D-Modules (Hodge modules seminar) 23 Feb 2017.

Defn: Let X be a smooth complex algebraic variety. Let T_X = tangent sheaf = $Der(\mathcal{O}_X)$. Then $\mathcal{D}_X \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ is the subsheaf of algebras generated by $\mathcal{O}_X$ & T_X

Alternatively, set $F_0 \mathcal{D}_X = \mathcal{O}_X$ and $F_i \mathcal{D}_X = \{ P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid [P, f] \in F_{i-1} \mathcal{D}_X$ for any $f \in \mathcal{O}_X \}$

This defines $\mathcal{D}_X$ together with the order filtration.

Locally, on some $\mathcal{U}$, we have $\mathcal{D}_{\mathcal{U}} = \bigoplus \mathcal{O}_{\mathcal{U}} \partial^{\alpha}$

Eg. $\mathbb{C}[x_1, \ldots, x_n] \leadsto \mathcal{D} = \mathbb{C}\langle x_1, \ldots x_n, \partial_1, \ldots \partial_n \rangle$ + relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$

$[\partial_i, x_j] = \delta_{ij}$.

A (left) $\mathcal{D}_X$-module is a sheaf of modules on $\mathcal{D}_X$. Given a quasi-coherent $\mathcal{O}_X$-module, it is enough to specify a connection $\nabla : T_X \to \mathcal{E}nd_{\mathbb{C}} M$ + flatness

Example: On $X = \mathbb{A}^1$, take $\mathcal{O}_X = M$ & set $\nabla_\partial(f) := \partial(f)$ (differentiate)

On $X = \mathbb{A}^1$, take $M = \mathbb{C}[x]/x \simeq \mathbb{C}$, supported at 0.

We need to define $\nabla_\partial(1) := c$, but $\nabla_\partial(f \cdot 1) = f \nabla_\partial(1) + (\partial f) \cdot 1$

LHS $= \nabla_\partial(f(0) \cdot 1) = f(0) \cdot c$; RHS $= f(0) \cdot c + (\partial f)(0) \to$ not possible!
We'll come back to this

A $\mathcal{D}_X$-module is coherent if it is a quasicoherent $\mathcal{O}_X$-module finitely generated over $\mathcal{D}$

E.g. $\mathbb{C}[x, x^{-1}]$, generated by x^{-1}. (there are many more of these)

Recall the order filtration $F_i \mathcal{D}_X$. Let M be a $\mathcal{D}_X$-module quasicoherent over $\mathcal{O}_X$

Then we define the notion of a good filtration $F_\bullet M$:

(i) $F_i M \subseteq F_{i+1} M$
(ii) $F_i M = 0$ for $i \ll 0$
(iii) $M = \bigcup_i F_i$
(iv) $(F_i \mathcal{D}_X)(F_j M) \subseteq F_{i+j} M$

} any filtration

$F_\bullet$ is good if:

(i) $F_i M$ coherent over $\mathcal{O}_X$ for any i, and $\exists\, i_0$ s.t. M is generated in degree at most i_0 as a $\mathcal{D}_X$-module
(ii) Equivalently, $gr^F M$ is coherent over $\pi_* \mathcal{O}_{T^*X}$.

Fact: Any coherent $\mathcal{D}_X$-module admits a good filtration.
It is not unique, but any two are commensurable:
$\exists\, i_0$ s.t. $F'_{i-i_0} M \subseteq F_i M \subseteq F'_{i+i_0} M$.

$SS(M) :=$ support of $gr^F M$ in T^*X (reduced)

Fact: It is independent of the choice of good filtration.

Thm: For any $\mathcal{D}$-module M, $\dim(SS(M)) \geq \dim X$.

Previous example: $M = \mathbb{C}[x]/x$: best you can do is take pushforward along $\bullet \hookrightarrow$ of $\mathbb{C}$, and we get $\mathbb{C}[\partial]$, defined as follows: $x \cdot 1 := 0$; $\partial_x \cdot 1 := \partial$

$= \mathcal{D}_X / \mathcal{D}_X \cdot x$

This has $SS(M) = y$-axis.

Kashiwara's thm: Let $Y \xrightarrow{f} X$ closed embedding. If M is a $\mathcal{D}_X$-module supported (as an $\mathcal{O}_X$-module) on Y, then
$M = \int_f M_Y$ for some M_Y.

Functors:

Given $f: X \to Y$ & M a $\mathcal{D}_Y$-module, we have:

$$f^* M := \underbrace{(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)}_{\mathcal{D}_{X\to Y}} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}(M)$$

$$f^+ M := Lf^* [\dim X - \dim Y]$$

Pushforward: If M a right $\mathcal{D}_X$-module, $f_* M := f_*[M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}]$

If M a left $\mathcal{D}_X$-module, then $\int_f M := f_*((\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}$

Set $\mathcal{D}_{Y\leftarrow X} = (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1})$; get $\int_f M = f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{O}_X} M)$

If $M^\cdot$ is a complex, $\int_f M^\cdot = Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{O}_X} M)$: i^+ & $\int_i$ are inverse for closed embedding.

$$\mathbb{D} M^\cdot := R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1} [d_X]$$

Holonomic $\mathcal{D}$-modules & the solutions functor

Recall M is holonomic if $\dim SS(M)$ is minimal.
If $f: X \to Y$, then direct & inverse image take hol. modules to hol. modules, and $\mathbb{D}$ takes hol. mods to hol. modules.

Define $\int_{f!} := \mathbb{D}_Y \int_f \mathbb{D}_X$ & $f^* := \mathbb{D}_X f^+ \mathbb{D}_Y$; adj to f^+ & $\int_f$

Minimal extension

If $i: Z \hookrightarrow Y$, locally closed embedding, then

$$\int_{i!} M \to \int_i M \quad \text{if } M \text{ a } \mathcal{D}_Z\text{-mod}$$

The image of this map is the minimal ext.

Example : $\mathcal{U} = \mathbb{A}^1 \setminus \{0\} \overset{j}{\hookrightarrow} \mathbb{A}^1$

$\int_j \mathcal{O}_{\mathcal{U}} = \mathbb{C}[x, x^{-1}]$; $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, x^{-1}] \twoheadrightarrow \mathbb{C}[\partial] \to 0$

$\int_{j!} \mathcal{O}_{\mathcal{U}} = \mathcal{D}_X / (\mathcal{D}_X x\partial_x)$ & $0 \to \mathcal{D}/\mathcal{D}_X \cdot x \to \mathcal{D}_X/\mathcal{D}_X(x\partial_x) \to \mathcal{O}_X \to 0$

Minimal ext of $\mathcal{O}_{\mathcal{U}}$ along j is $\mathcal{O}_X$.

- Every holonomic module is finite length
- Simple hol $\Rightarrow$ min ext of int conn
- $0 \to M' \to M \to M'' \to 0$; any 2 hol $\Rightarrow$ 3rd hol.

$DR(M) = \Omega_X \overset{L}{\otimes}_{\mathcal{D}_X} M$

Regular holonomic $\overset{?}{\to}$