Perverse sheaves on curves / Vanishing and nearby cycles

$$0 \overset{i}{\hookrightarrow} \Delta \overset{j}{\hookleftarrow} \Delta^*$$

$F \in D^b(\Delta)$ is perverse if:

1) $i^* F \in {}^p D_0^{\leq 0}$
$i^! F \in {}^p D_0^{\geq 0}$
$\Updownarrow$
$(i^* \mathbb{D} F \in {}^p D^{\leq 0})$

2) $j^* F \in {}^p D_{\Delta^*}^{\leq 0}$
$j^! F \in {}^p D_{\Delta^*}^{\geq 0}$
$\underbrace{\hphantom{j^! F \in {}^p D_{\Delta^*}^{\geq 0}}}$
$j^* F = j^! F \in \mathrm{LocSys}[1]$

$\Rightarrow$ $i^* F = \; — \; \underset{-1}{*} \; \underset{0}{*} \; —$

	pt	Δ^*
H^{-2}	0	0
H^{-1}	$*$	$*$
H^{0}	$*$	0
H^{1}	0	0

So we have one exact functor
$$j^* : \mathcal{M}_0 \longrightarrow \mathrm{LocSys}_{\Delta^*}$$

We want another exact functor that measures the perverse sheaves at 0

$$?? : \mathcal{M}_0 \longrightarrow \mathrm{Vect}_0 = \mathrm{LocSys}_0$$

i^* Does not work.

Replacement – vanishing / nearby cycles.

① $\mathcal{L}$ a loc syst on Δ^*.

Then $\mathcal{L} \Longleftrightarrow$ L = a vector space

$T : L \xrightarrow{\sim} L$ monodromy.

Choose $\tilde{\Delta}^* \xrightarrow{p} \Delta^*$ univ. cover

Generator T of $\pi_1(\Delta^*)$ which acts on $\tilde{\Delta}^*$.

$$L = \Gamma(\tilde{\Delta}^*, p^{-1}\mathcal{L}) \circlearrowleft T$$

$$= i_0^* j_* p_* p^{-1}\mathcal{L} \circlearrowleft T \quad \text{—— } ⊛$$

② Let us see ⊛ for $F = j_*\mathcal{L}$.

$$i_0^* j_* \mathcal{L} = \operatorname{Ker}(T - \mathrm{id} : L \to L)$$

$$\underbrace{i_0^* \mathcal{F}} \to \underbrace{i_0^* (j \circ p)_* (j \circ p)^{-1} \mathcal{F}}$$

$$i_0^* \mathcal{F} \longrightarrow L \circlearrowleft T$$

③ For any $\mathcal{F} \in D_c^b(\Delta)$ we have

$$i_0^* \mathcal{F} \longrightarrow \Psi_t \mathcal{F}$$

$$\Psi_t \mathcal{F} = i_0^* R(j \circ p)_* (j \circ p)^{-1} \mathcal{F}$$

It comes equipped with a monodromy action.

$$i_0^* \mathcal{F} \longrightarrow \Psi_t \mathcal{F} \longrightarrow \phi_t \mathcal{F}$$

"nearby cycles" $\qquad \boxed{\text{Def}} \qquad$ "Vanishing cycles".

ex. $\mathcal{F} = \hat{j}_* \mathcal{L}$

Then

$$L^T \longrightarrow L \longrightarrow L/L^T.$$

Def: $\quad {}^p\Psi_t = \Psi_t[-1]$

$\qquad {}^p\varphi_t = \varphi_t[-1].$

Main point: If $\mathcal{F}$ is perverse, then ${}^p\Psi_t$ and ${}^p\varphi_t$ are also perverse.

Examples

(i) $F = i_* \mathbb{C}_0$

Then ${}^p\Psi_t F = 0 \circlearrowleft \mathrm{id}$

$\qquad {}^p\varphi_t F = \mathbb{C}_0 \circlearrowleft \mathrm{id}$

$\qquad i^* F = \mathbb{C}_0$

(2) $F = j_! \mathbb{C}_{\Delta^*}[1]$

$$i^* F = 0$$
$${}^p\psi_t F = \mathbb{C}_0 \quad \circlearrowleft \mathrm{id}$$
$${}^p\varphi_t F = \mathbb{C}_0 \quad \circlearrowleft \mathrm{id}$$

(3) $j_* \mathbb{C}_{\Delta^*}[1] = F$ (this is a perv sh).

Then $\underline{\underline{i^* F}} = \mathbb{C}_{-1} \oplus \mathbb{C}_0$

$$\underline{\underline{{}^p\psi_t}} = \mathbb{C}_0 \quad \circlearrowleft \mathrm{id}$$

$$\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \\ & & \mathbb{C} & \oplus \ \mathbb{C} & \quad i^*F[-1] \\ & & \downarrow \mathrm{id} & & \\ & & \mathbb{C} & & \quad {}^p\psi_t F \end{array}$$

$${}^p\varphi[-1] \to i^*F[-1] \to {}^p\psi_p F$$
$$\begin{array}{c} {}^p\varphi[-1] \\ \text{II} \\ \mathbb{C}_1 \end{array}$$

$$\Rightarrow \underline{\underline{{}^p\varphi = \mathbb{C}_0}}.$$

(4) $\mathbb{C}_{\Delta}[1] = F$

$$i^* F = \mathbb{C}_{-1}$$

o

$${}^{1}\psi_t F = \mathbb{C}_0$$

$${}^{p}\varphi_t F = 0$$

Geometric Example

$$f : \Delta_1 \to \Delta_2$$
$$z \mapsto z^2$$

$$F = f_* \mathbb{C}_{\Delta_1}$$

Then
$$i^* F = \mathbb{C}_0$$
$${}^{p}\psi_t F = \mathbb{C}_0^2$$
$${}^{p}\varphi_t F = \mathbb{C}_0$$

$$i^* F \to {}^{p}\psi_t F$$
$$\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$$
$$1 \mapsto (1, 1)$$