b} \cdot \deg D. \end{aligned}$$

Using (4.5.2) and Proposition 4.5.2, we get

$$K_V = -\frac{b+2}{2b} \cdot (3T + \delta) + 2T.$$

Substituting  $T$  and  $\delta$  from Proposition 4.5.3 and using  $b = 2g + 4$  yields the result.  $\square$

#### 4.6. The ample cones

In this section, we identify the cone of ample divisors on  $\overline{T}_{g;1}^l$ . We retain the notation, especially the list of divisors, introduced at the beginning of Section 4.5.

Define the divisor  $D_l$  by the formula

$$D_l = (4c_2 - c_1^2) + \left(\frac{2l}{b}\right)^2 c_1^2,$$

where  $b = 2g + 4$ , as usual. Recall that a  $\mathbf{Q}$ -Cartier divisor on a space is *nef* if it intersects non-negatively with all complete curves in that space. The bulk of the section is devoted to proving that certain divisors on  $\overline{T}_{g;1}^l$  are nef.

**THEOREM 4.6.1.** *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . A divisor is nef on  $\overline{T}_{g;1}^l$  if and only if it is a non-negative linear combination of  $D_l$  and  $D_{l+2}$ .*

The proof is the content of Subsection 4.6.1 and Subsection 4.6.2.

Using the Nakai–Moishezon criterion for ampleness, we then deduce projectivity.

**THEOREM 4.6.2.** *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . A divisor is ample on  $\overline{T}_{g;1}^l$  if and only if it is a positive linear combination of  $D_l$  and  $D_{l+2}$ . In particular, the algebraic space  $\overline{T}_{g;1}^l$  is a projective scheme.*

The proof is the content of Subsection 4.6.3.

In terms of the more standard generators  $\lambda$  and  $\delta$ , the divisor  $D_l$  admits the following expression:

$$\left(\frac{g-3}{2}\right) D_l = ((7g+6)\lambda - g\delta) + \frac{l^2}{g+2} \cdot (9\lambda - \delta).$$

**4.6.1. Positivity for families with  $\text{Br}^2 \geq 0$ .** We prove that  $D_l$  and  $D_{l+2}$  are non-negative on one-parameter families in  $\overline{T}_{g;1}^l$  with non-negative  $\text{Br}^2$ .

PROPOSITION 4.6.3. *Let  $B$  be a smooth projective curve and  $\pi: P \rightarrow B$  a  $\mathbf{P}^1$  bundle with a section  $\sigma: B \rightarrow P$ . Let  $\phi: C \rightarrow P$  be a triple cover, étale over  $\sigma$ . Assume that on the generic point of  $B$ , we have*

$$\phi_*O_C/O_P \cong O_P(-m) \oplus O_P(-n),$$

where  $m \leq n$  are positive integers. Assume, furthermore, that  $\text{br}(\phi)^2 \geq 0$ . Set  $c_i = c_i(\phi_*O_C)$ . Then

$$\left( 4c_2 - c_1^2 + \left( \frac{n-m}{n+m} \right)^2 c_1^2 \right) [P] \geq 0.$$

COROLLARY 4.6.4. *Assume that  $0 < l < g$  and  $l \equiv g \pmod{2}$ . Let  $B$  be a projective curve and  $f: B \rightarrow \overline{T}_{g;1}^l$  a morphism such that  $f^*(\text{Br}^2) \geq 0$ . Then*

$$f^*D_{l+2} \geq f^*D_l \geq 0.$$

PROOF. By replacing  $B$  by a finite cover and normalizing, we may assume that we have a map  $B \rightarrow \overline{T}_{g;1}^l$  and  $B$  is smooth. Then Proposition 4.6.3 applies. Since  $\text{Br}^2 = 4c_1^2$ , and  $\text{Br}^2 \geq 0$ , we have  $c_1^2 \geq 0$ . Hence  $f^*D_{l+2} \geq f^*D_l$ . Putting  $n+m = b/2$  in the conclusion of Proposition 4.6.3 and using  $l \leq n-m$  (as the curves are  $l$ -balanced) we conclude that  $f^*D_l \geq 0$ .  $\square$

We give two proofs of Proposition 4.6.3. We freely use the divisor relations from Proposition 4.5.3.

4.6.1.1. *First proof.* The underlying tool in the first proof is the following result, coupled with an amusing “balancing trick” (Lemma 4.6.6).

PROPOSITION 4.6.5. *Let  $B$  be a smooth projective curve,  $P \rightarrow B$  a  $\mathbf{P}^1$  bundle, and  $E$  a vector bundle of rank  $r$  on  $P$ . If the restriction of  $E$  to any fiber of  $P \rightarrow B$  is balanced, then the class  $\pi_*(2rc_2(E) - (r-1)c_1^2(E))$  on  $B$  is non-negative.*

Recall that  $E$  is balanced if  $E \cong O(d)^{\oplus r}$  for some  $d$ . The result is a special case of a result of Moriwaki [28, Theorem A]. Stankova-Frenkel [38, Proof of Proposition 12.2] proves the particular case ( $r = 2$ ) that we need. Nevertheless, here is an independent proof.



PROOF. Observe that  $2rc_2(E) - (r-1)c_1^2(E)$  is unaffected if  $E$  is replaced by  $E \otimes L$  for any line bundle  $L$ . Thus, possibly after replacing  $B$  by a finite cover, assume that  $\det E \cong O_P$ . We must conclude that  $\pi_*c_2(E) \geq 0$ .

That  $E$  is generically balanced and  $\det E = O_P$  forces  $E_b \cong O_{P_b}^{\oplus r}$  for a generic  $b \in B$ . Then  $\pi_*E$  is a vector bundle of rank  $r$  on  $B$ . Since  $\pi^*\pi_*E \rightarrow E$  is generically an isomorphism and  $\det E = O_P$ , it follows that  $c_1(\pi_*E) \leq 0$ . Since  $R^1\pi_*E$  is supported on finitely many points, we evidently have  $c_1(R^1\pi_*E) \geq 0$ . Therefore,  $c_1(R\pi_*E) \leq 0$ . But Grothendieck–Riemann–Roch shows that  $c_1(R\pi_*E) = -\pi_*c_2(E)$ .  $\square$

The following lemma lets us cook up a balanced vector bundle from a possibly unbalanced cover. We will use the lemma for  $L = O_C$ . It is formulated more generally because the proof is by induction.

LEMMA 4.6.6. *Let  $\phi: C \rightarrow \mathbf{P}^1$  be a triple cover,  $s \in \mathbf{P}^1$  a point over which  $\phi$  is étale,  $\{t_1, t_2, t_3\}$  an ordering of  $\phi^{-1}(s)$  and  $L$  a line bundle on  $C$ . Assume that*

$$\phi_*L \cong O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n),$$

*for some positive integers  $m \leq n$ . Then there is an effective divisor  $D$  of degree  $n - m$  supported on  $\{t_1, t_2, t_3\}$  such that*

$$\phi_*(L(D)) \cong O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-m).$$

PROOF. The proof is by induction on  $n - m$ . If  $n = m$ , take  $D = 0$ .

Let  $n - m \geq 1$ . Set  $O_C(1) = \phi^*O_{\mathbf{P}^1}(1)$ . Consider the chain of inclusions

$$L \hookrightarrow L(t_1) \hookrightarrow L(t_1 + t_2) \hookrightarrow L(t_1 + t_2 + t_3) = L \otimes O_C(1),$$

and the induced inclusions on global sections

$$\begin{aligned} H^0(L \otimes O_C(m-1)) &\xrightarrow{\iota_1} H^0(L(t_1) \otimes O_C(m-1)) \xrightarrow{\iota_2} H^0(L(t_1 + t_2) \otimes O_C(m-1)) \\ &\xrightarrow{\iota_3} H^0(L(t_1 + t_2 + t_3) \otimes O_C(m-1)) = H^0(L \otimes O_C(m)). \end{aligned}$$

From  $\phi_*L = O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-m) \oplus O_{\mathbf{P}^1}(-n)$ , and  $n > m$ , we see that

$$h^0(L \otimes O_C(m)) - h^0(L \otimes O_C(m-1)) = 2.$$

Therefore, at least one of the  $\iota_j$  is an isomorphism. For such a  $j$ , the inclusion

$$H^0(L \otimes \mathcal{O}_C(m-1)) \hookrightarrow H^0(L(t_j) \otimes \mathcal{O}_C(m-1))$$

must be an isomorphism. This isomorphism implies that

$$\phi_* L(t_j) \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n+1).$$

The induction step is thus complete.  $\square$

REMARK 4.6.7. In Lemma 4.6.6, the support of  $D$  cannot be all of  $\{t_1, t_2, t_3\}$ . Otherwise,  $\phi_*(L(D))$  will contain  $\phi_* L \otimes \mathcal{O}_{\mathbf{P}^1}(1)$  and hence also  $\mathcal{O}_{\mathbf{P}^1}(1)$ , contradicting the conclusion of the lemma.

FIRST PROOF OF PROPOSITION 4.6.3. After a base change if necessary, assume that we have three distinct sections  $\tau_i: B \rightarrow C$  lying over  $\sigma$ , for  $i = 1, 2, 3$ . For a generic  $b \in B$ , we have

$$\phi_* \mathcal{O}_{C_b} \cong \mathcal{O}_{P_b} \oplus \mathcal{O}_{P_b}(-m) \oplus \mathcal{O}_{P_b}(-n).$$

By Lemma 4.6.6, there is an effective linear combination  $D$  of the sections  $\tau_i(B)$  of degree  $n-m$  such that

$$\phi_* \mathcal{O}_{C_b}(D) = \mathcal{O}_{P_b} \oplus \mathcal{O}_{P_b}(-m) \oplus \mathcal{O}_{P_b}(-m).$$

Consider the map  $\mathcal{O}_P \rightarrow \phi_* \mathcal{O}_C(D)$  adjoint to the map  $\mathcal{O}_C \xrightarrow{D} \mathcal{O}_C(D)$ . The map of vector bundles  $\mathcal{O}_P \rightarrow \phi_* \mathcal{O}_C(D)$  on  $P$  is fiberwise nonzero, since  $D$  does not contain any fiber of  $C \rightarrow P$ , by Remark 4.6.7. Define  $V$  as the cokernel

$$0 \rightarrow \mathcal{O}_P \rightarrow \phi_* \mathcal{O}_C(D) \rightarrow V \rightarrow 0.$$

Then  $V$  is locally free of rank 2. The restriction of  $V$  to  $P_b$  is balanced. Hence, by Proposition 4.6.5, we have

$$(4.6.1) \quad 4c_2(V) - c_1^2(V) \geq 0.$$

We compute the Chern classes of  $V$  in terms of those of  $\phi_* \mathcal{O}_C$ . Throughout, we abbreviate  $c_i(\phi_* \mathcal{O}_C)$  by  $c_i$ , and omit writing pullback or pushforward symbols where they are clear by context. Note that  $c_1 \cdot \sigma = 0$  since  $\phi$  is étale over  $\sigma$ .

Say  $D = a\tau_1 + b\tau_2$ , where  $a$  and  $b$  are positive integers with  $a + b = n - m$ . We have the exact sequences

$$\begin{aligned} 0 \rightarrow O_C \rightarrow O_C(a\tau_1) \rightarrow O_{a\tau_1}(a\tau_1) \rightarrow 0, \text{ and} \\ 0 \rightarrow O_C(a\tau_1) \rightarrow O_C(D) \rightarrow O_{b\tau_2}(D) \rightarrow 0. \end{aligned}$$

Here  $a\tau_1$  denotes the subscheme of  $C$  defined by the  $a$ th power of the ideal of  $\tau_1$ , and likewise for  $b\tau_2$ . Pushing forward along  $\phi$ , we get a relation in the Grothendieck group of  $P$ :

$$\phi_*O_C(D) = \phi_*O_C + \phi_*O_{a\tau_1}(a\tau_1) + \phi_*O_{b\tau_2}(D).$$

Since  $\phi$  is étale over  $\sigma$ , and  $\tau_1, \tau_2$  are disjoint, we have  $\phi_*O_{a\tau_1}(a\tau_1) \cong O_{a\sigma}(a\sigma)$  and  $\phi_*O_{b\tau_2}(D) \cong O_{b\sigma}(b\sigma)$ . Therefore,

$$\begin{aligned} c(V) &= c(\phi_*O_C(D)) \\ &= c(\phi_*O_C) \cdot (1 + a\sigma) \cdot (1 + b\sigma) \\ &= 1 + (c_1 + (n - m)\sigma) + (ab \cdot \sigma^2 + c_2). \end{aligned}$$

Using (4.6.1), we obtain

$$(4.6.2) \quad 4c_2 + ab \cdot \sigma^2 - c_1^2 - (n - m)^2 \sigma^2 \geq 0.$$

Since  $ab \geq 0$  and  $\sigma^2 = \text{Br}^2/b^2 \geq 0$ , we conclude that

$$4c_2 - c_1^2 + \left(\frac{n - m}{n + m}\right)^2 c_1^2 \geq 0.$$

□

4.6.1.2. *Second proof.* The second proof resembles the proof of positivity in the  $\text{Br}^2 < 0$  case. Although it is somewhat less elegant, it is more transparent. The underlying idea is to use, in some form, the morphism to  $\mathbf{P}^1$  given by the cross-ratio.

SECOND PROOF OF PROPOSITION 4.6.3. Assume, possibly after a finite base change, that we have three disjoint sections  $\tau_i : B \rightarrow C$  lying over  $\sigma$ , for  $i = 1, 2, 3$ . For a generic  $b$ , we have

$$\phi_*O_{C_b} \cong O_{P_b} \oplus O_{P_b}(-m) \oplus O_{P_b}(-n).$$

If  $m = n$ , then by Proposition 4.6.5, we get  $4c_2 - c_1^2 \geq 0$ . Since  $\text{Br}^2 = 4c_1^2 \geq 0$ , this implies the desired result. Henceforth, assume that  $m < n$ .

Set  $F = \phi_* O_C / O_P$ , and consider the map

$$\chi: \pi_*(F \otimes O_P(m\sigma)) \rightarrow \pi_*(F|_\sigma \otimes O_P(m\sigma)).$$

Note that  $\pi_*(F \otimes O_P(m\sigma))$  is a line bundle on  $B$ . Since  $\phi^{-1}\sigma = \tau_1 \sqcup \tau_2 \sqcup \tau_3$ , the bundle  $F|_\sigma$  is trivial. Clearly, over the points  $b \in B$  where  $F \cong O(-m) \oplus O(-n)$ , the map  $\chi$  is injective. Hence, there is a map  $p: O_\sigma^{\oplus 2} = F|_\sigma \rightarrow O_\sigma$  such that the induced map

$$p \circ \chi: \pi_*(F \otimes O_P(m\sigma)) \rightarrow \pi_*(O_\sigma(m\sigma))$$

is an isomorphism at the generic point of  $B$ .

Denote by  $(n-m)\sigma$  the scheme defined by the  $(n-m)$ th power of the ideal of  $\sigma$ . Consider the diagram of vector bundles of rank  $(n-m-1)$  on  $B$ :

$$(4.6.3) \quad \begin{array}{ccc} \pi_*(F \otimes O_P(m)) \otimes \pi_* O_P((n-m-1)\sigma) & \longrightarrow & \pi_* O_\sigma(m\sigma) \otimes \pi_* O_P((n-m-1)\sigma) \\ \downarrow & & \downarrow \\ \pi_*(F \otimes O_P((n-1)\sigma)) & \longrightarrow & \pi_* O_{(n-m)\sigma}((n-1)\sigma) \end{array} .$$

The top and the left maps are clear. The bottom one is the composition

$$\pi_*(F \otimes O_P((n-1)\sigma)) \rightarrow \pi_*(F|_{(n-m)\sigma} \otimes O_P((n-1)\sigma)) \xrightarrow{p} \pi_*(O_{(n-m)\sigma}((n-1)\sigma)).$$

The one on the right is induced by the map of rings

$$O_\sigma \rightarrow O_{(n-m)\sigma}$$

dual to the projection  $(n-m)\sigma \rightarrow \sigma$ . The map on the right is an isomorphism; the rest are isomorphisms generically on  $B$ . In particular, we conclude that

$$(4.6.4) \quad \deg \pi_* O_{(n-m)\sigma}((n-1)\sigma) \geq \deg \pi_*(F \otimes O_P((n-1)\sigma)).$$

We compute both sides in terms of  $c_i(\phi_* O_C) = c_i(F)$ , henceforth abbreviated by  $c_i$ . The left side is easy from the exact sequences

$$0 \rightarrow O_\sigma((m+i)\sigma) \rightarrow O_{(n-m-i)\sigma}((n-1)\sigma) \rightarrow O_{(n-m-i-1)\sigma}((n-1)\sigma) \rightarrow 0,$$

for  $0 \leq i \leq n - m - 1$ . So we have

$$\begin{aligned}
 \deg \pi_* O_{(n-m)\sigma}((n-1)\sigma) &= ((n-1)\sigma + (n-2)\sigma + \cdots + m\sigma) \cdot \sigma \\
 (4.6.5) \qquad \qquad \qquad &= \left( \frac{(n+m-1)(n-m)}{2} \right) \sigma^2 \\
 &= - \left( \frac{(n+m-1)(n-m)}{2(n+m)^2} \right) c_1^2.
 \end{aligned}$$

For the right side, apply Grothendieck–Riemann–Roch, keeping in mind

$$\omega_\pi \cdot \sigma = -\sigma^2, \quad \sigma^2 = -\frac{c_1^2}{2(n+m)}, \quad \omega_\pi \cdot c_1 = -\frac{c_1^2}{(n+m)}, \quad \text{and } c_1 \cdot \sigma = \omega_\pi^2 = 0.$$

The result is

$$\begin{aligned}
 &\text{ch}(R\pi_*(F \otimes O_P((n-1)\sigma))) \\
 &= \pi_* (\text{ch } F \cdot \text{ch } O_P((n-1)\sigma) \cdot \text{td}_{P/B}) \\
 &= \pi_* \left( \left( 2 + c_1 + \frac{c_1^2}{2} - c_2 \right) \cdot \left( 1 + (n-1)\sigma + \frac{(n-1)^2\sigma^2}{2} \right) \cdot \left( 1 - \frac{\omega_\pi}{2} \right) \right), \\
 (4.6.6) \quad &\text{so that}
 \end{aligned}$$

$$\begin{aligned}
 &c_1(R\pi_*(F \otimes O_P((n-1)\sigma))) \\
 &= \left( \frac{(n+m)^2 + (n+m) - 2(n-1)^2 - 2(n-1)}{2(n+m)^2} \right) c_1^2 - c_2 \\
 &= - \left( \frac{(n-m)(n+m-1) - 2mn}{2(n+m)^2} \right) c_1^2 - c_2.
 \end{aligned}$$

See that  $R^1\pi_*(F \otimes O_P((n-1)\sigma))$  is supported on finitely many points of  $B$ . Hence

$$c_1(\pi_*(F \otimes O_P((n-1)\sigma))) \geq c_1(R\pi_*(F \otimes O_P((n-1)\sigma))).$$

Combining with (4.6.4), we arrive at

$$c_1(\pi_* O_P((n-1)\sigma)|_{(n-m)\sigma}) \geq c_1(R\pi_*(F \otimes O_P((n-1)\sigma))).$$

Substituting the left side from (4.6.5) and the right side from (4.6.6), we get

$$\begin{aligned} - \left( \frac{(n+m-1)(n-m)}{2(n+m)^2} \right) c_1^2 &\geq - \left( \frac{(n-m)(n+m-1) - 2mn}{2(n+m)^2} \right) c_1^2 - c_2 \\ \implies 4c_2 - \left( \frac{4mn}{(n+m)^2} \right) c_1^2 &= (4c_2 - c_1^2) + \left( \frac{n-m}{n+m} \right)^2 c_1^2 \geq 0. \end{aligned}$$

The proof is thus complete.  $\square$

REMARK 4.6.8. Let us examine the proof to determine when equality holds. This is the case if and only if  $\pi_*(F \otimes O_P((n-1)\sigma)) \rightarrow \pi_* O_{(n-m)\sigma}((n-1)\sigma)$  is an isomorphism and  $R^1\pi_*(F \otimes O_P((n-1)\sigma)) = 0$ . Then the splitting type of  $F_b$  is  $(m, n)$  for all  $b \in B$ . Therefore, all the maps in (4.6.3) are isomorphisms. In particular, we have an isomorphism  $\pi_*(F \otimes O_P(m)) \cong O_\sigma(m\sigma)$ . Hence the map

$$\pi_*(F \otimes O_P(m)) \rightarrow \pi_*(F|_\sigma \otimes O_P(m)),$$

which defines the cross-ratio, is equivalent to a (nonzero) global section of

$$\mathcal{H}om_B(\pi_*(F \otimes O_P(m)), \pi_*(F|_\sigma \otimes O_P(m))) \cong O_B^{\oplus 2}.$$

The upshot is that  $C \rightarrow P$  is a family of triple covers with a constant Maroni invariant  $l = n - m$  and a constant cross-ratio.

Retracing the steps, it is easy to see that for such a family  $C \rightarrow P$  with a constant Maroni invariant  $l = n - m$  and a constant cross-ratio, equality holds; that is, the pullback of  $D_l$  is zero.

**4.6.2. Positivity for families with  $\text{Br}^2 < 0$ .** Having taken care of families with  $\text{Br}^2 \geq 0$ , we now consider families with  $\text{Br}^2 < 0$ .

PROPOSITION 4.6.9. *Let  $B$  be a smooth projective curve and  $\pi: P \rightarrow B$  a  $\mathbf{P}^1$  bundle with a section  $\sigma: B \rightarrow P$ . Assume that  $\text{Br}^2 < 0$  and let  $\zeta$  be the unique section of  $\pi$  of negative self-intersection. Let  $\phi: C \rightarrow P$  be a triple cover étale away from  $\zeta$ . Assume that the splitting type of the singularity of  $C \rightarrow P$  over  $\zeta$  is  $(m, n)$  over a generic point of  $B$ , where  $m < n$  are positive integers. Then*

$$\left( 4c_2 - c_1^2 + \left( \frac{n-m}{n+m} \right)^2 c_1^2 \right) [P] \geq 0.$$

COROLLARY 4.6.10. *Assume that  $0 < l < g$  and  $l \equiv g \pmod{2}$ . Let  $B$  be a projective curve and  $f: B \rightarrow \overline{T}_{g,1}^l$  a morphism such that  $f^* \text{Br}^2 < 0$ . Then*

$$f^* D_l \geq f^* D_{l+2} \geq 0.$$

PROOF. Since  $4c_1^2 = \text{Br}^2$  and  $\text{Br}^2 < 0$ , we have  $c_1^2 < 0$ . Therefore  $f^* D_l \geq f^* D_{l+2}$ . Hence, it suffices to prove that  $f^* D_{l+2} \geq 0$ .

By replacing  $B$  by a finite cover and normalizing if necessary, we may assume that  $f$  lifts to a map  $f: B \rightarrow \overline{T}_{g,1}^l$  and  $B$  is smooth. Say  $f$  is given by  $(P; \sigma; \phi: C \rightarrow P)$ . Since  $4c_1^2 = \text{Br}^2 < 0$ , the branch divisor of  $\phi$  must be supported on the section  $\zeta$  of  $P \rightarrow B$  of negative self-intersection. Thus, Proposition 4.6.9 applies. Since our family consists of  $l$ -balanced covers, we have  $n - m > l$ , and hence  $n - m \geq l + 2$  because  $n - m \equiv l \equiv g \pmod{2}$ . Using  $n + m = b/2$ ,  $c_1^2 < 0$  and  $n - m \geq l + 2$  in the conclusion of Proposition 4.6.9, we conclude that

$$f^* D_{l+2} \geq 0.$$

□

PROOF OF PROPOSITION 4.6.9. By making a base change if necessary, assume that we have three sections  $\tau_i: B \rightarrow C$  over  $\sigma$ , for  $i = 1, 2, 3$ . Let  $\tilde{C} \rightarrow C$  be the normalization and  $\tilde{\phi}: \tilde{C} \rightarrow P$  the corresponding map. By [39], the fibers  $\tilde{C} \rightarrow B$  are normalizations of the fibers of  $C \rightarrow B$ . Therefore,  $\tilde{C}$  is the disjoint union of three copies of  $P$ , each containing one section  $\tau_i$ . Set  $F = \phi_* O_C / O_P$ , and  $E = \tilde{\phi}_* O_{\tilde{C}} / O_P$ . Note that  $E \cong O_P^{\oplus 2}$ . The inclusion  $\phi_* O_C \hookrightarrow \tilde{\phi}_* O_{\tilde{C}}$  induces an inclusion  $F \hookrightarrow E$ , which is an isomorphism away from  $\zeta$ . We think of  $F$  as a subsheaf of  $E$  via this inclusion.

Since the generic spitting type of the singularity of the fibers of  $C \rightarrow B$  over  $\zeta$  is  $(m, n)$ , we have the inclusions

$$I_\zeta^n \cdot E \subset F \subset I_\zeta^m \cdot E.$$

Let  $\overline{F} = F / (I_\zeta^n \cdot E)$  and  $\overline{E} = (I_\zeta^m / I_\zeta^n) \cdot E$ . Both  $\overline{F}$  and  $\overline{E}$  are supported on  $\zeta$ , are  $\pi$ -flat, and their  $\pi$ -fibers have lengths  $(n - m)$  and  $2(n - m)$  respectively. Pushing forward  $\overline{F} \rightarrow \overline{E}$ , we get

$$i: \pi_* \overline{F} \rightarrow \pi_* \overline{E},$$

a map of locally free sheaves on  $B$  of rank  $(n - m)$  and  $2(n - m)$ , respectively. The target  $\pi_*\overline{E}$  is isomorphic to  $\pi_*(I_\zeta^m/I_\zeta^n) \otimes (O_B^{\oplus 2})$ .

We examine  $i$  explicitly over a point  $b \in B$  where the splitting type of the singularity is  $(m, n)$ . Let  $x$  be a local coordinate for  $P_b$  near  $\zeta(b)$ . Since the singularity of  $C_b \rightarrow P_b$  is of type  $(m, n)$ , the subalgebra  $O_{C_b} \subset O_{\tilde{C}_b} = O_P^{\oplus 3}$  is generated as an  $O_{\mathbf{P}^1}$  module, locally around  $x$ , by  $\langle 1, x^m f, x^n O_{P_b}^{\oplus 3} \rangle$ , where the image of  $f$  in  $E_b$  is nonzero modulo  $x$ . Therefore,

$$\overline{F}_b = k\langle x^m f, x^{m+1} f, \dots, x^{n-1} f \rangle.$$

Since the image of  $f$  in  $E|_{\zeta(b)}$  is nonzero, it is nonzero in one of the projections  $p: O_{\zeta(b)}^{\oplus 2} = E|_{\zeta(b)} \rightarrow O_{\zeta(b)}$ . It follows that the composite  $j_b = p \circ i_b$  gives an isomorphism

$$j_b: \overline{F}_b \xrightarrow{\sim} I_{\zeta(b)}^m/I_{\zeta(b)}^n.$$

Consequently, the composition  $j = p \circ i$  is an isomorphism on the generic fiber:

$$j: \pi_*\overline{F} \rightarrow \pi_*(I_\zeta^m/I_\zeta^n).$$

We conclude that

$$(4.6.7) \quad \deg \pi_*(I_\zeta^m/I_\zeta^n) \geq \deg \pi_*\overline{F}.$$

We compute both sides in terms of  $c_i(\phi_*O_C)$ , abbreviated henceforth by  $c_i$ . Since  $P \rightarrow B$  is a  $\mathbf{P}^1$  bundle with two disjoint sections  $\sigma$  and  $\zeta$  of positive and negative self-intersection, respectively, we have

$$\omega_{P/B} = -\sigma - \zeta.$$

By Grothendieck–Riemann–Roch,

$$\begin{aligned} \text{ch}(\pi_*(I_\zeta^m/I_\zeta^n)) &= \pi_*(\text{ch}(I_\zeta^m/I_\zeta^n) \cdot \text{td}_{P/B}) \\ &= \pi_*((\text{ch } I_\zeta^m - \text{ch } I_\zeta^n) \cdot \text{td}_{P/B}) \\ &= \pi_*\left(\left((n - m)\zeta + \frac{(m^2 - n^2)\zeta^2}{2}\right) \cdot \left(1 + \frac{\sigma + \zeta}{2}\right)\right); \\ c_1(\pi_*\overline{E}) &= \left(\frac{m^2 - n^2 + n - m}{2(n + m)^2}\right) c_1^2. \end{aligned}$$

The last equality uses  $\zeta^2 = -\sigma^2 = c_1^2/(n + m)^2$ .



Similarly, using  $c_1 = -(m+n)\zeta$  and Grothendieck–Riemann–Roch,

$$\begin{aligned} \text{ch}(\pi_*\bar{F}) &= \pi_*(\text{ch}\bar{F} \cdot \text{td}_{P/B}) \\ &= \pi_*((\text{ch}F - 2\text{ch}I_\zeta^n) \cdot \text{td}_{P/B}) \\ &= \pi_*\left(\left(c_1 + 2n\zeta + \frac{c_1^2}{2} - c_2 - n^2\zeta^2\right) \cdot \left(1 + \frac{\zeta + \sigma}{2}\right)\right); \\ c_1(\pi_*\bar{F}) &= \left(\frac{m^2 - n^2 + 2mn + n - m}{2(m+n)^2}\right)c_1^2 - c_2. \end{aligned}$$

Substituting into (4.6.7), we get

$$4c_2 - \left(\frac{4mn}{(m+n)^2}\right)c_1^2 = 4c_2 - c_1 + \left(\frac{n-m}{m+n}\right)^2 c_1^2 \geq 0.$$

□

REMARK 4.6.11. Let us examine the proof to determine when equality holds. This is the case if and only if the map  $\bar{F} \rightarrow I_\zeta^m/I_\zeta^n$  is an isomorphism. Then fiber  $C_b \rightarrow P_b$  has a singularity of splitting type  $(m, n)$  for all  $b \in B$ . Furthermore, the map  $\bar{F}|_\zeta \rightarrow \bar{E}|_\zeta$ , which defines the principal part, is equivalent to a (nonzero) global section of

$$\mathcal{H}om(\bar{F}|_\zeta, \bar{E}|_\zeta) \cong O_\zeta^{\oplus 2}.$$

The upshot is that the family  $C \rightarrow P$  has singularities of a constant  $\mu$  invariant  $l = n - m$  and a constant principal part.

Retracing the above steps, it is easy to see that for such a family  $C \rightarrow P$  of covers with concentrated branching with singularities of a constant  $\mu$  invariant  $l = n - m$  and a constant principal part, equality holds; that is, the pullback of  $D_l$  is zero.

We have essentially finished the proof of Theorem 4.6.1; it is now a matter of collecting the pieces. We recall the statement for the convenience of the reader.

THEOREM 4.6.1. *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . A divisor is nef on  $\bar{T}_{g,1}^l$  if and only if it is a non-negative linear combination of  $D_l$  and  $D_{l+2}$ .*

PROOF. By Corollary 4.6.4 and Corollary 4.6.10 we conclude that  $D_l$  and  $D_{l+2}$  are non-negative on any complete curve in  $\bar{T}_{g,1}^l$ . Hence, every non-negative linear combination of  $D_l$  and  $D_{l+2}$  is nef.

It remains to show that  $D_l$  and  $D_{l+2}$  are indeed the edges of the nef cone. For that, it suffices exhibit curves on which  $D_l$  is zero and  $D_{l+2}$  is positive, and vice-versa. Remark 4.6.8 and Remark 4.6.11 tell us how to construct such curves. We explain the constructions briefly. Let  $m < n$  be such that  $n + m = g + 2$  and  $n - m = l$ .

For the edge  $\langle D_l \rangle$ , we construct a family of covers with constant Maroni invariant  $l$  and constant cross-ratio. Such a family can be constructed, for example, as follows. Let  $C \rightarrow \mathbf{P}^2$  be a connected, generically étale triple cover with

$$O_C/O_{\mathbf{P}^2} \cong O_{\mathbf{P}^2}(-m) \oplus O_{\mathbf{P}^2}(-n).$$

Let  $p \in \mathbf{P}^2$  be a point over which  $C \rightarrow \mathbf{P}^2$  is étale. Let  $P = \text{Bl}_p \mathbf{P}^2$  and set  $C' = C \times_{\mathbf{P}^2} \text{Bl}_p \mathbf{P}^2$ . Then  $P \rightarrow \mathbf{P}^1$  is a  $\mathbf{P}^1$  bundle with a section  $\sigma$  given by the exceptional divisor. The family  $(P; \sigma; C' \rightarrow P)$  gives a curve in  $\overline{T}_{g;1}^l$ . The pullback of  $D_l$  to this curve is zero and the pullback of  $D_{l+2}$  is positive.

For the edge  $\langle D_{l+2} \rangle$ , we construct a family of covers with concentrated branching having  $\mu$ -invariant  $l + 2$  and constant principal part. To construct such a family, consider the two-parameter family of sub-algebras  $S(a, b)$  of  $O_{\mathbf{P}^1}^{\oplus 3}$ , where  $S(a, b)$  is generated locally around 0 as an  $O_{\mathbf{P}^1}$  module by

$$1, (x^{m-1} + ax^{n-1} + bx^n, 0, 0), \text{ and } x^{n+1}O_{\mathbf{P}^1}^{\oplus 3}.$$

Via the scaling  $x \mapsto tx$ , we have an isomorphism

$$S(a, b) \xrightarrow{\sim} S(t^{n-m}a, t^{n-m+1}b).$$

The resulting family on  $(k^{\oplus 2} \setminus 0)/\mathbf{G}_m$  gives a curve in  $\overline{T}_{g;1}^l$ . The pullback of  $D_{l+2}$  to this curve is zero and the pullback of  $D_l$  is positive.  $\square$

**4.6.3. Projectivity.** In this section, we prove that divisors in the interior of the nef cone of  $\overline{T}_{g;1}^l$  are indeed ample. This would follow from Kleiman's criterion if we knew that  $\overline{T}_{g;1}^l$  is a scheme. However, it is a priori only an algebraic space. Kleiman's criterion can fail for algebraic spaces, as pointed out by Kollár [24, § VI, Exercise 2.19.3]. Recall, however, that Nakai–Moishezon's criterion is still true.

**THEOREM 4.6.12** (Nakai–Moishezon criterion for ampleness). *[23, Theorem 3.11] Let  $X$  be an algebraic space proper over an algebraically closed field and  $H$  a Cartier divisor on  $X$ . Then  $H$  is*

ample if and only if for every irreducible closed subspace  $Y \subset X$  of dimension  $n$ , the number  $H^n \cdot Y$  is positive.

To deduce that divisors in the interior of the nef cone are ample, we need a mild extension of a result of Fedorchuk and Smyth [11, Lemma 4.12].

LEMMA 4.6.13. *Let  $X$  be an algebraic space proper over an algebraically closed field. Suppose  $X$  satisfies the following: for every irreducible subspace  $Y \subset X$ , there is a finite surjective map  $Z \rightarrow Y$  and a Cartier divisor  $D$  on  $X$  whose pullback to  $Z$  is numerically equivalent to a nonzero and effective divisor. Then any Cartier divisor in the interior of the nef cone of  $X$  is ample.*

PROOF. The proof follows the proof of [11, Lemma 4.12] almost verbatim. In what follows, “divisor” means a  $\mathbf{Q}$ -Cartier divisor. Let  $H$  be a divisor in the interior of the nef cone. By the Nakai–Moishezon criterion, it suffices to prove that for every  $n$ -dimensional closed subspace  $Y \subset X$ , the number  $H^n \cdot Y$  is positive. We induct on  $n$ ; the case  $n = 0$  is trivial.

Let  $Z \rightarrow Y$  be as in the hypothesis and denote by  $f$  the finite map  $Z \rightarrow X$ . Say

$$D \cdot [Z] \equiv \sum_i a_i [Z_i],$$

where  $a_i > 0$  and  $Z_i \subset Z$  are reduced and irreducible divisors. Let  $f_i: Z_i \rightarrow X$  be the restriction of  $f$ . Then  $f_i$  is also a finite map.

Since  $H$  is in the interior of the nef cone, for a sufficiently small  $\epsilon > 0$ , the divisor  $H - \epsilon D$  is nef. Since  $H$  and  $H - \epsilon D$  are nef, we have

$$H^{n-1}(H - \epsilon D) \cdot [Z] \geq 0.$$

Therefore,

$$\begin{aligned} (\deg f) \cdot H^n \cdot [Y] &= H^n \cdot [Z] \geq \epsilon H^{n-1} D \cdot [Z] \\ &= \epsilon \sum_i a_i H^{n-1} \cdot [Z_i] \\ &= \epsilon \sum_i a_i (\deg f_i) \cdot H^{n-1} [f_i(Z_i)] > 0, \end{aligned}$$

where the last inequality is by the induction hypothesis. The induction step is complete.  $\square$

We now have the tools to prove Theorem 4.6.2. We recall the statement for the convenience of the reader.

**THEOREM 4.6.2.** *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . A divisor is ample on  $\overline{T}_{g;1}^l$  if and only if it is a positive linear combination of  $D_l$  and  $D_{l+2}$ . In particular, the algebraic space  $\overline{T}_{g;1}^l$  is a projective scheme.*

**PROOF.** We check that  $X = \overline{T}_{g;1}^l$  satisfies the hypothesis of Lemma 4.6.13. Let  $Y \subset X$  be an irreducible closed subspace. Choose a finite surjective map  $Z \rightarrow Y$  such that  $Z$  is a normal scheme and  $Z \rightarrow X$  lifts to  $Z \rightarrow \overline{\mathcal{T}}_{g;1}^l$ , given by a family  $(\pi: P \rightarrow Z; \sigma; \phi: C \rightarrow P)$ . Set  $\Sigma = \text{br}(\phi) \subset P$ . Then  $\Sigma$  is a  $\pi$ -flat divisor of degree  $b$ , disjoint from  $\sigma$ . By passing to a finite cover of  $Z$  if necessary, assume that we have three disjoint sections  $\tau_1, \tau_2, \tau_3: Z \rightarrow C$  over  $\sigma$  and sections  $\sigma_1, \dots, \sigma_b: Z \rightarrow P$  such that

$$\Sigma = \sigma_1(Z) + \dots + \sigma_b(Z)$$

as divisors on  $P$ . Observe that all the  $\sigma_i$  are disjoint from  $\sigma$ , and hence are linearly equivalent to each other. Furthermore,  $\pi_*[\sigma_i^2] = -\pi_*[\sigma^2]$ .

We exhibit a divisor class on  $X$  whose pullback to  $Z$  is nonzero and effective.

**Case 1:  $\sigma_i$  are not all coincident.** Without loss of generality,  $\sigma_1 \neq \sigma_2$  at a generic point of  $Z$ . If  $\sigma_1(z) \neq \sigma_2(z)$  for all  $z \in Z$ , then we have three disjoint sections  $\sigma, \sigma_1$  and  $\sigma_2$  of the  $\mathbf{P}^1$  bundle  $P \rightarrow Z$ . Hence  $P \rightarrow Z$  is trivial and  $\Sigma \subset P$  is a constant family. In other words, the map  $Z \rightarrow \overline{\mathcal{T}}_{g;1}^l$  lies in a geometric fiber of  $\text{br}: \mathcal{H}^3 \rightarrow \mathcal{M}$ . By Lemma 1.6.2, the pullback of  $-\lambda$  is ample on  $Z$ . In particular, some multiple of  $-\lambda$  pulls back to a nonzero and effective divisor.

If  $\sigma_1(z) = \sigma_2(z)$  for some  $z \in Z$ , then  $\pi_*(\sigma_1 \cdot \sigma_2)$  is a nonzero and effective divisor on  $Z$ . Since  $\sigma_1 \sim \sigma_2 \sim -\sigma$ , the divisor  $\pi_*(\sigma_1 \cdot \sigma_2)$  is equivalent to the pullback of the divisor  $-\sigma^2$  on  $\overline{T}_{g;1}^l$ .

**Case 2:  $\sigma_i$  are all coincident.** Say  $\sigma_i = \zeta$  for  $i = 1, \dots, b$ . In this case, we have a family of covers with concentrated branching. We begin as in the proof of Proposition 4.6.9.

Let the splitting type of the singularity over a generic  $z \in Z$  be  $(m, n)$ , with  $m < n$ . Let  $\tilde{C} \rightarrow C$  be the normalization. By [39], the fibers of  $\tilde{C} \rightarrow Z$  are the normalizations of the corresponding fibers of  $C \rightarrow Z$ . In particular,  $\tilde{C} \cong P \sqcup P \sqcup P$ . Set

$$F = \phi_* O_C / O_P \text{ and } E = \tilde{\phi}_* O_{\tilde{C}} / O_P \cong O_P^{\oplus 2}.$$

We have inclusions

$$I_\zeta^n E \subset F \subset I_\zeta^m E.$$

Set

$$\bar{F} = F/I_\zeta^n E \text{ and } \bar{E} = I_\zeta^m E/I_\zeta^n E.$$

Then we have an induced map  $\bar{F} \rightarrow \bar{E}$ . Also, see that  $\bar{E} \cong (I_\zeta^m/I_\zeta^n)^{\oplus 2}$ . Since the generic splitting type of the singularity is  $(m, n)$ , there is a projection  $E \rightarrow I_\zeta^m/I_\zeta^n$  such that  $\bar{F} \rightarrow I_\zeta^m/I_\zeta^n$  is an isomorphism over the generic point of  $Z$ . Suppose  $\bar{F} \rightarrow I_\zeta^m/I_\zeta^n$  is not an isomorphism over all of  $Z$ . Then we get a map of line bundles on  $Z$ :

$$\det \pi_* \bar{F} \rightarrow \det \pi_* (I_\zeta^m/I_\zeta^n)$$

whose vanishing locus is a nonzero effective divisor. The class of this vanishing locus can be expressed as a pullback of  $c_1^2$  and  $c_2$ ; in fact, it is precisely  $D_{n-m}$ , as computed in the proof of Proposition 4.6.9.

We are thus left with the case where  $\bar{F} \rightarrow I_\zeta^m/I_\zeta^n$  is an isomorphism. In this case,  $\bar{F}$  and  $\bar{E}$  are vector bundles on  $(n-m)\zeta$  of rank one and two respectively. The map  $\bar{F} \rightarrow \bar{E}$  is just a global section  $f$  of  $\bar{V} = \mathcal{H}om_{(n-m)\zeta}(\bar{F}, \bar{E}) \cong O_{(n-m)\zeta}^{\oplus 2}$ . The restriction  $f|_\zeta$  is a section of  $O_\zeta^{\oplus 2}$ , and hence it is constant. We are thus dealing with a family of covers with concentrated branching, a constant  $\mu$ -invariant and a constant principal part.

Set  $G = \text{Isom}_Z((P, \zeta, \sigma), (\mathbf{P}^1, 0, \infty))$ . Then  $G \rightarrow Z$  is a  $\mathbf{G}_m$  torsor. We have a canonical isomorphism

$$(P_G, \zeta_G, \sigma_G) \xrightarrow{\sim} (\mathbf{P}^1 \times G, 0 \times G, \infty \times G).$$

Let  $x$  be a uniformizer of  $\mathbf{P}^1$  around 0. Then  $\bar{V}_G \cong (k[x]/x^{n-m})^{\oplus 2} \otimes_k O_G$ . We can interpret the global section  $f_G$  of  $\bar{V}_G$  as a  $\mathbf{G}_m$  equivariant map  $\phi_G: G \rightarrow (k[x]/x^{n-m})^{\oplus 2}$ , where  $\mathbf{G}_m$  acts on the  $k$  vector space  $(k[x]/x^{n-m})^{\oplus 2}$  by  $t: x^i \mapsto t^i x^i$ . Since the restriction of  $f_G$  to  $\zeta$  is constant,  $\phi_G$  has the form  $\phi_G = (c + \psi_G)$ , where  $c \in k^{\oplus 2}$  is a constant and  $\psi_G: G \rightarrow (xk[x]/x^{n-m})^{\oplus 2}$ . Explicitly, over a point  $b \in G$ , the subalgebra  $O_{C_b} \subset O_{\tilde{C}_b}$  is generated as an  $O_{P_b}$  module, locally around 0, by

$$1, x^n O_{\tilde{C}_b} \text{ and } x^m(c + \psi_G(b)).$$

Since the Maroni invariant of the resulting cover is less than  $n-m$ , we conclude that  $\psi_G(b) \neq 0$  for any  $b \in G$ . Furthermore, since the map  $Z \rightarrow \bar{T}_{g,1}^l$  is quasi-finite, so is the map  $\psi_G: G \rightarrow (xk[x]/x^{n-m})^{\oplus 2}$ . We thus have a finite map

$$\psi: Z \rightarrow [(xk[x]/x^{n-m})^{\oplus 2} \setminus 0 / \mathbf{G}_m],$$

where the right side is a weighted projective stack. We conclude that  $\psi^*O(1)$  is ample. But  $\psi^*O(-1)$  is the line bundle associated to  $G \rightarrow Z$ , which is  $P \setminus \sigma \rightarrow Z$ . Therefore,

$$\begin{aligned} c_1(\psi^*O(1)) &= c_1(\zeta^*O_P(-\zeta)) \\ &= \pi_*(-\zeta^2) = \pi_*(\sigma^2). \end{aligned}$$

In particular, the pullback to  $Z$  of some multiple of the divisor  $\sigma^2$  on  $\overline{T}_{g;1}^l$  is effective.

The proof of Theorem 4.6.2 is complete.  $\square$

As a result of the positivity of the interior of the cone spanned by  $D_l$  and  $D_{l+2}$ , we can easily deduce the following.

**PROPOSITION 4.6.14.** *Let  $0 < l < g$  and  $l \equiv g \pmod{2}$ . Consider the map  $\beta_l: \overline{T}_{g;1}^l \dashrightarrow \overline{T}_{g;1}^{l-2}$ . Then  $\text{Exc}(\beta_l)$  is covered by  $K$ -negative curves. If  $l > 0$ , then  $\text{Exc}(\beta_l^{-1})$  is covered by  $K$ -positive curves.*

**PROOF.**  $\text{Exc}(\beta_l)$  is the locus of  $(P; \sigma; \phi: C \rightarrow P)$  where  $\phi$  has Maroni invariant  $l$ . This locus is covered by curves  $S$  in which the cross-ratio is constant. Similarly  $\text{Exc}(\beta_l^{-1})$  is the locus of  $(P; \sigma; \phi: C \rightarrow P)$  where  $\phi$  has concentrated branching and  $\mu$  invariant  $l+2$ . For  $l > 0$ , this locus is covered by curves  $T$  in which the principal part is constant. By Remark 4.6.8 (resp. Remark 4.6.11), the divisor  $D_l$  (resp.  $D_{l+2}$ ) is zero on such  $S$  (resp.  $T$ ). Since  $D_{l+2}$  and  $K$  are on the opposite sides of the line spanned by  $D_l$  in  $\text{Pic}_{\mathbf{Q}}$ , the claim follows.  $\square$

## 4.7. The final model

In this section, we prove that for even  $g$ , the final model  $\overline{T}_{g;1}^0$  is Fano and for odd  $g$ , the final model  $\overline{T}_{g;1}^1$  is a Fano fibration over  $\mathbf{P}^1$ .

**4.7.1. The case of even  $g$ .** Let  $g = 2(h-1)$ , where  $h \geq 1$ . Fix an identification  $\mathbf{P}^1 = \text{Proj } k[X, Y]$  and set  $0 = [0 : 1]$  and  $\infty = [1 : 0]$ . Let  $G = \text{Aut}(\mathbf{P}^1, \infty)$ ; this is the group of affine linear transformations  $\mu_{\alpha, \beta}: (X, Y) \mapsto (X + \beta Y, \alpha Y)$ , where  $\alpha \in k^*$  and  $\beta \in k$ . Let  $\Lambda$  be the two dimensional  $k$  vector space  $\Lambda = \ker(\text{tr}: k^{\oplus 3} \rightarrow k)$ , where  $\text{tr}$  is the sum of the three coordinates.  $\Lambda$  should be thought of as the space of traceless functions on  $\{1, 2, 3\} \times \text{Spec } k$ .

Set  $\Gamma = \Lambda \otimes_k O_{\mathbf{P}^1}(h)$  and

$$\begin{aligned} V &= H^0(\mathrm{Sym}^3 \Gamma \otimes_{\mathbf{P}^1} \det \Gamma^\vee) \\ &= (\mathrm{Sym}^3(\Lambda) \otimes_k \det \Lambda^\vee) \otimes_k H^0(O_{\mathbf{P}^1}(h)). \end{aligned}$$

By the structure theorem of triple covers (Theorem 4.1.1), balanced triple covers of  $\mathbf{P}^1$  of arithmetic genus  $g$  correspond precisely to elements of  $V$ . Note that  $V$  admits a natural action of  $\mathrm{Gl}(\Lambda) \times G$ . Indeed,  $\mathrm{Gl}(\Lambda) = \mathrm{Gl}_2$  acts naturally on the first factor in  $V$ , whereas  $G$  acts on the second factor by

$$\mu_{\alpha,\beta}: p(X, Y) \mapsto p \circ \mu_{\alpha,\beta}^{-1}(X, Y).$$

Let  $v_1, v_2 \in V$  be two points and  $C_1 \rightarrow \mathbf{P}^1$  and  $C_2 \rightarrow \mathbf{P}^1$  the corresponding balanced triple covers. Using the point  $\infty \in \mathbf{P}^1$  as the additional marked point, treat them as marked triple covers  $(C_i \rightarrow \mathbf{P}^1; \infty)$ . We observe that these two marked covers are isomorphic if and only if  $v_1$  and  $v_2$  are related by the action of  $\mathrm{Gl}_2 \times G$ . Thus, we might expect  $\overline{\mathcal{T}}_{g;1}^0$  to be the quotient  $[V/\mathrm{Gl}_2 \times G]$ . However, this is not quite true since not all elements of  $V$  give an element of  $\overline{\mathcal{T}}_{g;1}^0$ . Firstly, the cover must be étale over  $\infty$  and secondly, it must not have  $\mu$ -invariant 0. Thus, we expect

$$\overline{\mathcal{T}}_{g;1}^0 = [U/\mathrm{Gl}_2 \times G],$$

for a suitable open  $U \subset V$ . In what follows, we prove that this is indeed the case. Along the way, we also simplify the presentation  $[U/\mathrm{Gl}_2 \times G]$ .

The first step is to exhibit a morphism  $\overline{\mathcal{T}}_{g;1}^0 \rightarrow [V/\mathrm{Gl}_2 \times G]$ . Let  $S$  be a scheme and  $S \rightarrow \overline{\mathcal{T}}_{g;1}^0$  a morphism given by  $(P; \sigma; \phi: C \rightarrow P)$ . Let  $E = (\phi_* O_C / O_P)^\vee$ . By the structure theorem for triple covers (Theorem 4.1.1), the cover  $C \rightarrow P$  gives a global section  $v$  of  $\mathrm{Sym}^3(E) \otimes \det E^\vee$ . Set

$$T = \mathrm{Isom}_S(\sigma^* E, \Lambda) \times_S \mathrm{Isom}_S((P, \sigma), (\mathbf{P}^1, \infty)).$$

Then  $T \rightarrow S$  is a  $\mathrm{Gl}_2 \times G$  torsor. Over  $T$ , we have canonical identifications

$$\sigma_T^* E_T \xrightarrow{\sim} \Lambda \otimes_k O_T \text{ and } (P_T, \sigma_T) \xrightarrow{\sim} (\mathbf{P}^1 \times T, \infty \times T).$$

Since  $C \rightarrow P$  is a family of balanced triple covers,  $E$  is fiberwise isomorphic to  $\Gamma$ . On  $T$ , the isomorphism  $\sigma^* E_T \xrightarrow{\sim} \Lambda \otimes_k O_T$  gives a canonical isomorphism

$$E_T \xrightarrow{\sim} \Gamma_T.$$

Thus, we may treat  $v_T$  as a global section of  $\mathrm{Sym}^3 \Gamma_T \otimes \det \Gamma_T^\vee$ , or equivalently as a map  $T \rightarrow V$ . By construction, this map is  $\mathrm{Gl}_2 \times G$  equivariant. We thus have a morphism

$$(4.7.1) \quad q: \overline{\mathcal{T}}_{g;1}^0 \rightarrow [V/\mathrm{Gl}_2 \times G].$$

PROPOSITION 4.7.1. *The morphism  $q$  in (4.7.1) is representable and an injection on  $k$ -points.*

Recall that  $k$ -points of  $[X/H]$  are just orbits of the action of  $H(k)$  on  $X(k)$ .

PROOF. Let  $p: \mathrm{Spec} k \rightarrow \overline{\mathcal{T}}_{g;1}^0$  be a point. For representability, we must show that the map  $\mathrm{Aut}_p \rightarrow \mathrm{Aut}_{q(p)}$  is injective. Say  $p$  is given by  $(P; \sigma; \phi: C \rightarrow P)$ . Set  $E = (\phi_* O_C / O_P)^\vee$  and pick identifications  $E|_\sigma \cong \Lambda$  and  $(P, \sigma) \cong (\mathbf{P}^1, \infty)$ . Consider an element  $\psi \in \mathrm{Aut}_p$ . Then  $\psi$  consists of  $(\psi_1, \psi_2)$ , where  $\psi_1: (\mathbf{P}^1; \infty) \rightarrow (\mathbf{P}^1; \infty)$  is an automorphism and  $\psi_2: C \rightarrow C$  is an automorphism over  $\psi_1$ . To understand the image of  $\psi$  in  $\mathrm{Aut}_{q(p)}$ , consider the map on algebras

$$\psi_2^\# : \psi_1^* \phi_* O_C \rightarrow \phi_* O_C$$

dual to  $\psi_2: C \rightarrow C$ . The map  $\psi_2^\#$  induces a map  $\alpha: E \rightarrow \psi_1^* E$ . Then the image of  $\psi = (\psi_1, \psi_2)$  is just  $(\alpha|_\sigma, \psi_1)$ .

Suppose that  $(\alpha|_\sigma, \psi_1) = \mathrm{id}$ . Then  $\psi_1 = \mathrm{id}$ . Furthermore, the fact that  $E$  is balanced and  $\alpha|_\sigma = \mathrm{id}$  implies that  $\alpha = \mathrm{id}$ . From the sequence

$$0 \rightarrow O_P \rightarrow \phi_* O_C \rightarrow E^\vee \rightarrow 0,$$

it follows that  $\psi_2 = \mathrm{id}$ . Thus  $\mathrm{Aut}_p \rightarrow \mathrm{Aut}_{q(p)}$  is injective.

It is clear that  $q$  is injective on  $k$ -points—two sections  $v_1, v_2$  in the same orbit of  $\mathrm{Gl}_2 \times G$  clearly give isomorphic marked covers.  $\square$

THEOREM 4.7.2. *Let  $g = 2(h - 1)$ , where  $h \geq 1$ . Then*

$$\overline{\mathcal{T}}_{g;1}^0 \cong [(\mathbf{A}^{2g+3} \setminus 0) / (\mathbf{S}_3 \times \mathbf{G}_m)],$$

where  $\mathbf{G}_m$  acts by weights

$$1, 2, \dots, h, 1, 2, \dots, h, 1, 2, \dots, h, 2, 3, \dots, h.$$



The space  $\overline{T}_{g;1}^0$  is the quotient of the weighted projective space

$$\mathbf{P}(1, \dots, h, 1, \dots, h, 1, \dots, h, 2, \dots, h)$$

by an action of  $\mathbf{S}_3$ . In particular,  $\overline{T}_{g;1}^0$  is a unirational, Fano variety.

We use the following lemma in the proof to simplify a group action.

LEMMA 4.7.3. *Let  $X$  be a normal variety over  $k$  with the action of a connected algebraic group  $H$ . Let  $X' \subset X$  be a reduced and irreducible subvariety and  $H' \subset H$  a subgroup such that the action of  $H'$  restricts to an action on  $X'$ . If  $[X'/H'] \rightarrow [X/H]$  is a bijection on  $k$ -points, then it is an isomorphism.*

PROOF. The map  $[X'/H'] \rightarrow [X/H]$  is representable. Set  $Y = X \times_{[X/H]} [X'/H']$ . It suffices to prove that  $Y \rightarrow X$  is an isomorphism. We have the diagram

$$\begin{array}{ccc} X' \times H & \longrightarrow & X' \\ \downarrow & \square & \downarrow \\ Y & \longrightarrow & [X'/H'] \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & [X/H] \end{array}$$

The smooth morphism  $Y \rightarrow [X'/H']$  shows that  $Y$  is reduced and the surjective morphism  $X' \times H \rightarrow Y$  shows that it is irreducible. Furthermore,  $Y \rightarrow X$  induces a bijection on  $k$ -points. The quasi-finite map  $Y \rightarrow X$  can be factored as  $Y \hookrightarrow \overline{Y} \rightarrow X$ , where the first is a dense open inclusion and the second a finite map. Since  $X$  is normal, Zariski's main theorem implies that  $\overline{Y} \rightarrow X$  is an isomorphism. But then  $Y \hookrightarrow \overline{Y}$  is a bijection on  $k$ -points. It follows that  $Y = \overline{Y}$ .  $\square$

PROOF OF THEOREM 4.7.2. Retain the notation at the beginning of Subsection 4.7.1. Let  $U \subset V$  be the open subset consisting of  $v \in V$  whose associated triple cover is étale over  $\infty$ . Then  $U$  is invariant under the  $\mathrm{Gl}_2 \times G$  action and  $q: \overline{\mathcal{T}}_{g;1}^0 \rightarrow [V/\mathrm{Gl}_2 \times G]$  lands in  $[U/\mathrm{Gl}_2 \times G]$ .

We now simplify the presentation  $[U/\mathrm{Gl}_2 \times G]$ . Choose coordinates on  $\Lambda$ : say  $s = \overline{(1, -1, 0)}$  and  $t = \overline{(0, 1, -1)}$ . Write points of  $V$  explicitly as

$$v_{s,t} = (as^3 + bs^2t + cst^2 + dt^3) \otimes (s^* \wedge t^*),$$

where  $a, b, c, d \in H^0(O_{\mathbf{P}^1}(h))$ . Let  $a = \sum a_i X^{h-i} Y^i$ , where  $a_i \in k$ , and similarly for  $b, c$  and  $d$ . Let  $W \subset U$  be the closed subvariety defined by

$$v_{s,t}(1, 0) = st(t+s) \otimes (s^* \wedge t^*) \text{ and } 2a_1 + 2d_1 - b_1 - c_1 = 0.$$

The first equation specifies  $v$  over  $\infty$ ; the second is the result of imposing the condition that the branch divisor of the triple cover given by  $v$  be centered around 0. Explicitly, the points of  $W$  have the form

$$\begin{aligned} & \left( \sum_{i=1}^h a_{h-i} X^i Y^{h-i} \right) s^3 \otimes (s^* \wedge t^*) \\ & + \left( X^h + \sum_{i=1}^h b_{h-i} X^i Y^{h-i} \right) s^2 t \otimes (s^* \wedge t^*) \\ & + \left( X^h + \sum_{i=1}^h c_{h-i} X^i Y^{h-i} \right) st^2 \otimes (s^* \wedge t^*) \\ & + \left( \sum_{i=1}^h d_{h-i} X^i Y^{h-i} \right) t^3 \otimes (s^* \wedge t^*). \end{aligned}$$

where  $2a_1 + 2d_1 - b_1 - c_1 = 0$ . The action of  $\mathbf{S}_3 \subset \mathrm{Gl}_2$  by permuting the three coordinates of  $\Lambda \subset k^3$  and the action of  $\mathbf{G}_m \subset G$  by scaling  $(X, Y) \mapsto (X, tY)$  restrict to actions on  $W$ .

CLAIM. *The map  $i: [W/\mathbf{S}_3 \times \mathbf{G}_m] \rightarrow [U/\mathrm{Gl}_2 \times G]$  is an isomorphism*

PROOF. By Lemma 4.7.3, it suffices to check that it is a bijection on  $k$ -points.

We first check that  $i$  is injective on  $k$ -points. Said differently, we want to check that two points of  $W$  that are related by the action of  $\mathrm{Gl}_2 \times G$  are in fact related by the action of  $\mathbf{S}_3 \times \mathbf{G}_m$ . Let  $w_1, w_2 \in W$  and  $\psi = (\psi_1, \psi_2) \in \mathrm{Gl}_2 \times G$  be such that  $w_1 = \psi w_2$ . Then the linear isomorphism  $\psi_1: k\langle s, t \rangle \rightarrow k\langle s, t \rangle$  takes  $st(s+t) \otimes s^* \wedge t^*$  to itself. It is easy to check that it must lie in the  $\mathbf{S}_3 \subset \mathrm{Gl}_2$ . Secondly, observe that for  $\psi_2: (X, Y) \mapsto (X + \beta Y, \alpha Y)$ , we have

$$(4.7.2) \quad \psi_2^{-1}(2a_1 + 2d_1 - b_1 - c_1) = \alpha(2a_1 + 2d_1 - c_1 - d_1) - h\beta.$$

By the second defining condition for  $W$ , we must have  $\beta = 0$ . In other words,  $\psi \in \mathbf{S}_3 \times \mathbf{G}_m$ .

We now check that  $i$  is surjective on  $k$ -points. Said differently, we want to check that every  $\mathrm{Gl}_2 \times G$ -orbit of  $U$  has a representative in  $W$ . Take a point  $v \in U$ . Since  $v$  describes a cover étale over  $\infty$ , the homogeneous cubic in  $v_{s,t}(1, 0)$  has distinct roots. By a suitable linear automorphism of

$k\langle s, t \rangle$ , the element  $v_{s,t}(1, 0)$  can thus be brought into the form  $st(s+t) \otimes s^* \wedge t^*$ . Secondly, (4.7.2) shows that we can make  $2a_1 + 2d_1 - b_1 - c_1 = 0$  after a suitable translation  $(X, Y) \mapsto (X + \beta Y, Y)$ .  $\square$

Returning to the main proof, we have a morphism

$$q: \overline{\mathcal{T}}_{g;1}^0 \rightarrow [U/\mathrm{Gl}_2 \times G] = [W/\mathbf{S}_3 \times \mathbf{G}_m],$$

which is representable and injective on  $k$ -points. Denote by  $0 \in W$  the point corresponding to  $a_i = b_i = c_i = d_i = 0$  for all  $i = 1, \dots, h$ . Its corresponding cover has concentrated branching over  $0 \in \mathbf{P}^1$  and  $\mu$  invariant zero. Hence  $q$  factors through

$$(4.7.3) \quad q: \overline{\mathcal{T}}_{g;1}^0 \rightarrow [(W \setminus 0)/(\mathbf{S}_3 \times \mathbf{G}_m)].$$

The right side  $[(W \setminus 0)/(\mathbf{S}_3 \times \mathbf{G}_m)]$  is smooth and proper over  $k$ . Indeed, it is a weighted projective stack modulo an action of  $\mathbf{S}_3$ . Thus, the morphism  $q$  in (4.7.3) is a representable, proper morphism between two smooth stacks of the same dimension which is an injection on  $k$ -points. By Zariski's main theorem, it must be an isomorphism.

Finally, note that

$$W \cong \mathbf{A}^{2g+3} = \mathbf{A}(a_1, \dots, a_h, b_1, \dots, b_h, c_1, \dots, c_h, d_2, \dots, d_h);$$

the  $d_1$  can be dropped owing to the condition  $2a_1 + 2d_1 - b_1 - c_1 = 0$ . The  $\mathbf{G}_m$  acts by weight  $i$  on  $a_i, b_i, c_i, d_i$ . The proof is thus complete.  $\square$

**4.7.2. The case of odd  $g$ .** Let  $g = 2h - 1$ . In this case, we do not have quite as explicit a description of the final model as in the case of even  $g$ . Nevertheless, we prove that it is a Fano fibration over  $\mathbf{P}^1$ .

The morphism to  $\mathbf{P}^1$  is defined by the cross-ratio as in Subsection 4.4.1. We quickly recall the construction. Set  $V = k^{\oplus 3}/k$ , to be thought of as the space of functions on  $\{1, 2, 3\} \times \mathrm{Spec} k$  modulo the constant functions. Then there is an action of  $\mathbf{S}_3$  on  $V$  and an induced action of  $\mathbf{S}_3$  on  $\mathbf{P}_{\mathrm{sub}}V$ . The cross-ratio map

$$\chi: \overline{\mathcal{T}}_{g;1}^1 \rightarrow [\mathbf{P}_{\mathrm{sub}}V/\mathbf{S}_3]$$

is defined by the following procedure. Let  $S \rightarrow \overline{\mathcal{T}}_{g;1}^1$  be a morphism given by the family  $(\pi: P \rightarrow S; \sigma; \phi: C \rightarrow P)$ . Set  $F = \phi_*O_C/O_P$ . Since  $F$  is fiberwise isomorphic to  $O(-h) \oplus O(-h-1)$ , we

see that  $\pi_*(F \otimes O_P(h\sigma))$  is a line bundle on  $S$ . Consider the map

$$(4.7.4) \quad \pi_*(F \otimes O_P(h\sigma)) \otimes \sigma^*O_P(-h\sigma) \rightarrow \sigma^*F.$$

It is easy to see that this remains injective at every point of  $S$ . Moreover, passing to  $Z = \text{Isom}_S(C|_\sigma, \{1, 2, 3\})$ , we have an identification  $\sigma_Z^*F_Z \xrightarrow{\sim} V \otimes_k O_Z$ . Hence (4.7.4) yields a map  $Z \rightarrow \mathbf{P}_{\text{sub}}V$ , which is by construction  $\mathbf{S}_3$  equivariant. We thus get a map  $S \rightarrow [\mathbf{P}_{\text{sub}}V/\mathbf{S}_3]$ .

The map  $\chi: \overline{T}_{g;1}^1 \rightarrow [\mathbf{P}_{\text{sub}}V/\mathbf{S}_3]$  induces a map on the coarse spaces:

$$(4.7.5) \quad \chi: \overline{T}_{g;1}^1 \rightarrow \mathbf{P}_{\text{sub}}V/\mathbf{S}_3 \cong \mathbf{P}^1.$$

**THEOREM 4.7.4.** *Consider the cross-ratio map  $\chi: \overline{T}_{g;1}^1 \rightarrow \mathbf{P}^1$  as in (4.7.5). Then,*

- (1)  $\chi^*O_{\mathbf{P}^1}(1) = \frac{3D_1}{2}$ ;
- (2) *fibers of  $\chi$  are Fano varieties of Picard rank one.*

**PROOF.** We use the setup introduced above, assuming furthermore that  $S$  is a smooth curve. On  $Z$ , we have

$$\pi_*(F \otimes O_P(h\sigma)) \otimes \sigma^*O_P(-h\sigma) \rightarrow \sigma^*F = V \otimes_k O_Z.$$

For the first relation, note that  $\mathbf{P}_{\text{sub}}V \rightarrow \mathbf{P}_{\text{sub}}V/\mathbf{S}_3 = \mathbf{P}^1$  is a degree six cover. Hence, it suffices to prove that the pullback of  $O_{\mathbf{P}_{\text{sub}}V}(1)$  to  $Z$  has class  $D_1/4$ . But the class of this pullback is just

$$(4.7.6) \quad -c_1(\pi_*F \otimes O_P(h\sigma)) - c_1(\sigma^*O_P(-h\sigma)).$$

Since  $R^1\pi_*(F \otimes O_P(h\sigma)) = 0$ , the first summand in (4.7.6) is a simple calculation using Grothendieck–Riemann–Roch (see (4.6.6) in Subsection 4.6.1.2 for this calculation in a more general case). The result is

$$c_1(\pi_*F \otimes O_P(h\sigma)) = \frac{h^2}{(2h+1)^2}c_1^2 - c_2.$$

The second summand in (4.7.6) is simply  $-h\pi_*[\sigma^2]$ . Using Proposition 4.5.3, we get

$$\begin{aligned} -c_1(\pi_*F \otimes O_P(h\sigma)) - c_1(\sigma^*O_P(-h\sigma)) &= c_2 - \frac{h^2 + h}{(2h+1)^2}c_1^2 \\ &= \frac{D_1}{4}. \end{aligned}$$

The first relation is thus proved.

Finally, set  $K = K_{\overline{T}_{g;1}^1}$  and let  $Y \subset \overline{T}_{g;1}^1$  be a fiber of  $\chi$ . Then all curves in  $Y$  have intersection number zero with  $D_1$ ; it follows that they are (numerically) rational multiples of each other. Hence the Picard rank of  $Y$  is one.

Since  $\langle D_1 \rangle$  and  $\langle D_3 \rangle$  bound the ample cone of  $\overline{T}_{g;1}$  and  $D_1|_Y \cong 0$ , the ray  $\langle D_3 \rangle$  must be positive on  $Y$ . But  $D_3$  and  $K$  are on the opposite sides of the line spanned by  $D_1$ ; hence  $K|_Y = K_Y$  is anti-ample.  $\square$

Finally, we collect the pieces together for the chamber decomposition. Recall that the Mori chamber  $\text{Mor}(\beta)$  of a birational contraction  $\beta: X \dashrightarrow Y$  is the cone spanned by the pullback of the ample cone of  $Y$  and the exceptional divisors of  $\beta$ . If the birational map  $X \dashrightarrow Y$  is clear, we abuse notation and call  $\text{Mor}(\beta)$  the Mori chamber of  $Y$ .

**THEOREM 4.7.5.** *Let  $0 \leq l < g$  be such that  $l \equiv g \pmod{2}$ .*

- (1) *For  $l > 0$ , the interior of the cone  $\langle D_l, D_{l+2} \rangle$  is the Mori chamber of the model  $\overline{T}_{g;1}^l$ .*
- (2) *For even  $g$ , the cone  $\langle D_0, D_2 \rangle$  is the Mori chamber of the model  $\overline{T}_{g;1}^0$ .*
- (3) *For even (resp. odd)  $g$ , the ray  $\langle D_0 \rangle$  (resp.  $\langle D_1 \rangle$ ) is an edge of the effective cone.*

**PROOF.** Since the  $\overline{T}_{g;1}^l$  are isomorphic to each other away from codimension two for  $0 < l < g$ , the Mori chamber of  $\overline{T}_{g;1}^l$  is simply its ample cone. Hence, (1) follows from Theorem 4.6.2.

In the case of  $l = 0$ , consider the Maroni contraction  $\overline{T}_{g;1}^2 \rightarrow \overline{T}_{g;1}^0$ . It is easy to check that the pullback of the ample ray of  $\overline{T}_{g;1}^0$  is the ray  $\langle D_2 \rangle$  and the class of the Maroni divisor is a positive multiple of  $D_0$  (in fact  $D_0/4$ ); both statements follow from a simple test curve calculation, which we omit. Hence  $\langle D_0, D_2 \rangle$  is the Mori chamber associated to  $\overline{T}_{g;1}^2 \rightarrow \overline{T}_{g;1}^0$ .

For the last statement, first consider the case of odd  $g$ . By Theorem 4.7.4, some positive multiple of  $D_1$  is the pullback of  $\mathcal{O}_{\mathbf{P}^1}(1)$  along the cross-ratio map. Hence the ray  $\langle D_1 \rangle$  must be an edge of the effective cone.

For even  $g$ , some positive multiple of  $D_0$  is the class of the Maroni divisor. Since the Maroni divisor is the exceptional locus of the birational morphism  $\overline{T}_{g;1}^2 \rightarrow \overline{T}_{g;1}^0$ , it follows that the ray  $\langle D_0 \rangle$  must be the edge of the effective cone.  $\square$

## Spaces of trigonal curves with a marked (ramified) fiber

The spaces of weighted admissible covers and the spaces of  $l$ -balanced covers together provide a beautiful picture of the birational geometry of the space of trigonal curves with a marked unramified fiber. The spaces of weighted admissible covers  $\overline{T}_{g;1}(\epsilon)$  give a sequence of divisorial contractions

$$(5.0.7) \quad \overline{T}_{g;1}(1) \rightarrow \cdots \dashrightarrow \overline{T}_{g;1}(1/j) \dashrightarrow \overline{T}_{g;1}(1/(j+1)) \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}(1/(b-1)),$$

which is followed by a sequence of flips, given by the spaces of  $l$ -balanced covers

$$(5.0.8) \quad \overline{T}_{g;1}(1/(b-1)) = \overline{T}_{g;1}^g \dashrightarrow \overline{T}_{g;1}^{g-2} \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1}^{1 \text{ or } 0},$$

culminating in a Fano-fibration, as expected from the Minimal Model Program.

In this chapter, we indicate how to generalize the above picture to the case of the space of trigonal curves with a marked fiber of a given ramification type. To mark such a fiber, we use pointed orbi-curves, as explained in Remark 1.3.7. Let  $1 \leq r \leq 3$ , and let  $\mathcal{T}_{g;1/r} \subset \mathcal{H}^3$  be the open and closed substack whose geometric points parametrize covers  $(\mathcal{P} \rightarrow P; \sigma_1; \phi: \mathcal{C} \rightarrow \mathcal{P})$ , where  $\text{Aut}_{\sigma_1}(\mathcal{P}) = \mu_r$  and  $\mathcal{C}$  is a connected curve of genus  $g$ . If  $r = 1$ , then we recover the previous case. Consider the case  $r > 1$ . Let  $p: \text{Spec } k \rightarrow \mathcal{T}_{g;1/r}$  be a geometric point given by the cover  $(\mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$ . Let  $C \rightarrow P$  be the coarse space and consider the induced cover  $\phi: C \rightarrow P$ . Since  $\mathcal{C} \rightarrow \mathcal{P}$  is étale around  $\sigma$  and  $\sigma$  is in the smooth locus of  $\mathcal{P}$ , the curve  $C$  is nonsingular over a neighborhood of  $\sigma$ . Furthermore, since  $\text{Aut}_{\sigma}(\mathcal{P}) = \mu_r$ , the monodromy of  $C \rightarrow P$  around  $\sigma$  is an  $r$ -cycle in  $\mathbf{S}_3$ . Thus, if  $r = 2$ , then  $C \rightarrow P$  has ramification type  $(2, 1)$  over  $\sigma$  and if  $r = 3$ , then  $C \rightarrow P$  has ramification type  $(3)$  over  $\sigma$ . By the Riemann–Hurwitz formula, we see that  $\text{deg } \phi + r - 1 = 2g + 4$ . Set  $b = 2g + 5 - r$ . Then we have a morphism

$$\mathcal{T}_{g;1/r} \rightarrow \mathcal{M}_{0;b,1},$$

which is proper by Theorem 1.3.8.

The recipe for constructing the first sequence of divisorial contractions analogous to (5.0.7) is straightforward. Recall that  $\overline{\mathcal{M}}_{0;b,1}(\epsilon) \subset \mathcal{M}_{0;b,1}$  is the open substack of  $\epsilon$ -stable marked rational curves in the sense of Hassett [18]. In analogy with  $\overline{\mathcal{T}}_{g;1}(\epsilon)$ , set

$$\overline{\mathcal{T}}_{g;1/r}(\epsilon) = \overline{\mathcal{M}}_{0;b,1}(\epsilon) \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1/r}.$$

Then  $\overline{\mathcal{T}}_{g;1/r}(\epsilon)$  is a smooth, proper, Deligne–Mumford stack which projective coarse space  $\overline{T}_{g;1/r}(\epsilon)$ . We thus get a sequence of divisorial contractions

$$\overline{T}_{g;1/r}(1) \rightarrow \cdots \dashrightarrow \overline{T}_{g;1/r}(1/j) \dashrightarrow \overline{T}_{g;1/r}(1/(j+1)) \dashrightarrow \cdots \dashrightarrow \overline{T}_{g;1/r}(1/(b-1)),$$

analogous to (5.0.7).

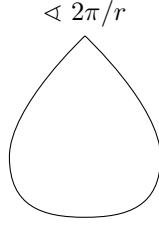
It is natural to wonder if we can continue this sequence by constructing a sequence of flips as in (5.0.8). In the case of an unramified fiber, such a sequence displayed an interplay between the global geometry of covers and the local geometry of tri-branch triple points. For the ramified case, should we expect an analogous interplay between the global geometry of covers and the local geometry of unibranch or bi-branch triple points? It turns out that this is indeed the case! Again, the key is to look at the splitting type of the structure sheaf  $\phi_*\mathcal{O}_{\mathcal{C}}$ , encoded by a refined Maroni invariant and the splitting type of a singularity of a cover with concentrated branching, encoded by the  $\mu$  invariant.

In this short chapter, we construct proper moduli stacks  $\overline{\mathcal{T}}_{g;1/r}^l$ , which generalize  $\overline{\mathcal{T}}_{g;1}^l$  and whose coarse spaces  $\overline{T}_{g;1/r}^l$  provide the analogue of (5.0.8). The birational geometry of the resulting spaces and rational maps can be studied by the same methods as used for the study of (5.0.8) in Chapter 4; we do not undertake this task.

The chapter is organized as follows. In Section 5.1, we recall some facts about the orbi-curve  $(\mathcal{P}; \sigma)$  with  $\text{Aut}_{\sigma} \mathcal{P} = \mu_r$ , which is the base in our families of triple covers. In Section 5.2, we define the Maroni invariant for a cover of  $(\mathcal{P}; \sigma)$  and relate it to the classical geometry of the induced cover on the coarse spaces. In this section, we also recall the  $\mu$  invariant of a cover with concentrated branching. In Section 5.3, we define  $l$ -balanced covers and prove the main theorem. The proof is by a formal reduction to the case  $r = 1$ ; there is little extra work.

### 5.1. The teardrop curve $\mathcal{P}$

In this section, we work over an algebraically closed  $\mathbf{K}$  field  $k$ .

FIGURE 1. The “teardrop curve”  $\mathbf{P}^1(\sqrt[r]{\infty})$  over  $\mathbf{C}$ 

Consider the orbi-curve  $\mathcal{P}$  with coarse space  $\rho: \mathcal{P} \rightarrow \mathbf{P}^1$ , where the local picture of  $\rho$  over  $\infty \in \mathbf{P}^1$  is given by

$$(5.1.1) \quad [\mathrm{Spec} k[v]/\mu_r] \rightarrow \mathrm{Spec} k[x],$$

where  $\mu_r$  acts by  $v \mapsto \zeta v$  and  $x = v^r$ . The curve  $\mathcal{P}$  can also be described as the root stack

$$\mathcal{P} = \mathbf{P}^1(\sqrt[r]{\infty})$$

or as a weighted projective stack

$$\mathcal{P} = [\mathbf{A}^2 \setminus 0 / \mathbf{G}_m],$$

where  $\mathbf{G}_m$  acts by weights 1 and  $r$ . The name “teardrop curve” is inspired by the picture for  $k = \mathbf{C}$ , where  $\mathcal{P}$  is imagined to be a ‘pinching’ of the Riemann sphere to make it have conformal angle  $2\pi/r$  at  $\infty$  (see Figure 1).

Let  $\xi \subset \mathcal{P}$  be the reduced preimage of  $\infty \in \mathbf{P}^1$ . In the explicit description of  $\mathcal{P}$  in (5.1.1), this is the closed substack defined by  $v = 0$ . Let  $L$  be the dual of the ideal sheaf of  $\xi$  in  $\mathcal{P}$  and set

$$O_{\mathcal{P}}(d) = L^{\otimes dr},$$

for  $d \in \frac{1}{r}\mathbf{Z}$ . Thus,  $L = O_{\mathcal{P}}(1/r)$ .

PROPOSITION 5.1.1. *With the notation above,*

- (1)  $\mathrm{Pic}(\mathcal{P})$  is generated by  $O_{\mathcal{P}}(1/r)$ .
- (2) The degree of  $O_{\mathcal{P}}(d)$  is  $d$ , for every  $d \in \frac{1}{r}\mathbf{Z}$ .
- (3) The canonical sheaf (which is also the dualizing sheaf) of  $\mathcal{P}$  is  $O_{\mathcal{P}}(-1 - 1/r)$ .
- (4) We have

$$\rho^*O_{\mathbf{P}^1}(n) = O_{\mathcal{P}}(n), \text{ for } n \in \mathbf{Z},$$



and

$$\rho_* O_{\mathcal{P}}(d) = O_{\mathbf{P}^1}(\lfloor d \rfloor), \text{ for } d \in \frac{1}{r}\mathbf{Z}.$$

(5) Every vector bundle on  $\mathcal{P}$  is isomorphic to a direct sum of line bundles.

PROOF. All of these statements are easy to see, except possibly the last one. The proof for the case of  $\mathbf{P}^1$  sketched by Hartshorne [16, V.2, Exercise 2.6] works verbatim. We present the details for lack of a reference.

Let  $E$  be a vector bundle on  $\mathcal{P}$ . Since the degree of subsheaves of  $\rho_* E$  is bounded above, the degree of subsheaves of  $E$  is also bounded above. Let  $L \subset E$  be a line bundle of maximum degree. Then the quotient  $E' = E/L$  is locally free. We claim that  $\text{Ext}^1(E', L) = 0$ . Then  $E = E' \oplus L$ , and the statement follows by induction on the rank.

By duality,  $\text{Ext}^1(E', L) = \text{Hom}(L(1 + 1/r), E')^\vee$ . Since we have an inclusion  $\text{Hom}(L(1 + 1/r), E') \subset \text{Hom}(L(1/r), E')$ , proving that the latter vanishes implies that the former vanishes. On one hand, we have the exact sequence

$$\text{Hom}(L(1/r), E) \rightarrow \text{Hom}(L(1/r), E') \rightarrow \text{Ext}^1(L(1/r), L) = \text{Hom}(L, L(-1))^\vee = 0.$$

On the other hand, we know that  $\text{Hom}(L(1/r), E) = 0$  by the maximality of  $\deg L$ . We conclude that  $\text{Hom}(L(1/r), E') = 0$ .  $\square$

## 5.2. The refined Maroni invariant and the $\mu$ invariant

We continue working over an algebraically closed  $\mathbf{K}$ -field  $k$ .

**5.2.1. The refined Maroni invariant.** Denote by  $\mathcal{P}$  the teardrop curve  $\mathbf{P}^1(\sqrt[r]{\infty})$  as in Section 5.1. Proposition 5.1.1 (5) gives us a way to define the Maroni invariant for triple covers of  $\mathcal{P}$ .

DEFINITION 5.2.1. Let  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  be a triple cover. Set  $F = \phi_* O_{\mathcal{C}}/O_{\mathcal{P}}$ . Then we have

$$F \cong O_{\mathcal{P}}(-m) \oplus O_{\mathcal{P}}(-n),$$

for some  $m, n \in \frac{1}{r}\mathbf{Z}$ . Define the Maroni invariant of  $\phi$  to be the difference

$$M(\phi) = |m - n|.$$

Note that the Maroni invariant lies in  $\frac{1}{r}\mathbf{Z}$ .

The refined Maroni invariant can be read off from the usual Maroni invariant of a new cover associated to  $\phi: \mathcal{C} \rightarrow \mathcal{P}$ . We now explain this procedure. Choose a point  $p \in \mathcal{P}$ , away from  $\xi$  and define  $\psi: \tilde{\mathcal{P}} \rightarrow \mathcal{P}$  to be the cyclic cover of degree  $r$  branched over  $p$ . Explicitly,  $\tilde{\mathcal{P}}$  is given by

$$(5.2.1) \quad \tilde{\mathcal{P}} = \text{Spec}_{\mathcal{P}} \left( \bigoplus_{i=0}^{r-1} O_{\mathcal{P}}(-i/r) \right),$$

where the ring structure is given by a section of  $O_{\mathcal{P}}(1)$  vanishing at  $p$ . It is easy to see that  $\tilde{\mathcal{P}} \cong \mathbf{P}^1$ . Set

$$\tilde{\mathcal{C}} = \mathcal{C} \times_{\mathcal{P}} \tilde{\mathcal{P}}$$

with the induced map  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{P}}$ .

**PROPOSITION 5.2.2.** *With the above notation, we have  $M(\phi) = M(\tilde{\phi})/r$ .*

**PROOF.** By construction, we have  $\psi^*O_{\mathcal{P}}(d) = O_{\tilde{\mathcal{P}}}(dr)$ . Since  $\tilde{\phi}_*O_{\tilde{\mathcal{C}}} = \psi^*\phi_*O_{\mathcal{C}}$ , the statement follows.  $\square$

**5.2.2. Relation with the gap sequence.** At first sight, the refined Maroni invariant seems to be an artifact of the stacky way of keeping track of ramification. However, it can be described purely in terms of the geometry of the cover of the coarse spaces. We now describe this connection.

As before, let  $\rho: \mathcal{P} = \mathbf{P}^1(\sqrt[r]{\infty}) \rightarrow \mathbf{P}^1$  be the teardrop curve. Consider a  $k$ -point of  $\mathcal{T}_{g;1/r}$  given by  $(\mathcal{P} \rightarrow \mathbf{P}^1; \infty; \mathcal{C} \rightarrow \mathcal{P})$  and let  $C \rightarrow \mathbf{P}^1$  be the induced cover on the coarse spaces. To lighten notation, we treat  $O_{\mathcal{C}}$  and  $O_C$  as sheaves on  $\mathcal{P}$  and  $P$  respectively, omitting the pushforward symbols. Then  $O_C = \rho_*O_{\mathcal{C}}$ . Therefore, if we have the splitting

$$O_{\mathcal{C}} \cong O_{\mathcal{P}} \oplus O_{\mathcal{P}}(-m) \oplus O_{\mathcal{P}}(-n),$$

then we deduce the splitting

$$O_C \cong O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}([-m]) \oplus O_{\mathbf{P}^1}([-n]).$$

Thus, the splitting type of  $\mathcal{C} \rightarrow \mathcal{P}$  determines the splitting type of  $C \rightarrow \mathbf{P}^1$ . For  $r > 1$ , however, it carries a bit more information. Let us understand what sort of extra information is contained in this refinement. To that end, observe that the data of the splitting type of  $C \rightarrow \mathbf{P}^1$  is equivalent to the data of the sequence  $\langle h^0(O_C(l)) \mid l \in \mathbf{Z} \rangle$ .

First consider the case  $r = 3$ . In this case, the map  $C \rightarrow \mathbf{P}^1$  is totally ramified over  $\infty$ . Denoting by  $x \in C$  the unique point over  $\infty \in \mathbf{P}^1$ , we have  $O_C(1) \cong O_C(3x)$ . Therefore, the data of the splitting type of  $C \rightarrow \mathbf{P}^1$  is the data of the sequence  $\langle h^0(O_C(3lx)) \mid l \in \mathbf{Z} \rangle$ . On the other hand, the data of the splitting type of  $C \rightarrow \mathcal{P}$  is precisely the data of the sequence  $\langle h^0(O_C(lx)) \mid l \in \mathbf{Z} \rangle$ . Thus the refined Maroni invariant encodes the so-called *Weierstrass gap sequence* of the point  $x$  on  $C$ .

Now consider the case  $r = 2$ . In this case, the map  $C \rightarrow \mathbf{P}^1$  has ramification type  $(2, 1)$  over  $\infty$ . Let the preimage of  $\infty$  be  $2x + y$ , where  $x, y \in \mathbf{P}^1$ . As before, the data of the splitting type of  $C \rightarrow \mathbf{P}^1$  is the data of the sequence  $\langle h^0(O_C(l(2x + y))) \mid l \in \mathbf{Z} \rangle$ . On the other hand, the splitting type of  $C \rightarrow \mathcal{P}$  encodes, in addition, the data of  $h^0(O_C(l(2x + y) - x))$  and  $h^0(O_C(l(2x + y) + x))$ , for  $l \in \mathbf{Z}$ .

**5.2.3. The  $\mu$  invariant.** Let  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  be a triple cover, étale except possibly over a point  $p \in \mathcal{P}$  different from  $\xi$ . In this case, we say that  $\phi$  has *concentrated branching*. Define the  $\mu$  invariant of  $\phi$  to be the  $\mu$  invariant of the singularity of  $\mathcal{C} \rightarrow \mathcal{P}$  over  $p$  as in Subsection 3.2.2.

We recall the definition in the current context. Let  $\Delta = \text{Spec } \widehat{O_{\mathcal{P}, p}}$  be the formal disk around  $p$  and set  $C = \mathcal{C} \times_{\mathcal{P}} \Delta$ . Let  $\tilde{C} \rightarrow C$  be the normalization. Then  $\tilde{C} \rightarrow \Delta$  is not necessarily étale. We choose a cover  $\Delta' \rightarrow \Delta$  of degree  $d$  such that the normalization of  $C' = C \times_{\Delta'} \Delta$  is étale over  $\Delta'$ . Then, by definition, we have

$$(5.2.2) \quad \mu(C \rightarrow \Delta) = \frac{1}{r} \mu(C' \rightarrow \Delta').$$

Note that the  $\mu$  invariant lies in  $\frac{1}{r}\mathbf{Z}$ .

As in the case of the refined Maroni invariant, the  $\mu$  invariant can be read off from that of the modified cover  $\tilde{\phi}: \tilde{C} \rightarrow \tilde{\mathcal{P}}$ . We recall the procedure. Define the cyclic cover  $\tilde{P} \rightarrow \mathcal{P}$  ramified only over  $p$ , as in (5.2.1). Set

$$\tilde{C} = \mathcal{C} \times_{\mathcal{P}} \tilde{P}$$

with the induced map  $\tilde{\phi}: \tilde{C} \rightarrow \tilde{P}$ . Let  $q \in \tilde{P}$  be the unique point over  $p \in \mathcal{P}$ . See that  $\tilde{\phi}: \tilde{C} \rightarrow \tilde{P}$  has concentrated branching over  $q$ .

**PROPOSITION 5.2.3.** *With the above notation, we have  $\mu(\phi) = \frac{1}{r} \mu(\tilde{\phi})$ .*

**PROOF.** This follows immediately from (5.2.2). □

### 5.3. The stack $\overline{\mathcal{T}}_{g;1/r}^l$ of $l$ -balanced covers

Having defined the Maroni invariant and the  $\mu$  invariant for covers of the teardrop curve, we are ready to formulate and prove the analogue of Theorem 3.3.4. We begin by defining  $l$ -balanced covers. The definition follows Definition 3.2.4 almost verbatim.

DEFINITION 5.3.1. Let  $l \in \frac{1}{r}\mathbf{Z}$  be non-negative and  $\mathcal{P} \cong \mathbf{P}^1(\sqrt[r]{\infty})$  the teardrop curve with the stacky point  $\xi$  as in Section 5.1. Let  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  be a triple cover, étale over  $\xi$ . We say that  $\phi$  is  $l$ -balanced if the following two conditions are satisfied.

- (1) The Maroni invariant of  $\phi$  is at most  $l$ :

$$M(\phi) \leq l.$$

- (2) If  $\phi$  has concentrated branching, then its  $\mu$  invariant is greater than  $l$ :

$$\mu(\phi) > l.$$

We can reduce Definition 5.3.1 to the case of an unramified fiber, namely Definition 3.2.4, by looking at a modified cover  $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{P}}$ . Let  $p \in \mathcal{P}$  be a point contained in  $\text{br}(\phi)$  and  $\tilde{\mathcal{P}} \rightarrow \mathcal{P}$  the cyclic cover of degree  $r$  branched over  $p$  as in (5.2.1). Set

$$\tilde{\mathcal{C}} = \mathcal{C} \times_{\mathcal{P}} \tilde{\mathcal{P}},$$

with the induced map  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{P}}$ .

PROPOSITION 5.3.2. *With the above notation, the cover  $\phi$  is  $l$ -balanced if and only if the cover  $\tilde{\phi}$  is  $rl$ -balanced in the sense of Definition 3.2.4.*

PROOF. Combine Proposition 5.2.2 and Proposition 5.2.3. □

Recall that  $\mathcal{T}_{g;1/r} \subset \mathcal{H}^3$  is the open and closed substack whose geometric points parametrize covers  $(\mathcal{P} \rightarrow P; \sigma_1; \phi: \mathcal{C} \rightarrow \mathcal{P})$ , where  $\text{Aut}_{\sigma_1}(\mathcal{P}) = \mu_r$  and  $\mathcal{C}$  is a connected curve of genus  $g$ .

DEFINITION 5.3.3. Define  $\overline{\mathcal{T}}_{g;1/r}^l$  to be the category whose objects over a  $\mathbf{K}$  scheme  $S$  are

$$\overline{\mathcal{T}}_{g;1/r}^l(S) = \{(P \rightarrow S; \mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})\},$$

such that

- (1)  $(P \rightarrow S; \mathcal{P} \rightarrow P; \sigma; \phi: \mathcal{C} \rightarrow \mathcal{P})$  is an object of  $\mathcal{T}_{g;1/r}(S)$ ;
- (2)  $P \rightarrow S$  is smooth, that is, a  $\mathbf{P}^1$  bundle;
- (3) For all geometric points  $s \rightarrow S$ , the cover  $\phi_s: \mathcal{C}_s \rightarrow \mathcal{P}_s$  is  $l$ -balanced.

Just as in the case of  $r = 1$  (Proposition 3.2.6), it is easy to see that the morphism  $\overline{\mathcal{T}}_{g;1/r}^l \rightarrow \mathcal{T}_{g;1/r}$  is an open immersion.

We now come to the main theorem of this chapter.

**THEOREM 5.3.4.** *Let  $l \in \frac{1}{r}\mathbf{Z}$  be non-negative. Then  $\overline{\mathcal{T}}_{g;1/r}^l$  is an irreducible Deligne–Mumford stack, smooth and proper over  $\mathbf{K}$ .*

By the jugglery of modifying a cover  $\mathcal{C} \rightarrow \mathcal{P}$  to get a cover  $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{P}}$  used many times in Section 5.2, the major steps in the proof can be reduced to the analogous steps in the case of  $r = 1$ . As a result, little hard work goes into the proof of Theorem 5.3.4.

**PROOF.** We divide the proof into steps.

**That  $\overline{\mathcal{T}}_{g;1/r}^l$  is smooth and of finite type.** We have an open immersion

$$\overline{\mathcal{T}}_{g;1/r}^l \hookrightarrow \mathcal{T}_{g;1/r}.$$

Denote by  $\mathcal{M}_{0;b,1}^s \subset \mathcal{M}_{0;b,1}$  the open substack parametrizing  $(P; \Sigma; \sigma)$  with  $P$  smooth. It is easy to see that  $\mathcal{M}_{0;b,1}^s$  is of finite type over  $\mathbf{K}$ . By the definition of  $\overline{\mathcal{T}}_{g;1/r}^l$ , the open immersion  $\overline{\mathcal{T}}_{g;1/r}^l \hookrightarrow \mathcal{T}_{g;1/r}$  factors as

$$\overline{\mathcal{T}}_{g;1/r}^l \hookrightarrow \mathcal{M}_{0;b,1}^s \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1/r} \hookrightarrow \mathcal{T}_{g;1/r}.$$

Since  $\mathcal{M}_{0;b,1}^s \times_{\mathcal{M}_{0;b,1}} \mathcal{T}_{g;1/r}$  is smooth and of finite type over  $\mathbf{K}$ , so is  $\overline{\mathcal{T}}_{g;1/r}^l$ . The irreducibility follows from the irreducibility of  $\mathcal{T}_{g;1/r}$ .

**That  $\overline{\mathcal{T}}_{g;1/r}^l$  is separated.** We use the valuative criterion. Let  $\Delta = \text{Spec } R$  be the spectrum of a DVR, with special point 0, generic point  $\eta$  and residue field  $k$ . Consider two morphisms  $\Delta \rightarrow \overline{\mathcal{T}}_{g;1/r}^l$  given by  $(\mathcal{P}_i \rightarrow P_i \rightarrow \Delta; \sigma_i; \phi_i: \mathcal{C}_i \rightarrow \mathcal{P}_i)$  for  $i = 1, 2$ . Let  $\psi_\eta$  be an isomorphism of this data over  $\eta$ . We must show that  $\psi_\eta$  extends to an isomorphism over all of  $\Delta$ . We may replace  $\Delta$  by a finite cover, if we so desire.

Let  $\Sigma_i \subset P_i$  be the branch divisor of  $\phi_i$ . By passing to a cover of  $\Delta$  if necessary, assume that we have sections  $p_i: \Delta \rightarrow \Sigma_i$  which agree over  $\eta$ , that is  $\psi_\eta^P \circ p_1|_\eta = p_2|_\eta$ . Denote by  $\xi_i \subset \mathcal{P}_i$  the

reduced preimage of  $\sigma_i$  and consider the cyclic triple cover  $\tilde{P}_i \rightarrow \mathcal{P}_i$  defined by

$$\tilde{P}_i = \operatorname{Spec}_{\mathcal{P}_i} \left( \bigoplus_{j=0}^{r-1} O_{\mathcal{P}_i}(-\xi_i) \right),$$

where the ring structure is given by a section of  $O_{\mathcal{P}_i}(r\xi_i) \cong O_{\mathcal{P}_i}(p_i)$  vanishing along  $p_i$ . Set

$$\tilde{C}_i = \mathcal{C}_i \times_{\mathcal{P}_i} \tilde{P}_i,$$

with the induced map  $\tilde{\phi}_i: \tilde{C}_i \rightarrow \tilde{P}_i$ . The reduced preimage  $\tilde{\sigma}_i$  of  $\sigma_i$  gives a section  $\tilde{\sigma}_i: \Delta \rightarrow \tilde{P}_i$ . By Proposition 5.3.2,  $(\tilde{P}_i; \tilde{\sigma}_i; \tilde{\phi}_i: \tilde{C}_i \rightarrow \tilde{P}_i)$  is a family of  $rl$ -balanced covers, for  $i = 1, 2$ . We have an isomorphism  $\tilde{\psi}_\eta$  of this data over  $\eta$ . By the separatedness in Theorem 3.3.4,  $\tilde{\psi}_\eta$  extends to an isomorphism  $\tilde{\psi}$  over all of  $\Delta$ . By descent, we conclude that  $\psi_\eta$  extends to an isomorphism over all of  $\Delta$ .

**That  $\overline{\mathcal{T}}_{g;1/r}^l$  is Deligne–Mumford.** Since we are in characteristic zero, it suffices to prove that a  $k$ -point  $(\mathcal{P} \rightarrow \mathbf{P}^1; \infty; \phi: \mathcal{C} \rightarrow \mathcal{P})$  of  $\overline{\mathcal{T}}_{g;1/r}^l$  has finitely many automorphisms. We have a morphism of algebraic groups

$$\tau: \operatorname{Aut}(\mathcal{P} \rightarrow \mathbf{P}^1; \infty; \phi: \mathcal{C} \rightarrow \mathcal{P}) \rightarrow \operatorname{Aut}(\mathbf{P}^1),$$

where the group on the left is proper because  $\overline{\mathcal{T}}_{g;1/r}^l$  is separated and the group on the right is affine. It is clear that  $\ker \tau$  is finite. Hence the group on the left is finite.

**That  $\overline{\mathcal{T}}_{g;1/r}^l$  is proper.** Let  $\Delta = \operatorname{Spec} R$  be as in the proof of separatedness. Let  $(\mathcal{P}_\eta \rightarrow \mathcal{P}_\eta; \sigma_\eta; \phi_\eta: \mathcal{C}_\eta \rightarrow \mathcal{P}_\eta)$  be an object of  $\overline{\mathcal{T}}_{g;1/r}^l$  over  $\eta$ . We need to show that, possibly after a finite base change, it extends to an object of  $\overline{\mathcal{T}}_{g;1/r}^l$  over  $\Delta$ .

By replacing  $\Delta$  by a finite cover if necessary, assume that we have a section  $p: \eta \rightarrow \operatorname{br} \phi_\eta$ . Define the cyclic cover  $\tilde{P}_\eta \rightarrow \mathcal{P}_\eta$  of degree  $r$  branched over  $p$ , as before. Set

$$\tilde{C}_\eta = \mathcal{C}_\eta \times_{\mathcal{P}_\eta} \tilde{P}_\eta,$$

with the induced map  $\tilde{\phi}: \tilde{C}_\eta \rightarrow \tilde{P}_\eta$  and the sections  $\tilde{\sigma}_\eta: \eta \rightarrow \tilde{P}_\eta$  and  $\tilde{p}_\eta: \eta \rightarrow \tilde{P}_\eta$  given by the reduced preimages of  $\sigma_\eta$  and  $p_\eta$ , respectively. Then  $(\tilde{P}_\eta; \tilde{\sigma}_\eta; \tilde{\phi}_\eta: \tilde{C}_\eta \rightarrow \tilde{P}_\eta)$  is a family of  $rl$ -balanced covers. By the properness in Theorem 3.3.4, it extends to a family of  $rl$ -balanced covers  $(\tilde{P}; \tilde{\sigma}; \tilde{\phi}: \tilde{C} \rightarrow \tilde{P})$  over  $\Delta$ . The idea is to descend  $\tilde{C} \rightarrow \tilde{P}$  down to  $\mathcal{C} \rightarrow \mathcal{P}$ , extending  $\mathcal{C}_\eta \rightarrow \mathcal{P}_\eta$ .

We first extend  $(\mathcal{P}_\eta, \sigma_\eta, p_\eta)$  over  $\eta$  to  $(\mathcal{P}, \sigma, p)$  over  $\Delta$  so that  $\tilde{P}$  is the cyclic cover of degree  $r$  of  $\mathcal{P}$  branched over  $p$ . For this, note that  $P_\eta \rightarrow \eta$  is a  $\mathbf{P}^1$ -bundle. Possibly after replacing  $\Delta$  by a ramified cover of degree  $r$ , identify  $(P_\eta, \sigma_\eta, p_\eta)$  with  $(\mathbf{P}_\eta^1, \infty, 0)$  via an isomorphism that induces an isomorphism  $(\tilde{P}, \tilde{\sigma}, p) \xrightarrow{\sim} (\tilde{\mathbf{P}}_\Delta^1, \infty, 0)$ , where the latter  $\tilde{\mathbf{P}}^1 \cong \mathbf{P}^1$  covers the former  $\mathbf{P}^1$  by  $[X : Y] \mapsto [X^r : Y^r]$ . Set  $P = \mathbf{P}_\Delta^1$  with two sections  $\sigma$  and  $p$  given by  $\infty$  and  $0$ , respectively. Then  $(P, \sigma, p)$  is an extension of  $(P_\eta, \sigma_\eta, p_\eta)$ . Setting  $\mathcal{P} = P(\sqrt[r]{\sigma})$ , we get an extension of  $\mathcal{P}_\eta$ . Note that the covering  $\tilde{P} \rightarrow P$  factors through  $\tilde{P} \rightarrow \mathcal{P}$ , extending  $\tilde{P}_\eta \rightarrow \mathcal{P}_\eta$ , and exhibiting  $\tilde{P} \rightarrow \mathcal{P}$  as a cyclic triple cover of degree  $r$  branched over  $p$ .

Let  $\Sigma \subset P$  be the unique flat extension of the divisor  $\text{br}(\phi_\eta) \subset P_\eta$ . Then the preimage of  $\Sigma$  in  $\tilde{P}$  is  $\text{br } \tilde{\phi}$ . Since the divisor  $\text{br } \tilde{\phi}$  is disjoint from  $\tilde{\sigma}$ , the divisor  $\Sigma$  is disjoint from  $\sigma$ . Thus  $(P; \Sigma; \sigma)$  is an object of  $\mathcal{M}_{0;b,1}(\Delta)$ . By the properness of  $\mathcal{T}_{g;1/r} \rightarrow \mathcal{M}_{0;b,1}$ , we have a unique extension  $\phi: \mathcal{C} \rightarrow \mathcal{P}$ . It remains to prove that the central fiber is  $l$ -balanced.

We claim that we have an isomorphism

$$(5.3.1) \quad \tilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C} \times_{\mathcal{P}} \tilde{P} \text{ over } \tilde{P}.$$

Indeed, by construction, we have such an isomorphism over  $\eta$ . Note that  $\tilde{\mathcal{C}} \rightarrow \tilde{P}$  and  $\mathcal{C} \times_{\mathcal{P}} \tilde{P}$  are covers of  $\tilde{P}$ , isomorphic over  $\eta$ , and they have the same branch divisor. By the separatedness of  $\mathcal{H}^3 \rightarrow \mathcal{M}$ , we conclude that the isomorphism  $\tilde{\mathcal{C}}_\eta \rightarrow \mathcal{C}_\eta \times_{\mathcal{P}_\eta} \tilde{P}_\eta$  extends over  $\Delta$ , yielding (5.3.1).

Finally, since the central fiber of  $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{P}$  is  $rl$ -balanced, we conclude that the central fiber of  $\phi: \mathcal{C} \rightarrow \mathcal{P}$  is  $l$ -balanced using Proposition 5.3.2.

The proof is now complete. □

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