The Riemann-Hilbert correspondence

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Introduction

The 21st of Hilbert's problems was about finding a *proof of the existence of linear differential equations having a prescribed monodromic group.* The problem itself was solved very early (a first solution appeared in 1908), but in recent times the theory of D-modules brought attention to the topic again, and provided a generalization to every dimension and in the derived context, the Riemann-Hilbert correspondence. This amazing result connecting algebraic and analytic geometry was proved independently by Kashiwara ([3]) and Mebkhout ([4]) in 1980, and it is stated in terms of an equivalence of categories between derived categories of regular holonomic D-modules and of constructible sheaves. In this minor thesis, we will define all such objects, and give a sketch of the proof of the correspondence in the case of an algebraic variety, illustrating everything with some examples (but missing many proofs on the way). Notations and structure will follow those from [2].

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1 The theory of algebraic D-modules

In this section we will describe the theory of D-modules on a smooth algebraic variety X, following step by step the theory of coherent sheaves. In that case, the central role is played by the sheaf of commutative rings $\mathcal{O}_X$; let's define the main object in this case, the sheaf of differential operators.

1.1 Definitions and examples

Definition 1.1. *On a smooth algebraic variety X, the sheaf of differential operators is the sheaf of (noncommutative) graded rings D_X generated inside $End_{\mathbb{C}_X}(O_X(U))$ by $\mathcal{O}_X$ and the sheaf of derivations (that is, the tangent sheaf).*

This sheaf is quasicoherent, and there is a natural grading on it, given by the filtration

$$\mathcal{O}_X = F_0 D_X \subseteq F_1 D_X \subseteq \ldots \subseteq F_n D_X \subseteq \ldots \subset D_X$$

where $F_k D_X$ has sections ξ such that $[f_k, \dots, [f_1, [f_0, \xi]]] = 0$ where $f_0, \dots, f_k \in \mathcal{O}_X$. The associate graded sheaf of commutative rings grD_X is canonically isomorphic to $\pi_* \mathcal{O}_{T^*X}$, for π the projection $T^*X \to X$ from the cotangent bundle of X.

Example 1.2. If $X = \mathbb{A}^n_{x_i}$, the sheaf of derivation (the tangent sheaf) is free of rank n, generated by n sections $\partial_1, \partial_2, \dots, \partial_n$; as $\mathcal{O}_X$-module, so, D_X is generated by monomials in $\partial_1, \partial_2, \dots, \partial_n$; its global sections are given by the ring

$$\mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$$

with commutation relationships $[x_i, x_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$ and $[\partial_i, \partial_j] = 0$. We can see (and this is a general fact) that the filtration is just given by the order of differential operators; the graded module has space of global section the commutative ring

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$$

that is, exactly coming from $\mathbb{A}^{2n}_{x_i, y_i}$, the total space of T^*X.

Definition 1.3. *A left (resp. right) D-module is a sheaf M of left (right) D_X-modules; in fact, it is sufficient to specify the action of $\mathcal{O}_X$ and Θ_X. We will denote by $Mod(D_X)$ the category of left D-modules.*

Key example 1.4. Let ξ be a differential operator, and let's consider the left D-module $D_X/(D_X \cdot \xi)$; in this case, the sheaf $\mathcal{H}om_{D_X}(D_X/(D_X \cdot \xi), \mathcal{O}_X)$ is the sheaf of solutions of the differential equation $\xi f = 0$ as sections of $\mathcal{O}_X$. Riemann-Hilbert correspondence is just a generalization of this fact; there is a "solution functor" that serves as an equivalence of categories between "sufficiently nice" D-modules and some "sheaves of solutions".

Key example 1.5. If a D-module M is locally free of finite rank (as $\mathcal{O}_X$-module), we will call it *integrable connection*, and the category of such objects will be called $Conn(X)$; indeed, this is precisely the algebraic counterpart of vector bundles with a flat connection. In this case, we can describe explicitly the Riemann-Hilbert correspondence; given an integrable connection M, we can construct a sheaf of vector spaces, also called *local system*, corresponding to *horizontal* or *parallel sections*, that are, sections on which the connection vanishes; conversely, given such a sheaf L (that is *not* an $\mathcal{O}_X$-module, just of $\mathbb{C}$-modules), we can get an integrable connection considering $\mathcal{O}_X \otimes_{\mathbb{C}_X} L$, that inherits a natural D_X-module structure; this two functors gives us the equivalence between $Conn(X)$ and $Loc(X)$, the category of local systems (this is indeed not entirely correct; the definitions and a precise statement will follow in chapter 3).

Remark 1.6. As the structure sheaf $\mathcal{O}_X$ has a natural structure of left D-module, the canonical line bundle $\Omega_X = \bigwedge^{dim(X)} T^*X$ has a natural structure of right D-module; in this way, if M is a left D-module we have a right module structure of $\Omega_X \otimes_{\mathcal{O}_X} M$; this gives in fact an equivalence of categories between left and right D-modules.

Definition 1.7. *A* ***coherent*** *D-module is a D-module that is finitely generated over D_X; a* ***quasi coherent*** *D-module is one that is quasi-coherent as $\mathcal{O}_X$-module (that is, a quasi coherent sheaf). We will denote by $Mod_c(D_X)$ (resp. $Mod_{qc}(D_X)$) the category of coherent (resp. quasi-coherent) D-modules, and by $D^b_c(D_X)$ (resp. $D^b_{qc}(D_X)$) the derived category of bounded complexes having as cohomology coherent D-modules (resp. quasi-coherent) D-modules.*

We will see that coherent D-modules behave, in some sense, better that coherent sheaves.

1.2 Functors

Given a map between smooth algebraic varieties $f : X \to Y$, we will define functors of inverse image and direct image of D-modules; a little bit of attention between left and right modules should here be paid. Let's first see what happens to the sheaf of differentials.

Definition 1.8. *We will denote by $D_{X\to Y}$ the sheaf*

$$f^*D_Y = O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y.$$

This is a left D_X module, and a right $f^{-1}D_Y$ module.

This double module structure allows us to pass from a D_Y-module structure to a D_X-module structure; in fact, this module is called transfer module. As an example, if $X \to Y$ is a closed embeding, then $D_{X\to Y}$ is the D_X-module containing $\mathcal{O}_X$ and formal derivatives in the normal directions in Y.

Definition 1.9. *Given $M \in Mod(D_X)$, his inverse image is defined as $f^*M = D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}M$.*

The left D_X-module structure on f^*M is given by the left action on $D_{X\to Y}$, and the tensor product "cancels" out the two (right and left) $f^{-1}D_Y$ actions.

Proposition 1.10. *This functor is right exact, and his derived functor Lf^* sends $D^b_{qc}(D_Y)$ to $D^b_{qc}(D_X)$. Moreover, we will call*

$$f^\dagger = Lf^*[dimX - dimY].$$

Lf^* does not send $D^b_c(D_Y)$ to $D^b_c(D_X)$, as for instance happens for closed embeddings; the derivatives in the normal directions gives here an infinite set of generators, so in this case $D_{X\to Y} = f^*D_Y$ is not D_X-coherent.

To define direct image, a little more work is needed. The main issue here is that direct image is more naturally defined as functor between right modules; a reason for this can be found thinking at what happens in the C^∞ setting: functions (a natural left C^∞-module) can be pulled back, distributions (a natural right C^∞-module) can be pushed forward integrating along fibers; this will be reflected in the notation we will use for direct image. What we need to do for a left module is to reduce to the right module case, taking the natural direct image, and then going back to a left module, using remark 1.6.

Definition 1.11. *If M is a right D_X module, then his direct image is $f_*(M \otimes_{D_X} D_{X\to Y})$.*

If M is a left D_X module, then his direct image is

$$\int_f M = f_*((\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y}) \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}.$$

Calling $D_{Y\leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$, and using the projection formula, we get finally

$$\int_f M = f_*(D_{Y\leftarrow X} \otimes_{D_X} M)$$

If $M^\cdot$ is a complex, then we define

$$\int_f M^\cdot = Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M)$$

Example 1.12. If $f : \{0\} \to \mathbb{A}^1_x$, let's consider $\int_f \mathcal{O}_{\{0\}}$; its direct image can be seen as a distribution supported in the point, and is the left D-module $D_{\mathbb{A}^1}/(D_{\mathbb{A}^1} \cdot x)$; in fact, the formal solution in distribution theory to the equation $x \cdot f = 0$ (it would be better to express it as $f \cdot x = 0$) is Dirac's delta δ_0.

Key example 1.13. Let $i : Z \to X$ be a closed embedding; then the functors $i^\dagger$ and $\int_i$ are inverse one each other, giving an equivalence of categories between $D_h^b(D_X, Z)$ of D_X-modules supported in Z, and $D_h^b(D_Z)$: this is called **Kashiwara equivalence**. This allows us also to define D-modules on singular algebraic varieties, after an embedding in smooth ones, as D-modules with constrained support.

The proof of the following fact is surprisingly nontrivial.

Proposition 1.14. *$\int_f$ sends $D_{qc}^b(D_Y)$ to $D_{qc}^b(D_X)$; if f is proper, then $\int_f$ sends $D_c^b(D_Y)$ to $D_c^b(D_X)$.*

Last functor in this section will be duality. A duality functor is not really present in the theory of coherent sheaves; it would be useful as part of a theory including also a functor $f_!$ of direct image with compact support, to get a Verdier duality type statement $f_! = \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X$. The problem is that in the category of $\mathcal{O}$-modules the functor f_* maps coherent sheaves into quasicoherent sheaves, and duality is not defined for quasicoherent sheaves, and we would not be able to compose $\mathbb{D}_Y$ to $f_!$. In the case of D-modules, however, we do have a partial result in this direction: we just saw that if f is proper f_* does preserve coherent objects in this category. We will see in next section that refining more the condition on D-modules (considering holonomic ones instead of just coherent) we will get all of this straight.

Definition 1.15. *Let $M^\cdot$ is an element of $D^-(D_X)$, then its dual is the element of $D^+(D_X)$ defined as*

$$\mathbb{D}M^\cdot = R\mathcal{H}om_{D_X}(M^\cdot, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X].$$

The reason for the shifting relies again in the theory of holonomic D-modules, that we are about to present. The reason of the tensor product by Ω_X^{-1} is, of course, because otherwise we land in right D-modules; the left module structures on $M^\cdot$ and D_X get canceled out by the $R\mathcal{H}om$ functor, so the only module structure remaining is the right one on D_X. The following proposition is now not totally unexpected.

Proposition 1.16. *Duality functor preserves $D_c^b(D_X)$, and on $D_c^b(D_X)$ we have $\mathbb{D}^2 = Id$.*

The following is a tecnical fact that we will need in the future, that relates the derived homomorphism functor to the dual just defined.

Lemma 1.17. *We have the following isomorphisms*

$$R\mathcal{H}om_{D_X}(M^{\cdot}, N^{\cdot}) \cong ((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} N^{\cdot})[-d_x].$$

Note that on this sheaf is not anymore defined any D_X-module structure or $\mathcal{O}_X$-module structure, because they get canceled out; this is in fact just a sheaf of vector spaces. This will come out again in chapter 3.

1.3 Holonomic D-modules

The definitions in this subsection are fundamental in order to achieve Riemann-Hilbert correspondence. First, we need to define a subclass of objects, that are, in some sense, those on which it makes more sense to consider their "solutions", that are, holonomic D-modules. To define them we first need to introduce the notion of singular support.

Definition 1.18. *Given M a coherent D-module, we can consider its graded module grM; this is naturally a coherent sheaf over the total space of the cotangent bundle T^*X. Its support is called the* ***singular support*** *$SS(M)$, a subvariety of T^*X. A D-module is called* ***holonomic*** *if the dimension of all irreducible components of $SS(M)$ is the same as the dimension of X, or if $SS(M)$ is empty. We will call $Mod_h(X)$ the category of holonomic D-modules, and $D^b_h(D_X)$ the category of complexes having as cohomology holonomic modules.*

Example 1.19. If M is an integrable connection, then $SS(M)$ is just the zero section of T^*X (potentially with a nonreduced structure), that is, X, so it is holonomic.

Example 1.20. For the D-module $D_{\mathbb{A}^1}/(D_{\mathbb{A}^1} \cdot x)$ from example 1.12, the singular support is the whole fiber of $T^*\mathbb{A}^1_{\mathbb{C}}$ over the origin; the D-module is then holonomic.

Example 1.21. If $Z \xrightarrow{i} X$ is a closed embedding and M is an integrable connection on Z, then $\int_i M$ is an holonomic D-module; in fact, its singular support is Z and his whole conormal bundle inside T^*X; the dimension is then $d_Z + (d_X - d_Z) = d_X$.

One fundamental property of holonomic D-modules is that they are a "very stable" subcategory.

Proposition 1.22. *Direct and inverse image preserve $Mod_h(D_X)$ and $D^b_h(D_X)$. Duality preserves $D^b_h(D_X)$, and in particular the dual complex of an holonomic D-module has only cohomology in degree 0, that is another holonomic D-module.*

Note that the last statement of this proposition is the reason for which we added the shifting $[d_X]$ to the definition of dual D-module; in this way, the dual is a functor $Mod_h(D_X) \to Mod_h(D_X)$.

Example 1.23. We will show an example in which holonomic D-modules behave better than coherent sheaves. Consider the open embedding

$$U = \mathbb{A}^1 \setminus \{0\} \xrightarrow{j} \mathbb{A}^1 = X,$$

and let's consider $\int_j \mathcal{O}_U$. Given that $D_{X \leftarrow U} = D_U$, we have that the direct image as D-module correspond, as a sheaf, to the direct image as coherent $\mathcal{O}$-module, that is, $\mathcal{O}_X[x^{-1}]$; now, this is not anymore a coherent $\mathcal{O}$-module, but it is a coherent D-module; in fact, using notation of the example 1.4, this module is the cyclic left D-module $D_X \cdot x^{-1} \cong D_X/(D_X \cdot \partial_x x)$; one can see it by the fact that the function x^{-1} is actually killed by the operator $\partial_x x$.

Given this, we actually are in a better setting than coherent sheaves; we can give two more straight definitions.

Definition 1.24. *Let $f : X \to Y$ a morphism of smooth algebraic varieties; we define functors*

$$\int_{f^!} = \mathbb{D}_Y \int_f \mathbb{D}_X : D_h^b(D_X) \to D_h^b(D_Y)$$

$$f^\star = \mathbb{D}_X f^\dagger \mathbb{D}_Y : D_h^b(D_Y) \to D_h^b(D_X)$$

These functors provide adjoint functors to $f^\dagger$ and $\int_f$.

Proposition 1.25. *For $M^\cdot \in D_h^b(D_X)$ and $N^\cdot \in D_h^b(D_Y)$, we have natural isomorphism*

$$R\mathcal{H}om_{D_Y}\left(\int_{f^!} M^\cdot, N^\cdot\right) \xrightarrow{\sim} Rf_* R\mathcal{H}om_{D_X}(M^\cdot, f^\dagger N^\cdot)$$

$$Rf_* R\mathcal{H}om_{D_X}(f^\star N^\cdot, M^\cdot) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}\left(N^\cdot, \int_f M^\cdot\right).$$

Proof. We will prove this fact, because the proof of Riemann-Hilbert correspondence will be a proof of this kind. We have the following chain of isomorphisms

$$
\begin{aligned}
& Rf_* R\mathcal{H}om_{D_X}(M^\cdot, f^\dagger N^\cdot)[-d_X] \cong \\
& \cong Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot) \otimes^L_{D_X} f^\dagger N^\cdot)[-d_X] \cong && (1.17)\\
& \cong Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot) \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} N^\cdot)[-d_Y] \cong && (1.9)\\
& \cong Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot) \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} N^\cdot[-d_Y] \cong \\
& \cong (\Omega_Y \otimes^L_{\mathcal{O}_Y} \textstyle\int_f \mathbb{D}_X M^\cdot) \otimes^L_{D_Y} N^\cdot[-d_Y] \cong && (1.11)\\
& \cong (\Omega_Y \otimes^L_{\mathcal{O}_Y} \mathbb{D}_Y \mathbb{D}_Y \textstyle\int_f \mathbb{D}_X M^\cdot) \otimes^L_{D_Y} N^\cdot[-d_Y] \cong && (1.16)\\
& \cong (\Omega_Y \otimes^L_{\mathcal{O}_Y} \mathbb{D}_Y \textstyle\int_{f!} M^\cdot) \otimes^L_{D_Y} N^\cdot[-d_Y] \cong && (1.24)\\
& \cong R\mathcal{H}om_{D_Y}(\textstyle\int_{f!} M^\cdot, N^\cdot). && (1.17)
\end{aligned}
$$

The second statement can be obtained by dualizing the first. □

1.4 Minimal extensions

Let's conclude this introduction giving a more concrete description of holonomic D-modules, in order to get generators of the category $D^b_h(D_X)$. Let's start with a theorem characterizing holonomic modules in two other ways.

Theorem 1.26. *Let $M \in Mod_c(D_X)$ be a coherent D-module; then the following are equivalent:*

- *$M \in Mod_c(D_X)$;*
- *there exists a stratification of X*

  $$X = X_0 \supset X_1 \supset \ldots \supset X_m \supset X_{m+1} = \emptyset$$

  by closed subvarieties such that the spaces $X_k \setminus X_{k+1}$ are smooth and the sheaves $H^j(i_k^\dagger M)$ are all integrable connection, where i_k is the inclusion of $X_k \setminus X_{k+1}$ into X;
- *for every point $x \in X$ the cohomologies $H^j(i_x^\dagger M)$ are finite dimensional vector spaces, where i_x is the inclusion of x in X.*

So, for every holonomic D-module M, we have an open set $U \subseteq X$ such that the restriction $j^\dagger M$ is an integrable connection; more in general, considering the support Y (as a sheaf) of M, we have an open set $V \subseteq Y$ such that the restriction to V is a positive rank integrable connection.

Now, we can try to work in the other way, trying to build up D-modules by pushing forward integrable connections from locally closed subsets. This fact comes out to be true, after the definition of a third kind of direct image of D-modules, that in some sense lies between $\int_f$ and $\int_{f!}$; to define it, we need a preparatory lemma.

Lemma 1.27. *Given a locally closed embedding $i : V \hookrightarrow X$ and an holonomic D-module M on V, we have a natural map*

$$\int_{i!} M \to \int_i M.$$

Proof. We have isomorphisms

$$Hom_{D^b_h(D_Y)}(\int_{f!} M^\cdot, \int_f M^\cdot) \cong Hom_{D^b_h(D_Y)}(M^\cdot, i^\dagger \int_f M^\cdot) \cong Hom_{D^b_h(D_Y)}(M^\cdot, M^\cdot)$$

where the first one is given by the adjunction formula 1.25; about the second one, it comes from $f^\dagger \int_f M^\cdot \cong M^\cdot$ that is true both for closed embeddings (by Kashiwara equivalence, example 1.13) and for open embeddings (by coherent sheaf theory), and indeed a locally closed embedding is a composition of them. Then, a natural element can be chosen corresponding to the identity in the right hand side. □

Definition 1.28. *In this setting, the image of this natural map will be called **minimal extension** of M in X.*

Example 1.29. Let's consider again the pushforward

$$\int_j \mathcal{O}_U = D_X/(D_X \cdot \partial_x x)$$

in Example 1.23. In this way we get a decomposition

$$0 \to \mathcal{O}_X \to D_X/(D_X \cdot \partial_x x) \to D_X/(D_X \cdot x) \to 0$$

telling us that this module is not simple, and has a submodule isomorphic to $\mathcal{O}_X$, that still restricts to $\mathcal{O}_U$ in U; note that the cokernel is the "Dirac's delta" D-module in example 1.20, supported in 0 (hence, outside U). So, if we are looking for an extension of $\mathcal{O}_U$ that is "minimal", we should take $\mathcal{O}_X$ instead of $\int_j \mathcal{O}_U$; in fact, we will see now that $\mathcal{O}_X$ is the minimal extension we have just defined. It is easy to see that

$$\int_{j!} \mathcal{O}_U = D_X/(D_X \cdot x\partial_x),$$

and that we have a decomposition

$$0 \to D_X/(D_X \cdot x) \to D_X/(D_X \cdot x\partial_x) \to \mathcal{O}_X \to 0.$$

Now, the map $\int_{j!} \mathcal{O}_U \to \int_j \mathcal{O}_U$ is the one having Dirac's module as both kernel and cokernel, and its image is precisely $\mathcal{O}_X$, as we wanted to show.

So, we are ready to give a structure theorem for holonomic D-modules.

Theorem 1.30. *Let X be a smooth variety. Then*

(i) Every holonomic D-module M is of finite lenght, meaning that every sequence of submodules of M is finite.

(ii) For an exact sequence of coherent D-modules

$$0 \to M' \to M \to M'' \to 0$$

we have that M is holonomic if and only if M' and M'' are.

(iii) Every simple holonomic D-module is the minimal extension of an integrable connection on a locally closed subset V of X.

Remark 1.31. We saw that for the embedding of the punctured affine line in the affine line, the minimal extension of the structure sheaf is still the structure sheaf. This holds more in general; if the closure $\bar{V}$ of a locally closed V in X is smooth, than the minimal extension of $\mathcal{O}_V$ in X is the same as the minimal extension of its closure, $\mathcal{O}_{\bar{V}}$. If $\bar{V}$ is not smooth, otherwise, the minimal extension of $\mathcal{O}_V$ on $\bar{V}$ will be a complex, whose name is **intersection cohomology sheaf**.

2 Analytic D-modules

In this section we are going to say something about D-modules in the analytic setting; these will serve as "bridge" in the Riemann-Hilbert correspondence between algebraic D-modules and "sheaves of solutions" that will be defined in the following section. The present section is meant to be a gentle introduction (avoiding as much as possible technical details) to the analytic setting for people with a more algebraic background. Let's start with one example, that should show why it is really necessary to go in the analytic category.

Key example 2.1. Let's consider the algebraic D-modules on the punctured affine line

$$D_X/(D_X \cdot (x\partial_x - \lambda)) \qquad \lambda \in \mathbb{C} \setminus \mathbb{Z}.$$

Notice that varying λ these modules will not be isomorphic each other. The associated differential equation is $xf' = \lambda f$, whose local solutions are the determinations of the function x^λ; the sheaf

$$\mathcal{H}om_{D_X}(D_X/(D_X \cdot (x\partial_x - \lambda)), \mathcal{O}_X)$$

is only the zero sheaf, because there are no regular functions satisfying the differential equation, and futhermore they wouldn't be defined over Zariski open sets. So, to have this solution functor to really remember the structure of the original D-module, we have to include analytic solutions, and pass to the analytic topology on X; for this reason, we are going to show in this chapter the theory of analytic D-modules. We will come back to this example later.

2.1 Definitions and examples

Let X be a complex manifold, with topology inherited from the Euclidean topology in $\mathbb{C}^n$. On X we have the sheaf of holomorphic functions $\mathcal{O}_X$, and the sheaf of holomorphic tangent vector bundles Θ_X; as in the previous section, we define D_X as the subsheaf of the sheaf of rings $End_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X. We will see that the theory of D-modules built on this objects is surprisingly similar to the algebraic one.

Example 2.2. If $X = \mathbb{C}$, then global sections of $\mathcal{O}_X$ are converging power series in one variable $\mathbb{C}[[z]]$; we can then find also holomorphic functions like e^z and $sin(z)$ that are not global sections in the algebraic case, because they are not *regular*, meaning, algebraic; the problem of finding regular functions among holomorphic ones is a task that can be better achieved compactifying X (in this case, to the projective line, and checking which one have a pole of finite order at infinity); we will see again such an issue when defining regular D-modules, that will be the main aim of section 4.

Definition 2.3. *Given a complex manifold X, a left (resp. right) analytic D-module on X is a sheaf of left (resp. right) D_X-modules. The category of left D-modules will be denoted again by $Mod(D_X)$.*

We have the following facts/definition in parallel with the algebraic case.

- There is a filtration $\{F_iD_X\}_{i\geq 0}$ of D_X such that the graded module is the same as the (sheaf) pushforward $\pi_*\mathcal{O}_{T^*X}$; for every D-module, we have a filtration $\{F_iM\}_{i\geq 0}$, and the support $SS(M)$ in T^*X of the graded module grM is called the **singular support** of M; if the dimension of $SS(M)$ if either 0 or d_X, and M is coherent, than we will call the module **holonomic**.

- A D-module that is locally free of finite rank will be called **integrable connection**.

- The tensor product (over $\mathcal{O}_X$) by $\Omega_X = \bigwedge^{d_X} \Theta_X^*$ gives an equivalence of categories between left and right D-modules.

- Given a morphism of complex manifold $f : X \to Y$, we define as in the previous section the modules $D_{X \to Y}$ and $D_{Y \leftarrow X}$, and functors

$$f^* : Mod(D_Y) \to Mod(D_X)$$
$$\int_f : Mod(D_X) \to Mod(D_Y).$$

- Coherent D-modules are defined in the same way, as well as $Mod_c(D_X)$, $Mod_h(D_X)$, $D_c^b(D_X)$, $D_h^b(D_X)$ and the derived functors $f^\dagger = Lf_*[d_X - d_Y]$ and $\int_f$

$$f^\dagger = Lf_*[d_X - d_Y] : D^b(D_Y) \to D^b(D_X)$$
$$\int_f : D^b(D_X) \to D^b(D_Y).$$

- Dual D-modules are defined in the same way, and in $D_c^b(D_X)$ duality is an involution.

In the analytic setting, holonomic modules are preserved under duality and inverse image.

Theorem 2.4. *Let M be an holonomic D-module on a complex manifold X. Then the dual $\mathbb{D}_X M$ has only cohomology in degree 0, and is another holonomic D-module. More in general, duality preserves $D_h^b(D_X)$.*

Theorem 2.5. *Let $f : X \to Y$ be a morphism of complex manifolds, then f^* preserves $Mod_h(D_X)$ and $f^\dagger$ preserves $D_h^b(D_X)$.*

For direct image, this is not true anymore, as the following example shows.

Example 2.6. Let again $j : \mathbb{C}^* \to \mathbb{C}$ be the open embeddinge, and let's pushforward the sheaf of holomorphic functions on $\mathbb{C}^*$. Now, the pushforward is *not* anymore $\mathcal{O}_\mathbb{C}[z^{-1}]$, because in global sections we can find also functions with essential singularity at 0, such as $e^{1/z}$; the module $\int_j \mathcal{O}_{\mathbb{C}^*}$ is so not even a coherent D-module anymore (as it would be the D-module $\mathcal{O}_\mathbb{C}[z^{-1}]$, for the same reason as in example 1.23).

We actually have a partial result in this direction, if the morphism is proper and another technical hypothesis holds (the module has to admit a so called *good filtration* for the map).

2.2 Algebraic to analytic

As we saw, we are going to use analytic D-modules as bridge to Riemann-Hilbert correspondence; so, let's describe how one can pass from algebraic D-modules to analytic D-modules.

Let X be an algebraic variety, X^{an} the complex manifold of its closed points. Despite the very different topologies, we have a continuous map of topological spaces $\iota : X^{an} \to X$, and a morphism of sheaves (on X) $\mathcal{O}_X \to \iota_* \mathcal{O}_{X^{an}}$, that injects regular sections into holomorphic sections in Zariski open subsets; the data of these two morphisms is exactly the definition of a map between locally ringed spaces $(X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$. In the other direction, consider the sheaf $\iota^{-1}\mathcal{O}_X$ on X^{an}: sections of it are harder to describe, because on every (Euclidean) open set U, sections are direct limits of sections of $\mathcal{O}_X$ in Zariski open sets containing U. It's easy to see that we also have a canonical morphism $\iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{an}}$ of sheaves on X^{an}, and in the same way a canonical morphism $\iota^{-1}D_X \to D_{X^{an}}$. Using this morphisms, we get functors

$$\begin{array}{rcl} Mod(\mathcal{O}_X) & \xrightarrow{an} & Mod(\mathcal{O}_{X^{an}}) \\ F & \longmapsto & F^{an} = \mathcal{O}_{X^{an}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}F \end{array}$$

$$\begin{array}{rcl} Mod(D_X) & \xrightarrow{an} & Mod(D_{X^{an}}) \\ M & \longmapsto & M^{an} = D_{X^{an}} \otimes_{\iota^{-1}D_X} \iota^{-1}M \end{array}.$$

This functors turn out to be exact, because of the flatness of $\iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{an}}$ and $\iota^{-1}D_X \to D_{X^{an}}$; we then get functors

$$(\cdot)^{an} : D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_{X^{an}})$$

$$(\cdot)^{an} : D^b(D_X) \to D^b(D_{X^{an}}).$$

Let's analyze now properties of this last functor; again, there's a little failure regarding direct image.

Proposition 2.7. *If $M^{\cdot} \in D^b_c(D_X)$, then we have $(\mathbb{D}_X M^{\cdot})^{an} \cong \mathbb{D}_{X^{an}}(M^{\cdot})^{an}$.*

Proposition 2.8. *If $f : X \to Y$ is a morphism of smooth algebraic varieties, then*

i) if $M^{\cdot} \in D^b(D_Y)$, we have $(f^\dagger M^{\cdot})^{an} \cong (f^{an})^\dagger (M^{\cdot})^{an}$;

ii) if $M^{\cdot} \in D^b(D_X)$, we have a canonical morphism $(\int_f M^{\cdot})^{an} \to \int_{f^{an}} (M^{\cdot})^{an}$, and this is an isomorphism if f is proper and $M \in D^b_c(D_X)$.

The failure of analytification functor to commute with direct image can be checked again in our usual example.

Example 2.9. Let again $j : U \hookrightarrow X$ be the open embedding of the punctured affine line in the affine line, such that we have $j^{an} : \mathbb{C}^* \hookrightarrow \mathbb{C}$; let's consider the structure sheaf of U. On one hand, we have that

$$\left(\int_j \mathcal{O}_U\right)^{an} \cong (\mathcal{O}_X[x^{-1}])^{an} \cong \mathcal{O}_{\mathbb{C}}[z^{-1}] \cong D_{\mathbb{C}}/D_{\mathbb{C}} \cdot (\partial_z z).$$

On the other hand, we have already seen in example 2.6 that $\int_{j^{an}} \mathcal{O}_{\mathbb{C}^*}$ is much bigger, so we only can get an injective map.

2.3 Serre's GAGA

We will recall now (part of) Serre's GAGA for coherent sheaves.

Theorem 2.10. *Let X, Y be smooth algebraic varieties, and X^{an} and Y^{an} their complex manifolds of closed points.*

i) If $f : X \to Y$ is a proper morphism, and F a coherent sheaf on X then $(f_* F)^{an}$ is isomorphic to $f^{an}{}_* F^{an}$; more in general, we have

$$(R^i f_* F)^{an} \cong R^i f^{an}{}_* F^{an}.$$

ii) If X is proper and R is an analytic coherent sheaf on X^{an}, then there exists an algebraic coherent sheaf F on X such that $F^{an} \cong R$.

iii) If X is proper, F and G are algebraic coherent sheaves on X, and $r : F^{an} \to G^{an}$ a morphism between their analytifications, then there exists a morphism $f : F \to G$ in the category of algebraic coherent sheaves such that $f^{an} \cong r$.

Remark 2.11. By point *i*), if Y is a point, we get that $H^i(X, F) \cong H^i(X^{an}, F^{an})$ for every i; in particular, if F is a line bundle, meromorphic functions with given pole along a (algebraic) divisor D are the same.

Remark 2.12. Points *ii*) and *iii*) may be stated in a shorter way, that is the following. If X is a proper algebraic varieties, then analytification gives an equivalence of categories

$$Mod_c(\mathcal{O}_X) \cong Mod_c(\mathcal{O}_{X^{an}}).$$

We could try to extend these results to D-modules; we will see tough in section 4.3 that we are going to need it only for integrable connections.

3 Local systems and constructible sheaves

Let's now finally describe the "sheaves of solutions" we longed so much, and describe the functor that will give us Riemann-Hilbert correspondence.

3.1 Definitions

Let X be a complex manifold, and let $\mathbb{C}_X$ be the costant sheaf with complex coefficients. We will build a theory of $\mathbb{C}_X$-modules, that is, just sheaves of vector spaces. This is indeed what we expect for a sheaf of solutions to a differential equation: over an open set, being a finite dimensional vector space. So, let's start again with our tour of definitions.

We will indicate by $Mod(\mathbb{C}_X)$ the category of such sheaves. Let $f : X \to Y$ a morphism of analytic spaces. Then we have functors

$$f^{-1} : Mod(\mathbb{C}_Y) \to Mod(\mathbb{C}_X)$$

$$f_* : Mod(\mathbb{C}_X) \to Mod(\mathbb{C}_Y)$$

$$f_! : Mod(\mathbb{C}_X) \to Mod(\mathbb{C}_Y)$$

directly from the definitions in sheaf theory, the first being exact, the latters left exact. In this way we get functors on derived categories:

$$f^{-1} : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$$

$$Rf_* : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$$

$$Rf_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y).$$

As right adjoint of $Rf_!$, we get also the functor

$$f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X).$$

About duality, we need a further object, namely the dualizing sheaf.

Definition 3.1. *Given an analytic space X, let's consider the map onto a point $q : X \to \{pt\}$; the* ***dualizing complex*** *of X is $\omega^{\cdot}_X = q^!\mathbb{C}$.*

Definition 3.2. *Let $F^{\cdot}$ be a complex in $D^b(\mathbb{C}_X)$; then the* ***dual complex*** *is defined by*

$$\mathbb{D}_X F^{\cdot} = R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, \omega^{\cdot}_X)$$

Now, let's give the definition that in some sense corresponds to that of holonomic D-modules.

Definition 3.3. *Given a complex manifold X, a* ***local system*** *is a $\mathbb{C}_X$ module that is locally free of finite rank; we will call $Loc(X)$ the category of such objects. A $\mathbb{C}_X$-module F is called* ***constructible*** *if there is a stratification of X by locally closed analytic smooth subsets such that all restrictions to strata are local systems. If X carries also an algebraic structure and the stratification is algebraic, the sheaf will be called* ***algebraically constructible****. We will denote by $D^b_c(X)$ the category of bounded complexes having as cohomology algebraically constructed sheaves; note that here X is the algebraic space, even if $D^b_c(X)$ is defined as subcategory of $D^b_c(\mathbb{C}_{X^{an}})$.*[1]

Example 3.4. Let's consider again, as in key example 2.1, the differential equation over $\mathbb{C}^*$

$$zf'(z) - \lambda f(z) = 0$$

with λ is not an integer; we will show that its sheaf of solutions is indeed a local system. This equation has as solutions the function z^λ only on open sets on which such a function is defined, that are, sets not winding around the origin; to be precise, open sets $U \subset \mathbb{C}^*$ such that $\pi_1(U) \to \pi(\mathbb{C}^*)$ is the zero map. The "sheaf" of solutions of such an equation is then a sheaf being the vector space $\mathbb{C}z^\lambda$ on such open sets, and zero otherwise. So, this sheaf is actually locally free of rank one (as sheaf of vector spaces) and so is a local system.

Remark 3.5. Let's consider again the sheaf in the previous example. The function z^λ may be defined maximally over $\mathbb{C}^*$ minus one half line coming out from the origin; the function may be extended further, but taking different values from the original one; more precisely, such a function is globally defined only on the universal cover; so, we can ask how the fundamental group, exchanging the sheets of the universal cover, changes the value of the function; in this case, "winding around" the origin changes the value of the function by a multiplication by $e^{2\pi\lambda i}$. More in general, whenever we have a local system of rank k on $\mathbb{C}^*$, there is an element of $GL_k(\mathbb{C})$ obtained in the same way; this linear transformation is called the **monodromy** of the system; more in general, for an arbitrary manifold X, monodromy is defined as a map $\pi_1(X) \to GL_k$, that, is, a representation of $\pi_1(X)$. This gives us the formal setting in which Hilbert's 21st problem was stated: he asked whether every local system with a prescribed monodromy around the origin in $\mathbb{C}^*$ might be obtained as a solution of a linear differential equation. In the case of rank one linear systems, any monodromy action is just the multiplication by a scalar, so varying λ we can obtain all such monodromy actions as sheaf

[1] If this last sentence seems obscure, it may become clearer right before theorem 5.2.

of solutions of the differential equation $zf' - \lambda f = 0$: Hilbert's 21st problem is then proved, in the case of rank one linear systems on $\mathbb{C}^*$.

Again, let's prove that constructible sheaves are a "stable" subcategory.

Proposition 3.6. *(i) Let X be a complex manifold. Then duality preserves $D^b_c(X)$, and is an involution on it.*

(ii) Let $f : X \to Y$ be a morphism between complex manifold; then the functors f^{-1}, $f^!$ preserve constructible sheaves, and $f^! = \mathbb{D}_X \circ f^{-1} \circ \mathbb{D}_Y$.

(iii) Let $f : X \to Y$ be a proper morphism between complex manifold; then the functors Rf_*, $Rf_!$ preserve constructible sheaves, and $Rf_! = \mathbb{D}_Y \circ Rf_* \circ \mathbb{D}_X$.

(iv) If f is actually an algebraic morphism of smooth algebraic varieties, all functors preserve also algebraically constructible sheaves, and we have the result of (iii) without the properness hypothesis.

(v) Tensor product of two constructible sheaves is again a constructible sheaf; the same holds for the left derived tensor product in $D^b_c(X)$.

3.2 Solution and de Rham functors

We are now ready to define the functor of solutions of an analytic D-module.

Definition 3.7. *Let X be a complex manifold, and $M^\cdot \in D^b(D_X)$; then the **solution complex** is*

$$Sol_X(M^\cdot) = R\mathcal{H}om_{D_X}(M^\cdot, \mathcal{O}_X)$$

where $\mathcal{O}_X$ is given the usual left D_X action.

Note that this sheaf has only the structure of $\mathbb{C}_X$-module. For Riemann-Hilbert correspondence, anyways, we are going to use the following functor, that we will see is not so far from the previous one.

Definition 3.8. *Let X be a complex manifold, and $M^\cdot \in D^b(D_X)$; then the **de Rham complex** is*

$$DR_X(M^\cdot) = \Omega_X \otimes^L_{D_X} M^\cdot.$$

where Ω_X is given the usual right D_X action.

Remark 3.9. By a slight abuse of notation, when X is an algebraic variety and $M^\cdot \in D^b_c(D_X)$, we define

$$DR_X(M^\cdot) = \Omega_{X^{an}} \otimes^L_{D_{X^{an}}} (M^\cdot)^{an}$$

and we will denote this functor by the de Rham functor (thus including the analytification functor); from here to the end of the section though, X will be a complex manifold.

The following proposition, given the properties of tensor product and Hom, is not completely unexpected.

Proposition 3.10. *For $M^\cdot \in D^b_c(D_X)$, we have*

$$DR_X(M^\cdot) \cong Sol_X(\mathbb{D}_X M^\cdot)[d_X]$$

The following proposition, giving a resolution of Ω_X in free right D-modules, turns out to be very useful for explicit calculations.

Proposition 3.11. *We have an exact sequence of right D_X-modules*

$$0 \to \Omega^0_X \otimes_{\mathcal{O}_X} D_X \to \Omega^1_X \otimes_{\mathcal{O}_X} D_X \to \ldots \to \ldots \Omega^{d_X}_X \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

where Ω^i_X is the (coherent) sheaf of degree i differential forms, and the action of D_X is given by right multiplication on the right factor.

Using this proposition, we have the following isomorphisms in the derived category; given M a D-module, we have

$$DR_X(M) = \Omega_X \otimes_{D_X} M \cong [(\Omega^\cdot_X \otimes_{\mathcal{O}_X} D_X) \otimes_{D_X} M][d_X] \cong [\Omega^\cdot_X \otimes_{\mathcal{O}_X} M][d_X].$$

This tells us that if M is a coherent D-module, its de Rham complex will have cohomology only in degrees $-d_X$ to 0.

Example 3.12. Let's make an example explicitly; let X be $\mathbb{C}$, and M the holonomic D-module $D_X/(D_X \cdot \partial_z z)$, the analytic analogue of examples 1.23 and 1.29; let's apply the de Rham functor. We have $\Omega_X \cong \mathcal{O}_X$, so by the previous contruction the de Rham complex of M is

$$0 \to M \xrightarrow{\partial_z \cdot} M \to 0$$

where the central map is the *left* multiplication by ∂_z, so that the map is not an either left or right D-modules map. Working very explicitly, the only operators in $D_X/(D_X \cdot \partial_z z)$ killed by left multiplication by ∂_z are scalar multiples of z, so we have $H^{-1}(DR_X(M)) = \mathbb{C}_X$. In the same way, looking at the cokernel of the map, one can check that $H^0(DR_X(M))$ is the sheaf $i_*\mathbb{C}_p$ where $i : p \hookrightarrow X$ is the immersion of the origin.

Remark 3.13. From the previous example we saw that if we start from an holonomic D-module, the de Rham complex may have many nonzero cohomologies; if we want it to be an equivalence of categories then, this can hold only if we take the whole derived categories (or restrict further to integrable connections, as we will see in the following example). This complexes, tough, are quite special, and we can describe them more precisely. We saw that in our case we had the dimension of the support of H^{-1} is 1-dimensional, and that of H^0 is 0; this is true more in general: let's denote by **perverse sheaves** the complexes $F^{\cdot} \in D^b_c(X)$ such that

$$dimsupp(H^{-i}(F^{\cdot})) \leq i$$

$$dimsupp(H^{-i}(\mathbb{D}_X F^{\cdot})) \leq i.$$

Then, on a smooth algebraic variety X, analytification and the de Rham functor sends holonomic D-modules into perverse sheaves (see also section 5.2); once we restrict to regular holonomic D-modules (the definition is in the next section) this functor will be an equivalence.

Key example 3.14. Let's use the same construction to give a better insight in the Riemann-Hilbert correspondence in case of integrable connections, as in key example 1.5. Let's consider now an integrable connection M; from what we have seen, the sheaf of solution (without taking the right derived functor) is the kernel of the map

$$\nabla : M \cong \Omega^0_X \otimes_{\mathcal{O}_X} M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$$

that is exactly how a connection on a vector bundle is defined; kernel of this map is then the sheaf of sections that are *horizontal* for this section, and by the classical Frobenius theorem this is actually a local system (that means, is locally free of finite rank). So, we have a functor

$$H^{-d_X}(DR_X(\cdot)) : Conn(X) \to Loc(X).$$

As seen in 1.5, we have an inverse candidate, given by

$$L \longmapsto \mathcal{O}_X \otimes_{\mathbb{C}_X} L$$

with D_X-action given by the left action on $\mathcal{O}_X$. It's easy to see that these are actually inverse one each other, so that the de Rham actually induces a first version of Riemann-Hilbert correpondence. We then proved the following theorem.

Theorem 3.15. *On a complex manifold X, we have an equivalence of categories*

$$H^{-d_X}(DR_X(\cdot)) : Conn(X) \xrightarrow{\sim} Loc(X).$$

4 Regularity of meromorphic connections

So, we described all the categories involved, and we defined the functors; in particular our strategy is to start with an algebraic D-module, and taking the de Rham complex (note: of its analitification). Sadly, this is not enough to give an equivalence of categories, because of the following example.

Key example 4.1. Let X be the affine line $\mathbb{A}^1$, and let M be the holonomic D-module $D_X/D_X \cdot (\partial_x - 1)$. It's easy to see that the solutions of the associated differential equation are just scalar multiples of the (analytic) function e^z, that is globally defined. So, analytifying M to M^{an}, and taking the de Rham complex, gives just the constant sheaf $\mathbb{C}_X$ (in degree -1); this is a problem, because this is the same answer that we would get performing the same process on the algebraic holonomic D-module $\mathcal{O}_X$, that is not isomorphic to M. So, this process can't definitely give a correspondence. Note that in this example the analytifications M^{an} and $(\mathcal{O}_X)^{an} = \mathcal{O}_{X^{an}}$ are indeed isomorphic as analytic D-modules.

What we are going to do is to define a subcategory of the category of holonomic algebraic D-modules (that is, *regular* holonomic D-modules), on which this process will actually give an equivalence of categories. In particular, in the previous example, the module M is not regular.

For sake of brevity, in this section a higher portion of the tecnical details is going to be spared; we will instead try to explain as much as possible in words and through examples what is going on.

4.1 Regularity of algebraic integrable connections

In the first part of this section, we are going to define regularity only for integrable connections, and as it often happened, see what happens in that case.

Regularity of an integrable connection is something that has to be checked going into a compactification; on a compatification, an integrable connection extends to a *meromorphic connection.* Let's reduce again the problem, and consider the case of the space being a curve.

Let C be a smooth algebraic curve, p a point, $\mathcal{O}_{C,p}$ the local ring at p, and $K_{C,p}$ its quotient field; let x be a local parameter around p. Locally, an integrable connection looks just like a free $\mathcal{O}_{C,p}$ module. To have it meromorphic, we are going to use $K_{C,p}$ instead.

Definition 4.2. *At a point p of a curve C, a* ***local algebraic meromorphic connection*** *is a finite dimensional $K_{C,p}$-vector space M, with a $\mathbb{C}$-linear map $\nabla : M \to M$ satisfying the Leibniz rule, that is,*

$$\nabla(fu) = \frac{df}{dx} \otimes u + f\nabla u \qquad \forall f \in K_{C,p}, \forall u \in M.$$

Basically, a situation like this is supposed to happen whenever we have an integrable connection in $C \setminus p$ and we push forward it to the whole C, so that we can't expect it to keep being locally free.

We can now give the definition of regularity in this case.

Definition 4.3. *Let M be a meromorphic connection as before. We will say M is* ***regular*** *if there exist a $\mathcal{O}_{C,p}$-submodule L of M such that*

- *L is finitely generated as $\mathcal{O}_{C,p}$ module;*
- *L generates M, meaning, $K_{C,p}L = M$;*
- *L is stable under the operator $x\nabla$.*

The last condition is the one that more than everything characterize regular connections. Roughly speaking, this means that the order of pole of ∇ at p can't be more than 1, so that the operator $x\nabla$ still behaves "holomorphically" (see example below for a better understanding of this).

We can give now the definition of a regular integrable connection on a curve. Let C be a smooth curve, let M be an integrable connection, and let $\bar{C} \supseteq C$ be a smooth completion of C. We can extend M to $\int_j M$ by the open embedding $j : C \hookrightarrow \bar{C}$; on the points in $\bar{C} \setminus C$, the stalk of $\int_j M$ is going to be a local algebraic meromorphic connection.

Definition 4.4. *In this setting, we say M is a* ***regular integrable connection*** *on C if for every point $p \in \bar{C} \setminus C$ the stalk $(\int_j M)_{\bar{C},p}$ is a regular local meromorphic connection.*

Before proceeding to the case of a smooth algebraic variety in general, let's go back to example 4.1.

Example 4.5. Let's see then why the integrable connection on the affine line $\mathbb{A}^1$ in example 4.1 is not regular. Compatifying the affine line to $\mathbb{P}^1$, taking coordinates in which the extra point is in the origin, and taking the direct image, we get the D-module

$$D^1_{\mathbb{A}}/D^1_{\mathbb{A}} \cdot (x^2\partial_x + 1)$$

Considering it as a local meromorphic connection on the origin means considering $K_{\mathbb{A}^1,0}$, with a connection given by

$$\nabla(1) = -1/x^2$$

(because $x^2\partial + 1 = 0$ on a generator) and extending using the Leibniz rule, getting the meromorphic connection

$$\nabla(f) = \frac{df}{dx} - \frac{f}{x^2}.$$

This connection is then not regular, because there's not a nonzero coherent $\mathcal{O}_{\mathbb{A}^1,0}$-module stable for the operator $x\nabla$, because $x\nabla$ "goes down in powers of x".

For a higher dimensional algebraic variety, we do not have either a local picture or a canonical smooth completion to work with; we are going to define regularity using a curve-testing criterion.

Definition 4.6. *Let X be a smooth algebraic variety, and M an integrable connection on it; then M is* ***regular*** *if for every immersion of a smooth curve $i : C \hookrightarrow X$ the inverse image i^*M is a regular integrable connection. We will call $Conn^{reg}(X)$ the category of such objects.*

We have now defined regular integrable connections. Next aim will be to prove what we suspected in example 4.1, that is, the equivalence between regular algebraic and analytic, in the case of integrable connections.

Theorem 4.7. *Let X be a smooth algebraic variety, X^{an} the complex manifold of its complex points. Then the analytification functor induces an equivalence of categories*

$$(\cdot)^{an} : Conn^{reg}(X) \xrightarrow{\sim} Conn(X^{an}).$$

In this proving this, together with theorem 3.15, will give us what is called *Deligne's Riemann-Hilbert correspondence.*

Theorem 4.8 (Deligne, 1970)**.** *Let X be an algebraic smooth variety. Then the de Rham functor gives an equivalence of categories*

$$Conn^{reg}(X) \xrightarrow{\sim} Loc(X^{an}).$$

The proof of theorem 4.7 will be based on Serre's GAGA; to work with it in its full power, we have to lie on a compact manifold, so we will consider a smooth algebraic completion $j : X \hookrightarrow \bar{X}$ where $D = \bar{X} \setminus X$ is a divisor; we now just reverted the problem, and we will work with couples $(\bar{X}, D)$; on them, we consider integrable connections *meromorphic along D*.

Definition 4.9. *Let $X, \bar{X}, D$ be as before, and let $\mathcal{O}_{\bar{X}}[D]$ be the sheaf $j_*\mathcal{O}_X$. A $D_{\bar{X}}$-module that is isomorphic to a locally free $\mathcal{O}_{\bar{X}}[D]$-module, is called an* ***algebraic meromorphic connection*** *along D. We will say it's* ***regular*** *if the restriction to X is a regular integrable connection. We will denote by $Conn(\bar{X}, D)$ (resp. $Conn^{reg}(\bar{X}, D)$) the category of algebraic meromorphic connections (resp. regular ones).*

There is a reason we introduced these objects: the following lemma.

Lemma 4.10. *Restriction gives equivalences of functors*

$$j^* : Conn(\bar{X}, D) \xrightarrow{\sim} Conn(X)$$
$$j^* : Conn^{reg}(\bar{X}, D) \xrightarrow{\sim} Conn^{reg}(X).$$

So, on the algebraic side, we can talk about meromorphic connection on $\bar{X}$ along D; if we had it for the analytic side, we could relate structures on $\bar{X}$ and $\bar{X}^{an}$, and invoke Serre's GAGA. There is a problem tough: there is nothing like lemma 4.10 in the anaytic setting; more deeply, is not possible even to give a definition such as definition 4.9, because in the analytic setting the direct image for an open embedding behaves very badly, as seen in example 2.6, because of functions having essential singularities along D. To solve this issues, we have give a new definition for *regular analytic meromorphic connections* along a divisor, that will give us a category $Conn^{reg}(\bar{X}^{an}, D^{an})$ to fit in the square

$$\begin{array}{ccc} Conn^{reg}(\bar{X}, D) & \longrightarrow & Conn^{reg}(\bar{X}^{an}, D^{an}) \\ \downarrow & & \downarrow \\ Conn^{reg}(X) & \longrightarrow & Conn(X^{an}). \end{array}$$

In this square, to prove that the lower arrow is an equivalence, we will prove that the other three are. The left one is lemma 4.10, and we will see in the next subsection the upper (that will be some GAGA argument) and right arrows; in fact, next subsection will be about the refinition of regularity in the analytic case.

4.2 Regularity of analytic meromorphic connections along a divisor

Let's start with the local picture, again. On the complex plane $\mathbb{C}$, let's consider the stalk of the sheaf of holomorphic functions at zero $(\mathcal{O}_{\mathbb{C}})_0$, and its function field K.

Definition 4.11. *A **local analytic meromorphic connection** is a finite dimensional K-vector space M with a $\mathbb{C}$-linear map $\nabla : M \to M$ satisfying the Leibniz rule, as in definition 4.2.*

Now, we are not going to define what regularity means in this case, it would require too much further analytical work about (meromorphic) linear differential equations, and solutions with moderate growth. So we will take a leaf of faith and we will keep going supposing to have a notion of regularity in this context too, related to the fact that the singularity can't be too bad, but substantially different from the algebraic one (basically, because of example 4.1). We will though describe it explicitly in one situation in remark 4.14.

Let's now take a complex manifold X, and a compactification $\bar{X}$ by the divisor D. Remember that our aim is to define regular connections meromorphic along D. Following definition 4.9, let's define the sheaf $\mathcal{O}_{\bar{X}}[D]$ again; in this case, this *can't* be defined as the direct image $j_*(\mathcal{O}_X)$ by the open embedding $j : X \hookrightarrow \bar{X}$, because the resulting sheaf is not coherent. In this case, by $\mathcal{O}_{\bar{X}}[D]$ we mean the sheaf that locally is $\mathcal{O}_{\bar{X}}[f_D^{-1}]$ where f_D is a function vanishing with order one on D.

Definition 4.12. *Let $X, \bar{X}, D, \mathcal{O}_{\bar{X}}[D]$ be as before. A $D_{\bar{X}}$-module that is isomorphic to a locally free $\mathcal{O}_{\bar{X}}[D]$-module, is called an **analytic meromorphic connection** along D.*

Definition 4.13. *Let M be a meromorphic connection on $\bar{X}$ along D, and let B be the unit ball*

$$B = \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

*We will say that M is **regular** is for every embedding $i : B \hookrightarrow \bar{X}$ such that $i^{-1}(D) = \{0\}$, we have that the stalk $(i^*M)_0$ is a regular local analytic connection. We will call the category of such objects $Conn^{reg}(\bar{X}, D)$.*

Remark 4.14. If D is a normal crossing divisor, We can give a description of what regularity means, in terms of *logarithmic poles*. Let M be a meromorphic connection on $\bar{X}$ along D, and suppose there exist an holomorphic vector bundle L on $\bar{X}$ such that $M \cong \mathcal{O}_{\bar{X}}[D] \otimes_{\mathcal{O}_{\bar{X}}} L$; let p be a point of D, with local coordinates $z_1, \dots, z_n$ in which the local equation of D is given by $z_1 z_2 \cdots z_r = 0$, and let $e_1, \dots, e_s$ be local coordinates of L; then, we can write the connection ∇ of M locally as

$$\nabla e_i = \sum_{1 \le j \le s}^{1 \le k \le n} a_{ij}^k dz_k \otimes e_j$$

where a_{ij}^k are functions in $\mathcal{O}_{\bar{X}}[D] = \mathcal{O}_{\bar{X}}[z_1^{-1}, \ldots, z_r^{-1}]$. The meromorphic connection M is said to have a **logarithmic pole** along D for the line bundle L if the functions $z_k a_{ij}^k$ $(1 \leq k \leq r)$ and a_{ij}^k $(r < k \leq n)$ are holomorphic. We have, in fact, that all meromorphic connections with logarithmic poles for a given L are regular.

Using this description, one can formulate the following theorem, in our humble opinion one of the deepest in the whole theory, one of the main reasons for all of this to be true.

Theorem 4.15 (Deligne)**.** *Let $\bar{X}$ be a compactification of a compact manifold X by a normal crossing divisor D; let M be an integrable connection on X. Then there exist a vector bundle L on $\bar{X}$ such that $\mathcal{O}_{\bar{X}}[D] \otimes_{\mathcal{O}_{\bar{X}}} L$ is a meromorphic connection along D with logarithmic pole for the line bundle L, and has the structure of a regular D-module such that on X restricts to M. Furthermore, this L is unique once chosen a (discountinuous) determination of the logarithm on the complex plane (that means, a discountinuous function $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ that is a section of the projection).*

The last part of this statement is very misterious (the image of τ determines where certain eigenvalues are going to lie), but we are not going to spend time explaining what is going on. The only thing the we would like to remark, is that this statement does not give a uniqueness statement for L, that is something we would want to prove an equivalence of categories. But, as changing the section τ we get many regular D-modules on $\bar{X}$, they turn out to be all isomorphic (because twisting by components of D keeps the restriction to X the same). We have in fact the following.

Corollary 4.16. *Let X be a complex manifold, and $\bar{X}$ a compactification by a (non necessarily normal crossing) divisor D; then restriction gives an equivalence of categories*

$$Conn^{reg}(\bar{X}, D) \xrightarrow{\sim} Conn(X).$$

The proof is based on a reduction to the normal crossing case by a resolution $\bar{X}' \to \bar{X}$, and then using theorem 4.15.

4.3 Proof of Deligne's Riemann-Hilbert corespondence

We are now ready to prove Deligne's Riemann-Hilbert correspondence.

Theorem 4.8 (Deligne, 1970). *Let X be an algebraic smooth variety. Then the de Rham functor gives an equivalence of categories*

$$Conn^{reg}(X) \xrightarrow{\sim} Loc(X^{an}).$$

By what we saw earlier in this section, we need to prove that in the commutative square the upper row is an equivalence:

$$\begin{array}{ccc} Conn^{reg}(\bar{X}, D) & \xrightarrow{an} & Conn^{reg}(\bar{X}^{an}, D^{an}) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle res} \\ Conn^{reg}(X) & \xrightarrow{an} & Conn(X^{an}) \end{array}$$

to get that the bottom one is an equivalence too. Now, in order to do it, we have to invoke Serre's GAGA, and are in the right setting because now we are dealing with the complete algebraic variety $\bar{X}$; there is a problem tough, that is the fact that for the first time we have to effectively relate the two different definitions of regularity (indeed, we haven't even given the one in the analytic setting); we are going to do it using the following proposition, that basically says that the two definitions are strictly related one each other; we will not prove it, because the proof is based on the reduction to the local case, and we decided not to go deep in details about local regularity in the analytic setting.

Proposition 4.17. *Let X be a smooth algebraic variety (non necessarily complete), M an integrable connection; then the following are equivalent:*

- *M is regular;*
- *for one completion $j : X \hookrightarrow \bar{X}$ by a divisor, the analytification of the direct image $(j_*M)^{an}$ is a regular meromorphic connection.*
- *for any completion $j : X \hookrightarrow \bar{X}$ by a divisor, the analytification of the direct image $(j_*M)^{an}$ is a regular meromorphic connection.*

So, if we prove that the analytification

$$Conn(\bar{X}, D) \xrightarrow{an} Conn(\bar{X}^{an}, D^{an})$$

is an equivalence, than by proposition 4.17 the subcategories of regular objects will be equivalent as well. Unfortunately, this functor to be an equivalence is too much to hope (more or less, for the same reason for which this does not work for integrable connections on X); we have again to define a subcategory of $Conn(\bar{X}^{an}, D^{an})$ to serve as target category to have an equivalence.

Definition 4.18. *Let M meromorphic connection on $\bar{X}^{an}$ along D^{an}; we will call it **effective** if as $\mathcal{O}_{\bar{X}^{an}}[D^{an}]$-module it is generated by a coherent $\mathcal{O}_{\bar{X}^{an}}$-module. We will denote the category of such objects by $Conn^e(\bar{X}^{an}, D^{an})$.*

Remark 4.19. Note that such a definition does not make sense in the algebraic setting, because an algebraic meromorphic connection is always generated as $\mathcal{O}_{\bar{X}}[D]$-module by a coherent $\mathcal{O}_{\bar{X}}$-module. It is true then that analytification functor maps $Conn(\bar{X}, D)$ into $Conn^e(\bar{X}^{an}, D^{an})$.

Remark 4.20. Note also that any regular analytic meromorphic connection is indeed effective, by theorem 4.15; so, we get a square

$$\begin{array}{ccc} Conn^{reg}(\bar{X}, D) & \xrightarrow{an} & Conn^{reg}(\bar{X}^{an}, D^{an}) \\ \downarrow & & \downarrow \\ Conn(\bar{X}, D) & \xrightarrow{an} & Conn^e(\bar{X}^{an}, D^{an}) \end{array}$$

where the lower arrow is well defined from the previous remark, and the vertical arrows are embeddings of subcategories (the right one because of the first sentence of this remark). Moreover, the square is commutative, because of proposition 4.17; so, proving that the lower arrow is an equivalence does imply that the upper one is too; this is what we are going to do, in order to prove Deligne's Riemann-Hilbert correspondence.

Proposition 4.21. *In the setting as above, analytification gives an equivalence of categories*

$$Conn(\bar{X}, D) \xrightarrow{\sim} Conn^e(\bar{X}^{an}, D^{an})$$

We will prove actually a slightly different statement, and leave out some technical details about differential operators. Let's consider the categories $Mod(\mathcal{O}_{\bar{X}}[D])$ and $Mod^e(\mathcal{O}_{\bar{X}^{an}}[D^{an}])$, consisting of algebraic (resp. analytic effective) coherent $\mathcal{O}_{\bar{X}}[D]$-modules (resp. $Mod^e(\mathcal{O}_{\bar{X}^{an}}[D^{an}])$-modules). Remember that an algebraic (resp. analytic regular) meromorphic connection is the data of such a module *and* a connection ∇; we have then forgetful functors

$$Conn(\bar{X}, D) \to Mod(\mathcal{O}_{\bar{X}}[D])$$
$$Conn^e(\bar{X}^{an}, D^{an}) \to Mod^e(\mathcal{O}_{\bar{X}^{an}}[D^{an}])$$

"forgetting" the connection. We are only going to prove the equivalence

$$Mod(\mathcal{O}_{\bar{X}}[D]) \xrightarrow{\sim} Mod^e(\mathcal{O}_{\bar{X}^{an}}[D^{an}])$$

given by analytification; to conclude the proof, one should also take into account the connections, and we will not get into the details about it.

Proof. Let's prove essential surjectivity; given $\tilde{M} \in Mod^e(\mathcal{O}_{\bar{X}^{an}}[D^{an}])$, we have a coherent $\mathcal{O}_{\bar{X}^{an}}$-module $\tilde{L}$ such that $\tilde{M} \cong \mathcal{O}_{\bar{X}^{an}}[D^{an}] \otimes_{\mathcal{O}_{\bar{X}^{an}}} \tilde{L}$; by Serre's GAGA, now, we have an algebraic coherent sheaf L on $\bar{X}$ such that $L^{an} \cong \tilde{L}$. So, if we consider $M \cong \mathcal{O}_{\bar{X}}[D] \otimes_{\mathcal{O}_{\bar{X}}} L$, we have

$$M^{an} = \mathcal{O}_{\bar{X}^{an}} \otimes_{\mathcal{O}_{\bar{X}}} (\mathcal{O}_{\bar{X}}[D] \otimes_{\mathcal{O}_{\bar{X}}} L) \cong$$

$$\cong (\mathcal{O}_{\bar{X}^{an}} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{X}}[D]) \otimes_{\mathcal{O}_{\bar{X}^{an}}} (\mathcal{O}_{\bar{X}^{an}} \otimes_{\mathcal{O}_{\bar{X}}} L) \cong \mathcal{O}_{\bar{X}^{an}}[D^{an}] \otimes_{\mathcal{O}_{\bar{X}^{an}}} \tilde{L} \cong \tilde{M}.$$

Let's now prove fully faithfulness, that is, given two algebraic coherent $\mathcal{O}_{\bar{X}}[D]$-modules M and N, we have an isomorphism

$$Hom_{\mathcal{O}_{\bar{X}}[D]}(M, N) \xrightarrow{\sim} Hom_{\mathcal{O}_{\bar{X}^{an}}[D]}(M^{an}, N^{an}).$$

Here, of course, the fact that we take homomorphisms as $\mathcal{O}_{\bar{X}}[D]$-modules is pretty far from considering homomorphism in the category $Conn(\bar{X}, D)$; here we see that a substantial amount of further work is needed to pass to these categories (considering the morphisms "respecting" also the connections). Now, let's consider a coherent sheaf M_0 such that $M \cong \mathcal{O}_{\bar{X}}[D] \otimes_{\mathcal{O}_{\bar{X}}} M_0$; we have isomorphisms

$$Hom_{\mathcal{O}_{\bar{X}}[D]}(M, N) \cong Hom_{\mathcal{O}_{\bar{X}}}(M_0, N)$$

$$Hom_{\mathcal{O}_{\bar{X}^{an}}[D]}(M^{an}, N^{an}) \cong Hom_{\mathcal{O}_{\bar{X}^{an}}}(M_0^{an}, N^{an})$$

that move us closed to a possible application of Serre's GAGA again. The only issue now is that N is not coherent; so, we consider a filtration of N by coherent $\mathcal{O}_{\bar{X}}$ modules obtained by the order of pole along D

$$N_0 \subset N_1 \subset \ldots \subset N_k \subset \ldots \subset N$$

so that we get, to conclude the proof

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\bar{X}}}(M_0, N) &\cong \bigcup_{i \geq 0} Hom_{\mathcal{O}_{\bar{X}}}(M_0, N_i) \cong \\ &\cong \bigcup_{i \geq 0} Hom_{\mathcal{O}_{\bar{X}^{an}}}(M_0^{an}, N_i^{an}) \cong \quad \text{(by Serre's GAGA)} \\ &\cong Hom_{\mathcal{O}_{\bar{X}^{an}}}(M_0^{an}, N^{an}). \end{aligned}$$

□

5 Riemann-Hilbert correspondence

Let's prove now the Riemann-Hilbert correspondence in its full generality, bringing in back algebraic holonomic D-modules, and defining regularity in this setting.

5.1 Regular holonomic D-modules

Remember that an holonomic D-module is composed by composition factors that are minimal extensions of integrable connections from locally closed subvarieties.

Definition 5.1. *Let X be a smooth algebraic variety, and M an holonomic D-module. We will call it* ***regular*** *if all composition factors are minimal extensions of regular integrable connections. The category of regular holonomic D-modules will be called $Mod_{rh}(D_X)$; the category of bounded complexes with cohomology in $Mod_{rh}(D_X)$ will be denoted by $D^b_{rh}(D_X)$.*

There is a definition of regularity also in the analytic setting, that as it happened in the previous chapter is harder to define (we have not in fact either a structure theorem to use to extend the definition from integrable connections); there is also a (harder) version of Riemann-Hilbert correspondence involving only analytic objects, that we will not show. We are now ready to state the main theorem of this paper. Remember that the category of bounded complexes of sheaves on X^{an} having as cohomology algebraically constructible sheaves is denoted by $D^b_c(X)$ (even if it is actually related with the topology of X^{an} rather than that of X).

Theorem 5.2 (Riemann-Hilbert correpondence for D-modules)**.** *Let X be a smooth algebraic variety. Then the de Rham functor give an equivalence of categories*

$$D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X).$$

The rest of the section is devoted to a sketch of the proof of this results. We first need some further technical results. As holonomic D-modules, also regular holonomic ones are stable under the usual operations.

Theorem 5.3. *Let X be a smooth algebraic manifold; then duality functor preserves $D^b_{rh}(D_X)$. Let $f : X \to Y$ be a morphism of smooth algebraic variery; then $\int_f$, $\int_{f!}$ send $Mod_{rh}(D_X)$ to $Mod_{rh}(D_Y)$, and $f^\dagger$ and $f^\star$ send $Mod_{rh}(D_Y)$ to $Mod_{rh}(D_X)$; the same holds for the derived categories $D^b_{rh}(D_X)$ and $D^b_{rh}(D_Y)$.*

Proof of this, that we will not give because it is basically a long sequence of reductions, deals again strictly with the definition of regularity; in particular, is strictly linked with the proof of the following structure theorem.

Theorem 5.4. *Let M be an holonomic D-module on a smooth algebraic variety X; then the following are equivalent:*

(i) *M is regular;*

(ii) *$i^\dagger M$ is regular for every locally closed embedding of a curve $i : C \hookrightarrow X$.*

The same holds for $M^\cdot \in D^b_h(D_X)$; in fact, the following are equivalent

(i) *$M^\cdot \in D^b_{rh}(D_X)$ is regular;*

(ii) *$i^\dagger M^\cdot \in D^b_{rh}(D_C)$ is regular for every locally closed embedding of a curve $i : C \hookrightarrow X$.*

5.2 Kashiwara contructibility theorem and perverse sheaves

At first, we have to prove that the functor is well defined; given a complex $M^\cdot \in D^b_{rh}(D_X)$, the de Rham complex $DR_X(M^\cdot)$ is bounded, and has as cohomology constructibe sheaves. To prove it, we will show it for an holonomic D-module; as seen in remark 3.13, the objects in $D^b_c(X)$ arising as image of holonomic modules are called perverse sheaves; let's give again the definition.

Definition 5.5. *Let X be a smooth algebraicvariety; an element $F^\cdot \in D^b_c(X)$ is called perverse sheaf if*

$$dimsupp(H^{-i}(F^\cdot)) \leq i$$

$$dimsupp(H^i(\mathbb{D}_X F^\cdot)) \leq i.$$

We will denote the category of perverse sheaves by $Perv(X)$.

Proposition 5.6 (Kashiwara's constructibility)**.** *Let X be a smooth algebraic variety, and M an holonomic D-module; then, the de Rham complex $DR_X(M)$ is in $D^b_c(X)$, and in particular is a perverse sheaf.*

The proof of this fact is a local check; the algebraic stratification of the complex of constructible sheaves $DR_X(M)$ is of course going to be determined by the composition factors of the D-module M, and more in particular by the structure of the closures of the locally closed subsets they come from.

As corollary of the full Riemann Hilbert correspondence, so, we get the following.

Corollary 5.7. *The de Rham functor gives an equivalence of categories*

$$Mod_{rh}(X) \to Perv(X).$$

5.3 Facts about the de Rham functor

We will need some technical facts about the de Rham functor, that will constitute the proof of the main theorem; in particular, that commutes with all the functors we described so far.

Let's recall all the functors we have; first, on a smooth algebraic variety X, we have the two functors

$$\mathbb{D}_X : D^b_c(D_X) \to D^b_c(D_X) \quad \mathbb{D}_X : D^b_c(X) \to D^b_c(X)$$

Proposition 5.8. *For $M^\cdot \in D^b_c(D_X)$ we have a canonical morphism*

$$DR_X(\mathbb{D}_X M^\cdot) \to \mathbb{D}_X(DR_X(M^\cdot)).$$

If $M^\cdot \in D^b_h(D_X)$, then this morphism is actually an isomorphism.

Proof. Given proposition 3.10, a morphism like the one in the proposition is equivalent to the following

$$Sol_X(M^\cdot)[d_X] \to \mathbb{D}_X(DR_X(M^\cdot))$$

$$DR_X(M^\cdot) \otimes_{\mathbb{C}^{an}} Sol_X(M^\cdot) \to \mathbb{C}^{an}[d_X]$$

$$\mathbb{D}_X(Sol_X(\mathbb{D}_X M^\cdot)) \otimes_{\mathbb{C}^{an}} Sol_X(M^\cdot) \to \mathbb{C}^{an}$$

$$\mathbb{D}_X(R\mathcal{H}om(\mathbb{D}_X(M^\cdot)^{an}, \mathcal{O}_{X^{an}})) \otimes_{\mathbb{C}^{an}} R\mathcal{H}om((M^\cdot)^{an}, \mathcal{O}_{X^{an}}) \to \mathbb{C}^{an}$$

$$R\mathcal{H}om(\mathcal{O}_{X^{an}}, (M^\cdot)^{an}) \otimes_{\mathbb{C}^{an}} R\mathcal{H}om((M^\cdot)^{an}, \mathcal{O}_{X^{an}}) \to R\mathcal{H}om(\mathcal{O}_{X^{an}}, \mathcal{O}_{X^{an}}).$$

The last one of which is actually canonical. Now, this morphism is obviously an isomorphism for integrable connections; working a little further, one can prove that it is an isomorphism for every $M^\cdot \in D^b_h(D_X)$. □

Furthermore, given a morphism of smooth algebraic varieties $f : X \to Y$, we have functors

$$\int_f : D^b_c(D_X) \to D^b_c(D_Y)$$

$$\int_{f!} = \mathbb{D}_Y \circ \int_f \circ \mathbb{D}_X : D^b_c(D_X) \to D^b_c(D_Y)$$

$$f^\dagger = Rf^*[d_X - d_Y] : D^b_c(D_Y) \to D^b_c(D_X)$$
$$f^\star = \mathbb{D}_X \circ f^\dagger \circ \mathbb{D}_{\mathbb{Y}} : D^b_c(D_Y) \to D^b_c(D_X)$$
$$Rf_* : D^b_c(X) \to D^b_c(Y)$$
$$Rf_! : D^b_c(X) \to D^b_c(Y)$$
$$f^{-1} : D^b_c(Y) \to D^b_c(X)$$
$$f^! = \mathbb{D}_X \circ f^{-1} \circ \mathbb{D}_{\mathbb{Y}} : D^b_c(Y) \to D^b_c(X)$$

Let's try to match them, and have some commutivity relationships; note that the shift that we have in $f^\dagger$ isn't in the constructible sheaves counterparts f^{-1} or $f^!$.

Proposition 5.9. *Given $M^\cdot \in D^b_c(D_X)$ have a canonical morphism*

$$DR_Y\left(\int_f M^\cdot\right) \to Rf_*(DR_X(M^\cdot))$$

and if $M^\cdot \in D^b_{rh}(X)$ then this is an isomorphism.

Proof. We have

$$DR_Y\left(\int_f M^\cdot\right) = DR_{Y^{an}}\left(\int_f M^\cdot\right)^{an} \to DR_{Y^{an}}\int_{f^{an}}(M^\cdot)^{an} \cong$$
$$\cong \Omega_{Y^{an}} \otimes^L_{D_{Y^{an}}} Rf_*(D_{Y^{an}\leftarrow X^{an}} \otimes^L_{D_{X^{an}}} (M^\cdot)^{an}) \cong$$
$$\cong Rf_*(f^{-1}\Omega_{Y^{an}} \otimes^L_{f^{-1}D_{Y^{an}}} D_{Y^{an}\leftarrow X^{an}} \otimes^L_{D_{X^{an}}} (M^\cdot)^{an}) \cong$$
$$\cong Rf_*(\Omega_{X^{an}} \otimes^L_{D_{X^{an}}} M^\cdot) \cong Rf_* DR_{X^{an}}(M^\cdot)^{an} = Rf_* DR_X M^\cdot.$$

The only arrow comes from proposition 2.8 ii), and it is an isomorphisms if the morphism is proper. Then, the result is true for *regular* integrable connections (note that this is the only place in which we will use regularity), and as usual one can conlude using the structure theorem for regular holonomic D-modules. □

Corollary 5.10. *Given $M^\cdot \in D^b_c(D_X)$, we also have a canonical morphism*

$$Rf_!(DR_X(M^\cdot)) \to DR_Y\left(\int_{f!} M^\cdot\right)$$

and if $M^\cdot \in D^b_{rh}(D_X)$ then this is an isomorphism too.

Proof. Everything follows from proposition 5.9 by taking duals, and multiple use of proposition 5.8. □

Let's go also in the other direction.

Proposition 5.11. *If $N^\cdot \in D^b_c(D_Y)$ have a canonical morphism*

$$DR_X(f^\dagger N^\cdot) \to f^! DR_Y(N^\cdot)$$

that is an isomorphism if $N^\cdot \in D^b_h(D_Y)$.

Proof. We have the following morphisms

$$Hom_{D^b_c(D_Y)}(f^\dagger N^\cdot, f^\dagger N^\cdot) \cong Hom_{D^b_c(D_Y)}\left(\int_{f!} f^\dagger N^\cdot, N^\cdot\right) \to$$

$$\to Hom_{D^b_c(Y)}\left(DR_Y\left(\int_{f!} f^\dagger N^\cdot\right), DR_Y N^\cdot\right) \to$$

$$\to Hom_{D^b_c(Y)}\left(Rf_! DR_X\left(f^\dagger N^\cdot\right), DR_Y N^\cdot\right) \cong$$

$$\cong Hom_{D^b_c(X)}\left(DR_X\left(f^\dagger N^\cdot\right), f^! DR_Y N^\cdot\right)$$

where we used adjunction formulas, so that we have a canonical element that is image of the identity in the first. The proof of the regular holonomic case is a little more tricky, and is based on a reduction to the case of f begin a closed embedding. □

Again, we have a corollary obtained considering duals and applying proposition 5.8.

Corollary 5.12. *Given $N^\cdot \in D^b_c(D_Y)$, we also have a canonical morphism*

$$f^{-1}(DR_Y(N^\cdot)) \to DR_X\left(f^\star N^\cdot\right)$$

and if $N^\cdot \in D^b_h(D_Y)$ then this is an isomorphism too.

We need some further work; in particular, we would like de Rham functor to commute also with Hom functor, and tensor products. This is very hard to prove directly; we will use instead the box product, that is more manageable.

Definition 5.13. *Let M, N be D-modules on a smooth algebraic variety X; their **box product** (also called exterior tensor product) is the D-module on $X \times X$ given by*

$$M \boxtimes N = D_{X\times X} \otimes_{p_1^{-1}D_X \otimes_{\mathbb{C}} p_2^{-1}D_X} (p_1^{-1}M \otimes_{\mathbb{C}} p_2^{-1}N).$$

Intuitively, this product should be better than the usual tensor product, because it keeps separated the structure of the two modules. But there's more.

Lemma 5.14. *The functor $\boxtimes : Mod(D_X) \times Mod(D_X) \to Mod(D_{X\times X})$ is exact in each factor. In particular, it induces the functor*

$$\boxtimes : D^b(D_X) \times D^b(D_X) \to D^b(D_{X\times X}).$$

The two products are related tough, by the following proposition (that follows from the equivalent for $\mathcal{O}_X$-modules).

Proposition 5.15. *Let $M^\cdot, N^\cdot \in D^b(D_X)$, and let $\Delta : X \to X \times X$ be the diagonal embedding. Then we have canonical isomorphism*

$$M^\cdot \otimes^L_{\mathcal{O}_X} N^\cdot \cong L\Delta^*(M^\cdot \boxtimes N^\cdot)$$

As corollary, we have the following, that will be then core of the proof of fully faithfulness in the main theorem.

Corollary 5.16. *In the same setting, if $M^\cdot, N^\cdot \in D^b_h(D_X)$ we have also an isomorphism*

$$RHom_{D^b_c(D_X)}(M^\cdot, N^\cdot) \cong \int_q (\Delta^\dagger(\mathbb{D}_X M^\cdot \boxtimes N^\cdot))$$

where q is the projection of X onto a point.

Remark 5.17. We have an analogous isomorphism for constructible sheaves; namely, if $F^\cdot, G^\cdot \in D^b_c(X)$, then we define

$$F^\cdot \boxtimes_{\mathbb{C}} G^\cdot = p_1^{-1}F^\cdot \otimes_{\mathbb{C}} p_2^{-1}G^\cdot$$

and we have an isomorphism

$$RHom_{D^b_c(X)}(F^\cdot, G^\cdot) \cong Rq_*(\Delta^!(\mathbb{D}_X M^\cdot \boxtimes_{\mathbb{C}} N^\cdot)).$$

Now that we have this new functor, we can prove that the de Rham functor commutes with this too.

Proposition 5.18. *Let again $M^\cdot, N^\cdot \in D^b_h(D_X)$, then we have a canonical moprhism*

$$DR_X M^\cdot \boxtimes_{\mathbb{C}} DR_X N^\cdot \to DR_{X\times X}(M^\cdot \boxtimes N^\cdot)$$

that is an isomorphism if one between $M^\cdot$ and $N^\cdot$ is in $D^b_h(D_X)$.

5.4 Proof of the main theorem

We are now ready to prove the main theorem of the paper.

Theorem 5.2 (Riemann-Hilbert correpondence for D-modules). *Let X be a smooth algebraic variety. Then the de Rham functor give an equivalence of categories*

$$D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X).$$

Proof. Let's prove essential surjectivity first; let's take a generator in the category $D^b_c(X)$ of constructible sheaves, that is, the extension Ri_*L of a local system L for the embedding of a locally closed subvariety $i : Z \to X$; by Deligne Riemann-Hilbert correspondence, we have a regular integrable connection N on Z such that $DR_Z(N) = L[d_Z]$. Then if we take $M = \int_i N[-d_Z]$ we have

$$DR_X(\int_i N[-d_Z]) \cong Ri_*(DR_Z(N[-d_Z])) \cong Ri_*L$$

by the commutativity of the de Rham functor with direct image, proposition 5.9.

Let's prove now fully faithfulness; let $M^\cdot, N^\cdot \in D^b_{rh}(D_X)$. We need to prove that the natural map

$$Hom_{D^b_c(D_X)}(M^\cdot, N^\cdot) \to Hom_{D^b_C(X)}(DR_X M^\cdot, DR_X N^\cdot)$$

is a bijection; we will actually prove something further, that is, that

$$RHom_{D^b_c(D_X)}(M^\cdot, N^\cdot) \cong RHom_{D^b_C(X)}(DR_X M^\cdot, DR_X N^\cdot).$$

Let's prove this isomorphism; we will omit to prove that this isomorphism is actually given by the de Rham functor. We have

$$\begin{aligned}
RHom_{D^b_c(D_X)}(M^\cdot, N^\cdot) \cong & \\
\cong \textstyle\int_q(\Delta^\dagger(\mathbb{D}_X M^\cdot \boxtimes N^\cdot)) \cong & \quad \text{(proposition 5.16)}\\
\cong DR_{pt} \textstyle\int_q(\Delta^\dagger(\mathbb{D}_X M^\cdot \boxtimes N^\cdot)) \cong & \quad (DR_{pt} = Id)\\
\cong Rq_* DR_X(\Delta^\dagger(\mathbb{D}_X M^\cdot \boxtimes N^\cdot)) \cong & \quad \text{(proposition 5.9)}\\
\cong Rq_* \Delta^! DR_{X\times X}(\mathbb{D}_X M^\cdot \boxtimes N^\cdot) \cong & \quad \text{(proposition 5.11)}\\
\cong Rq_* \Delta^! DR_{X\times X}(\mathbb{D}_X M^\cdot \boxtimes N^\cdot) \cong & \quad \text{(proposition 5.11)}\\
\cong Rq_* \Delta^!(DR_X \mathbb{D}_X M^\cdot \boxtimes DR_X N^\cdot) \cong & \quad \text{(proposition 5.18)}\\
\cong Rq_* \Delta^!(\mathbb{D}_X(DR_X M^\cdot) \boxtimes DR_X N^\cdot) \cong & \quad \text{(proposition 5.8)}\\
\cong Hom_{D^b_c(X)}(DR_X M^\cdot, DR_X N^\cdot) & \quad \text{(remark 5.17)}
\end{aligned}$$

that completes the proof. □

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