Perverse sheaves in Algebraic Geometry

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Lecture notes by Tony Feng

Contents

1 The Decomposition Theorem

Perhaps the most successful application of perverse sheaves, and the motivation for their introduction, is the Decomposition Theorem. That is the subject of this section.

The decomposition theorem is a generalization of a 1968 theorem of Deligne's, from a smooth projective morphism to an arbitrary proper morphism.

1.1 The smooth case

Let $X = Y \times F$. Here and throughout, we use $\mathbb{Q}$-coefficients in all our cohomology theories.

Theorem 1.1 (Künneth formula)**.** *We have an isomorphism*

$$H^\bullet(X) \cong \bigoplus_{q \geq 0} H^{\bullet - q}(Y) \otimes H^q(F).$$

In particular, this implies that the pullback map $H^\bullet(X) \twoheadrightarrow H^\bullet(F)$ is surjective, which is already rare for fibrations that are not products.

Question. Suppose that $f\colon X \to Y$ is a smooth projective morphism (in particular $f_x\colon T_x X \to T_{f(x)} Y$ is surjective). If $\overline{X} \supset X$ is some (smooth) compactification of X, then what is the relationship between the images of $H^\bullet(X)$ and $H^\bullet(\overline{X})$ in $H^\bullet(F)$?

Theorem 1.2 (Cohomological Deligne Decomposition Theorem)**.** *Let $f\colon X \to Y$ be a smooth projective map of complex algebraic manifolds. Then*

$$H^\bullet(X) \cong \bigoplus_{q \geq 0} H^{\bullet - q}(Y, R^q f_* \mathbb{Q}).$$

Remark 1.3. This is surprising because it fails if one takes any smooth bundle: as we know, the failure is measured by the Leray spectral sequence. The content of this theorem is that the spectral sequence degenerates on the E_2 page. It is specific to the realm of complex algebraic geometry (and fails for real algebraic geometry or complex geometry, for example).

The answer to the question posed above is furnished by the *Global invariant cycle Theorem*, which in turn follows from this theorem of Deligne.

Here $R^q f_* \mathbb{Q}$ is the qth direct image sheaf, which is locally constant but not constant in general. It is the sheafification of the presheaf $U \mapsto H^q(f^{-1}(U))$.

Theorem 1.4 (Ehresmann)**.** *If f is smooth and proper, then f is a fiber bundle.*

Ehresmann's Theorem implies that on a small enough open subset (where the bundle is trivialized) $R^q f_* \mathbb{Q}$ is a constant sheaf. Therefore, $R^q f_* \mathbb{Q}$ is a locally constant sheaf. On the other hand, a locally constant sheaf is equivalent to the data of a representation of $\pi_1(Y)$. Intuitively, for a loop on Y we have an action on the fiber obtained by tracing out the loop. This is a linear action, hence defines a representation of $\pi_1(Y)$.

By Deligne's Theorem, we have a surjective map $H^q X \twoheadrightarrow H^0(Y, R^q)$. By a simple exercise, the restriction map identifies $H^0(Y, R^q) \cong (R^q_y)^{\pi_1(Y)} \subset R^q_y$. In particular, the image of $H^\bullet(X)$ in $H^\bullet(F)$ is π_1-invariant.

Theorem 1.5 (Global invariant cycle theorem)**.** *Under the maps $H^\bullet(\overline{X})) \to H^\bullet(X) \to H^\bullet(F)$ induced by the inclusions $F \hookrightarrow X \hookrightarrow \overline{X}$, we have that* Im $H^\bullet(\overline{X})$ *coincides with* Im $H^\bullet(X)$ *in $H^\bullet(F)$ as $H^\bullet(F)^{\pi_1(Y)}$.*

While the first characterization follows from Deligne's theorem, this follows from that plus input from the theory of mixed Hodge structures. Since $\overline{X}$ admits a p, q Hodge decomposition, this implies that $H^\bullet(F)^{\pi_1}$ *also* has a p, q Hodge decomposition. This is quite striking because that is a *topological* construction.

1.2 Generalization to singular maps

We now want to state a more general cohomological decomposition theorem. To do so, we need to introduce the *intersection cohomology* groups, denoted $IH^\bullet(X)$. These groups agree with $H^\bullet(X)$ when X is smooth, but not in general.

There is a battery of results when X is projective, generalizing the usual nice properties from the smooth case.

- Poincaré duality: a perfect pairing $IH^\bullet(X) \times IH^{n-\bullet}(X) \to IH^n(X)$.
- Lefschetz hyperplane theorem: a relation between intersection cohomology of X and that of a hyperplane section.
- Hard Lefschetz theorem: the action of cupping with $\eta \in H^2(X)$ the first chern class of an ample line bundle is an isomorphism from $IH^{\bullet - n}$ to $IH^{\bullet + n}$.

This theory is very flexible: it involves the input of a locally constant sheaf. If $X^\circ \subset X_{\text{reg}}$ is an open smooth subset and L is a locally constant sheaf on X°, then we have intersection cohomology groups $IH^\bullet(X, L)$. These satisfy again Poincaré duality, Lefschetz hyperplane theorem, and Hard Lefschetz theorem under suitable hypotheses on L.

One can't hope for a naïve version of the decomposition theorem.

Exercise 1.6. To see why, resolve the cone over an elliptic curve.

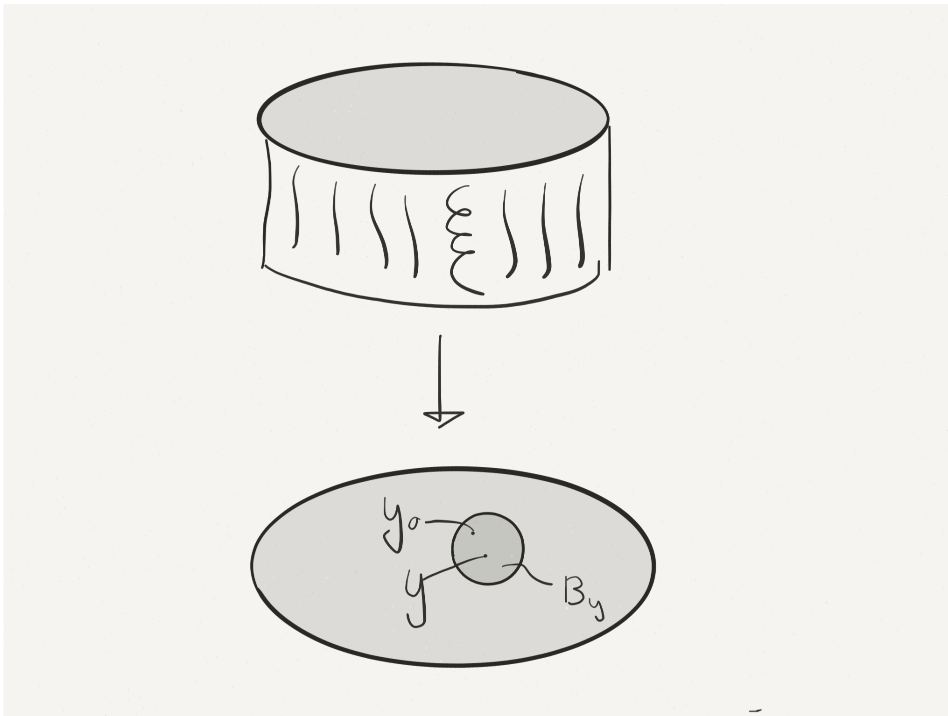
Theorem 1.7 (Cohomological Decomposition Theorem)**.** *We have an isomorphism*

$$IH^\bullet(X) \cong \bigoplus_q \bigoplus_{(S,L) \in \mathcal{E}_q} IH^{\bullet - q}(S, L)$$

where $\mathcal{E}_q$ ranges over "enriched subvarieties" consisting of pairs (S, L) where $S \subset Y$ is a closed subvariety and L a local system on S°.

Deligne's cohomological decomposition theorem is a special case where the (S, L) are (Y, R^q).

Exercise 1.8. See what this says in the case of a blowup.



If you take a small ball B_y centered at $y \in Y$, then $H^\bullet(f^{-1}B_y) \cong H^\bullet(f^{-1}y)$. Take a good point $y_0 \in Y^0$ lying in the image of the ball, so this admits a map $H^\bullet(f^{-1}B_y) \to H^\bullet(f^{-1}y_0)$.

Theorem 1.9 (Local invariant cycle theorem)**.** *The image of $H^\bullet(f^{-1}y_0)$ in $H^\bullet(f^{-1}y)$ consists of the $\pi_1(Y, y)$ invariants.*

This follows from the cohomological theorem. The proof of the first theorem is purely cohomological, but the only known proofs of the second theorem use the derived category and perverse sheaves.

1.3 The derived category

Let $f\colon X \to Y$ be a projective morphism and K a complex of sheaves on X. Then Rf_*K is a complex on Y. Consider the cohomology sheaves $\mathcal{H}^q(Rf_*K) = R^qf_*K$ as before.

Why might one study this? Because we have the (tautological) identity $H^\bullet(Y, Rf_*K) = H^\bullet(X, K)$. For example, if the direct image complex splits, then we get:

Theorem 1.10 (Deligne's Theorem in the derived category)**.** *If $f\colon X \to Y$ is smooth and projective, then*

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{q \geq 0} R^qf_*\mathbb{Q}_X[-q].$$

Furthermore, the sheaves $R^q f_* \mathbb{Q}_X[-q]$ *are semisimple.*

Remark 1.11. Deligne's proof is by a trick showing that Hard Lefschetz on fibers splits the direct image sheaf. There is another proof by Deligne's theory of weights, by considering the triangle

$$R^0 f_* \mathbb{Q} \to R f_* \mathbb{Q} \to R^1 f_* \mathbb{Q}[-1].$$

This is classified by an element of $\mathrm{Ext}^1(R^1 f_* \mathbb{Q}[-1], R^0 f_* \mathbb{Q})$. The weight of $R^0 f_* \mathbb{Q}$ is 0 and the weight of $R^1 f_* \mathbb{Q}$ is 1, but after shifting we have that the weight of $R^1 f_* \mathbb{Q}[-1]$ is 0. So the weight of $\mathrm{Ext}^1(R^1 f_* \mathbb{Q}[-1], R^0 f_* \mathbb{Q})$ is 1. (We are passing to a finite field situation.) On the other hand, since this can all be defined over some finite field, it is preserved by some power of Frobenius, meaning that it also has weight 0. That is only possible if the extension is trivial.

Now we discuss the decomposition theorem in the derived category. There are intersection cohomology groups $IH(S, L) = H^\bullet(S, \mathcal{IC}_S(L))$ where $\mathcal{IC}_S(L)$, the *intersection complex of* S *with coefficients in* L, is some complex of sheaves on S.

Theorem 1.12 (Decomposition Theorem)**.** *Let* $f\colon X \to Y$ *be proper. Then*

$$R f_* \mathcal{IC}_X \cong \bigoplus_{q \geq 0} \bigoplus_{(S,L) \in \mathcal{E}_q} \mathcal{IC}_S(L)[-q].$$

with L *semisimple.*

2 Perverse Sheaves

2.1 The constructible category

Let X be a complex variety. Denote by $\mathbf{Shv}(X, \mathbb{Q})$ be the category of sheaves of $\mathbb{Q}$-vector spaces on X, and by $D^b(\mathbf{Shv}(X, \mathbb{Q}))$ the bounded derived category of $\mathbf{Shv}(X, \mathbb{Q})$.

Definition 2.1. A sheaf $\mathcal{F}$ on X is *constructible* if there exists a stratification $X = \coprod X_\alpha$ into locally closed non-singular subsets such that $F|_{X_\alpha}$ is locally constant of finite rank (i.e. a local system).

Definition 2.2. The *constructible derived category* of sheaves of $\mathbb{Q}$-vector spaces on X is the full subcategory $D(X) \subset D^b(\mathbf{Shv}(X, \mathbb{Q}))$ whose objects are complexes K with $\mathcal{H}^i(K)$ constructible for all i.

Stability. One reason we like the constructible derived category is that it is stable under all the reasonable operations on sheaves that we meet in algebraic geometry.

- If $f : X \to Y$ is a morphism, then $Rf_*, Rf_! : D(X) \to D(Y)$ preserve constructibility.

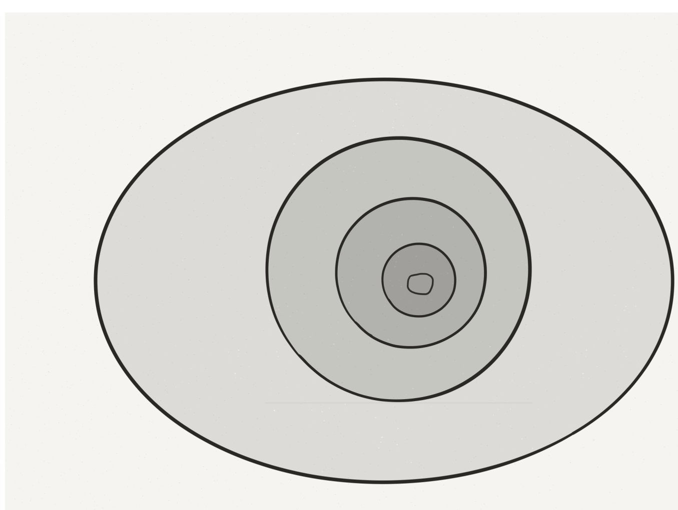
  Remark 2.3. This was implicitly used last time when we discussed why for a proper map $f : X \to Y$,

$$\begin{array}{ccc} f^{-1}U_y & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_y & \longrightarrow & Y \end{array}$$

  then for $y \in Y$, there exists a neighborhood U_y of y such that

$$H^\bullet(f^{-1}U_y) = H^\bullet(f^{-1}y).$$

  Constructibility allows us to talke about "standard neighborhoods $U_y(\epsilon)$" such that $H^i(U_y(\epsilon), K|_{U_y(\epsilon)})$ is "constant in ϵ."



- Another form of stability is under *Verdier duality*. Namely, there is a duality $D(X)^{\text{opp}} \xrightarrow{\sim} D(X)$. If $K \in D(X)$, then we denote by $K^\vee$ the dual of K, and it satisfies

$$H^\bullet(U, K^\vee) \cong H_c^{-\bullet}(U, K)^\vee.$$

  Exercise 2.4. Using this, guess for a smooth manifold X what $K^\vee$ should be for $K = \mathbb{Q}_X$ (answer: $\mathbb{Q}^\vee = \mathbb{Q}[2 \dim X]$.)

  Examine a singular space to see that this guess cannot hold in general. The reason is that cohomology is invariant under homothety, so the cohomology around a singular point is concentrated in degree 0. However, cohomology with compact supports *is* sensitive to singularities.

- Also, we have stability under $RHom, \otimes$, etc.

2.2 Perverse sheaves

We will build a (full) subcategory $P(X) \subset D(X)$, the *category of perverse sheaves*. Before describing them formally, we state some of their properties.

Exercise 2.5. Check that $D(X)$ is not abelian, because it doesn't admit kernels (since kernels are forced to split in the derived category, it suffices to produce an indecomposable object with a morphism with non-zero kernel).

Before we begin the formalism, we discuss some properties that perverse sheaves enjoy over constructible sheaves.

Artinian. There is an analogy of $P(X)$ with the category of finite dimensional vector spaces. Indeed, the latter is abelian and noetherian, and so is the category of constructible sheaves (and hence also $P(X)$). However, the category of constructible sheaves is *not* artinian, whereas $P(X)$ and the category of finite dimensional vector spaces are. This is an important property which is gained by passing to perverse sheaves.

Artin Vanishing Theorem. Let X be an affine smooth variety of dimension n. Then $H^\bullet(X, \mathbb{Q}) = 0$ if $\bullet \notin [0, n]$. Artin generalized this to *constructible* sheaves $\mathcal{F}$: if X is affine then $H^\bullet(X, \mathcal{F})$ has the same property. However, there is no vanishing theorem for compactly supported cohomology. But there is a salvage of Artin's vanishing theorem to $P(X)$, for both ordinary and compactly supported cohomology.

Lefschetz Hyperplane Theorem. It is known that the Lefschetz hyperplane theorem follows formally from Artin's vanishing theorem (which we shall see later). Since Artin's vanishing theorem fails for singular spaces, we don't get Lefschetz for such spaces, but we do if we restrict our attention to perverse sheaves.

Poincaré Duality. We get a form of Poincaré duality for perverse sheaves.

Now we are ready to start describing the definition of perverse sheaves.

Example 2.6. We have $\mathbb{C}^0 \subset \mathbb{C}^1 \subset \mathbb{C}^2$. Then a perverse sheaf on $\mathbb{C}^2$ is

$$\mathcal{O}_{\mathbb{C}^2}[2] \oplus \mathcal{O}_{\mathbb{C}^1}[1] \oplus \mathcal{O}_{\mathbb{C}^0}[0].$$

Definition 2.7. $K \in D(X)$ satisfies the *conditions of support* if

$$\dim \operatorname{supp} \mathcal{H}^{-i}(K) \leq i$$

Example 2.8. The complex in Example 2.6 above wouldn't be perverse if we put in a summand of $\mathcal{O}_{\mathbb{C}^2}[1]$, since its $\mathcal{H}^{-1}$ is supported on all of $\mathbb{C}^2$.

Definition 2.9. A constructible complex of sheaves K is in $P(X)$ if K and $K^\vee$ satisfy the conditions of support. By Verdier duality, this is equivalently to K satisfying the conditions of support and also

$$\dim \operatorname{supp} \mathcal{H}^i_c(K) \leq n - i.$$

Example 2.10. In $\mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^n}[n] \in P(\mathbb{C}^n)$, and no other shift is perverse.

Exercise 2.11. If X is the cone over a projective manifold, verify that $\mathcal{O}_X[n]$ has conditions of support, but not conditions of cosupport (by computing the compactly supported cohomology in the neighborhood of the cone point). ♠♠♠ TONY: [TODO]

2.3 Proof of Lefschetz Hyperplane Theorem

Theorem 2.12 (Artin Vanishing theorem)**.** *Let X be affine and $P \in P(X)$. Then $H^{>0}(X, P) = 0$ and $H_c^{<0}(X, P) = 0$.*

Exercise 2.13. Assume the Artin Vanishing theorem for constructible sheaves. Using the Grothendieck spectral sequence $H^p(X, \mathcal{H}^q(P)) \implies H^{p+q}(X, P)$, deduce the Artin Vanishing theorem for perverse sheaves.

Theorem 2.14. *Let X be quasiprojective and $H \subset X$ a general hyperplane section. Then for $P \in P(X)$ we have that the map induced by restriction*

$$H^i(X, P) \to H^i(H, P|_H)$$

is an isomorphism if $i < -1$ and injective if $i = -1$.

Proof. The long exact sequence of a pair in algebraic geometry is promoted in the derived category to a distinguished triangle. Namely, if we have $j \colon U \hookrightarrow X \hookleftarrow Z \colon i$, with U open and Z closed, then we get a distinguished triangle

$$Rj_! j^! \to \mathrm{Id} \to Ri_* i^*$$

so on cohomology we get the sequence

For simplicity, we will assume that X is projective. Then in the long exact sequence of cohomology, we get

$$H^\bullet(X, Rj_! j^! P) \to H^\bullet(X, P) \to H^\bullet(H, P|_H).$$

Let's change to cohomology with compact supports, which doesn't have any effect on the sequence above because X is compact. However, it is still useful because it highlights the compatibility with $Rf_!$: just as $H^\bullet(Y, Rf_*K) \cong H^\bullet(X, K)$ (cleary with the derived functor definition) we have $H_c^\bullet(Y, Rf_!K) = H_c^\bullet(X, K)$. So the long exact sequence becomes

$$\begin{array}{c} H_c^\bullet(Y, Rj_!j^!P) \longrightarrow H^\bullet(X, P) \longrightarrow H^\bullet(H, P|_H) \xrightarrow{\delta} \\ H_c^{\bullet+1}(Y, Rj_!j^!P) \longrightarrow \ldots \end{array}$$

Now, $H_c^\bullet(X \backslash H, P|_{X-H})$ vanishes in negative degrees by Artin's Vanishing Theorem, because $X - H$ is affine. That implies the result. □

Exercise 2.15. Show that $\mathbf{Shv}_c(X)$ is noetherian. Show that this implies that $P(X)$ is noetherian. Since $P(X)$ is closed under duality $(-)^\vee$, which exchanges the ACC and DCC, $P(X)$ is Artinian.

This means that we have the Jordan-Hölder theorem.

Theorem 2.16. *For all $P \in P(X)$, there exists a finite increasing filtration*

$$0 \subset \ldots \subset P_i \subset P_{i+1} \subset \ldots \subset P$$

such that P_{i+1}/P_i are simple in $P(X)$.

Perhaps the single most important property of perverse sheaves is that we can even characterize the simple objects, which are the *intersection cohomology* sheaves.

Theorem 2.17. *If $P \in P(X)$ is simple, then there exists a unique pair (S, L) with $S \subset X$ closed and irreducible and $S^\circ \subset S^{reg} \subset S$ and L a local system on S such that $P = IC_S(L)$. Conversely, if L is simple then $IC_S(L)$ is simple.*

Remark 2.18. There are some confusing notation conventions. We denote $IC_S(L) := \mathcal{IC}_S(L)[\dim S]$. The complex $\mathcal{IC}_S(L)$ is in positive degrees, but is not perverse; $IC_S(L)$ is perverse.

2.4 Perverse t-structure

Let $K \in D^b(X)$. There are *truncation functors* τ_i such that

$$\tau_{\leq i+1}K \underbrace{\rightarrow}_{\mathcal{H}^i(K)[-i]} \tau_{\leq i}K \rightarrow \ldots$$

For $A \to B \to C$ a distinguished triangle, we get a long exact sequence in cohomology

$$\ldots \to \mathcal{H}^i(A) \to \mathcal{H}^i(B) \to \mathcal{H}^i(C) \to \mathcal{H}^{i+1}(A) \to \ldots$$

A *t-structure* is an abstraction of a truncation functor. There is a *perverse* *t*-structure

$$ {}^{\mathrm{p}}\tau_{\leq i+1}K \underbrace{\to}_{{}^{\mathrm{p}}\mathcal{H}^i(K)[-i]} {}^{\mathrm{p}}\tau_{\leq i}K \to \dots $$

leading to *perverse cohomology sheaves* ${}^{\mathrm{p}}\mathcal{H}^i(K)[-i]$. For a distinguished triangle

$$ A \to B \to C, $$

we again get a long exact sequence of perverse cohomology sheaves

$$ \dots \to {}^{\mathrm{p}}\mathcal{H}^i(A) \to {}^{\mathrm{p}}\mathcal{H}^i(B) \to {}^{\mathrm{p}}\mathcal{H}^i(C) \to {}^{\mathrm{p}}\mathcal{H}^{i+1}(A) \to \dots $$

such that ${}^{\mathrm{p}}\mathcal{H}^i(K[j]) = {}^{\mathrm{p}}\mathcal{H}^{i+j}(K)$.

Example 2.19. If $K = \bigoplus P^i[-i]$, then ${}^{\mathrm{p}}\mathcal{H}^i(K) = P^i$.

We know by semisimplicity that

$$ Rf_*IC_X(M) \cong \bigoplus_{i\in\mathbb{Z}} \left(\bigoplus_{(S,L)} IC_S(L) \right) [-i] $$

and the Decomposition Theorem tells us that

$$ Rf_*IC_X(M) \cong \bigoplus_{i\in\mathbb{Z}} {}^{\mathrm{p}}\mathcal{H}^i(Rf_*IC_X[m])[-i] $$

which is in complete analogy to Deligne's theorem. In particular, if X is non-singular then we recover

$$ Rf_*\mathbb{Q}_X \cong \bigoplus_{q\geq 0} {}^{\mathrm{p}}\mathcal{H}^i(Rf_*\mathbb{Q})[-i] $$

2.5 Intersection cohomology

Now, we move on to defining the intersection cohomology sheaves. $\mathcal{IC}_X$ was originally defined using a Whitney stratification of X. Starting with a local system on an open stratum, you take a (derived) pushforward across the next layer of strata and then truncate, to get $\tau_{\leq ?}Rj_*(?)$. Then you keep doing this, with ? changing according to some precise recipe. This was the original definition of MacPherson and Goresky, following a suggestion of Deligne.

$IC_S(L)$ **as an intermediate extension.** There is another construction of $IC_S(L)$.

1. Start with (S°, L) and consider $j\colon S^\circ \hookrightarrow S$. Then you can take $Rj_*L[?]$, with the shift arranged to make the result perverse. To simplify notation, we're going to ignore the shift from now on (but it's essential).

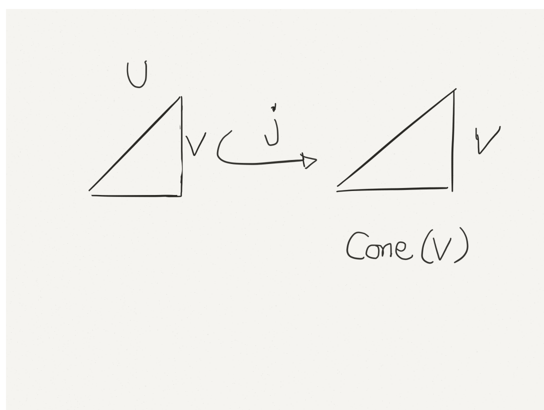
2. There is a map $Rj_!L \to Rj_*L$, since we always have a morphism of functors $j_! \to j_*$. Then by passing to an injective resolution, you get a morphism of functors $Rj_! \to Rj_*$.

3. Taking cohomology, you get a map $\mathcal{H}^0(Rj_!L) \to \mathcal{H}^0(Rj_*L)$. Since we are in an abelian category, we get a factorization

$$\begin{array}{ccc} H^0(Rj_!L) & \xrightarrow{\quad a \quad} & H^0(Rj_*L) \\ & \searrow \qquad \nearrow & \\ & \mathrm{Im}\ a & \end{array}$$

   We then define $IC_S(L) := \mathrm{Im}\ a$.

Example 2.20. Let $V^d \subset \mathbb{P}^N$ be a projective variety, and Y its projective cone. If $U \subset Y$ is the complement of the cone point, then U is smooth and $\tau_{\leq d} Rj_*\mathbb{Q}_U = \mathcal{IC}_Y$.



By one of the exercises, $R^0 j_*\mathbb{Q}_U = Q_Y$ and $R^i j_*\mathbb{Q}_U = H^i(U)_{\text{prim}}$. You can compute that $\mathcal{H}^i \mathcal{IC}_Y$ is the primitive cohomology of V for $i = 1, \ldots, d$, and then vanishes.

Exercise 2.21. Show that if you truncate one step later, the sheaf is not self-dual. The truncation is precisely what's needed for a self-dual object.

3 Semi-small Maps

3.1 Examples and properties

Semi-small maps are a class of maps that behave especially nicely with respect to push-fowards of perverse sheaves.

Example 3.1. The Springer resolution $\widetilde{N} \to \mathcal{N}$ is semi-small.

This is a special case of a general theorem of Kaledin that holomorphic symplectic resolutions are automatically semi-small.

$$\begin{array}{c} X \text{ holomorphic symplectic} \\ \downarrow \\ Y \text{ singular} \end{array}$$

In the case of the Springer resolution, $\widetilde{N}$ is the cotangent bundle of the flag variety, hence is symplectic.

Example 3.2. Hilbert schemes of surfaces $X^{[n]} \xrightarrow{\pi} X^{(n)}$ are semi-small.

Definition 3.3 (First definition). Let X be smooth and $f\colon X \to Y$ a proper surjection. Then we say that f is *semi-small* if $\dim X \times_Y X \leq \dim X$.

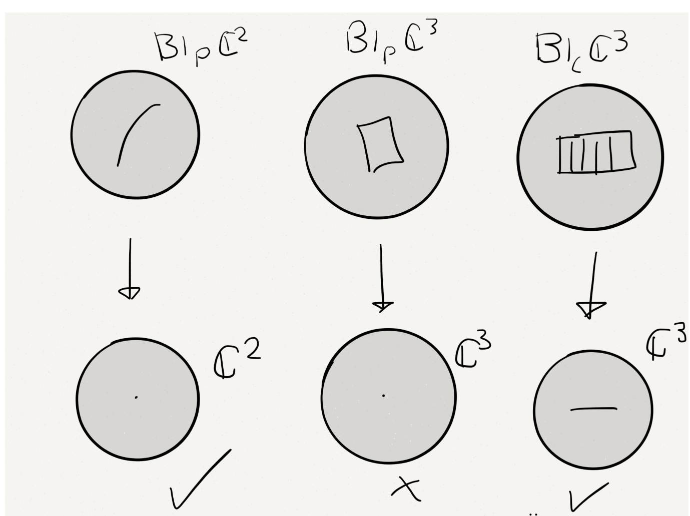
This is a slick but somewhat opaque definition.

Definition 3.4 (Second definition). Let $S_k \subset Y$ be the locally closed subscheme of fiber dimension k:

$$S_k = \{y \in Y \mid \dim f^{-1}y = k\}.$$

Then f is semi-small if $\dim S_k + 2k \leq \dim X$, i.e. $\operatorname{codim} S_k \geq 2k$.

Example 3.5.



The blowup of a point in $\mathbb{C}^2$ is semi-small. The blowup of a point in $\mathbb{C}^3$ is not semi-small. The blowup of a line in $\mathbb{C}^3$ is semismall.

The important property is of semi-small maps is: that *the pushforward of perverse sheaves along semi-small maps is perverse.*

Example 3.6. Let X be the blowup of $\mathbb{P}^2$ at a point σ. Consider $Rf_*\mathbb{Q}_X[2]$. You can compute that

$$\begin{aligned} R^{-2}f_*\mathbb{Q}_X[2] &\cong \mathbb{Q}_Y \\ R^{-1}f_*\mathbb{Q}_X[2] &= 0, \\ R^{0}f_*\mathbb{Q}_X[2] &\cong \mathbb{Q}_\sigma \\ R^{i}f_*\mathbb{Q}_X[2] &= 0 \quad i > 2. \end{aligned}$$

So is the support condition satisfied? We want to know if

$$\dim \operatorname{supp} \mathcal{H}^{-i}(Rf_*\mathbb{Q}_X[2]) \leq i?$$

The only point with non-trivial higher cohomology is σ, and indeed that has dimension ≤ 0, as required.

Also, $Rf_*\mathbb{Q}_X[2]$ is self-dual because we had the duality $(\mathbb{Q}_X[2])^\vee \cong \mathbb{Q}_X[2]$ *before* push-forward (as X was smooth), and duality exchanges Rf_* with $Rf_!$, which are the same if f is proper:

$$(Rf_*\mathcal{F})^\vee \cong Rf_!(\mathcal{F}^\vee) \overset{f \text{ proper}}{=} Rf_*(\mathcal{F}^\vee)$$

Proposition 3.7. *If $f\colon X \to Y$ is semi-small, then $Rf_*\mathbb{Q}_X[\dim X] \in P(Y)$.*

3.2 Lefschetz Hyperplane and Hard Lefschetz

Recall that we proved the following form of the Lefschetz Hyperplane Theorem:

Theorem 3.8 (Lefschetz Hyperplane Theorem)**.** *If Y is quasiprojective and H is a general hyperplane in Y, then for $P \in P(Y)$ we have that*

$$H^i(Y, P) \to H^i(H, P|_H)$$

is an isomorphism for $i \leq -2$ and an injection for $i = -1$.

We can get more in the setting of a semi-small map. Let $f\colon X \to Y$ be semi-small and H a general hyperplane section. Then $P := Rf_*\mathbb{Q}_X[\dim X]$ is perverse by Proposition 3.7, so we have that

$$H^i(Y, Rf_*\mathbb{Q}_X[\dim X]) \to H^i(H, Rf_*\mathbb{Q}_X[\dim X]|_H).$$

is an isomorphism for $i < -1$ and an injection for $i = -1$. Now what is $Rf_*\mathbb{Q}_X[\dim X]|_H$? Consider the fiber diagram

$$\begin{array}{ccc} X_H & \hookrightarrow & X \\ \big\downarrow g & & \big\downarrow f \text{ proper} \\ H & \hookrightarrow & Y \end{array}$$

so by proper base change we have $Rg_*(|_{X_H}) = (Rf_*)|_H$. Then we have a commutative diagram

$$
\begin{array}{ccc}
H^i(Y, Rf_*\mathbb{Q}_X) & \longrightarrow & H^i(H, Rf_*\mathbb{Q}_X|_H) \\
\| & & \| \\
H^i(X, \mathbb{Q}) & \longrightarrow & H^i(H, Rg_*\mathbb{Q}_{X_H}) \\
\| & & \| \\
H^i(X, \mathbb{Q}) & \longrightarrow & H^i(X|_H, \mathbb{Q})
\end{array}
$$

Now, the top map is an isomorphism for $i < \dim Y - 1$ and an injection for $i = \dim Y - 1$, so we deduce the same of the bottom map. This gives a Lefschetz Hyperplane Theorem for semi-small maps.

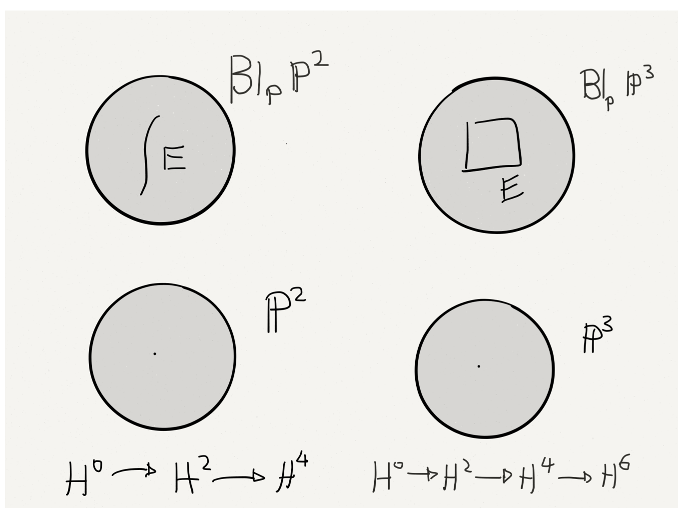
Theorem 3.9. *Let $f\colon X \to Y$ be a semi-small map where Y is quasiprojective. If H denotes a generic hyperplane section of Y and X_H the pre-image in X, then the restriction map*

$$H^i(X, \mathbb{Q}_X[\dim X]) \to H^i(X_H, \mathbb{Q}_X[\dim X])$$

is an isomorphism for $i < -1$ and an injection for $i = -1$.

Since we have just proved that semi-small maps enjoy a version of the Lefschetz Hyperplane Theorem, why not go on and ask if they satisfy a version of the Hard Lefschetz Theorem?

Example 3.10. Consider the blowup of $\mathbb{P}^2$ at a point, and the blowup of $\mathbb{P}^3$ at a point. We ask if cupping with the ith power of the hyperplane class is an isomorphism $H^{n-i} \to H^{n+i}$.



In the case of $\mathrm{Bl}_p(\mathbb{P}^2)$, the assertion is vacuous on H^2 and works on $H^0 \to H^4$, since the square of the hyperplane class is the fundamental class of the point.

For $\mathrm{Bl}_p(\mathbb{P}^3)$, cupping does not induce an isomorphism $H^2 \to H^4$ since the class of the hyperplane misses the exception divisor, so $[H] \cdot [E] = 0$.

In the example we saw that Hard Lefschetz formula held when the map was semi-small, and not failed when it wasn't. This is a general phenomenon.

Theorem 3.11. *Let $f : X \to Y$ be a semi-small map and H an ample line bundle on Y. If $L = f^*H$, then Hard Lefschetz holds for L if and only if f is semi-small.*

Proof. We argue by induction. Fix a hyperplane section $H \subset Y$ and consider the pullback:

$$\begin{array}{ccc} X_H & \hookrightarrow & X \\ \downarrow & & \downarrow \\ H & \hookrightarrow & Y. \end{array}$$

This induces

$$\begin{array}{ccc} H^{d-i}X & \xrightarrow{L^i} & H^{d+i}X \\ {\scriptstyle i^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_!}\uparrow{\scriptstyle \cong} \\ H^{d-i}X_H & \dashrightarrow & H^{d+i-2}X_H \end{array}$$

The Lefschetz Hyperplane Theorem implies that i^* and $i^!$ are isomorphisms for $i \geq 2$. Thinking in homology, the map on the left is induced by intersecting with $X_H \leftrightarrow L$, and the upper map is induced by intersecting with L^i, so the bottom horizontal map is $L|_H^{i-1}$.

$$\begin{array}{ccc} H^{d-i}X & \xrightarrow{L^i} & H^{d+i}X \\ {\scriptstyle i^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_!}\uparrow{\scriptstyle \cong} \\ H^{d-i}X_H & \xrightarrow[\cong]{L|_H^{i-1}} & H^{d+i-2}X_H \end{array}$$

Note that the map $X_H \to H$ is still semi-small for general H, because cutting with a general hyperplane changes neither the fiber dimensions nor the codimension. Therefore, by the induction hypothesis the bottom map is an isomorphism, hence so is the top.

When we reach $i = 1$, we have an interesting coincidence because the bottom two groups become the same:

$$\begin{array}{ccccc} H^{d-1}X & & \xrightarrow{\quad L \quad} & & H^{d+1}X \\ & {\scriptstyle i^*}\searrow & & \nearrow{\scriptstyle i_!} & \\ & & H^{d-1}X_H & & \end{array}$$

We want to argue that L is an isomorphism. We've exhibited as a composition of an injection and a surjection, but of course this isn't enough: for instance, why isn't the composition $i_! \circ i_*$ zero? There are two ways to see this.

One is Deligne's theorem on monodromy. Indeed, consider moving H in a pencil. This presents X as a fibration over $\mathbb{P}^1$ minus a finite set of bad points, with fibers being the hyperplane sections. If we denote by π the projection map $\{X_H\} \to \mathbb{P}^1$, then Deligne's Theorem tells us that the monodromy $R^q\pi_*$ is semisimple. Therefore, $(R^q\pi_*)^{\pi_1}$ is a direct summand.

But we know another description of $(R^q\pi_*)^{\pi_1}$: it is Im $H^q(X) \subset H^q(X|_H)$ by the global invariant cycle theorem. So we have

$$R^q\pi_* = (R^q\pi_*)^{\pi_1} \oplus \dots.$$

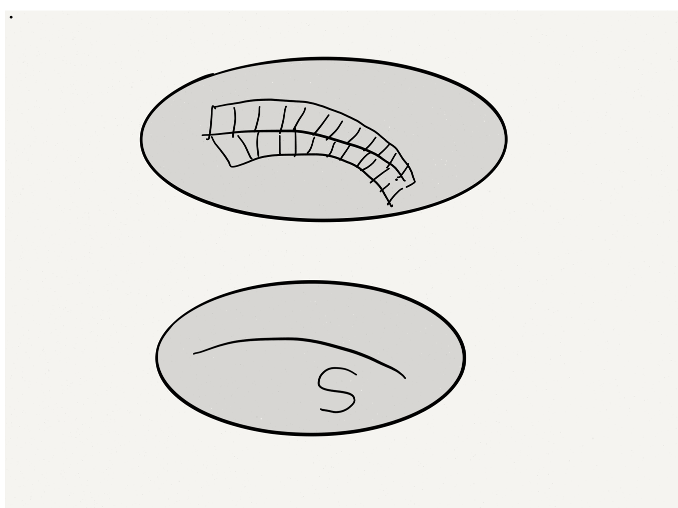
Now, consider the intersection form on $H^{d-1}X_H$, which is non-degenerate. The intersection form is compatible with this splitting, so by linear algebra the intersection form in non-degenerate on the image in $H^{d-1}(X) \hookrightarrow H^{d-1}(X_H)$. In particular, if if a_H is in the image then there is some $b \in \operatorname{Im} H^{d-1}(X) \hookrightarrow H^{d-1}(X_H)$ such that

$$0 \neq \int_{X_H} a|_{X_H} \smile b|_{X_H} = \int_X L \smile a \smile b.$$

This shows that cupping with L is injective, and its image admits a perfect pairing with $H^{d-1}(X)$ via the cup product, so it must be surjective too. □

3.3 Decomposition Theorem

Definition 3.12. Let $S_k \subset Y$. Suppose $\dim S_k + 2k \leq \dim X$. We say that S_k is *relevant* if equality holds.



Consider the higher direct image $R^{2k}f_*\mathbb{Q}_X$ with $2k = \dim X - \dim S_k$, i.e. $R^{\dim X - \dim S}f_*\mathbb{Q}_X$. By passing to a dense open subset on the target, we may assume that this sheaf is locally constant. Then

$$(R^{\dim X - \dim S}f_*\mathbb{Q}_X)_s \cong H^{2k}(f^{-1}(s))$$

and has a canonical basis (because it is top dimensional) consisting of the fundamental classes of irreducible components of dimension $2k$. (See the exercises.) In particular if S_k is not relevant, then this is 0.

Let A be the set of strata on Y and A_{rel} the relevant strata. For all $a \in A_{\text{rel}}$, we have an enriched variety $(S_a, R^{\dim X - \dim S_a} =: L^a)$.

Theorem 3.13 (Decomposition theorem for semi-small maps)**.** *If f is semi-small, then we have*

$$Rf_*\mathbb{Q}_x[\dim X] \cong \bigoplus_{a \in A_{rel}} IC_{\overline{S_a}}(L_a).$$

Remark 3.14. L_a is a semisimple representation of $\pi_1(S_a)$, which is actually a finite group. More precisely,

$$L_a = \bigoplus_{\chi \in \mathrm{Irr}(\pi_1(S_a))} L_{a,\chi} \otimes M_{a,\chi}.$$

This allows us to rewrite the conclusion of the theorem as

$$Rf_*\mathbb{Q}_x[\dim X] \cong \bigoplus_a \bigoplus_\chi IC_{\overline{S_a}}(L_{a,\chi} \otimes M_{a,\chi}).$$

This has the following important consequence.

Question. If $S, T \subset Y$ then what are maps $IC_S(L) \to IC_T(M)$?

Proposition 3.15 (Schur's Lemma for IC sheaves)**.** *If $S \neq T$, and L, M are local systems on S and T, respectively, then there are no non-zero maps $IC_S(L) \to IC_T(M)$. If $S = T$, then all such maps are induced by maps $L \to M$.*

Fact. Another property of intersection cohomology is that if $Z \subset T$ is a proper subvariety, then intersection cohomology on T has no quotient or subobjects supported on Z. (However, it could have subquotients supported on Z.)

Proof. Indeed, if we had any non-zero map $IC_S(L) \to IC_T(M)$ then the kernel or cokernel would be supported on $S \cap T$, hence 0.

If $S = T$, then by the same argument any map of IC sheaves is induced by a map $L \to M$. □

In particular, if $S = T$ and $L = M$ then we find that $\mathrm{End}(IC_S(L)) \cong \mathrm{End}(L)$. If L is simple, then $\mathrm{End}(L)$ is a $\mathbb{Q}$-division algebra.

Exercise 3.16. (Warm-up) Recall that B (the Borel subgroup of GL_2) is not semisimple. Use this to prove that for the projection map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we have $Rf_*\mathbb{Q}_{\mathbb{P}^1 \times \mathbb{P}^1} = K$ where $\mathrm{End}(K)$ is not semisimple. ♠♠♠ TONY: [TODO]

Theorem 3.13 tells us that $Rf_*\mathbb{Q}_X[\dim X]$ is semisimple. Therefore,

$$\mathrm{End}_{P(Y)}(Rf_*\mathbb{Q}_X[\dim X]) = \prod \mathrm{End}(L_{a,\chi}M_{a,\chi}).$$

Now $\mathrm{End}(L_{a,\chi})$ is a division algebra, so $\mathrm{End}(L_{a,\chi}M_{a,\chi})$ is a matrix algebra over a division algebra, hence semisimple by Artin-Wedderburn.

Here's an application. We know that the group algebra $\mathbb{Q}[W]$ is semisimple. In fact, it coincides with $\mathrm{End}(R\pi_*\mathbb{Q}_{\widetilde{N}})$, where $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is the Springer resolution. . The fact that they are the same contains the Springer correspondence (see the notes of Zhiwei Yun). The geometric interpretation is

$$\mathrm{End}(R\pi_*\mathbb{Q}_{\widetilde{N}}) \cong H^{BM}_{2\dim\mathcal{N}}(\widetilde{N} \times_N \widetilde{N}).$$

How does this work? Fix a cohomology class $\gamma \in H^{BM}_{2\dim\mathcal{N}}(\widetilde{N} \times_N \widetilde{N})$, which we want to interpret as an element of $\mathrm{End}(R\pi_*\mathbb{Q}_{\widetilde{N}})$. For $\alpha \in H^\bullet(\widetilde{N}, Q_{\widetilde{N}})$ we can pull back α to a cohomology class of $\widetilde{N} \times_N \widetilde{N}$ and then cap with γ to get another class in $H^\bullet(\widetilde{N}, Q_{\widetilde{N}})$. If we interpret this endomorphism as an element of the Weyl group algebra, then we obtain the Springer correspondence.

4 The Decomposition Theorem

4.1 Symmetries from Poincaré-Verdier Duality

Let $f\colon X \to Y$ be a proper map. Consider an intersection complex $IC_X(M)$, where M is *self-dual* local system (in the sense of representations). Then it is simple to observe that $IC_X(M)$ is also self-dual. There are many ways to see this, for instance: $IC_X(M)$ is characterized by being the unique (up to isomorphism) perverse sheaf extending M with the properties:

- $IC_X(M)|_{X^\circ} = M[\dim X]$
- $\dim \operatorname{supp} \mathcal{H}^{-i} IC_X(M) < i$ for $i \in [-\dim X + 1, \dots, -1, 0]$, and the support condition for the dual $IC_X(M)^\vee$.

Since the dual of $IC_X(M)$ also clearly satisfies the above properties, we see that $IC_X(M)$ is self-dual.

Therefore, $Rf_* IC_X(M)$ is self-dual (using that f is proper), by the functoriality of Verdier duality. The decomposition theorem tells us that

$$Rf_* IC_X(M) \cong \bigoplus_{b\in\mathbb{Z}} \bigoplus_{S\in V_b} IC_S(L)[-b]$$

Here we've conglomerated all the IC sheaves coming from local systems on S into L, so that L is semisimple (not necessarily simple). We know that the left hand side is self-dual, so the right hand side should be as well:

$$\begin{aligned} Rf_* IC_X(M)^\vee &\cong \bigoplus_{b\in\mathbb{Z}} \bigoplus_{S\in V_b} IV_S(L)[-b]^\vee \\ &\cong \left(\bigoplus_{b<0} \bigoplus_{V_b} IC_S(L)[-b]\right) \oplus \big(\bigoplus_{V_0} IC_S(L)\big) \oplus \left(\bigoplus_{b<0} \bigoplus_{V_b} IC_S(L^\vee)[b]\right) \end{aligned}$$

This presentation is called the "BBD symmetry" of the decomposition theorem.

Theorem 4.1. *Let $f\colon X \to Y$ be a proper, surjective morphism of pure relative dimension d with X smooth. Let S be a subvariety in Y appearing in the Decomposition Theorem for $Rf_*\mathbb{Q}_X[\dim X]$. Then $\operatorname{codim}_Y(S) \le d$.*

Proof. There is a largest $b \in \mathbb{Z}$, call it b_S^+, such that some local system supported on S appears in $Rf_*\mathbb{Q}_X[\dim X]$. By the BBD symmetry, this b_S^+ is non-negative (since if $IC_S(L)[-b]$ appears, then so does $IC_S(L^\vee)[b]$). So for some local system L, we have that $IC_S(L)[-b_S^+]$ is a direct summand of $Rf_*\mathbb{Q}_X[\dim X]$.

By definition, there is some dense open subset $S^\circ \subset S$ on which L is a local system. Then we can consider

$$\begin{array}{ccc} X_U & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \\ \uparrow & & \uparrow \\ S^\circ & \longrightarrow & S \end{array}$$

By proper base change, the decomposition is preserved by restriction to U. On U, we obviously have $IC_S(L)|_U \cong L$, so we have that $L[\dim S][-b_S^+]$ is a direct summand of $Rf_*\mathbb{Q}_{X_U}[\dim X]$. In particular the cohomology sheaf $\mathcal{H}^{b_s^+-\dim S} Rf_*\mathbb{Q}_{X_U}[\dim X]$ has L as a summand. By proper base change again, we have for $s \in S$

$$(\mathcal{H}^{b_s^+-\dim S} Rf_*\mathbb{Q}_{X_U}[\dim X])_s \cong H^{\dim X+b_s^+-\dim S}(f^{-1}(s))$$

and $\dim f^{-1}(s) = d$. Therefore, the index $\dim X + b_s^+ - \dim S$ is at most $2d$, so

$$\dim X - \dim S + b_S^+ \leq 2d.$$

Since b_S^+ is non-negative, and $\dim X = \dim Y + d$, we find that

$$\dim Y + d = \dim X - \dim S \leq \dim X - \dim S + b_s^+ \leq 2d.$$

□

Remark 4.2. What happens when we have equality? Then $S \supset S^\circ$ such that $R^{2d} f_*\mathbb{Q}$ has as a direct summand a sheaf supported on S. So what? The point is that the sheaf $R^{2d} f_*\mathbb{Q}$ is a sheaf of sets, whose stalks are the generated by the fundamental classes of the fibers, which is a fairly accessible description. Thus even if L has complicated monodromy, it can be realized inside a relatively simple sheaf.

If $\mathcal{M}$ is self-dual, then applying the perverse cohomology sheaves to the identity

$$Rf_! IC_X(M)^\vee \cong (Rf_* IC_X(M))^\vee$$

gives

$$\boxed{{}^{\mathrm{p}}\mathcal{H}^{-b}(Rf_! IC_X(M)) \cong {}^{\mathrm{p}}\mathcal{H}^{b}(Rf_* IC_X(M))^\vee}.$$

This is the classical expression of Poincaré-Verdier duality.

4.2 Hard Lefschetz

Let X be a smooth projective manifold and $\eta \in H^2(X, \mathbb{C})$ the class of an ample divisor. Then Poincaré duality says that

$$H^{d-i}(X) \cong (H^{d+i}(X))^\vee$$

and Hard Lefschetz says that

$$H^{d-i}(X) \xrightarrow{\eta^i} H^{d+i}(X).$$

These offers the same identification of Betti numbers, but the second theorem tells us more: by the primitive decomposition, it implies unimodality of the Betti numbers (i.e. they increase monotonically until half the dimension, and then decrease monotonically).

Now suppose $f\colon X \to Y$ is a projective morphism and $\eta \in H^2(X, \mathbb{Q})$ is the first Chern class of a line bundle $\mathcal{N}$ on X which is ample on the fibers of f. Then η corresponds to a map

$$\mathbb{Q}_X \xrightarrow{\eta} \mathbb{Q}_X[2].$$

Tensoring with K, we get a map

$$K \xrightarrow{\eta} K[2].$$

That induces $Rf_*K \xrightarrow{\eta} Rf_*K[2]$, hence ${}^{\mathrm{p}}H^i(Rf_*K) \xrightarrow{\eta} {}^{\mathrm{p}}H^{i+2}(Rf_*K)$. Iterating, we get

$${}^{\mathrm{p}}\mathcal{H}^{-b}(Rf_*K) \xrightarrow{\eta^b} {}^{\mathrm{p}}\mathcal{H}^{b}(Rf_*K) \text{ for all } b \geq 0.$$

Theorem 4.3 (Relative Hard Lefschetz)**.** *The map*

$${}^{\mathrm{p}}\mathcal{H}^{-b}(Rf_*) \xrightarrow{\eta^b} {}^{\mathrm{p}}\mathcal{H}^{b}(Rf_*)$$

is an isomorphism for all $b \geq 0$.

Taking f to be the projection to a point, we obtain the Hard Lefschetz Theorem for $IH^\bullet(X, \mathbb{Q})$ for projective X.

Surprise application. There is a surprising application of intersection cohomology to a conjecture of McMullen. To a rational, simplicial polytope P in $\mathbb{R}^d$ one can associate a *face vector* $f(P) = (f_0, f_1, \ldots, f_{d-1})$, and to a face vector one can associate a corresponding h-vector $h(P) = (h_0, \ldots, h_d)$.

McMullen made a conjecture concerning which vectors can be the face vectors of a polytope. He conjectured that necessary and sufficient properties for an f-vector to be the f-vector associated to a polytope were that the corresponding h-vector satisfied $h_i = h_{d-i}$, and unimodality.

For any such vector, one can conjecture that there is a projective variety with these Betti numbers. It was proved by geometric methods that the conditions were sufficient. For the necessity, from P we can construct a simplicial, projective toric variety X_P. Then Stanley showed that this variety has $IC_{X_P} = \mathbb{Q}_{X_P}[\dim X_P]$, and that the h-vector is the vector of Betti numbers of X_P. This completes the proof of McMullen's conjecture.

4.3 An Important Theorem

Theorem 4.4. *Let $f\colon X \to Y$ be a proper morphism. Then $\mathcal{IC}_{f(X)}$ is a direct summand of $Rf_*\mathcal{IC}_X$.*

An important special case is when f is a resolution of singularities, where it says that $IH^\bullet(Y)$ is a direct summand of $H^\bullet(X)$.

Remark 4.5. The direct sum decomposition is *not* canonical. However, there is something canonical here, which we try to explain. Recall that we have

$$\tau_{\leq i}K \to K$$

inducing $H^\bullet(Y, \tau_{\leq i}K)) \subset H^\bullet(Y, K)$. You can do the same thing with perverse cohomology:

$$^{\mathrm{p}}\tau_{\leq i}K \to K$$

indces $H^\bullet(Y, \tau_{\leq i}K)) \hookrightarrow H(Y, K)$.

There is a perverse filtration P on $H^\bullet(K)$. If $K = Rf_*C$, then P is called the *perverse Leray* filtration (in analogy with the usual Leray filtration).

The decomposition theorem tells you the associated graded of the perverse filtration :

$$P_{i+1}/P_i = H^\bullet(Y, {}^{\mathrm{p}}H^i(Rf_*IC_X)).$$

The non-canonical nature of the direct sum decomposition is precisely the non-canonical nature of the vector space isomorphism between a filtered vector space and its associated graded.

This means that the intersection cohomology downstairs isn't canonically a summand of the cohomology upstairs, but a *subquotient*.

Proof. First reduction: by replacing Y by $f(X)$, we may assume that f is surjective. Then we need to show that $\mathcal{IC}_Y$ is a direct summand of $Rf_*\mathcal{IC}_X$.

There are three ingredients:

Step 1: *IC localization principle.* Consider $IC_S(L)$. We claim that for all open subset $U \subset Y$ meeting S, we have

$$IC_{S\cap U}(L|_{S^\circ \cap U}) \cong IC_S(L)|_U.$$

The content here is a theorem that if L is a local system on $S^\circ \supset (S^\circ)'$ and $L' = L|_{(S^\circ)'}$ then $IC_S(L) \cong IC_S(L')$. This is *not a tautology*, but it does follow immediately from the characterization of IC sheaves that we gave at the beginning of the section.

Step 2: *IC normalization principle.* If $\widehat{X} \to X$ is a normalization, then $\nu_*IC_{\widehat{X}} = IC_X$.

Since ν is finite, $R\nu_* = \nu_*$. This follows again from the conditions of support and by restricting the morphism to an open dense set where it is an isomorphism.

Step 3: *Decomposition Theorem localization principle.* $IC_S(L)$ appears as a summand in the decomposition theorem if and only if $IC_S(L)|_U$ appears on some U meeting S in the decomposition theorem applied to $f^{-1}(U) \to U$.

This is obvious from the decomposition theorem (including its uniqueness), but without it is very deep. (Go back to the example of the Hopf surface to see that it fails without delicate hypotheses.)

Now we commence the proof. By the Decomposition Theorem,

$$Rf_*\mathcal{I}C_X \cong \bigoplus_{q\geq 0} \bigoplus_{(S,L)\in\mathcal{E}_q} \mathcal{I}C_S(L)[-q].$$

Let's throw away all the proper closed subsets of Y, since they don't matter for our purpose. Replacing Y with the complement of all proper subvarieties S appearing in the decomposition theorem, we have

$$Rf_*\mathcal{I}C_X \cong \bigoplus_{q\geq 0} \bigoplus_{L} \mathcal{I}C_Y(L)[-q].$$

Shrinking Y again, we may assume that each $\mathcal{I}C_Y(L)$ is even a local system. So then

$$Rf_*\mathcal{I}C_X \cong \bigoplus_{q} L^q[-q].$$

In particular, we have that $\mathcal{I}C_Y = \mathbb{Q}_Y$. Now we can replace X by the normalization,

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\nu} & X \\ & {\scriptstyle\hat{f}}\searrow & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

since $R\hat{f}_*\mathcal{I}C_{\widehat{X}} = Rf_*R\nu_*\mathcal{I}C_{\widehat{X}} = Rf_*\mathcal{I}C_X$. This allows us to assume that X is normal, hence that $\mathbb{Q}_X \xrightarrow{\sim} \mathcal{H}^0\mathcal{I}C_X$ is an isomorphism (the normalization is necessary because this will fail if X has branches - consider the normalization of a nodal curve).

We always have a map $\mathbb{Q}_X \to \mathcal{I}C_X$, but because $Q_X \cong \mathcal{H}^0(\mathcal{I}C_X)$ we get that

$$\mathbb{Q}_X \to \mathcal{I}C_X \to \tau_{\geq 1}\mathcal{I}C_X$$

is a distinguished triangle, so pushing it down gives

$$Rf_*\mathbb{Q}_X \to Rf_*\mathcal{I}C_X \to Rf_*\tau_{\geq 1}\mathcal{I}C_X,$$

hence $R^0f_*\mathbb{Q}_X \cong \mathcal{H}^0Rf_*\mathcal{I}C_X$. This reduces to showing that $\mathbb{Q}_Y$ is a direct summand of $Rf_0\mathcal{I}C_X \cong R^0f_*\mathbb{Q}_X$. That is nice, because we can get a handle on $f_*\mathbb{Q}_X$.

Now Stein factorize $X \to Y$ into

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

where h is finite and g has connected fibers. This reduces to the case where f is finite or f has connected fibers. When f has connected fibers, we're done because then $(R^0 f_* \mathbb{Q}_X)_s \cong H^0(f^{-1}(s), \mathbb{Q}_X) \cong \mathbb{Q}_X$, which shows that $R^0 f_* \mathbb{Q}_X \cong \mathbb{Q}_Y$.

Finally, we consider the case where f is finite. We want to show that $f_* \mathbb{Q}_X$ has $\mathbb{Q}_Y$ as a direct summand. But if we can shrink again until $Z \to Y$ is a topological covering, this will be obvious. In characteristic 0, this is easy. In characteristic p, factor $Z \to Y$ into inseparable and étale parts, and notice that the inseparable part is irrelevant. The étale part then follows from a standard trace argument. □

5 The Perverse (Leray) Filtration

5.1 The classical Leray filtration.

Let $\pi\colon E \to B$ be a fiber bundle. Suppose we have a filtration on the base, perhaps coming from a CW complex structure. If B_p is the pth skeleton, and the filtration is

$$\ldots \leq B_p \leq B_{p+1} \leq \ldots$$

such that $B_p/B_{p-1} = \bigwedge S^p$, then we get a spectral sequence $H^\bullet(B_p, B_{p-1}) \implies H^\bullet(B)$. The differentials all vanish, so this actually degenerates to a complex, which is the familiar cellular complex.

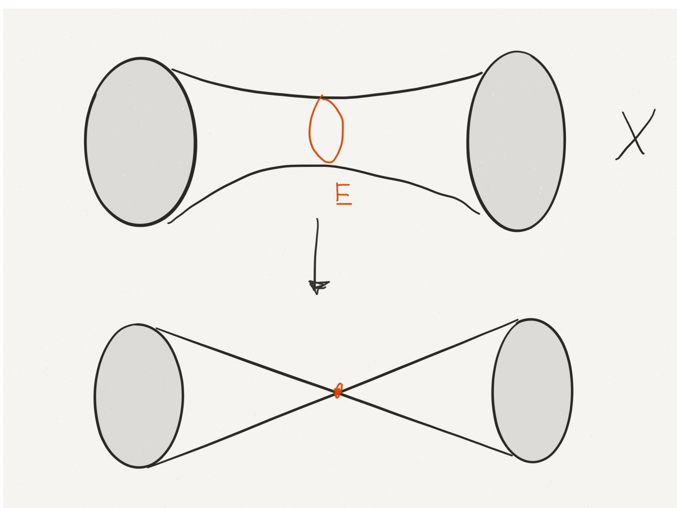
This is all more interesting if you pull the filtration back to the total space. Then you get an honest spectral sequence $H^\bullet(E_p, E_{p-1}) \implies H^\bullet(E)$. On the first page, one gets $H^\bullet(B_p, B_{p-1}) \otimes H^\bullet(F)$. On the second page one gets $H^\bullet(B, \underline{H}(F))$, which is the Leray spectral sequence.

The point is that you can show that the Leray filtration $L_? H^\bullet(E)$ can be described as the kernel of the restriction maps $H^\bullet E \to H^\bullet E_?$. We won't figure out the indices explicitly. We just want to emphasize the story of how the filtration arose: by finding a filtration on the base, which lifted to a filtration on the total space.

5.2 The perverse filtration

We want to replicate this picture in the context of algebraic geometry and perverse sheaves.

Example 5.1. Let Y be the cone over $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ and X be the big blow up, so its exceptional fiber is $\mathbb{P}^1 \times \mathbb{P}^1$.



Then $Rf_*\mathbb{Q}_X \cong \mathbb{Q}_v[-2] \oplus \mathcal{IC}_Y \oplus \mathbb{Q}_v[-4]$. The first $\mathbb{Q}_X[-2]$ can be thought of the fundamental class of the exceptional divisor, and the other skyscraper is the H^4 of the exceptional divisor.

From an exercise on the problem sheets, we know that $H^2 X \cong \mathbb{Q}$, generated by the primitive second cohomology in $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Remark 5.2. This was all in non-perverse notation. To make things perverse, we would have to shift by $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y = 3$. Recall that $IC_Y[-3] = \mathcal{I}C_Y$.

So we have by the decomposition theorem,

$$Rf_* \mathcal{I}C_X \cong \bigoplus {}^{\mathrm{p}}\mathcal{H}^b[-b].$$

In Example 5.1 above we had $b = 2, 3, 4$.

From the truncation functors ${}^{\mathrm{p}}\tau_{\leq i}$, for all K on X we get a perverse Leray filtration on $P_\bullet \subset H^\bullet(X, K)$ by taking the images of $H^\bullet(Y, {}^{\mathrm{p}}\tau_{\leq i} Rf_* K) \subset H^\bullet(X, K)$.

Definition 5.3. If $K := Rf_* \mathcal{I}C_X$, and φ is a choice of decomposition then we define the *perverse Leray filtration* by

$$P_b H^\bullet(X, K) = \varphi\left(\bigoplus_{b' \leq b} H^{\bullet - b'}(Y, {}^{\mathrm{p}}\mathcal{H}^{b'}(Rf_* K))\right).$$

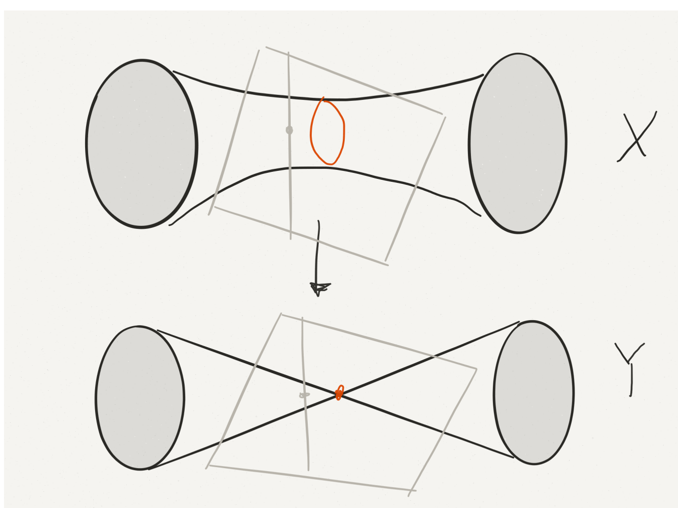
Although φ is not natural, *the filtration is*.

Going back to Example 5.1, the perverse filtration in degree 2 is morally given by $\mathbb{Q}_v[-2]$, and in degree 3 by $\mathbb{Q}_v[-2] \oplus \mathcal{I}C_Y$.

	2	3	4
$Rf_* \mathbb{Q}_X$	$\mathbb{Q}_v[-2]$	$\mathcal{I}C_Y$	$\mathbb{Q}_v[-4]$

How do we actually describe the filtration intrinsically? In the classical Leray filtration, we lifted a filtration on the base to a filtration on the total space. Can we try to do something similar here?

Example 5.4. Consider as in Example 5.1 Y = the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{C}^4$.



Choose general $\mathbb{A}^3, \mathbb{A}^2, \mathbb{A}^1, \mathbb{A}^0$ inside $\mathbb{A}^4$, and filter by cutting:

$$(Y = Y_0) \supset (Y_1 = Y \cap \mathbb{A}^3) \supset (Y_2 = Y \cap A^2) \supset (Y \cap \mathbb{A}^1) \supset Y_4 = \emptyset.$$

Now let's try pulling this all back to X, so we get a filtration

$$X = X_0 \supset X_1 \supset X_2 \supset X_3 \supset X_4 = \emptyset.$$

We know that $P^2 H^2(X) \cong H^2(Y, \mathbb{Q}_v[-2])$ and $P^3 H^2(X) \cong H^2(Y, \mathbb{Q}_v[-2]) \oplus H^2(Y, \mathcal{IC}_Y)$.

We want to describe $P_3 H^2(X)$ as the kernel of some map, which must be the zero map. Looking above at the H^2 of the filtration, we see that ♠♠♠ TONY: [TODO: verify this]

	X_0	X_1	X_2	X_3	X_4
$H^2(X_i)$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$	0	0	0

So if we want $P_3 H^2(X)$ to be realized as the kernel of a restriction to some filtered piece, a natural choice is X_2: we set

$$P_3 H^2(X) := \ker(H^2(X) \to H^2(X_2)).$$

This suggests a guess that $P_b H^2(X) = \ker(H^2(X) \to H^2(X_{b-2+1}))$. Let's check this in the next case: is $H^2(Y, \mathbb{Q}_v[-2]) \cong P_2 H^2(X) = \ker H^2(X) \to H^2(X_1)$? We claim so. Indeed, this restriction map may be identified with

$$H^2(Y, \mathbb{Q}_v[-2]) \oplus H^2(Y, \mathcal{IC}_Y) \to H^2(Y_1, \mathbb{Q}_v[-2]) \oplus H^2(Y_1, \mathcal{IC}|_{Y_1}).$$

The map on first factors is 0, because Y_1 is the intersection with a general hyperplane, which doesn't even meet v. The map on second factors is injective by the Lefschetz Hyperplane Theorem. So the kernel is indeed identified with $H^2(Y, \mathbb{Q}_v[-2])$.

We can extrapolate the general formula from this example:

$$\boxed{P_b H^\bullet = \ker(H^\bullet \to H^\bullet|_{X_{b-\bullet+1}}).}$$

Theorem 5.5. *Let Y be affine in $\mathbb{A}^N$ (or more generally, quasiprojective) and $f \colon X \to Y$ be **any** map. If $K \in D(X)$, then for all general flags $Y_\bullet$ of linear sections of Y, we have*

$$P_b H^\bullet(X, K) = \ker(H^\bullet(X, K) \to H^\bullet(X_{b-\bullet+1}, K|_{X_{b-\bullet+1}}).$$

Remark 5.6. We do *not* assume that f is proper or that K splits, and the result is non-trivial even if f is the identity morphism!

Proof sketch in a special case. Suppose $K = \bigoplus Q_b[-b]$ where $Q_b \in P(Y)$. Let $Y_\bullet \subset Y$ be a filtration on some $Y^n \subset \mathbb{A}^N$. Then

$$H^\bullet(Y, C) \cong \bigoplus_b H^{\bullet-b}(Y, Q_b).$$

Consider the filtered piece P_b, which is the image of the boxed summands below:

$$\boxed{\ldots \oplus H^{\bullet-b+1}(Y, Q_{b-1}) \oplus H^{\bullet-b}(Y, Q_b)} \oplus H^{\bullet-b-1}(Y, Q_{b+1}) \oplus \ldots \tag{1}$$

We want to show that P_b can be distinguished by the Artin Vanishing theorem and the Lefschetz Hyperplane Theorem. We have a restriction map from(1) to the cohomology of Y_K:

$$\boxed{\ldots \oplus H^{\bullet-b+1}(Y_k, Q_{b-1}|_{Y_k}) \oplus H^{\bullet-b}(Y_k, \mathbb{Q}_b|_{Y_k})} \oplus H^{\bullet-b-1}(Y_k, Q_{b+1}|_{Y_k}) \oplus \ldots$$

The Artin Vanishing theorem says that the restriction map kills the boxed groups to the left of a certain k. If we knew that $Q_{b+1}|_{Y_k}$ were perverse, then the Lefschetz hyperplane theorem implies the restriction map is injective on the groups right of a certain k. It turns out that the correct k is precisely $-\bullet+b+1$.

Thus, the proof will be completed if we can that the restriction of a perverse sheaf to a general hyperplane section is still perverse. For Q on Y, and a $H \subset Y$, does $Q|_H[-1]$ satisfy the conditions of support? It turns out that $Q|_H[-1]$ satisfies conditions of support in general, but not the conditions of cosupport. However, if H is general then you *do* get the conditions of cosupport. (This isn't easy to see.) Iterating, we find that for general choices $Q|_{Y_k}[-k] \in P(Y_k)$.

Remark 5.7. It is always true that the restriction of a *simple* perverse sheaf (admittedly a very strong condition, which doesn't apply here) to any divisor is perverse.

We want to show that restriction to $H^{\bullet-b+\epsilon}(Y_{b-\bullet+1}, \mathbb{Q}_Y)$ is 0 if $\epsilon \geq 0$. But this is just the cohomology of $\mathbb{Q}_{Y_{b-\bullet+1}}[\bullet - b + \epsilon][b - \bullet + 1]$, so Artin Vanishing implies that this whole group is 0 for any $\epsilon \geq 0$.

□

5.3 Hodge-theoretic implications

For compactly supported cohomology, there is a dual story.

Corollary 5.8. *For Y quasiprojective and $f\colon X \to Y$ any map, the spectral sequence*

$$E_2^{p,q} = H^p(Y, {}^{\mathfrak{p}}R^q f_* \mathbb{Q}) \implies H^{p+q}(X, \mathbb{Q})$$

is a spectral sequence in the category of mixed Hodge structures.

Then $P_b H(X, \mathbb{Q})$ has a sub-mixed Hodge structure, hence so does the quotient $P_b H(X, \mathbb{Q})/P_{b-1} H(X, \mathbb{Q})$. If f is proper and X is smooth, then this quotient is precisely

$$P_b H(X, \mathbb{Q})/P_{b-1} H(X, \mathbb{Q}) \cong H^\bullet(Y, {}^{\mathfrak{p}}\mathcal{H}^b) = \bigoplus IH^\bullet(S, L).$$

Since the left hand side has a mixed Hodge structure, a natural question if if we can put a mixed Hodge structure on each $IH^\bullet(S, L)$ such that this is even an isomorphism of mixed Hodge structures.

Said differently, can we choose $\varphi\colon Rf_*\mathcal{IC}_X \cong \bigoplus \mathcal{IC}_S(L)$ such that $\varphi\colon IH(S,L) \to IH(X)$ is a map of mixed Hodge structures? The answer is yes.

Let's go to the situation of the decomposition theorem: let $f\colon X \to Y$ be proper, so we know that there exists φ such that

$$\varphi\colon IH^\bullet(X) \cong \bigoplus_{q>0} \bigoplus_{(S,L)} IH^{\bullet-q}(S, L_q) \text{ of mixed Hodge structures.}$$

So we can form a projection map

$$\begin{array}{ccc} IH^\bullet(X) & \xrightarrow{\quad\pi\quad} & IH^\bullet(X) \\ & \searrow \quad\quad \nearrow & \\ & \varphi IH(S, L_Q) & \end{array}$$

Since X is smooth projective, the map $\pi\colon H^\bullet X \to H^\bullet X$ is equivalent to a cohomology class $\pi \in H^{2\dim X}(X \times X)$. Since the maps are compatible with the Hodge structures, furthermore know that π is a Hodge (n,n) class.

According to the Hodge conjecture, any such class is algebraic. We might ask if we can see this in this special construction. However, that appears to be very hard.

Remark 5.9. If $X \to Y$ is the resolution of a cone over a projective manifold, then one of these classes π being algebraic implies the Grothendieck standard conjecture of Lefschetz type. However:

1. The answer is known for semi-small maps.

2. Pick an automorphism $\sigma\colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ which induces $X^\sigma \xrightarrow{f^\sigma} Y^\sigma$, and also $H^\bullet(X \times X) \xrightarrow{\sim} X^\bullet(X^\sigma \times X^\sigma)$. This destroys the $\mathbb{Q}$-structure and the p, q decomposition.

   Deligne has introduced a notion of "absolute Hodge" classes which remain Hodge under Galois. One can ask if our classes π are absolute Hodge. One can show that φ can be chosen so that $\pi \mapsto \pi^\sigma$, so it is the case that π is absolute Hodge.