Degenerations of VHS

X smooth quasi-proj variety stratified $X^{\bullet}$
$\Delta \supset 0$ stratified disk

Recall $\mathbb{Q}$ VHS / X, consists of
① Vector bundle V with a flat connection ∇
② Filtration satisfying Griffiths transversality
$$\nabla F^k \subset F^{k+1}$$
③ Antiholomorphic filtration.
④ $\mathbb{Q}$-local system H
⑤ $\alpha : \operatorname{Ker} \nabla \xrightarrow{\sim} H \otimes \mathbb{C}$

Goal - understand singularities that arise in the above.

$V \rightsquigarrow$ D-module
$FV \rightsquigarrow$ filtered D-module
$H \rightsquigarrow$ perverse sheaf
$\alpha \rightsquigarrow$ de Rham complex. $(M) \xrightarrow{\sim} H \otimes \mathbb{C}$

$DR(M)$ is defined as follows —
$$\underbrace{[D \to \Omega^1 \otimes D \to \Omega^2 \otimes D \to \cdots \to \omega \otimes D]}_{} \simeq \omega$$
$$\otimes M \quad = DR(M).$$
$$= DR(\omega \otimes M).$$

e.g. $\Delta \supset 0$ Hodge module supported at 0.

Let $\mathcal{V} = (V, F_1^\bullet V, F_2^\bullet V, H)$ at 0

Then the push forward

$$i_+ \mathcal{V} = (i_+ V, \oplus \partial_t^k F^\bullet V, \ldots, i_* H)$$

$$i_+ V = \bigoplus_k \partial_t^k V$$

Some examples of Hodge modules $/\Delta$ are Honest $\mathbb{Q}$ variations of Hodge structures $/\Delta$.

<u>More interesting examples</u>

Perverse sheaves - $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$

$$j_{!*} : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$$

$$Rj_! \to Rj_*$$

Then take
$$\begin{array}{ccc} {}^p\mathcal{H}^0(Rj_!) & \longrightarrow & {}^p\mathcal{H}(Rj_*) \\ \| & & \| \\ {}^pj_! & & {}^pj_* \end{array}$$

$${}^pj_{!*} = \text{image of } {}^pj_! \text{ in } {}^pj_*.$$

<u>Example</u>: $\Delta \supset 0$

Here (luckily) ${}^p j_! = j_!$ and

${}^p j_* = j_*$.

because $j_* L[1]$ and $j_! L[1]$ are already perverse for a local system L on Δ^*.

$$0 \to j_! L \to j_* L \to (j_* L)_0 \to 0$$

in the usual t-structure.

In the perverse t-structure:

$$0 \to (j_* L)_0 \to j_! L[1] \to j_* L[1] \to 0$$

so $j_{!*} = j_*$ in this example.

$j_{!*}$ has the funny property that $j_{!*} L$ has no sub or quotients supported at 0.

Comes from ${}^p j_* L$ has no subs supported at 0

and ${}^p j_! L$ has no quotients supported at 0.

Def^n: $IC_X(L) = j_{!*}(L)$

Fact: Simple objects in $\mathrm{Perv}(X)$ are IC sheaves of Local systems on open strata.

Hodge modules on X: — On some open set of their support, they'll be honest VHS.

For the disks, HM with strict support Δ will have underlying perverse sheaf $= j_{!*}L$.

D-modules : Deligne extension.

(V, ∇) on $X \setminus D = U$.

∇ is regular if there is a holomorphic extension $\tilde{V}$ of V such that ∇ extends to a

$$\tilde{\nabla} : \tilde{V} \to \Omega^1(\log D) \otimes \tilde{V}$$

We can build a D-module by taking

$\tilde{V} \otimes \mathcal{O}(\infty D)$, where we have a D-module structure

Regularity means that there is some basis that generates a coherent $\mathcal{O}_X$-module and $\tilde{\nabla}$ has simple poles on this.

Deligne Given a VHS on Δ^*, and given a choice of multivalued section τ of
$$e : \mathbb{H} \to \Delta^*$$
There is a unique extension $(\tilde{V}, \tilde{\nabla})$ of (V, ∇) such that the eigenvalues of the residues of

$$\operatorname{Res} \tilde{\nabla} \in \tau(\Delta^*)$$

<u>Assume</u>: the eigenvalues of the residue are <u>real</u>.

$\rightsquigarrow$ extension is determined by $b \in \mathbb{R}$, τ lands in (strip $[b, b+1]$).

$e^* V$ is trivial and has $\circlearrowleft T$ monodromy

Assume T unipotent. so we can define

$$N = \frac{1}{2\pi i} \log(T)$$

Given $s \in H^0(e^* H)$ define

$$\tilde{s} = e^{2\pi i z N} s.$$

Then $\tilde{s}(z+1) = T \tilde{s}(z)$

$\Rightarrow \tilde{s}$ descends. Take a flat frame $e_1, \dots, e_n$ of V.

Then $\tilde{e}_1, \dots, \tilde{e}_n$ will be a frame of $\tilde{V}$.

Then ∇ extends to $\tilde{\nabla}$ with residue N.

The choice of b leads to

$\tilde{V}^b \subset \tilde{V} \leftarrow$ meromorphic v.b.

different holomorphic sub-bundles depending on b.

The choices of b defines a real filtration of $\tilde{V}^b$:

$$\mathrm{gr}^b \tilde{V} = \tilde{V}^b / \tilde{V}^{>b}$$

$\parallel$

b-eigenspace of $\tilde{V}(0)$ of residue N.

<u>Fact</u> – $DR(\tilde{V}, \tilde{\nabla}) = Rj_* V$ not $j_{!*} V$

$$\tilde{V}_{min} = \tilde{V}^{>-1}$$

<u>Fact</u> $DR(\tilde{V}_{min}, \tilde{\nabla}) = j_{!*} V.$

intermediate ext. of D-module.

$\nabla_{\partial t} \; \tilde{V}^b \longrightarrow \tilde{V}^{b-1}$. and

$t \nabla_{\partial t}$ acts as b on gr^b.