Overview - Ben Bakker

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Ref: ① Popa, Positivity for Hodge modules & Geometric applications
② Schnell, An overview of Morihiko Saito's theory of Mixed Hodge modules

Motivation -

$X/\mathbb{C}$ smooth proj variety

We have local systems $/X$, which we can study.
But when non-smooth maps show up, we cannot remain in the world of local systems.

$$\text{Sheaf of flat sections} \longleftarrow E$$
$$V \longmapsto V \otimes \mathcal{O}_X$$

$$\begin{tikzcd}
\text{Constr. sheaves} & \text{Loc Sys on } X \arrow[l, hook'] \arrow[r, leftrightarrow] \arrow[d] & \text{V.b with flat conn.} \arrow[d] & \text{VHS} \arrow[l, hook'] \arrow[d] \\
& \text{Perverse Sheaves} \arrow[r, leftrightarrow] & \mathcal{D}\text{-modules} & \boxed{\text{Hodge Modules}} \arrow[l, hook']
\end{tikzcd}$$

Riemann-Hilbert correspondence

Among {V.b. with flat connection}, we have Variations of Hodge Struct.
VHS = ($\mathbb{Q}$)- local system V with a filtration by holomorphic subbundles satisfying some properties.

Goal: (A) Generalize VHS to possibly singular situations.
(B) Provide unifying context for the following -

(1) Positivity: $X \xrightarrow{f} Y$, $f_*\omega$ has positivity properties
(2) Vanishing: analogous to Kodaira Vanishing.
(3) Decomposition: • Leray spectral seq degenerates

What is a Hodge Module?

$\mathcal{D}_X$ = Sheaf of differential operators
$= \mathcal{O}_X$-Subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by derivations of $\mathcal{O}_X$.

A left/right $\mathcal{D}$-module is a left/right module over $\mathcal{D}_X$.

Notation: $F_k\mathcal{D}_X$ = Order $\leq k$ diff ops.

A filtered $\mathcal{D}$-module M has an increasing filtration $F_\bullet M$ such that $F_k\mathcal{D}_X F_\ell M \subset F_{k+\ell} M$.

e.g. 1) $M = \mathcal{O}_X$. $F_k\mathcal{O}_X = \begin{cases} 0 & k<0 \\ \mathcal{O}_X & k \geq 0. \end{cases}$

trivial filtration.

2) E a flat vector bundle. The flatness implies that we get an action of $\mathcal{D}_X$ on E.

Why? Action of derivations is given by the connection

$$T_X \otimes E \xrightarrow{\nabla} E$$

Flatness $\Rightarrow$ $[\nabla_x, \nabla_y] = \nabla_{[x,y]}$

$\Rightarrow$ Action extends to an algebra action of $\mathcal{D}_X$.

3) ω_X is a right $\mathcal{D}_X$-module

$$\omega \cdot \xi = -L_\xi \omega$$
$$= -d(i_\xi \omega)$$

Allows us to go from left $\leftrightarrow$ right modules by taking $\mathrm{Hom}(\omega_X, -)$.

Let M be a filtered $\mathcal{D}$-module. We get a $\mathbb{C}$-linear complex

$$DR(M) = [M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to \cdots \to M \otimes \Omega^n]$$

↳ complex of abelian groups whose coh is constr.

Derived category of complexes whose coh is constr.

Turns out that this is a perverse sheaf.

We have a filtration —

$$\begin{array}{ccc} DR(M) & = & [M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to \cdots \to M \otimes \Omega^n] \\ \cup & & \\ F_k DR(M) & = & [F_k M \to F_k M \otimes \Omega^1 \to \cdots] \end{array}$$

The associated spec. seq. is the Hodge → DeRham seq.

In particular, $DR(\mathcal{O}_X) \cong \mathbb{C}[n]$.

A pure polarizable Hodge module of weight l consists of —

(1) Filtered $\mathcal{D}$-module M satisfying ....

(2) $\mathbb{Q}$-perverse sheaf P such that

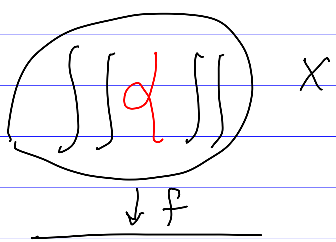
$$P \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} DR(M).$$

The main requirement is inductive:

* if $\dim \operatorname{supp} M = 0$, then this should be a pure polarizable Hodge structure.

* inductively, for any local holomorphic function f, the nearby and vanishing cycles of M wrt f must be a pure polarizable Hodge module.

these are supported on the special fiber.



Two ways to think about this:

1) A Hodge module is an extra gadget on a perverse sheaf such that any way of producing a vector space from a perverse sheaf now produces a Hodge structure

2) A very special type of D-module that has a filtration & $\mathbb{Q}$-structure given by a perverse sheaf.

e.g. $M = \mathcal{O}_X$. $DR(\mathcal{O}_X) = \mathbb{C}_X[n]$

$$\left.\begin{array}{c} DR(\mathcal{O}_X) = \mathbb{C}_X[n] \\ \cup \\ \mathbb{Q}_X[n] \end{array}\right\} \mathbb{Q}^H_X[n]$$

a Hodge module.

True facts & applications

1) Structure theorem - $Z \subset X$ irreducible.

(a) A pure polarizable Hodge structure on an open $U \subset Z$ extends uniquely to a ppH module on Z.

Analogous to

Every local system V on U extending to $IC_Z(V)$ on Z.

(b) Every ppHM on Z with strict support Z is obtained from a pp VHS on some open $U \subset Z$ as above.

2) Stability thm: $X \xrightarrow{f} Y$; X, Y smooth.
& M a ppHM on X. Then
$\mathcal{H}^i(f_+ M)$ underlies a ppHM and
$f_+ M$ is strict.

e.g. $X \xrightarrow{f} pt$. Then $\mathcal{H}^i(f_+ M) = H^{n-i}(X, \mathbb{C})$
$M = \mathbb{Q}^H_X[n]$ Strict $\Rightarrow$ Degeneration of Hodge to de Rham.

3) <u>Decomposition thm</u> $X \xrightarrow{f} Y$; X, Y sm. proj.

M a ppHM on X. Then

$$f_+ M \cong \bigoplus \mathcal{H}^i(f_+ M)[-i]$$

<u>Consequence</u>: $M = \mathbb{Q}_X^H[n]$.

Then one of the filtered pieces of f_+M is $Rf_* \omega$.

We then recover a theorem of Kollar:

$$Rf_* \omega \cong \bigoplus R^i f_* \omega [-i]$$

4) <u>Vanishing</u> – Mega version of Kodaira Vanishing

L ample on X. M ppHM on X.

Then

$$\mathbb{H}^i(X, \mathrm{gr}_k^F DR(M) \otimes L) = 0. \quad \text{for } i > 0$$

For $M = \mathbb{Q}_X^H[n]$, we get

$$H^q(X, \Omega^p \otimes L) = 0 \quad \text{for } p+q > n.$$

For $X \xrightarrow{f} Y$ and $M = f_+ \mathbb{Q}_X^H[n]$

Then vanishing above $\Rightarrow$ Kóllár vanishing

$$H^i(Y, R^j f_* \omega_X \otimes L) = 0$$

$\forall j, \forall i > 0$

Applications of Vanishing/Structure theorems to geometry

① Generic vanishing.

X sm proj; F a sheaf.

Define $V^i(F) = \{L \in \mathrm{Pic}^0 X \mid h^i(X, F \otimes L) \neq 0\}$.

<u>Thm</u>: $\quad \operatorname{codim} V^i(\mathcal{O}_X) \geq i - \dim a(X)$, where

$a : X \to \mathrm{alb}(X)$ is the map to albanese.

Green-Lazarsfeld proved this using deformation theory of sections + Hodge theory.

Hacon proved this by Fourier-Mukai transforms and Kollár decomp + vanishing.

With Hodge-modules, we can do much more;

only the ones like $\mathbb{Q}_X^H$ etc.

$$\operatorname{codim}(V^i(\Omega^p)) \geq |i+p-n| + n - \dim a(X).$$