Classical Hodge Theory

Ref: The MHM project, Chapter 0, Sabbah-Schnell.

X a smooth projective variety $/ \mathbb{C}$.

$$H^*(X, \mathbb{Z}) = \bigoplus_K H^K(X, \mathbb{Z}).$$

Basic Thm:

$$H^K(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^{p,q}(X)$$

$$H^{p,q}(X) = H^q(X, \Omega_X^p).$$

$$= \{\text{Classes of closed } (p,q) \text{ forms}\}$$

i.e. locally $f \cdot dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$

So $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Hodge Structure: Abstraction of this decomposition.

Def 0: A $(\mathbb{Z}, \mathbb{Q}, \mathbb{R})$ - Hodge structure of weight k is a free module H along with a $\mathbb{C}$-linear decomp.

$$H \otimes \mathbb{C} = \bigoplus_{p+q=K} H^{p,q} \quad \text{such that} \quad H^{p,q} = \overline{H}^{q,p}.$$

Families: $\pi : \mathcal{X} \to S$ a family of sm proj var.

Get $H_{\mathbb{Q}} = R^K \pi_*(\mathbb{Q})$, a local system on S.

$$H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C} \quad \text{a (flat) } \mathbb{C}\text{-v.b.}$$

Then $H = \bigoplus H^{p,q}$ where

$$H^{p,q} = R^q \pi_*(\Omega_X^p).$$

Abstract this idea:

A VHS of wt K on S is a flat $\mathbb{C}$-vector bundle H with a decomposition
$$H = \bigoplus_{p+q} H^{p,q} \quad + \quad \dots$$
(1)

A $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ structure on this is a local system $H_{\mathbb{Z}}/H_{\mathbb{Q}}/H_{\mathbb{R}}$ & an isomorphism
$$H \cong H_{\mathbb{Q}} \otimes \mathbb{C}$$
(2)

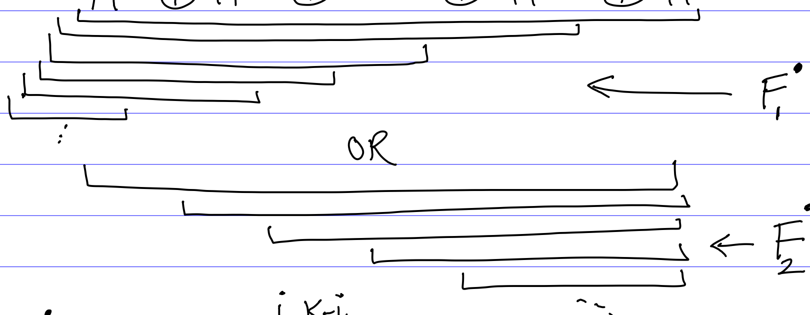
Part (1) & (2) generalize in separate ways.

(1) $\to$ $\mathbb{C}$ Hodgemodules (built from D-modules)

(2) $\longrightarrow$ Perverse sheaves $\xrightarrow{\text{Riemann-Hilb corr.}}$ Hodge module with $\mathbb{Q}$ str.

Rem: In the geometric setting, $H^{p,q} \subset H$ are not holomorphic sub-bundles. (only C^∞).

But:
$$H^{K,0} \oplus H^{K-1,1} \oplus \cdots \oplus H^{1,K-1} \oplus H^{0,K}$$



$$F_1^\bullet = \bigoplus_{i \geq \bullet} H^{i,K-i}$$
$$F_2^\bullet = \bigoplus_{j \geq \bullet} H^{K-j,j}$$

$F_1^i \cap F_2^j = 0$ if $i+j > K$

and

$F_1^i \oplus F_2^j \xrightarrow{\sim} H$ if $i+j = K$.

... ① $F_1^\bullet \subset H$ are holomorphic sub-bundles
$F_2^\bullet \subset H$ are anti-holomorphic sub-bundles.

② Griffiths transversality:

$$F_1^p \otimes T_S \xrightarrow{\nabla} F_1^{p-1} \quad \text{i.e.}$$

$$F_1^p \xrightarrow{\nabla} \Omega_1^S \otimes F_1^{p-1}$$

(Will be important in making H a $\mathcal{D}$-module.)

Back to Hodge structures:

Def 1: A $\mathbb{C}$-Hodge structure of weight ω is a $\mathbb{C}$ vector space H with two ω-opposite filtrations $F_1^\bullet$ and $F_2^\bullet$

$$F_1^i \cap F_2^j = 0 \quad \text{if} \quad i+j \geq K$$

$$\& \quad F_1^i + F_2^j \xrightarrow{\sim} H \quad \text{if} \quad i+j = K.$$

Rem: This is equivalent to prev. def. in terms of $H^{p,q}$.

Operations

① Tensor product

$$F^p(H_1 \otimes H_2) = \sum_{p_1+p_2=p} F^{p_1}(H_1) \otimes F^{p_2}(H_2)$$

② Hom:

$$F^p(\mathrm{Hom}(H_1,H_2)) = \{ f \mid f(F^i H_1) \subset F^{i+k} H_2 \}$$

③ Dual $\mathrm{Hom}(-,\mathbb{C})$. weight $-\omega$

④ Conjugation: $(\overline{H}, \overline{F_2}, \overline{F_1})$ same weight.

⑤ **Adjoint** : conjugate & dual.

⑥ **Shift** :

$$(H, F_1^\bullet[k], F_2^\bullet[\ell]) \quad \text{weight } \omega-k-\ell.$$

$$F^\bullet[k] = F^{\bullet+k}.$$

Effect on $\oplus$ decomp

$$H[k,\ell]^{p,q} = H^{p-k,\,q-\ell}.$$

Particular case: "Tate twist"

$$H \to H(k,k)$$

VHS :

$$\begin{tikzcd} & (\mathcal{H}, F_1^\bullet, F_2^\bullet) \arrow[dl] \arrow[d] \arrow[dr] & \\ \text{flat } \mathbb{C}\text{-v.b.} & \text{Hol filt} + \text{Griff transv.} & \text{Antihol filt} + \text{Anti Griff transv.} \end{tikzcd}$$

Polarization

Back to geometric situation $X/\mathbb{C}$. of dim n

$$\begin{tikzcd} H^*(X,\mathbb{Z}) \otimes H^*(X,\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{Z} & \text{bilinear} \\ H^*(X,\mathbb{C}) \otimes H^*(X,\mathbb{C}) \arrow[r, "\langle\,,\,\rangle_n"] & \mathbb{C} & \text{bilinear.} \end{tikzcd}$$

Then $\langle H^{p,q}, H^{r,s} \rangle = 0$ if $p+r \neq n$ & $q+s \neq n$.

$$\langle x, y \rangle_0 := \langle x, \bar{y} \rangle_n.$$

$\langle , \rangle_0$ is Hermitian $(-1)^n$ symmetric
Non deg pairing.
<u>Moreover</u>
restriction to each $H^{p,n-p}$ is non deg
because $\overline{H^{p,n-p}} = H^{n-p,p}$.

i.e. $\langle , \rangle_0 : H \otimes \overline{H} \to \mathbb{C}(-n)$.

or $\langle , \rangle_0 : H \xrightarrow{\sim} \overline{H}(-\omega)$.

<u>Def</u>: A <u>polarization</u> on a VHS $\mathcal{H}$ is a map of Hodge structures

$$\mathcal{H} \otimes \overline{\mathcal{H}} \longrightarrow \mathbb{C}(-w)$$

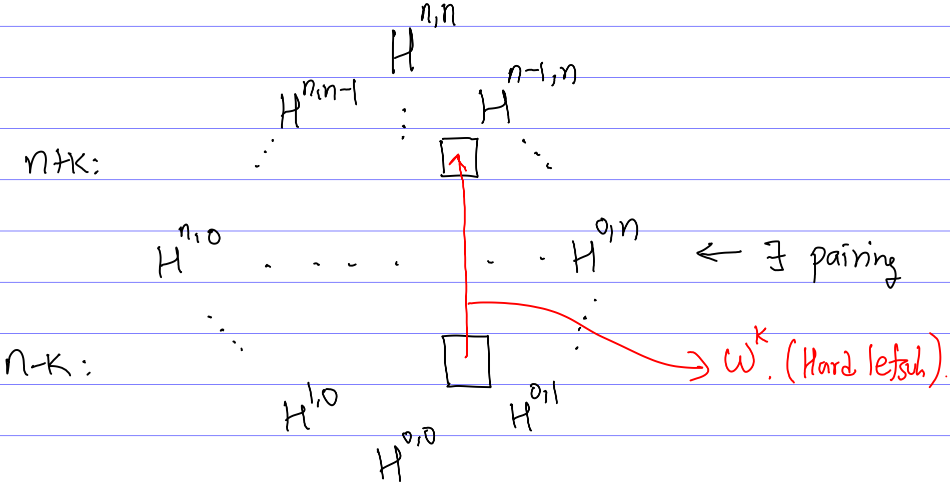
which is (fiberwise) a non-deg pairing. + ⊛.

<u>Rmk</u>: <u>Map of Hodge structures</u> includes ① <u>flat</u> map
(i.e. $e_1 \otimes e_2$ is locally constant if e_1 & e_2 are)
② plays well with filtrations.
⊛ : There is a positive definiteness condition, which I am suppressing.

<u>How to get a polarization in other degrees?</u>

Use the Hard Lefschetz theorem.
Let $\omega = c_1(L)$, where L is an ample line bundle.
Then $\omega \in H^{1,1}(X)$.

<u>Thm</u>: $\omega^k : H^{n-k}(X) \to H^{n+k}(X)$ is an iso.



So: if we want to pair $a, b \in H^{n-k}$, then we can set

$$\langle a,b \rangle_n = \langle a, \omega^k b \rangle_n$$

or to pair $a, b \in H^{p,q}$ with $p+q = n-k$ set

$$\langle a, b \rangle_0 = \langle a, \overline{\omega^k b} \rangle_n .$$

Rmk: There are usually some signs and powers of i to make the resulting form positive definite on primitive cohomology.