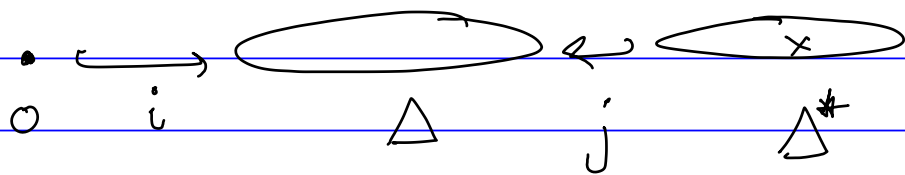


# Perverse sheaves on curves / Vanishing and nearby cycles



$F \in \mathcal{D}^b(\Delta)$  is perverse if:

1)  $i^* F \in \mathcal{D}_0^{\leq 0}$

$i^! F \in \mathcal{D}_0^{\geq 0}$



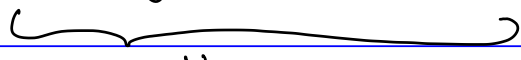
$(i^* F \in \mathcal{D}^{\leq 0})$



$i^* F = \begin{matrix} * & * \\ -1 & 0 \end{matrix}$

2)  $j^* F \in \mathcal{D}_{\Delta^*}^{\leq 0}$

$j^! F \in \mathcal{D}_{\Delta^*}^{\geq 0}$



$j^* F = j^! F \in \text{LocSys}[1]$

	pt	$\Delta^*$
$H^{-2}$	0	0
$H^{-1}$	*	*
$H^0$	*	0
$H^1$	0	0

So we have one exact functor

$j^*: \mathcal{M}_0 \longrightarrow \text{LocSys}_{\Delta^*}$

We want another exact functor that measures the perverse sheaves at 0

?? :  $\mathcal{M}_0 \longrightarrow \text{Vect}_0 = \text{LocSys}_0$

$i^*$  Does not work.

Replacement - vanishing / nearby cycles.

①  $\mathcal{L}$  a loc syst on  $\Delta^*$ .

Then  $\mathcal{L} \iff L =$  a vector space

$T: L \xrightarrow{\sim} L$  monodromy.

Choose  $\tilde{\Delta}^* \xrightarrow{p} \Delta^*$  univ. cover

Generator  $T$  of  $\pi_1(\Delta^*)$  which acts on  $\tilde{\Delta}^*$ .

$$L = \Gamma(\tilde{\Delta}^*, p^* \mathcal{L}) \cong T.$$

$$= i_0^* j_* p_* p^* \mathcal{L} \cong T \quad \text{--- } \textcircled{*}$$

② Let us see  $\textcircled{*}$  for  $F = j_* \mathcal{L}$ .

$$i_0^* j_* \mathcal{L} = \text{Ker}(T - \text{id} : L \rightarrow L)$$

$$i_0^* \mathcal{F} \rightarrow i_0^* (j_* p)_* (j_* p)^* \mathcal{F}$$

$$i_0^* \mathcal{F} \longrightarrow L \cong T$$

③ For any  $\mathcal{F} \in \mathcal{D}_c^b(\Delta)$  we have

$$i_0^* \mathcal{F} \longrightarrow \Psi_t \mathcal{F}$$

$\parallel$

$$i_0^* R(j_* p)_* (j_* p)^* \mathcal{F}$$

It comes equipped with a monodromy action.

$$i_0^* \mathcal{F} \rightarrow \Psi_t \mathcal{F} \rightarrow \phi_t \mathcal{F}$$

"nearby cycles"

Def

"Vanishing cycles"

ex.  $\mathcal{F} = j_* \mathcal{L}$

Then

$$L^T \rightarrow L \rightarrow L/L^T.$$

Def:  ${}^p\Psi_t = \Psi_t[-1]$   
 ${}^p\phi_t = \phi_t[-1].$

Main point: If  $\mathcal{F}$  is perverse, then  ${}^p\Psi_t$  and  ${}^p\phi_t$  are also perverse.

## Examples

(i)  $F = i_* \mathbb{C}_0$

Then  ${}^p\Psi_t F = 0 \supseteq \text{id}$

${}^p\phi_t F = \mathbb{C}_0 \supseteq \text{id}$

$i^* F = \mathbb{C}_0$

$$(2) F = j_{!} \mathbb{C}_{\Delta^*} [1]$$

$$\begin{aligned} i^* F &= 0 \\ p_{\psi_t} F &= \mathbb{C}_0 \quad \cong \text{id} \\ p_{\psi_t} F &= \mathbb{C}_0 \quad \cong \text{id} \end{aligned}$$

$$(3) j_* \mathbb{C}_{\Delta^*} [1] = F \quad (\text{this is a per sh.})$$

$$\text{Then } \underline{i^* F} = \mathbb{C}_{-1} \oplus \mathbb{C}_0$$

$$\underline{p_{\psi_t}} = \mathbb{C}_0 \quad \cong \text{id}$$

$$\begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & \end{array}$$

$$\mathbb{C} \oplus \mathbb{C} \quad i^* F[-1]$$

$$\downarrow \text{id}$$

$$\mathbb{C}$$

$$p_{\psi_t} F$$

$$p_{\varphi}[-1] \rightarrow i^* F[-1] \rightarrow p_{\psi_t} F$$

$$\parallel$$

$$\mathbb{C}_1$$

$$\Rightarrow \underline{p_{\varphi} = \mathbb{C}_0}$$

$$(4) \mathbb{C}_{\Delta} [1] = F$$

$$i^* F = \mathbb{C}_{-1}$$

$$0$$

$$i^* \Psi_t F = \mathbb{C}_0$$

$$p^* \varphi_t F = \mathbb{O}$$

### Geometric Example

$$f: \Delta_1 \rightarrow \Delta_2 \\ z_1 \mapsto z_1^2$$

$$F = f_* \mathbb{C}_{\Delta_1}$$

Then

$$i^* F = \mathbb{C}_0 \\ p^* \Psi_t F = \mathbb{C}_0^2 \\ p^* \varphi_t F = \mathbb{C}_0$$

$$i^* F \rightarrow p^* \Psi_t F \\ \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C} \\ 1 \mapsto (1, 1)$$

