Basic Facts on Sheaves

Definition 1. A **sheaf of abelian groups** $\mathcal{F}$ on a topological space X is the following collection of data:

- for each open set $U \subset X$, an abelian group $\mathcal{F}(U)$, with $\mathcal{F}(\varnothing) = 0$
- if $U \subset V$, a **restriction map** $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$, with $\rho_{UU} = \mathrm{id}$

such that

(1) (restriction) if $U \subset V \subset W$, then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$
(2) (gluing) if $\{V_i\}$ is an open covering of U, and we have $s_i \in \mathcal{F}(V_i)$ such that for all i, j, $\rho_{V_i,V_i\cap V_j}(s_i) = \rho_{V_j,V_i\cap V_j}(s_j)$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{U,V_i}(s) = s_i$ for all i.

If one omits the gluing condition from the above definition, one has a **presheaf of abelian groups**. Elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U, and elements of $\mathcal{F}(X)$ are called **global sections**.

One can also define (pre)sheaves of R-modules, vector spaces, *etc.*

Notation 2. $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. $s|_V := \rho_{VU}(s)$.

Lemma 3. *Let $\mathcal{F}$ be a sheaf. If $\{V_i\}$ is an open cover of U, and $s, t \in \mathcal{F}(U)$ are sections such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$. In particular, if $s|_{V_i} = 0$ for all i, then $s = 0$.*

Definition 4. Let $\mathcal{F}$ be a presheaf on X. The **stalk** of $\mathcal{F}$ at a point $x \in X$, denoted $\mathcal{F}_x$, is the group whose elements are equivalence classes of pairs

$$(U, s) \qquad \text{where } U \text{ is a neighborhood of } x \text{ and } s \in \mathcal{F}(U),$$

and the equivalence relation is

$$(U, s) \sim (V, t) \qquad \text{if there is an open set } W \subset U \cap V \text{ with } x \in W \text{ and } s|_W = t|_W$$

For any neighborhood U of x, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section $s \in \mathcal{F}(U)$ to the equivalence class of the pair (U, s). That equivalence class is denoted s_x and is called the **germ** of s at x.

Lemma 5. *A section is determined by its germs. That is, if $s, t \in \mathcal{F}(U)$, and if $s_x = t_x$ for all $x \in U$, then $s = t$.*

Definition 6. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. A **morphism** $f : \mathcal{F} \to \mathcal{G}$ is a collection of abelian group homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$, where U ranges over all open sets of X, that are compatible with restriction. That is, the following diagram must commute whenever $V \subset U$:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\
\mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

A morphism $f : \mathcal{F} \to \mathcal{G}$ is an **isomorphism** if there is another morphism $g : \mathcal{G} \to \mathcal{F}$ such that $f \circ g = \mathrm{id}_{\mathcal{G}}$ and $g \circ f = \mathrm{id}_{\mathcal{F}}$.

It is easy to check that $f : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all open sets $U \subset X$.

Definition 7. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, and let $x \in X$. The **induced homomorphism of stalks** at x is the map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ given by $f_x([(U, s)]) = [(U, f_U(s))]$. (Here, the notation $[(V, t)]$ for an element of a stalk denotes the equivalence class of the pair (V, t).)

One must check that f_x is well-defined; the proof uses the fact that the f_U are compatible with restriction.

Definition 8. A **sub(pre)sheaf** of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open set U, and whose restriction maps are the restrictions of the restriction maps of $\mathcal{F}$. That is, in the diagram

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, hook] \arrow[d, "\rho^{\mathcal{G}}_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho^{\mathcal{F}}_{UV}"] \\ \mathcal{G}(V) \arrow[r, hook] & \mathcal{F}(V) \end{tikzcd}$$

we must have $\rho^{\mathcal{G}}_{UV} = \rho^{\mathcal{F}}_{UV}|_{\mathcal{G}(U)}$.

Definition 9. Let $\mathcal{F}$ be a presheaf. The **sheafification** of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \text{for all } x \in U,\ s(x) \in \mathcal{F}_x, \text{ and there is a neighborhood } V \subset U \\ \text{of } x \text{ and a section } t \in \mathcal{F}(V) \text{ such that for all } y \in V,\ s(y) = t_y \end{array} \right\}$$

(Check that $\mathcal{F}^+$ really is a sheaf.)

When $\mathcal{F}$ happens to be a subpresheaf of a sheaf, there is a convenient alternate description of sheafification.

Lemma 10. *Suppose $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, and assume that $\mathcal{G}$ is a sheaf. Define a new sheaf $\mathcal{F}' \subset \mathcal{G}$ by*

$$\mathcal{F}'(U) = \{ s \in \mathcal{G}(U) \mid \textit{there is an open cover } \{V_i\} \textit{ of } U \textit{ such that } s|_{V_i} \in \mathcal{F}(U) \textit{ for all } i \}.$$

Then $\mathcal{F}' \simeq \mathcal{F}^+$.

Corollary 11. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq \mathcal{F}^+$.*

Lemma 12. *Let $\mathcal{F}$ be a presheaf. For all $x \in X$, $\mathcal{F}_x \simeq \mathcal{F}^+_x$.*

For any presheaf $\mathcal{F}$, there is a canonical morphism $\iota : \mathcal{F} \to \mathcal{F}^+$ defined as follows: given $s \in \mathcal{F}(U)$, we let $\iota_U(s) \in \mathcal{F}^+(U)$ be the function $U \to \coprod \mathcal{F}_x$ given by $\iota_U(s)(x) = s_x$.

Lemma 13 (Universal property of sheafification)**.** *Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ a sheaf. Given a morphism $f : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $f^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $f = f^+ \circ \iota$.*

In other words, this lemma asserts the existence of a morphism f^+ making the following diagram commute:

$$\begin{tikzcd} & \mathcal{F}^+ \arrow[dr, dotted, "f^+"] & \\ \mathcal{F} \arrow[ur, "\iota"] \arrow[rr, "f"'] & & \mathcal{G} \end{tikzcd}$$

Definition 14. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The **kernel** of f is the sheaf $\ker f$ given by $(\ker f)(U) = \ker f_U$. This is a subsheaf of $\mathcal{F}$.

(The assertion that $\ker f$ is a sheaf requires proof.)

Definition 15. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The **presheaf-image** of f is the presheaf $\operatorname{psim} f$ given by $(\operatorname{psim} f)(U) = \operatorname{im} f_U$. This is a subpresheaf of $\mathcal{G}$, but it is not in general a sheaf.

The **image** of f is the sheafification of its presheaf-image: $\operatorname{im} f = (\operatorname{psim} f)^+$. By Lemma 10, $\operatorname{im} f$ can be identified with a subsheaf of $\mathcal{G}$.

Definition 16. A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is **injective** if $\ker f = 0$, and **surjective** if $\operatorname{im} f = \mathcal{G}$.

Lemma 17. *A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U.*

WARNING: The analogue of the preceding lemma for surjective morphisms is false.

Lemma 18. *A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective, an isomorphism) if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, an isomorphism) for all $x \in X$.*

Operations on Sheaves; Adjointness Theorems

February 1, 2007

Definition 1. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{F}$ is a sheaf on X, the **push-forward** by f of $\mathcal{F}$, also called its **direct image**, and denoted $f_*\mathcal{F}$, is the sheaf on Y defined by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.

Remark 2. Note that specifying a sheaf on a one-point topological space is the same as specifying a single abelian group. One-point spaces occur in several examples below, and we will frequently treat abelian groups as sheaves on these spaces without further comment.

Example 3. Let $f : X \to \{x\}$ be the constant map to a one-point space. Then $f_*\mathcal{F} \simeq \Gamma(X, \mathcal{F})$ for any sheaf $\mathcal{F}$ on X.

Example 4. Let $x_0 \in X$, and let $i : \{x_0\} \hookrightarrow X$ be the inclusion of the point. Let A be an abelian group, thought of as a sheaf on $\{x_0\}$. Then i_*A is sheaf on X whose stalks are all 0 except at x_0, where its stalk is isomorphic to A. This kind of sheaf is called a **skyscraper sheaf**.

Definition 5. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Their **direct sum** is the sheaf $\mathcal{F} \oplus \mathcal{G}$ defined by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ for all open sets $U \subset X$.

Example 6. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $f : X \to X$ be the map $f(z) = z^2$. Then $f_*\underline{\mathbb{C}} \simeq \underline{\mathbb{C}} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is the square-root sheaf on X.

Definition 7. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{F}$ is a sheaf on Y, its **pull-back** or **inverse image**, denoted $f^{-1}\mathcal{F}$, is the sheafification of the presheaf $\mathrm{ps}f^{-1}\mathcal{F}$ defined by

$$\begin{aligned}(\mathrm{ps}f^{-1}\mathcal{F})(U) &= \varinjlim_{V \supset f(U)} \mathcal{F}(V) \\ &= \text{equivalence classes of pairs } (V, s) \text{ where } V \supset f(U) \text{ and } s \in \mathcal{F}(V).\end{aligned}$$

The equivalence relation is the same as for stalks: $(V, s) \sim (V', s')$ if there is an open set $W \subset V \cap V'$ such that $s|_W = s'|_W$.

Example 8. For any continuous map $f : X \to Y$ and any abelian group A, we have $f^{-1}\underline{A}_Y \simeq \underline{A}_X$.

Definition 9. Let $i : X \hookrightarrow Y$ be the inclusion map of a subspace. If $\mathcal{F}$ is a sheaf on Y, the pull-back $i^{-1}\mathcal{F}$ is also called the **restriction** of $\mathcal{F}$ to X, and is denoted $\mathcal{F}|_X$.

Notation 10. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. The set of all morphisms of sheaves $\mathcal{F} \to \mathcal{G}$ is denoted $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$, or simply $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. It is an abelian group.

Definition 11. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be functors. We say that S is **left-adjoint** to T and that T is **right-adjoint** to S, or simply that (S, T) is an **adjoint pair**, if

$$\mathrm{Hom}_{\mathcal{B}}(S(A), B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, T(B))$$

for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Example 12. Perhaps the best-known adjoint pair of functors is the following. Let $\mathcal{A} = \mathcal{B} =$ the category of abelian groups. For a fixed abelian group C, the functors $\cdot \otimes C$ and $\mathrm{Hom}(C, \cdot)$ are adjoint:

$$\mathrm{Hom}(A \otimes C, B) \simeq \mathrm{Hom}(A, \mathrm{Hom}(C, B)).$$

Theorem 13. *Let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on Y, and $\mathcal{G}$ a sheaf on X. Then* $\mathrm{Hom}_X(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_Y(\mathcal{F}, f_*\mathcal{G})$.

Definition 14. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Their **sheaf Hom** is the sheaf $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ (or simply $\mathcal{H}om(\mathcal{F}, \mathcal{G})$) defined by

$$\mathcal{H}om_X(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U).$$

Their **sheaf tensor product** is the sheafification of the presheaf

$$(\mathcal{F} \underset{\mathrm{ps}}{\otimes} \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U).$$

It is denoted $\mathcal{F} \otimes \mathcal{G} = (\mathcal{F} \otimes_{\mathrm{ps}} \mathcal{G})^+$.

Warning 15. It is tempting to define sheaf Hom by setting $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$, but this definition is not correct. Indeed, it does not make sense—there is no way to define restriction maps here, so this "definition" does not even specify a presheaf.

In the context of sheaves, many adjointness theorems come in ordinary and sheaf-theoretic versions. The sheaf-theoretic version of Theorem 13 says:

Theorem 16. $f_* \mathcal{H}om_X(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om_Y(\mathcal{F}, f_*\mathcal{G})$.

Here are two more:

Theorem 17. $\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$.

Theorem 18. $\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$.

Local Systems and Constructible Sheaves

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Convention. Henceforth, all sheaves will be sheaves of complex vector spaces. All topological spaces will be locally compact, Hausdorff, second-countable, locally path-connected, and semilocally simply connected. Unless otherwise specified, they will also be path-connected.

Remark 1. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of complex vector spaces, objects like $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes \mathcal{G}$ depend on whether one is working in the category of sheaves of abelian groups, or in the category of sheaves of vector spaces. Indeed, the same phenomenon is already visible with ordinary Hom and $\otimes$: in the category of complex vector spaces, we have $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C}$, while in the category of real vector spaces, $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{R}^4$. In the category of abelian groups, $\mathbb{C} \otimes \mathbb{C}$ is an uncountable-rank free $\mathbb{Z}$-module for which we cannot give an explicit basis.

Henceforth, all Hom-groups, sheaf Hom's, and tensor products are to be computed in the category of sheaves of complex vector spaces.

Definition 2. A sheaf $\mathcal{F}$ on X is **locally constant**, or $\mathcal{F}$ is a **local system**, if for all $x \in X$, there is a neighborhood U containing x such that $\mathcal{F}|_U$ is a constant sheaf.

Example 3. A constant sheaf is locally constant.

Example 4. The square-root sheaf $\mathcal{Q}$ on $\mathbb{C} \smallsetminus \{0\}$ is locally constant, but not constant.

Example 5. Let $\mathcal{F}$ be the sheaf of continuous functions on $X = \mathbb{C}$. This sheaf is *not* locally constant. In general, if U is a connected neighborhood of x, then over any smaller connected neighborhood $V \subset U$ there will be sections (continuous functions) that are not the restriction of any section over U. This situation does not occur in constant sheaves, so $\mathcal{F}|_U$ is not a constant sheaf for any open set U.

Definition 6. A sheaf $\mathcal{F}$ on X is **constructible** if there is a decomposition $X = \bigsqcup_{i=1}^n X_i$ of X into a disjoint union of finitely many locally closed subsets X_i such that $\mathcal{F}|_{X_i}$ is locally constant. (A set is **locally closed** if it is the intersection of an open set and a closed set.)

Note, in particular, that all open sets and all closed sets are locally closed.

Typically, the required decomposition of X will either be obvious or fixed in advance. Proving that a given sheaf is constructible usually consists only of showing that the restrictions $\mathcal{F}|_{X_i}$ are locally constant, and not of finding the decomposition of X.

Theorem 7. *There is a bijection*

$$\left\{ \begin{matrix} \textit{local systems on } X \\ \textit{up to isomorphism} \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \textit{representations of } \pi_1(X, x_0) \\ \textit{up to isomorphism} \end{matrix} . \right\}$$

In fact, more is true: there is an equivalence of categories between the two sides of this picture.

A subset $K \subset X$ is called *good* (for $\mathcal{F}$ if it is connected, and there is a connected open set V containing K such that $\mathcal{F}|_V$ is a constant sheaf. Note that if K is good, any connected subset $K' \subset K$ is also good.

If $\gamma, \gamma' : [0,1] \to X$ are two paths in X, we write $\gamma \sim \gamma'$ to indicate that they are homotopic.

Lemma 7.1. Let $x_1, \ldots, x_n$ be points in a good set K. There are natural isomorphisms of stalks $\mathcal{F}_{x_i} \xrightarrow{\sim} \mathcal{F}_{x_j}$ for all i and j. These isomorphisms are compatible with each other: for any i, j, k, the composition $\mathcal{F}_{x_i} \xrightarrow{\sim} \mathcal{F}_{x_j} \xrightarrow{\sim} \mathcal{F}_{x_k}$ coincides with $\mathcal{F}_{x_i} \xrightarrow{\sim} \mathcal{F}_{c_k}$.

Proof. Choose a good open set V containing K. Since $\mathcal{F}|_V$ is constant, the natural maps $\mathcal{F}(V) \to \mathcal{F}_{x_i}$ are all isomorphisms. Composing one of these with the inverse of another gives the desired isomorphisms: $\mathcal{F}_{x_i} \xleftarrow{\sim} \mathcal{F}(V) \xrightarrow{\sim} \mathcal{F}_{x_j}$. The compatibility is obvious. □

Lemma 7.2. If $\gamma : [0,1] \to X$ (resp. $H : [0,1]^2 \to X$) is a continuous map, there exist numbers $a_0 = 0 < a_1 < \cdots < a_n = 1$ (resp. and $b_0 = 0 < b_1 < \cdots < b_m = 1$) such that for all i, $\gamma([a_i, a_{i+1}])$ (resp. for all i and j, $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$) is good.

Proof. For every point $t \in [0,1]$ (resp. $t \in [0,1]^2$), there is a good open neighborhood of $\gamma(t)$ (resp. $H(t)$), so by continuity, there is an open neighborhood V of t such that $\gamma(V)$ (resp. $H(V)$) is good. Inside V, find an interval $[a, a']$ (resp. box $[a, a'] \times [b, b']$) containing t. The interiors of all these intervals (resp. boxes), as t ranges over all points of $[0,1]$ (resp. $[0,1]^2$), form an open cover. By compactness, we can select finitely of them that still form an open cover. From the remaining intervals (resp. boxes), write all a's and a''s in order as $a_0 < a_1 < \cdots < a_n$ (resp. and write all the b's and b''s as $b_0 < b_1 < \cdots < b_m$). Each interval $[a_i, a_{i+1}]$ (resp. box $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$) is contained in one of the original open sets (called "V" above), so $\gamma([a_i, a_{i+1}])$ (resp. $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$) is good. □

Proof of Theorem 7. The proof proceeds in five steps.

Step A. Given a path $\gamma : [0,1] \to X$, define an invertible linear transformation $\rho(\gamma) : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$. Invoke Lemma 7.2 to get a sequence of points $a_0 = 0 < a_1 < \cdots < a_n = 1$. Since $\gamma([a_i, a_{i+1}])$ is good for each i, by invoking Lemma 7.1, we obtain a number of isomorphisms as follows:

$$\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(a_1)} \xrightarrow{\sim} \mathcal{F}_{\gamma(a_2)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}.$$

We would like to define $\rho(\gamma)$ to simply be the composition of all of these, but at first glance it is not clear that that would be well-defined: it appears to depend on the a_i's that come out of the invocation of Lemma 7.2.

To deal with this problem, note first that if we add a new point a_{new} to our list, say between a_i and a_{i+1}, then $\gamma([a_i, a_{\text{new}}])$ and $\gamma([a_{\text{new}}, a_{i+1}])$ are good, so we can repeat the above construction. But the total composition $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$ will remain unchanged, because the triangle

$$\begin{tikzcd}
\mathcal{F}_{\gamma(a_i)} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{F}_{\gamma(a_{i+1})} \\
& \mathcal{F}_{\gamma(a_{\text{new}})} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

commutes by Lemma 7.1. Indeed, by induction adding finitely many points to the a_i's does not change the total composition $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$.

In general, to show that $\rho(\gamma)$ defined with respect to $a_0, \ldots, a_n$ coincides with the map defined by another set of points, say $a'_0, \ldots, a'_{n'}$, we simply note that both coincide with the map defined with respect to the set of all a_i's and a'_i's. So $\rho(\gamma)$ is well-defined. □

Step B. If $\gamma \sim \gamma'$, then $\rho(\gamma) = \rho(\gamma')$. Let $H : [0,1]^2 \to X$ be a homotopy between γ and γ'. That is, $H(t,0) = \gamma(t)$ and $H(t,1) = \gamma'(t)$. Let $a_0 < a_1 < \cdots < a_n$ and $b_0 < b_1 < \cdots < b_m$ be as given by Lemma 7.2. Let $\gamma_j : [0,1] \to X$ be the path $\gamma_j(t) = H(t, b_j)$. (In particular, $\gamma_0 = \gamma$ and $\gamma_m = \gamma'$.) To prove that $\rho(\gamma) = \rho(\gamma')$, it clearly suffices to prove $\rho(\gamma_j) = \rho(\gamma_{j+1})$ for all j.

Consider the diagram

$$\begin{tikzcd}
\mathcal{F}_{H(0,b_{j+1})} \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{F}_{H(a_1,b_{j+1})} \arrow[r, "\sim"] \arrow[d, leftrightarrow, "\wr"] & \mathcal{F}_{H(a_2,b_{j+1})} \arrow[r, "\sim"] \arrow[d, leftrightarrow, "\wr"] & \cdots \arrow[r, "\sim"] & \mathcal{F}_{H(1,b_{j+1})} \arrow[d, equal] \\
\mathcal{F}_{H(0,b_j)} \arrow[r, "\sim"] & \mathcal{F}_{H(a_1,b_j)} \arrow[r, "\sim"] & \mathcal{F}_{H(a_2,b_j)} \arrow[r, "\sim"] & \cdots \arrow[r, "\sim"] & \mathcal{F}_{H(1,b_j)}
\end{tikzcd}$$

By Lemma 7.2, every small square in this diagram commutes, because $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ is good. Since $\rho(\gamma_j)$ is the composition of all the maps along the bottom of this diagram, and $\rho(\gamma')$ is the composition of all maps along the top, we see that $\rho(\gamma) = \rho(\gamma')$. □

Corollary. There is a well-defined map $\rho : \pi_1(X, x_0) \to GL(\mathcal{F}_{x_0})$. This follows immediately from Steps A and B.

Step C. Given a representation $\tau : \pi_1(X, x_0) \to GL(E)$, construct a local system $\mathcal{G}$ on X. For each point $x \in X$, let us choose, once and for all, a path $\alpha_x : [0,1] \to X$ that joins x_0 to x. (That is, $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$.) In particular, let us take α_{x_0} to be the constant path at x_0.

Now, we define a sheaf $\mathcal{G}$ on X by:

$$\mathcal{G}(U) = \left\{ \begin{array}{c|c} \text{functions} & \text{for any path } \gamma : [0,1] \to U, \text{ we have} \\ k : U \to E & k(\gamma(1)) = [\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}] \cdot k(\gamma(0)) \end{array} \right\}$$

Here, "$*$" indicates composition of paths. Note that $\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}$ is a loop based at x_0, so its homotopy class $[\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}]$ is an element of $\pi_1(X, x_0)$. Via τ, this element acts on the vector $k(\gamma(0)) \in E$.

The verification that $\mathcal{G}$ is sheaf is routine, and we omit it. It does, however, remain to show that $\mathcal{G}$ is locally constant. Here we require the fact that X is semilocally simply connected. That is, every point has a neighborhood V with the property that every loop in V is null-homotopic in X.

Given such a V, we will now show that $\mathcal{G}(V) \simeq E$. Choose a point $x \in V$, and define a map $\phi : \mathcal{G}(V) \to E$ by $\phi(k) = k(x)$. It is easy to check that ϕ is injective. Indeed, a section $k \in \mathcal{G}(V)$ is determined by the vector $k(x) \in E$ as follows: for any $y \in V$, we have $k(y) = [\alpha_y^{-1} * \gamma * \alpha_x] \cdot k(x)$, where γ is any path joining x to y. To show that ϕ is surjective, given $e \in E$, we define a function $k \in \mathcal{G}(V)$ by

$$k(y) = [\alpha_y^{-1} * \gamma * \alpha_x] \cdot e \qquad \text{where } \gamma \text{ is a path joining } x \text{ to } y.$$

(In particular, if $y = x$, we take γ to be the constant path at x, so that $\phi(k) = k(x)$ is indeed e.) But now there is a well-definedness issue arising from the choice of γ. For a given $y \in V$, suppose γ and γ' are two different paths joining x to y. Then $\gamma^{-1} * \gamma'$ is a loop in V, which is, by assumption, null-homotopic in X. It follows from this that $\gamma \sim \gamma'$ in X, and therefore that $[\alpha_y^{-1} * \gamma * \alpha_x] = [\alpha_y^{-1} * \gamma' * \alpha(x)] \in \pi_1(X, x_0)$. Thus, $k(y)$ is independent of the choice of γ, and k is well-defined. We conclude that $\phi : \mathcal{G}(V) \xrightarrow{\sim} E$ is an isomorphism.

Now, the same argument can be repeated for any connected open set $V' \subset V$, so $\mathcal{G}(V') \simeq E$ for such sets as well. It follows then that in fact $\mathcal{G}|_V \simeq \underline{E}_V$. Since a neighborhood V of this type can be found around every point in X, we conclude that $\mathcal{G}$ is locally constant. □

At this point, we have a construction $\mathcal{F} \mapsto \rho$ assigning a representation to any local system, and another, $\tau \mapsto \mathcal{G}$, assigning a local system to any representation. It remains to show that these two assignments are inverses of one another.

Step D. Given $\mathcal{F}$ and ρ as in Steps A–B, take $\tau = \rho$ and construct $\mathcal{G}$ as in Step C. Then $\mathcal{F} \simeq \mathcal{G}$. Note that the vector space E of Step C is now identified with $\mathcal{F}_{x_0}$. We will actually work with the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ rather than $\mathcal{F}$ itself. (Of course, since $\mathcal{F}$ is already a sheaf, the two are isomorphic.) First, we define a morphism $\phi : \mathcal{F}^+ \to \mathcal{G}$. Let U be a connected open set such that $\mathcal{F}^+|_U$ is a constant sheaf. We define $\phi_U : \mathcal{F}^+(U) \to \mathcal{G}(U)$ as follows: given $s \in \mathcal{F}^+(U)$ (recall that s is a function $U \to \coprod_{x \in U} \mathcal{F}_x$), we define $\phi_U(s) \in \mathcal{G}(U)$ to be the function

$$\phi_U(s) : U \to \mathcal{F}_{x_0}, \qquad \phi_U(s)(x) = \rho(\alpha_x^{-1}) \cdot s(x).$$

(This makes sense: $s(x) \in \mathcal{F}_x$, and α_x^{-1} is a path from x to x_0, so $\rho(\alpha_x^{-1})$ is a linear transformation $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{x_0}$.) We need to check that $\phi_U(s)$ is a valid section of $\mathcal{G}$: given a path $\gamma : [0,1] \to U$, note first that $\gamma([0,1])$ is good, so the linear transformation $\rho(\gamma)$ constructed in Step A coincides with the canonical isomorphism $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$ of Lemma 7.1. The latter is defined by taking germs of a given section, so we see that $\rho(\gamma) \cdot s(\gamma(0)) = s(\gamma(1))$. Then,

$$\begin{aligned} \phi_U(s)(\gamma(1)) &= \rho(\alpha_{\gamma(1)}^{-1}) \cdot s(\gamma(1)) = \rho(\alpha_{\gamma(1)}^{-1})\rho(\gamma) \cdot s(\gamma(0)) \\ &= \rho(\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)})\rho(\alpha_{\gamma(0)}^{-1}) \cdot s(\gamma(0)) = \rho([\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}]) \cdot \phi_U(s)(\gamma(0)). \end{aligned}$$

So $\phi_U(s)$ is indeed a valid section of $\mathcal{G}(U)$.

Although we have only defined ϕ_U for certain open sets U, those sets suffice—ϕ_U is determined on other open sets by gluing. The verification of this is left as an exercise.

Next, we define a morphism $\psi : \mathcal{G} \to \mathcal{F}^+$. This time, let V be an open set in which all loops are null-homotopic in X (again, we are using the semilocally-simply-connectedness of X). Given $k \in \mathcal{G}(V)$ (a certain function $k : U \to \mathcal{F}_{x_0}$), define $\psi_V(k) \in \mathcal{F}^+(V)$ by

$$\psi_V(k) : U \to \coprod_{x \in V} \mathcal{F}_x, \qquad \psi_V(k)(x) = \rho(\alpha_x) \cdot k(x).$$

This time, we must check the local condition for $\psi_V(k)$ to be a valid section of $\mathcal{F}^+(V)$. Let $U \subset V$ be a connected neighborhood of x such that $\mathcal{F}|_U$ is constant, and let $s \in \mathcal{F}(U)$ be such that $s_x = \psi_V(k)(x)$. We will show that for all $y \in U$, $s_y = \psi_V(k)(y)$. As argued above, $s_y = \rho(\gamma) \cdot s_x$, where γ is any path in U joining x to y. Recall also that k satsifies $k(y) = \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot k(x)$. Now,

$$\begin{aligned}\psi_V(k)(y) &= \rho(\alpha_y) \cdot k(y) = \rho(\alpha_y) \cdot \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot k(x) \\ &= \rho(\gamma) \cdot \rho(\alpha_x) \cdot k(x) = \rho(\gamma) \cdot s_x = s_y.\end{aligned}$$

As before, it suffices to define ψ on a certain collection of open sets.

It remains to check that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity morphisms of $\mathcal{F}^+$ and $\mathcal{G}$, respectively. This is straightforward from the formulas. □

Step E. Given τ and $\mathcal{G}$ as in Step C, take $\mathcal{F} = \mathcal{G}$ and construct ρ as in Steps A–B. Then $\rho = \tau$. Note first that all stalks of $\mathcal{G}$ are copies of E, and the germ of a section k at x is simply its value $k(x)$. Let U be an open set such that $\mathcal{G}|_U$ is a constant sheaf, and let $\gamma : [0,1] \to U$ be a path in U. Following the proof of Lemma 7.1, we construct $\rho(\gamma) : E \to E$ as follows: given $e \in E$, take a function $k \in \mathcal{G}(U)$ such that $k(x) = e$; then $\rho(\gamma) \cdot e = k(y)$. We have $k(y) = \tau([\alpha_y^{-1} * \gamma * \alpha_x]) \cdot k(x)$, so $\rho(\gamma) = \tau([\alpha_y^{-1} * \gamma * \alpha_x])$.

Now, if γ is a loop based at x_0, let us follow the construction of $\rho(\gamma)$ in Step A. Take $a_0 = 0 < a_1 < \cdots < a_n = 1$ such that $\gamma([a_i, a_{i+1}])$ is good. The action of each restricted path $\gamma|_{[a_i, a_{i+1}]}$ is constructed as in the previous paragraph, and $\rho(\gamma)$ is the composition of all of these. Thus:

$$\begin{aligned}\rho(\gamma) &= \tau([\alpha_{x_0}^{-1} * \gamma|_{[a_{n-1},1]} * \alpha_{\gamma(a_{n-1})}]) \cdot \tau([\alpha_{\gamma(a_{n-1})}^{-1} * \gamma|_{[a_{n-2},a_{n-1}]} * \alpha_{\gamma(a_{n-2})}]) \cdots \tau([\alpha_{\gamma(a_1)}^{-1} * \gamma|_{[0,a_1]} * \alpha_{x_0}]) \\ &= \tau([\gamma|_{[a_{n-1},1]} * \gamma|_{[a_{n-2},a_{n-1}]} * \cdots * \gamma_{[0,a_1]}]) = \tau([\gamma]).\end{aligned}$$

Here we have used the fact that α_{x_0} is the constant path. This completes the proof of the theorem. □

Categories of Complexes; Distinguished Triangles

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Definition 1. An **abelian category** is a category satisfying the following four axioms:

(1) For any two objects A and B in $\mathcal{A}$, the set of morphisms $\mathrm{Hom}(A,B)$ is endowed with the structure of an abelian group, and composition of morphisms is biadditive.
(2) There is a "zero object" 0 with the property that $\mathrm{Hom}(0,0) = 0$. (This property implies that it is unique up to unique isomorphism, and that $\mathrm{Hom}(0,A) = \mathrm{Hom}(A,0) = 0$ for all other objects A.)
(3) For any two objects A and B, there is a "biproduct" $A \oplus B$. It is equipped with morphisms as shown below:

$$\begin{tikzcd} A \arrow[r, shift left, "i_1"] & A \oplus B \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

These morphisms satisfy the following identities:

$$p_1 \circ i_1 = \mathrm{id}_A \qquad p_1 \circ i_2 = 0 \qquad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A \oplus B}$$
$$p_2 \circ i_2 = \mathrm{id}_B \qquad p_2 \circ i_1 = 0$$

(4) Every morphism $\phi : A \to B$ gives rise to a diagram

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "i"'] \arrow[rr, bend left, "\phi"] & I \arrow[r, "j"'] & B \arrow[r, "c"] & C \end{tikzcd}$$

Here, $j \circ i = \phi$. Also, K is the kernel of ϕ, C is its cokernel, and I is both the cokkernel of $k : K \to A$ and the kernel of $c : B \to C$. Such a diagram is called the **canonical decomposition** of ϕ. The object I is called the **image** of ϕ.

A category satisfying the first three of these axioms is called an **additive category**.

Definition 2. The **category of complexes** over $\mathcal{A}$, denoted $C(\mathcal{A})$, is the category whose objects are **complexes** of objects of $\mathcal{A}$: that is, a sequence of objects (A^i) labelled by integers, together with morphisms (called **differentials**) $d^i : A^i \to A^{i+1}$, such that $d^i \circ d^{i-1} = 0$.

$$A^\bullet : \quad \cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \quad \text{where } d^i \circ d^{i-1} = 0 \text{ for all } i,$$

in which a morphism $f : A^\bullet \to B^\bullet$ is a family of morphisms in $\mathcal{A}$, $f^i : A^i \to B^i$, that are compatible with the differentials. That is, every square in the following diagram should commute:

$$\begin{tikzcd} \cdots \arrow[r] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"'] & A^0 \arrow[r] \arrow[d, "f^0"'] & A^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \\ \cdots \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots \end{tikzcd}$$

Definition 3. The ith **cohomology functor** $H^i : C(\mathcal{A}) \to \mathcal{A}$ is defined as follows: given an object $A^\bullet$ in $C(\mathcal{A})$, let I be image of d^{i-1}, and K the kernel of d^i. The natural morphism $I \to A^i$ factors through K (by the universal property of the kernel, and the fact that $d^i \circ d^{i-1} = 0$). $H^i(A^\bullet)$ is defined to be the cokernel of the induced morphism $I \to K$. (In short, $H^i(A^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$.) It is easy to check that a morphism $f : A^\bullet \to B^\bullet$ induces a morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$, so H^i is indeed a functor.

Definition 4. A **homotopy** between two morphisms $f, g : A^\bullet \to B^\bullet$ in $C(\mathcal{A})$ is a collection of morphisms $h^i : A^i \to B^{i-1}$, such that

$$dh + hd = f - g.$$

That is, in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r] \arrow[dl, "h^{-1}"'] \arrow[d, shift right, "f^{-1}"'] \arrow[d, shift left, "g^{-1}"] & A^0 \arrow[r] \arrow[dl, "h^0"'] \arrow[d, shift right, "f^0"'] \arrow[d, shift left, "g^0"] & A^1 \arrow[r] \arrow[dl, "h^1"'] \arrow[d, shift right, "f^1"'] \arrow[d, shift left, "g^1"] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

we have $d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i = f^i - g^i$ for each i. If there is a homotopy between f and g, then f and g are said to be **homotopic**.

Definition 5. The **homotopy category** over $\mathcal{A}$, denoted $K(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, but whose morphisms are given by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/(\text{morphisms homotopic to } 0).$$

In other words, a morphism in $K(\mathcal{A})$ is a homotopy class of morphisms in $C(\mathcal{A})$.

Lemma 6. *If $f, g : A^\bullet \to B^\bullet$ are two homotopic morphisms in $C(\mathcal{A})$, then $H^i(f) = H^i(g)$.*

Corollary 7. *The cohomology functors can be regarded as being functors $K(\mathcal{A}) \to \mathcal{A}$.*

Definition 8. A morphism $f : A^\bullet \to B^\bullet$ (in $C(\mathcal{A})$ or $K(\mathcal{A})$) is a **quasi-isomorphism** (often abbreviated to **qis**) if the morphisms $H^i(f)$ in $\mathcal{A}$ are isomorphisms for all i.

Definition 9. The **derived category** of $\mathcal{A}$, denoted $D(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, and in which a morphism from $A^\bullet$ to $B^\bullet$ is an equivalence class of "roofs"

$$\begin{tikzcd}
& X^\bullet \arrow[dl, "q"'] \arrow[dr, "u"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}$$

where q is a quasi-isomorphism. The equivalence relation is as follows: Two roofs

$$\begin{tikzcd}
& X^\bullet \arrow[dl, "q"'] \arrow[dr, "u"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& Y^\bullet \arrow[dl, "r"'] \arrow[dr, "v"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}$$

are equivalent if there exists a roof over $X^\bullet$ and $Y^\bullet$ making the following diagram commute:

$$\begin{tikzcd}
& & Z^\bullet \arrow[dl] \arrow[dr] & & \\
& X^\bullet \arrow[dl] \arrow[drrr] & & Y^\bullet \arrow[dlll] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

Definition 10. Let $n \in \mathbb{Z}$, and let $A^\bullet$ be a complex. The nth **translation** (or **shift**) is the complex $A[n]^\bullet$ defined by

$$A[n]^i = A^{i+n} \qquad \text{with differential} \qquad d^i = (-1)^n d_A^{i+n}.$$

Given a morphism $f : A^\bullet \to B^\bullet$, the translated morphism $f[n] : A[n]^\bullet \to B[n]^\bullet$ is defined by $f[n]^i = f^{i+n}$. In this way, translation by n is a functor $C(\mathcal{A}) \to C(\mathcal{A})$ (or $K(\mathcal{A}) \to K(\mathcal{A})$, or $D(\mathcal{A}) \to D(\mathcal{A})$).

Definition 11. Let $f : A^\bullet \to B^\bullet$ be a morphism in $C(\mathcal{A})$. The **cone** of f a complex, denoted $\mathrm{cone}^\bullet f$, defined as follows:

$$\mathrm{cone}^i f = A^{i+1} \oplus B^i \qquad \text{with differential} \qquad \begin{matrix} d^i : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1} \\ \text{given by } \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} \end{matrix}$$

It is easy to check that $d^{i+1} \circ d^i = 0$:

$$d^{i+1} \circ d^i = \begin{pmatrix} -d_A^{i+2} & 0 \\ f^{i+2} & d_B^{i+1} \end{pmatrix} \cdot \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} = \begin{pmatrix} d_A^{i+2} \circ d_A^{i+1} & 0 \\ -f^{i+2} \circ d_A^{i+1} + d_B^{i+1} \circ f^{i+1} & d_B^{i+1} \circ d_B^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

There are natural morphisms $B^\bullet \to \mathrm{cone}^\bullet f$ and $\mathrm{cone}^\bullet f \to A[1]^\bullet$:

$$B^\bullet \xrightarrow{\binom{0}{\mathrm{id}}} \mathrm{cone}^\bullet f \xrightarrow{(\mathrm{id}\ 0)} A[1]^\bullet$$

Definition 12. A sequence of morphisms $X^\bullet \to Y^\bullet \to Z^\bullet \to Z[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ is called a **distinguished triangle** (abbreviated to **d.t.**) if there is a commutative diagram

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d, "q"'] & Y^\bullet \arrow[r] \arrow[d, "r"'] & Z^\bullet \arrow[r] \arrow[d, "s"'] & X[1]^\bullet \arrow[d, "q[1]"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r] & \mathrm{cone}^\bullet f \arrow[r] & A[1]^\bullet \end{tikzcd}$$

where q, r, and s are isomorphisms. (Of course, the meaning of "isomorphism," and hence of "distinguished triangle," depends on whether one is working in $K(\mathcal{A})$ or $D(\mathcal{A})$.)

Theorem 13. *Distinguished triangles in $K(\mathcal{A})$ enjoy the following four properties:*

(1) **(Identity)** The triangle $A^\bullet \xrightarrow{\mathrm{id}} A^\bullet \to 0 \to A[1]^\bullet$ is distinguished.
(2) **(Rotation)** The triangle $A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet \to A[1]^\bullet$ is distinguished if and only if the **rotated** triangle $B^\bullet \to C^\bullet \to A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet$ is.
(3) **(Square Completion)** Any commutative square

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d, "u"'] & B^\bullet \arrow[d] \\ F^\bullet \arrow[r] & G^\bullet \end{tikzcd}$$

can be completed to a commutative diagram of distinguished triangles:

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d, "r"'] & B^\bullet \arrow[r, dashed] \arrow[d] & C^\bullet \arrow[r, dashed] \arrow[d, dashed] & A[1]^\bullet \arrow[d, "r[1]"] \\ F^\bullet \arrow[r] & G^\bullet \arrow[r, dashed] & H^\bullet \arrow[r, dashed] & F[1]^\bullet \end{tikzcd}$$

(4) **(Octahedral Property)** Given a commutative diagram of morphisms

$$\begin{tikzcd} & B^\bullet \arrow[dr, "g"] & \\ A^\bullet \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C^\bullet \end{tikzcd},$$

extend each of f, g, and h to a distinguished triangle:

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow F^\bullet \longrightarrow A[1]^\bullet$$

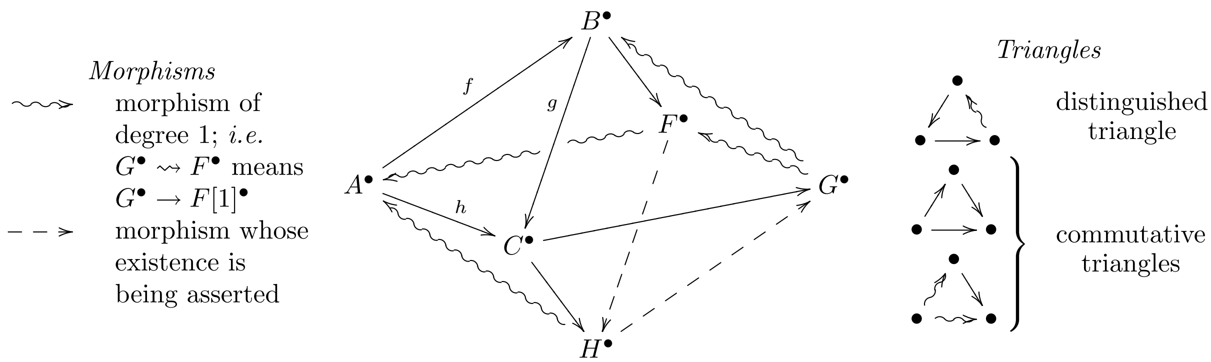
$$B^\bullet \xrightarrow{g} C^\bullet \longrightarrow G^\bullet \longrightarrow B[1]^\bullet$$

$$A^\bullet \xrightarrow{h} C^\bullet \longrightarrow H^\bullet \longrightarrow A[1]^\bullet$$

Arrange three triangles, together with the obvious morphism $G^\bullet \to F[1]^\bullet$ (composition of $G^\bullet \to B[1]^\bullet$ and $B[1]^\bullet \to F[1]^\bullet$) can be arranged in an octahedron as shown below. There exist morphisms $F^\bullet \to H^\bullet$ and $H^\bullet \to G^\bullet$ making

$$F^\bullet \longrightarrow H^\bullet \longrightarrow G^\bullet \longrightarrow F[1]^\bullet$$

a distinguished triangle, and making the neighboring faces of the octahedron commute.



Remark 14. This theorem essentially states that $K(\mathcal{A})$ satisfies the axioms for a **triangulated category**.

Proof. (1) The following diagram is clearly commutative:

$$\begin{tikzcd}
A^\bullet \arrow[r, "\mathrm{id}"] \arrow[d, equal] & A^\bullet \arrow[r] \arrow[d, equal] & \mathrm{cone}^\bullet \,\mathrm{id} \arrow[r] \arrow[d] & A[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r, "\mathrm{id}"] & A^\bullet \arrow[r] & 0 \arrow[r] & A[1]^\bullet
\end{tikzcd}$$

It remains to check that $\mathrm{cone}^\bullet \,\mathrm{id} \to 0$ is an isomorphism in $K(\mathcal{A})$. Its putative inverse, of course, is the zero morphism $0 \to \mathrm{cone}^\bullet \,\mathrm{id}$; we need to check that the compositions in both directions are equal (in $K(\mathcal{A})$, *i.e.*, homotopic) to the identity morphisms of the their respective objects. For the zero object, this is trivial. For $\mathrm{cone}^\bullet \,\mathrm{id}$, we now show that the zero morphism is homotopic to the identity morphism. Let $h^i : \mathrm{cone}^i \,\mathrm{id} \to \mathrm{cone}^{i-1} \,\mathrm{id}$ be the homotopy given by

$$h^i : A^{i+1} \oplus A^i \to A^i \oplus A^{i-1}, \qquad h^i = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$$

Then

$$dh + hd = \begin{pmatrix} -d_A & 0 \\ \mathrm{id} & d_A \end{pmatrix}\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -d_A & 0 \\ \mathrm{id} & d_A \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\mathrm{cone}^\bullet \,\mathrm{id}} - 0.$$

Thus, $\mathrm{cone}^\bullet \,\mathrm{id} \to 0$ is an isomorphism. □

(2) Starting with a distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} \mathrm{cone}^\bullet f \xrightarrow{r} A[1]^\bullet$, we must show that the rotated triangle is isomorphic to the distinguished triangle obtained by taking the cone of g. Note that $\mathrm{cone}^i f = A^{i+1} \oplus B^i$, and $\mathrm{cone}^i g = B^{i+1} \oplus \mathrm{cone}^i f = B^{i+1} \oplus A^{i+1} \oplus B^i$. Consider the following diagram:

$$\begin{tikzcd}
B^\bullet \arrow[r, "g"] \arrow[d, equal] & \mathrm{cone}^\bullet f \arrow[r, "r"] \arrow[d, equal] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, "\theta"'] & B[1]^\bullet \arrow[d, equal] \\
B^\bullet \arrow[r, "g"] & \mathrm{cone}^\bullet f \arrow[r, "s"] & \mathrm{cone}^\bullet g \arrow[r, "t"] & B[1]^\bullet
\end{tikzcd}$$

Here, g, r, s and t are the usual morphisms, and θ is defined as below:

$$g = \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}, \qquad r = \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}, \qquad t = \begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \theta = \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix}$$

For future reference, we note that the differentials are given by

$$d_{\mathrm{cone}^\bullet f} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \qquad \text{and} \qquad d_{\mathrm{cone}^\bullet g} = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix}$$

We must check first that the diagram is commutative, and then that θ is an isomorphism. It is clear that $t\theta = \mathrm{id} \circ -f[1]$, but not that $\theta r = s \circ \mathrm{id}$. For the latter, note that

$$s - \theta r = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} - \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

We define a homotopy $h^i : \mathrm{cone}^i f \to \mathrm{cone}^{i-1} g$ by

$$h = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$dh + hd = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} = s - \theta r.$$

Thus, s is homotopic to θr, so the diagram is commutative.

To show that θ is an isomorphism, define $\psi : \mathrm{cone}^\bullet g \to A[1]^\bullet$ by the matrix $\begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix}$. It is obvious that $\psi\theta = \mathrm{id}_{A[1]^\bullet}$. To show that $\theta\psi = \mathrm{id}_{\mathrm{cone}^\bullet g}$, we need the homotopy

$$k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\mathrm{id}_{\mathrm{cone}^\bullet g} - \theta\psi = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} - \begin{pmatrix} 0 & -f & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Then

$$\begin{aligned} dk + kd &= \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -d_B \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} + \begin{pmatrix} \mathrm{id} & f & d_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\mathrm{cone}^\bullet g} - \theta\psi. \end{aligned}$$

Thus, θ is an isomorphism, and the rotated triangle is distinguished.

(3) This is the easiest property to prove. We need to define $\theta : \mathrm{cone}^\bullet f \to \mathrm{cone}^\bullet g$ such that the following diagram commutes:

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "r"'] & B^\bullet \arrow[r] \arrow[d, "s"'] & \mathrm{cone}^\bullet f \arrow[r] \arrow[d, "\theta"'] & A[1]^\bullet \arrow[d, "r[1]"] \\ F^\bullet \arrow[r, "g"] & G^\bullet \arrow[r] & \mathrm{cone}^\bullet g \arrow[r] & F[1]^\bullet \end{tikzcd}$$

Define $\theta^i : A^{i+1} \oplus B^i \to F^{i+1} \oplus G^i$ by the matrix

$$\theta = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.$$

Using the fact that $sf = gr$, it is trivial to check that θ commutes with differentials (so that it is actually a well-defined morphism) and that it makes the above diagram commute.

(4) This, of course, is the hardest property to check, but only in terms of the number of calculations to be done. Assume that $F^\bullet = \text{cone}^\bullet f$, $G^\bullet = \text{cone}^\bullet g$, and $H^\bullet = \text{cone}^\bullet h$. Recall:

$$F^i = A^{i+1} \oplus B^i \qquad G^i = B^{i+1} \oplus C^i \qquad H^i = A^{i+1} \oplus C^i$$

$$d_F = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \qquad d_G = \begin{pmatrix} -d_B & 0 \\ g & d_C \end{pmatrix} \qquad d_H = \begin{pmatrix} -d_A & 0 \\ h & d_C \end{pmatrix}$$

Define the following morphisms:

$$r : F^\bullet \to H^\bullet \qquad s : H^\bullet \to G^\bullet \qquad t : G^\bullet \to F[1]^\bullet$$

$$r = \begin{pmatrix} \text{id} & 0 \\ 0 & g \end{pmatrix} \qquad s = \begin{pmatrix} f & 0 \\ 0 & \text{id} \end{pmatrix} \qquad t = \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix}$$

Next, let $K^\bullet = \text{cone}^\bullet r$, so that

$$K^i = F^{i+1} \oplus H^i = A^{i+1} \oplus B^{i+1} \oplus A^{i+1} \oplus C^i.$$

The differential of $K^\bullet$ and the natural maps $u : H^\bullet \to K^\bullet$ and $v : k^\bullet \to F[1]]^\bullet$ are given by:

$$d_K = \begin{pmatrix} d_A & 0 & 0 & 0 \\ -f & -d_B & 0 & 0 \\ \text{id} & 0 & -d_A & 0 \\ 0 & g & h & d_C \end{pmatrix} \qquad u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \qquad v = \begin{pmatrix} \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \end{pmatrix}$$

Next, define $\theta : G^\bullet \to K^\bullet$ and $\psi : K^\bullet \to G^\bullet$ by

$$\theta = \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \\ 0 & 0 \\ 0 & \text{id} \end{pmatrix} \qquad \psi = \begin{pmatrix} 0 & \text{id} & f & 0 \\ 0 & 0 & 0 & \text{id} \end{pmatrix}$$

Finally, define homotopies $k : H^i \to K^{i-1}$ and $l : K^i \to K^{i-1}$ by

$$k = \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad l = \begin{pmatrix} 0 & 0 & \text{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We need to show that the following diagram is well-defined and commutative to establish that the top row is a distinguished triangle:

$$\begin{tikzcd}
F^\bullet \arrow[r, "r"] \arrow[d, equal] & H^\bullet \arrow[r, "s"] \arrow[d, equal] & G^\bullet \arrow[r, "t"] \arrow[d, "\theta", shift left] & F[1] \arrow[d, equal] \\
F^\bullet \arrow[r, "r"] & H^\bullet \arrow[r, "u"] & K^\bullet \arrow[r, "v"] \arrow[u, "\psi", bend left] & F[1]^\bullet
\end{tikzcd}$$

To establish that this diagram makes sense, the following assertions need to be checked:

- r, s, t, θ, and ψ are well-defined (they commute with differentials).
- $\theta s = u$ (in fact, they are homotopic, and $dk + kd = u - \theta f$).
- θ is an isomorphism (in fact, $\psi\theta = \text{id}_G$, and $\theta\psi$ is homotopic to id_K: we have $dl + ld = \text{id}_K - \theta\psi$).

Finally, to check that this distinguished triangle is compatible with the octahedron, we must check:

- t coincides with the composition $G^\bullet \to B[1]^\bullet \to F[1]^\bullet$.
- The natural morphism $F^\bullet \to A[1]^\bullet$ coincides with the composition $F^\bullet \xrightarrow{r} H^\bullet \to A[1]^\bullet$.
- The natural morphism $C^\bullet \to G^\bullet$ coincides with the composition $C^\bullet \to H^\bullet \xrightarrow{s} G^\bullet$.

The proofs of all these assertions are straightforward matrix calculations. □

Derived Categories and Derived Functors

March 6, 2007

Recall that distinguished triangles in $K(\mathcal{A})$ have four important properties. These properties are essential for even showing that $D(\mathcal{A})$ is a well-defined category: specifically, composition of morphisms in $D(\mathcal{A})$ is not an obvious operation, and we have to use distinguished triangles in $K(\mathcal{A})$ to show how to do it.

Theorem 1. *Distinguished triangles in $D(\mathcal{A})$ enjoy the following properties:*

(0) ***(Existence)*** *Any morphism $A^\bullet \to B^\bullet$ can be completed to a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$.*

(1–4) *The Identity, Rotation, Completion, and Octahedral properties of distinguished triangles in $K(\mathcal{A})$.*

Proposition 2. *A distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ gives rise to a long exact sequence in cohomology*

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots .$$

The rest of this set of notes is devoted to discussing how a functor of abelian categories gives rise to a functor of derived categories. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then:

(1) F induces a functor of chain complexes $F : C(\mathcal{A}) \to C(\mathcal{B})$.

(2) F induces a functor of homotopy categories $F : K(\mathcal{A}) \to K(\mathcal{B})$. (The proof of this is just to check that $F : C(\mathcal{A}) \to C(\mathcal{B})$ takes homotopic morphisms to homotopic morphisms.)

Now, suppose F is actually an exact functor. Then one also has:

(3) F induces a functor of derived categories $F : D(\mathcal{A}) \to D(\mathcal{B})$. The proof requires showing that $F : K(\mathcal{A}) \to K(\mathcal{B})$ takes quasi-isomorphisms to quasi-isomorphisms; that fact is not true in general if F is not exact.

(4) $F : D(\mathcal{A}) \to D(\mathcal{B})$ takes distinguished triangles to distinguished triangles.

The harder, and more interesting, problem is that of defining a functor of derived categories when F is *not* exact. In this case, the constructions requires the concepts of *resolutions* and *adapted classes*.

Definition 3. The **bounded-below category of complexes** (resp. **bounded-below homotopy category**, **bounded-below derived category**) over $\mathcal{A}$ is the full subcategory $C^+(\mathcal{A})$ (resp. $K^+(\mathcal{A})$, $D^+(\mathcal{A})$) of $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, $D(\mathcal{A})$) containing those objects $A^\bullet$ for which there exists an integer N such that $H^i(A^\bullet) = 0$ for all $i < N$. (Here, N may depend on the object $A^\bullet$.) The **bounded-above** categories $C^-(\mathcal{A})$, $K^-(\mathcal{A})$, $D^-(\mathcal{A})$ are defined similarly.

Definition 4. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is **large enough** (or one says that $\mathcal{A}$ **has enough objects in** $\mathcal{R}$) if for every object A of $\mathcal{A}$, there is a monomorphism $A \to R$, where $R \in \mathcal{R}$.

Theorem 5 (Existence of Resolutions)**.** *If $\mathcal{R}$ is large enough, then for any complex $A^\bullet \in D^+(\mathcal{A})$, there exists a complex $R^\bullet \in D^+(\mathcal{A})$ with $R^i \in \mathcal{R}$ for all i, and a quasi-isomorphism $t : A^\bullet \to R^\bullet$.*

Definition 6. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is an **adapted class of objects** for F if the following two conditions are satisfied:

(1) F is exact on $\mathcal{R}$ (that is, F takes any short exact sequence of objects in $\mathcal{R}$ to a short exact sequence in $\mathcal{B}$).

(2) $\mathcal{R}$ is large enough.

Definition 7 (Derived Functor)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a derived functor, and let $\mathcal{R}$ be an adapted class for F. The **derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined as follows: for any object $A^\bullet$ in $D^+(\mathcal{A})$, choose an $\mathcal{R}$-resolution $R^\bullet$, and let $RF(A^\bullet)$ be the complex $F(R^\bullet)$. The latter is well-defined up to quasi-isomorphism (*i.e.*, up to isomorphism in $D(\mathcal{B})$) because $R^\bullet$ is unique up to quasi-isomorphism, and since F is exact on $\mathcal{R}$, it takes quasi-isomorphisms of complexes of objects in $\mathcal{R}$ to quasi-isomorphisms.

There is a further issue of well-definedness: the choice of the adapted class $\mathcal{R}$. That is, if $\mathcal{R}$ and $\mathcal{R}'$ are two adapted classes for F, do they both give rise to the same notion of derived functor? It can be shown that the answer is yes, *i.e.*, that derived functors are independent of the choice of adapted class, but we will not treat this question in full generality. Instead, there is a special case in which the answer is obviously yes, and this case covers all the examples we will meet. Specifically, if it happens that $\mathcal{R} \subset \mathcal{R}'$, then every $\mathcal{R}$-resolution is also an $\mathcal{R}'$-resolution, so the two RF's defined with respect to these two classes coincide.

In our examples, we will only work with adapted classes that contain the class of injective objects.

Definition 8. An object I of an abelian category $\mathcal{A}$ is **injective** if, for every morphism $f : A \to I$ and monomorphism $i : A \hookrightarrow B$, there exists a morphism $g : B \to I$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & B \arrow[d, dashed, "g"] \\ & I \end{tikzcd}$$

Lemma 9. *Any left-exact functor is exact on the class $\mathcal{I}$ of injective objects.*

Corollary 10. *If $\mathcal{A}$ has enough injectives, then the class $\mathcal{I}$ of injective objects is an adapted class for every left-exact functor $F : \mathcal{A} \to \mathcal{B}$.*

Definition 11. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. The ith **classical derived functor** $R^iF : \mathcal{A} \to \mathcal{B}$ is defined as follows: given an object $A \in \mathcal{A}$, regard it as a complex in $D(\mathcal{A})$ with a single nonzero term located in degree 0. Then $R^iF(A) = H^i(RF(A))$.

Example 12. Let C be an abelian group. The functor $\mathrm{Hom}(C, -) : \mathfrak{Ab} \to \mathfrak{Ab}$ is left-exact, and the category $\mathfrak{Ab}$ has enough injectives, so it gives rise to a derived functor $R\,\mathrm{Hom} : D^+(\mathfrak{Ab}) \to D^+(\mathfrak{Ab})$. The corresponding classical derived functors $R^i\,\mathrm{Hom}(C, -)$ are usually denoted $\mathrm{Ext}^i(C, -)$.

There is a parallel theory for right-exact functors, giving rise to left derived functors $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$. The class of *projectives* takes the place of the class of injectives in this setting. Unfortunately, the categories of greatest interest to us—categories of sheaves on topological spaces—do not, in general, have enough projectives.

Example 13. The category $\mathfrak{Ab}$ does have enough projectives, so for a fixed abelian group C, the right-exact functor $C \otimes -$ gives rise to a derived functor $C \otimes^L -$. The classical derived functors known as "Tor" are defined by $\mathrm{Tor}_i(C, D) = H^{-i}(C \otimes^L D)$. (Note the negative exponent: this comes from the fact that Tor is usually defined in terms of *homological* complexes (in which differentials reduce degree), whereas by convention all of our complexes are *cohomological* (differentials raise degree).)

The examples above, Ext and Tor, are probably the best-known classical examples of derived functors. They also illustrate a shortcoming of the theory we have developed so far: the functors $R\,\mathrm{Hom}(-, -)$ and $- \otimes^L -$ are asymmetric in that they take an object of $\mathfrak{Ab}$ in the first variable, but a complex from $D(\mathfrak{Ab})$ in the second. It would be more natural to allow complexes in both variables.

The correct way to handle this problem is to develop a full theory of derived *bifunctors* (functors of two variables); such a theory should incorporate the idea (familiar from the classical theory of Ext and Tor) that one should be able to compute a derived bifunctor by taking an appropriate resolution of either variable. Such a theory is indeed developed in the book of Kashiwara–Schapira, but we will not develop it here. Instead, we will simply show how to repair the definitions of $R\,\mathrm{Hom}$ and $\otimes^L$ to allow complexes in both variables.

Recall that in general, a left-exact functor gives rise to a sequence of four functors:

$$\begin{aligned} F &: \mathcal{A} \to \mathcal{B} \\ F &: C(\mathcal{A}) \to C(\mathcal{B}) \\ F &: K(\mathcal{A}) \to K(\mathcal{B}) \\ RF &: D^+(\mathcal{A}) \to D^+(\mathcal{B}) \end{aligned}$$

To repair $R\operatorname{Hom}$ and $\otimes^L$, we will interfere with these sequences at the second step: instead of simply considering the functors

$$\operatorname{Hom}(C,-) : C(\mathfrak{Ab}) \to C(\mathfrak{Ab}) \qquad \text{and} \qquad C \otimes - : C(\mathfrak{Ab}) \to C(\mathfrak{Ab})$$

that are induced by functors $\mathfrak{Ab} \to \mathfrak{Ab}$, we replace them by new functors (called "graded Hom" and "graded tensor product") defined only at the level of complexes and not on $\mathfrak{Ab}$ itself.

Let $C^\bullet$ and $D^\bullet$ be complexes of abelian groups. Their graded Hom, denoted $\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)$, is the complex given by

$$\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^i = \bigoplus_{k-j=i} \operatorname{Hom}(C^j, D^k).$$

The differential $d : \underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^i \to \underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^{i+1}$ can be thought of as a large collection of maps

$$\operatorname{Hom}(C^j, D^k) \to \operatorname{Hom}(C^m, C^n),$$

where $k - j = i$ and $n - m = i + 1$. These small portions of d are defined as follows:

$$(d)_{jk,mn} : \operatorname{Hom}(C^j, D^k) \to \operatorname{Hom}(C^m, D^n)$$

$$(d)_{jk,mn} = \begin{cases} f \mapsto d_D^k \circ f & \text{if } m = j \text{ and } n = k+1, \\ f \mapsto (-1)^j f \circ d_C^{j-1} & \text{if } m = j-1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

It is left as an exercise to verify that this differential makes $\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)$ into a complex (*i.e.*, that $d^2 = 0$).

Now, $R\operatorname{Hom}(C^\bullet, D^\bullet)$ is simply defined to be $\underline{\operatorname{Hom}}(C^\bullet, I^\bullet)$, where $I^\bullet$ is an injective resolution of $D^\bullet$. The proof that this is well-defined is the same as before.

The definition of the graded tensor product is similar: $C^\bullet \underline{\otimes} D^\bullet$ is the complex given by

$$(C^\bullet \underline{\otimes} D^\bullet)^i = \bigoplus_{j+k=i} (C^j \otimes D^k).$$

As above, the differential can be specified by specifying

$$(d)_{jk,mn} : C^j \otimes D^k \to C^m \otimes D^n$$

$$(d)_{jk,mn} = \begin{cases} \mathrm{id}_{C^j} \otimes d_D^k & \text{if } m = j \text{ and } n = k+1, \\ (-1)^j d_C^j \otimes \mathrm{id}_{D^k} & \text{if } m = j+1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

The verification that $d^2 = 0$ is again left as an exercise. Then $C^\bullet \otimes^L D^\bullet$ is defined to be $C^\bullet \underline{\otimes} P^\bullet$, where $P^\bullet$ is a projective (or flat) resolution of $D^\bullet$.

Derived Functors in Categories of Sheaves

March 8, 2007

In the philosophy expounded by Grothendieck, there are six important operations on sheaves, occuring in three adjoint pairs:

$$(\overset{L}{\otimes}, R\mathcal{H}om), \qquad (f^{-1}, Rf_*), \qquad (Rf_!, f^!).$$

All other operations can be built out of these. (For example, $R\Gamma$ is the same as Ra_*, where $a : X \to \{\mathrm{pt}\}$ is the constant map to a point, and $R\operatorname{Hom} = R\Gamma \circ R\mathcal{H}om$.) In this set of notes, we will review the definitions of all of these except $f^!$, list adapted classes, and collect various composition and adjointness theorems.

Convention. All statements below are correct for sheaves of abelian groups. However, some of them contain conditions that become superfluous or trivial in the category of sheaves of vector spaces. Specifically, all sheaves of vector spaces are flat, so $\otimes$ is already an exact functor. Nevertheless, we will use the usual notation $\otimes^L$ for the functor induced by $\otimes$ on the derived category.

Theorem 1. *The category $\mathfrak{Sh}_X$ of sheaves on X has enough injectives.*

As a consequence, we can form the derived functor of all left-exact functors on sheaves.

Refer to the previous set of notes for the definitions $\underline{\operatorname{Hom}}$ and $\underline{\otimes}$. Analogous definitions can be made for $\underline{\mathcal{H}om}$ and $\underline{\otimes}$ in the category of sheaves on a space X. Then, we define

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \qquad \text{where } \mathcal{I}^\bullet \text{ is an injective resolution of } \mathcal{G}^\bullet,$$

$$\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet = \mathcal{F}^\bullet \underline{\otimes} \mathcal{P}^\bullet \qquad \text{where } \mathcal{P}^\bullet \text{ is a flat resolution of } \mathcal{G}^\bullet.$$

(Because all sheaves of vector spaces are flat, one can always just take $\mathcal{P}^\bullet = \mathcal{G}^\bullet$.)

Given a continuous map $f : X \to Y$, the push-forward functor f_* is left-exact, so we can form its derived functor Rf_*. The pullback functor f^{-1} is exact and so automatically gives rise to a functor f^{-1} on the derived category.

We need to a bit of preparation to define $f_!$.

Definition 2. Let $\mathcal{F}$ be a sheaf on X, let $s \in \mathcal{F}(U)$. The **support** of s is defined to be

$$\operatorname{supp} s = \{x \in U \mid s_x \neq 0\}.$$

This is automatically a closed subset of U.

Definition 3. A continuous map $f : X \to Y$ is **proper** if, for every compact set $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.

Definition 4. Let $f : X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf on X. The **proper push-forward** of $\mathcal{F}$, denoted $f_!\mathcal{F}$, is the subsheaf of $f_*\mathcal{F}$ defined by

$$(f_!\mathcal{F})(U) = \{s \in f^{-1}(U) \mid f|_{\operatorname{supp} s} : \operatorname{supp} s \to U \text{ is proper}\}.$$

Note that the restriction of a proper map to a closed subset of its domain is always proper. If f is a proper map, then the functors $f_!$ and f_* coincide, because $f|_{\operatorname{supp} s}$ is always proper. In particular, if f is an inclusion of a closed subset, proper push-forward and ordinary push-forward coincide.

If f is an inclusion of an open subset, $f_!$ coincides with the "extension by zero" functor defined earlier.

Definition 5. A sheaf $\mathcal{F}$ on X is **flabby** (or **flasque**) if the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective for all U in X.

Definition 6. $\mathcal{F}$ is **soft** if for every compact set $K \subset X$, the natural map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}|_K)$, or, equivalently, the natural map

$$\mathcal{F}(X) \to \varinjlim_{U \supset K} \mathcal{F}(U),$$

is surjective.

Remark 7. Some sources use the term "c-soft" for sheaves satisfying the above definition, and use the term "soft" for a related notion in which "compact" is replaced by "closed." However, on locally compact, second countable Hausdorff spaces, the two notions coincide.

Proposition 8. *Every injective sheaf is flabby. Every flabby sheaf is soft.*

Theorem 9. *The various functors on sheaves are left-exact, right-exact, or exact as shown in the table below. For each functor, the classes of sheaves listed in the third column are adapted.*

Note that this theorem is *not* asserting that the largest class listed for a given functor is in fact the largest adapted class for that functor.

Functor	*Exactness*	*Adapted classes*	*Derived functor*	*Classical derived functors*
Γ	left	injective, flabby	$R\Gamma$	$H^i(X,-)$
f_*	left	injective, flabby	Rf_*	$R^i f_*$
Hom	left	injective	$R\,\mathrm{Hom}$	Ext^i
$\mathcal{H}om$	left	injective	$R\,\mathcal{H}om$	$\mathcal{E}xt^i$
$f_!$	left	injective, flabby, soft	$Rf_!$	$R^i f_!$
f^{-1}	exact	—	f^{-1}	
$\otimes$	right	flat	$\otimes^L$	$\mathcal{T}or_i$

Once again, recall that for sheaves of vector spaces, $\otimes$ is exact, and all sheaves are flat. Also, the classical derived functor $\mathcal{T}or_i$ (as well as the functor Tor_i for abelian groups) is usually defined *homologically*: it is defined in terms of a resolution whose differentials decrease degree. By convention, all our complexes have differentials that raise degree, so the classical $\mathcal{T}or_i$ is obtained from our $\otimes^L$ by:

$$\mathcal{T}or_i(\mathcal{F},\mathcal{G}) = H^{-i}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}).$$

Since the category $\mathfrak{Sh}_X$ has enough injectives, no other adapted classes would have been needed if all we wanted to do was define the derived functors of the various left-exact functors. However, we also want to be able to understand compositions of derived functors. The following fact tells us how to do this.

Proposition 10. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two left-exact functors. If there is an adapted class $\mathcal{R}$ for F and an adapted class $\mathcal{S}$ for G such that $F(\mathcal{R}) \subset \mathcal{S}$, then $R(G \circ F) = RG \circ RF$.*

Not every left-exact functor takes injective objects to injective objects, so in order to apply the above proposition, we usually need to have some additional adapted classes at hand. This is where flabby and soft sheaves are useful.

Proposition 11. *The functors f_*, $f_!$, and $\mathcal{H}om$ act on injective, flabby, and soft sheaves as shown here:*

$$\begin{array}{lll} f_* \text{ (injective)} = \text{injective} & \quad & \mathcal{H}om(\text{anything}, \text{injective}) = \text{flabby} \\ f_* \text{ (flabby)} = \text{flabby} & & \mathcal{H}om(\text{flat}, \text{injective}) = \text{injective} \\ f_! \text{ (soft)} = \text{soft} & & \end{array}$$

The following theorem on compositions is now immediate:

Theorem 12. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then*

$$Rf_* \circ Rg_* = R(f \circ g)_*$$
$$Rf_! \circ Rg_! = R(f \circ g)_!$$

If $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are complexes of sheaves on X, then

$$R\Gamma(R\,\mathcal{H}om(\mathcal{F}^\bullet, G^\bullet)) = R\,\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Finally, we obtain two of the three important adjointness theorems by using the above facts on compositions of derived functors. In both cases, the proof consists of reducing to the corresponding theorems in the nonderived setting.

Theorem 13. *If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-(\mathfrak{Sh}_X)$ and $\mathcal{H}^\bullet \in D^+(\mathfrak{Sh}_X)$, then we have:*

$$\operatorname{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq \operatorname{Hom}(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)),$$
$$R\,\mathcal{H}om(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\operatorname{Hom}(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)),$$
$$R\operatorname{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\,\mathcal{H}om(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)).$$

Theorem 14. *If $f : X \to Y$ is a continuous map, $\mathcal{F}^\bullet \in D^-(\mathfrak{Sh}_Y)$, and $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_X)$, then we have:*

$$\operatorname{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$
$$R\operatorname{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\operatorname{Hom}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$
$$Rf_*R\,\mathcal{H}om(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\,\mathcal{H}om(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$

Proper Pull-Back and Poincaré–Verdier Duality

March 13, 2007

Convention. We now add an assumption to our standing list of assumptions on topological spaces. For each topological space X, we assume there is a number r such that for any exact sequence

$$\mathcal{F}^0 \to \cdots \to \mathcal{F}^r \to 0$$

of sheaves on X, if $\mathcal{F}^0, \dots, \mathcal{F}^{r-1}$ are all soft, then so is $\mathcal{F}^r$. (This assumption is true for manifolds and for closed subsets of manifolds.)

We also add a hypothesis on sheaves. From now on, all sheaves will be assumed to be sheaves of *finite-dimensional* vector spaces, and $\mathfrak{Sh}_X$ will denote the category of such sheaves.

The main result in this set of notes is the following.

Theorem 1. *Let $f : X \to Y$ be a continuous map. There is a functor $f^! : D^+(\mathfrak{Sh}_Y) \to D^+(\mathfrak{Sh}_X)$, unique up to isomorphism, that is right-adjoint to $Rf_!$:*

$$\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

The proof relies on the following sequence of lemmas.

Lemma 2. *Any sheaf $\mathcal{F}$ on X has a soft resolution of length at most r.*

Lemma 3. *For any sheaf $\mathcal{F}$ on X, there exists a sheaf $\mathcal{P}$ and a surjective morphism $\mathcal{P} \to \mathcal{F}$ where $\mathcal{P}$ is of the form $\mathcal{P} = \prod_i \underline{\mathbb{C}}_{V_i}$. Here $\{V_i\}$ is some collection of open sets, and $\underline{\mathbb{C}}_{V_i}$ is regarded as a sheaf on X by extension by zero.*

(The proof of the preceding lemma is close to a general proof of the existence of flat resolutions in the category of sheaves of abelian groups.)

Lemma 4. *If $\mathcal{K}$ is a flat, soft sheaf on X, then for any sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{K}$ is soft.*

Corollary 5. *The functor $\mathcal{F} \mapsto f_!(\mathcal{F} \otimes \mathcal{K})$ is exact.*

Now, if $\mathcal{K}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, let $\mathcal{H}(\mathcal{K}, \mathcal{G})$ be the presheaf given by

$$\mathcal{H}(\mathcal{K}, \mathcal{G})(V) = \operatorname{Hom}(f_!(j_!(\mathcal{K}|_V)), \mathcal{G}) \qquad \text{where } j : V \hookrightarrow X \text{ is the inclusion map.}$$

Lemma 6. *If $\mathcal{K}$ is flat and soft, and $\mathcal{G}$ is injective, then*

(1) *$\mathcal{H}(\mathcal{K}, \mathcal{G})$ is a sheaf.*
(2) $\operatorname{Hom}(f_!(\mathcal{F} \otimes \mathcal{K}), \mathcal{G}) \simeq \operatorname{Hom}(\mathcal{F}, \mathcal{H}(\mathcal{K}, \mathcal{G}))$.
(3) *$\mathcal{H}(\mathcal{K}, \mathcal{G})$ is injective.*

The last step is to actually define $f^!$. Choose a resolution $\mathcal{K}^\bullet$ of $\underline{\mathbb{C}}_X$ by flat, soft modules. (If we had been working in the setting of sheaves of abelian groups, the "flat" part of the preceding sentence would require additional work, but for sheaves of vector spaces, it is automatic.) Let $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_X)$. Assume that $\mathcal{G}^\bullet$ is a complex of injective sheaves (*i.e.*, replace it by an injective resolution if necessary. Define the complex $f^!\mathcal{G}^\bullet$ by

$$(f^!\mathcal{G}^\bullet)^i = \bigoplus_{k-j=i} \mathcal{H}(\mathcal{K}^j, \mathcal{G}^k),$$

with differentials defined in the same way as for $\underline{\mathcal{H}om}$. Once the theorem is proved, we also know that $f^!\mathcal{G}^\bullet$ is independent of the choice of $\mathcal{K}^\bullet$ by the following lemma.

Lemma 7 (Uniqueness of adjoints). *If (F, G) and (F, H) are both adjoint pairs of functors, then $G \simeq H$.*

As usual, we also have derived and sheaf-theoretic versions of the adjointness theorem. Note that these cannot be proved by reducing to the nonderived case (as was done for the adjoint pairs $(\otimes^L, R\mathcal{H}om)$ and (f^{-1}, Rf_*)), because there is no nonderived version of $f^!$ in general.

Proposition 8. *Let $f : X \to Y$ be a continuous map. For any $\mathcal{F}^\bullet \in D^b(\mathfrak{Sh}_X)$ and $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_Y)$, we have:*

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\operatorname{Hom}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet),$$
$$R\mathcal{H}om(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq Rf_*R\mathcal{H}om\,\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

Proposition 9. $f^!R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(f^{-1}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$.

Definition 10. Let $a : X \to \{\mathrm{pt}\}$ be the constant map from X to a one-point space. The **dualizing complex** on X is the complex $\omega_X = a^!\underline{\mathbb{C}}$.

Definition 11. Let $\mathcal{F}^\bullet \in D^-(\mathfrak{Sh}_X)$. The **Verdier dual** of $\mathcal{F}^\bullet$ is the complex $\mathbb{D}\mathcal{F}^\bullet = R\mathcal{H}om(\mathcal{F}, \omega_X)$.

Proposition 12. $f^!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}f^{-1}\mathcal{F}^\bullet$.

Proposition 13. $\mathbb{D}\mathbb{D}\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet$.

(The preceding proposition is not true if one works with sheaves of possibly infinite dimension, essentially because taking the vector space dual twice gives the original vector space only if the original vector space is finite-dimensional.)

Corollary 14. $f^{-1}\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}f^!\mathcal{F}^\bullet$.

Proposition 15. *We have:*

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{G}^\bullet, \mathbb{D}\mathcal{F}^\bullet) \qquad Rf_!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_*\mathcal{F}^\bullet$$
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq \mathbb{D}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \qquad Rf_*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_!\mathcal{F}^\bullet$$

The following theorem is an easy consequence of the adjointness of $Rf_!$ and $f^!$ in the special case where $f : X \to \{\mathrm{pt}\}$ is the map to a point.

Theorem 16 (Verdier Duality). $H_c^{-i}(X, \mathcal{F}^\bullet)^* \simeq H^i(X, \mathbb{D}\mathcal{F}^\bullet)$.

Classical Poincaré duality can be recovered from this with the aid of the following fact.

Proposition 17. *Let X be a smooth, oriented n-dimensional manifold. Then $\omega_X \simeq \underline{\mathbb{C}}_X[n]$.*

It follows immediately from this that on a smooth, oriented n-dimensional manifold, we have $\mathbb{D}\underline{\mathbb{C}} \simeq \underline{\mathbb{C}}[n]$. Then the Verdier duality theorem above becomes the following.

Theorem 18 (Poincaré Duality). $H_c^{n-i}(X, \mathbb{C})^* \simeq H^i(X, \mathbb{C})$.

Historically, of course, Poincaré duality came long before Verdier duality, and Verdier duality can be seen as a generalization of Poincaré duality.

Indeed, Verdier duality holds in great generality, whereas Poincaré duality, a much more specific statement, fails for most spaces that are not manifolds. For most spaces, and most complexes of sheaves, the complexes $\mathcal{F}^\bullet$ and $\mathbb{D}\mathcal{F}^\bullet$ that appear in Verdier duality are different. What makes Poincaré duality work for manifolds is the fact that the constant sheaf is close to self-dual. (Its dual is just a shift of itself.)

If we could find self-dual complexes of sheaves on other spaces, then we could achieve a sort of intermediate generalization of Poincaré duality: a duality theorem that is closer in spirit to the original Poincaré duality, but yet that holds on many spaces that are not manifolds.

The search for such self-dual complexes of sheaves is one of the principal motivations for the development of the theory of perverse sheaves.

Triangulated Categories and t-Structures

March 27, 2007

Definition 1. A **triangulated category** is an additive category $\mathcal{C}$ equipped with (a) a **shift functor** $[1] : \mathcal{C} \to \mathcal{C}$ and (b) a class of triangles $X \to Y \to Z \to X[1]$, called **distinguished triangles**, satisfying the following axioms:

(-2) The shift functor is an equivalence categories. In particular, it has an inverse, denoted $[-1] : \mathcal{C} \to \mathcal{C}$.
(-1) Every triangle isomorphic to a distinguished triangle is a distinguished triangle.
(0) Every morphism $f : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.
(1) (Identity)
(2) (Rotation)
(3) (Square Completion)
(4) (Octahedral Property)

(The last four axioms are the properties of distinguished triangles in the derived category that were established in an earlier set of notes.)

Example 2. The derived category of an abelian category, with its usual shift functor and its usual notion of distinguished triangles, is a triangulated category. The same is true of the homotopy category of complexes over an abelian category. Indeed, these are essentially the only examples of triangulated categories that we will see.

Definition 3. Let $\mathcal{C}$ be a triangulated category. A ***t*-structure** on $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ satisfying the axioms below. (For any $n \in \mathbb{Z}$, we use the notation $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[-n]$, $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[-n]$.)

(1) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 0} \supset \mathcal{C}^{\geq 1}$.
(2) $\bigcap_{n \in \mathbb{Z}} \mathcal{C}^{\leq n} = \bigcap_{n \in \mathbb{Z}} \mathcal{C}^{\geq n} = \{0\}$.
(3) If $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$, then $\mathrm{Hom}(A, B) = 0$.
(4) For any object X in $\mathcal{C}$, there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

Although the last axiom does not say anything about uniqueness of the distinguished triangle, it turns out to be unique as a consequence of the other axioms. Specifically, we have:

Proposition 4. *The distinguished triangle in Axiom* (4) *above is unique up to isomorphism. Indeed, there are functors* ${}^t\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^{\leq 0}$ *and* ${}^t\tau_{\geq 1} : \mathcal{C} \to \mathcal{C}^{\geq 1}$ *such that for any object* X *of* $\mathcal{C}$,

$$ {}^t\tau_{\leq 0}X \to X \to {}^t\tau_{\geq 1}X \to ({}^t\tau_{\leq 0}X)[1] $$

is that distinguished triangle.

Proposition 5. *Let* $\iota_{\leq 0} : \mathcal{C}^{\leq 0} \to \mathcal{C}$ *be the inclusion functor. Then* $(\iota_{\leq 0}, {}^t\tau_{\leq 0})$ *is an adjoint pair.*
Similarly, let $\iota_{\geq 1} : \mathcal{C}^{\geq 1} \to \mathcal{C})$ *be the inclusion functor. Then* $({}^t\tau_{\geq 0}, \iota_{\geq 0})$ *is an adjoint pair.*

When doing calculations in a triangulated category with a t-structure, the preceding proposition typically comes up in the following way: if $A \in \mathcal{C}^{\leq 0}$ and $f : A \to X$ is any morphism in $\mathcal{C}$, then f factors through ${}^t\tau_{\leq 0}X$. That is, there is a unique morphism f' making the following diagram commute:

$$\begin{tikzcd} A \arrow[dr, dashed, "f'"'] \arrow[drr, bend left, "f"] & & \\ & {}^t\tau_{\leq 0}X \arrow[r] & X \end{tikzcd}$$

Similarly, if $B \in \mathcal{C}^{\geq 1}$ and $g : X \to B$ is any morphism, then g factors through ${}^t\tau_{\geq 1}X$:

$$\begin{tikzcd} & & B \\ X \arrow[r] \arrow[urr, bend left, "g"] & {}^t\tau_{\geq 1}X \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

Of course, although the last two propositions were stated in terms of "≤ 0" and "≥ 1," there are corresponding statements with "$\leq n$" and "$\geq n$" for any $n \in \mathbb{Z}$, obtained by shifting. In particular, there are truncation functors ${}^t\tau_{\leq n}$ and ${}^t\tau_{\geq n}$ for all n, and there are distinguished triangles

$$ {}^t\tau_{\leq n}X \to X \to {}^t\tau_{\geq n+1}X \to ({}^t\tau_{\leq n}X)[1] $$

for all n. The relationship between truncation and shifting is given by:

$$ {}^t\tau_{\leq n}X = ({}^t\tau_{\leq 0}(X[n]))[-n]. $$

Proposition 6. *Suppose $n \leq m$. Then we have*

$$ \begin{aligned} &{}^t\tau_{\leq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\leq n} = {}^t\tau_{\leq n}, && {}^t\tau_{\geq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\geq n} \\ &{}^t\tau_{\geq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\geq n} = {}^t\tau_{\geq m}, && {}^t\tau_{\leq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\leq n} = 0 \quad \text{if } n < m. \end{aligned} $$

In particular, all truncation functors commute with each other.

Definition 7. Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. The category $\mathcal{T} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is called the **heart** (or **core**) of the t-structure.
The functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ defined by ${}^tH^0 = {}^t\tau_{\leq 0}{}^t\tau_{\geq 0} = {}^t\tau_{\geq 0}{}^t\tau_{\leq 0}$ is called **(zeroth) t-cohomology**. Moreover, for any $i \in \mathbb{Z}$, the functor ${}^tH^i$ defined by ${}^tH^i(X) = {}^tH^0(X[i])$ (or, equivalently, ${}^tH^i = {}^t\tau_{\leq i}{}^t\tau_{\geq i} = {}^t\tau_{\geq i}{}^t\tau_{\leq i}$ is called the i**th t-cohomology**.

The following theorem is the reason we want to introduce the notion of t-structures.

Theorem 8. *Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, and let $\mathcal{T}$ be its heart. $\mathcal{T}$ is an abelian category. Moreover, the functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ with the following properties:*

(1) *For any object $A \in \mathcal{T}$, ${}^tH^0(A) \simeq A$.*
(2) *${}^tH^0$ takes distinguished triangles in $\mathcal{C}$ to long exact sequences in $\mathcal{T}$.*
(3) *A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism if and only if the morphisms ${}^tH^i(f)$ are isomorphisms in $\mathcal{T}$ for all $i \in \mathbb{Z}$.*
(4) *We have*

$$ \begin{aligned} \mathcal{C}^{\leq 0} &= \{X \in \mathcal{C} \mid {}^tH^i(X) = 0 \text{ for all } i > 0\}, \\ \mathcal{C}^{\geq 0} &= \{X \in \mathcal{C} \mid {}^tH^i(X) = 0 \text{ for all } i < 0\}. \end{aligned} $$

Gluing t-Structures; Perverse Sheaves

March 29, 2007

Notation 1. Henceforth, the derived category of sheaves on a topological space X will be denoted $D(X)$ instead of $D(\mathfrak{Sh}_X)$. Its various bounded versions will similarly be denoted $D^+(X)$, $D^-(X)$, and $D^b(X)$. The standard t-structure on $D^b(X)$ will be denoted $({}^{\mathrm{std}}D^b(X)^{\leq 0}, {}^{\mathrm{std}}D^b(X)^{\geq 0})$.

Let X be a topological space; let $U \subset X$ be an open set, and $Z = X \smallsetminus U$ its complement. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Consider the bounded derived categories $D^b(U)$, $D^b(Z)$, and $D^b(X)$. We have the following six functors:

$$\begin{aligned} Rj_* &: D^b(U) \to D^b(X) & Ri_* &: D^b(Z) \to D^b(X) \\ Rj_! &: D^b(U) \to D^b(X) & i^! &: D^b(X) \to D^b(Z) \\ j^{-1} &: D^b(X) \to D^b(U) & i^{-1} &: D^b(X) \to D^b(Z) \end{aligned}$$

(Note that $Ri_! = Ri_*$, since a closed inclusion is a proper map, and $j^! = j^{-1}$ by Exercise 4 on Problem Set 7.)

The following facts are useful to keep in mind: for any sheaf $\mathcal{F}$, there is a distinguished triangle

$$Rj_!j^{-1}\mathcal{F} \to \mathcal{F} \to Ri_*i^{-1}\mathcal{F} \to (Rj_!j^{-1}\mathcal{F})[1].$$

Using the Verdier duality functor $\mathbb{D}$, we also obtain a distinguished triangle

$$Ri_*i^!\mathcal{F} \to \mathcal{F} \to Rj_*j^{-1}\mathcal{F} \to (Ri_*i^!\mathcal{F})[1].$$

(Take the first distinguished triangle for the sheaf $\mathbb{D}\mathcal{F}$, then apply $\mathbb{D}$ to the whole triangle, keeping in mind the interaction of $\mathbb{D}$ with the various push-forward and pull-back functors.)

Also, it is obvious that

$$i^{-1}Rj_!\mathcal{F} = 0$$

for any sheaf $\mathcal{F}$ on U, by the definition of "extension by zero." Another Verdier duality argument yields that

$$i^!Rj_*\mathcal{F} = 0$$

as well.

Now, let D_U be a full subcategory of $D^b(U)$, D_Z a full subcategory of $D^b(Z)$, and D a full subcategory of $D^b(X)$, such that the following conditions hold:

- D_U, D_Z, and D are triangulated categories. (In particular, they are stable under the shift operation and under formation of distinguished triangles.)
- The six functors above still "work." (That is, $Rj_*|_{D_U}$ should take values in D, *etc.*)

As an example, one can certainly take $D_U = D^b(U)$, $D_Z = D^b(Z)$, and $D = D^b(X)$, but later on we will make a different choice for these categories.

Theorem 2 (Gluing t-structures)**.** *Given t-structures $(D_U^{\leq 0}, D_U^{\geq 0})$ and $(D_Z^{\leq 0}, D_Z^{\geq 0})$ on D_U and D_Z, respectively, there is a t-structure on D defined by*

$$\begin{aligned} D^{\leq 0} &= \{\mathcal{F} \mid j^{-1}\mathcal{F} \in D_U^{\leq 0} \text{ and } i^{-1}\mathcal{F} \in D_Z^{\leq 0}\} \\ D^{\geq 0} &= \{\mathcal{F} \mid j^{-1}\mathcal{F} \in D_U^{\geq 0} \text{ and } i^!\mathcal{F} \in D_Z^{\geq 0}\} \end{aligned}$$

The main idea in the definition of perverse sheaves is that using the above theorem, we want to define a t-structure on the derived category of sheaves on a stratified space by gluing together various shifts of the standard t-structure on each stratum.

Definition 3. A **stratification** of a topological space X is a finite set $\mathcal{S}$ of subspaces (called **strata**) of X such that:

- X is the disjoint union of all the strata.

- Each stratum $S \in \mathcal{S}$ is a manifold.
- The closure of a stratum $\overline{S}$ is a union of strata.

We also call X a **stratified space**.

(Later, we will impose an additional condition on our stratifications.)

If X is a stratified space, then its set of strata $\mathcal{S}$ carries a natural partial order: we say that $S \leq T$ if and only if $S \subset \overline{T}$. Since $\mathcal{S}$ is finite, there must obviously be strata that are minimal with respect to this partial order, and others that are maximal. Evidently, a stratum is minimal if and only if it is closed. A stratum is maximal if and only if it is open.

Definition 4. An ordinary sheaf $\mathcal{F}$ on a stratified space X is **constructible** with respect to the stratification $\mathcal{S}$ if for all $S \in \mathcal{S}$, the restriction $\mathcal{F}|_S$ is locally constant.

A complex of sheaves $\mathcal{F}$ is said to be **constructible** if all of its cohomology sheaves $H^i(\mathcal{F})$ are constructible in the above sense.

The full subcategory of $D^b(X)$ consisting of constructible sheaves is denoted $D^b_c(X)$.

The category $D^b_c(X)$ is often casually referred to as "the derived category of constructible sheaves," although this is somewhat of a misnomer: it is a subcategory of the derived category of the category of ordinary sheaves, and it is not equivalent to the derived category of the category of ordinary constructible sheaves.

Definition 5. Let X be a stratified space, with set of strata $\mathcal{S}$. A **perversity function** is simply a function $p : \mathcal{S} \to \mathbb{Z}$.

For each stratum $S \in \mathcal{S}$, let $i_S : S \hookrightarrow X$ be the inclusion map. The **perverse *t*-structure** on $D = D^b(X)$ with respect to the perversity p is given by

$$
\begin{aligned}
{}^pD^{\leq 0} &= \{\mathcal{F} \mid i_S^{-1}\mathcal{F} \in {}^{\mathrm{std}}D^b(S)^{\leq p(S)} \text{ for all } S \in \mathcal{S}\} \\
{}^pD^{\geq 0} &= \{\mathcal{F} \mid i_S^{!}\mathcal{F} \in {}^{\mathrm{std}}D^b(S)^{\geq p(S)} \text{ for all } S \in \mathcal{S}\}
\end{aligned}
$$

(This is a slightly nonstandard definition; the issue will be clarified later.) The associated truncation and t-cohomology functors are denoted ${}^p\tau_{\leq 0}$, ${}^p\tau_{\geq 0}$, and ${}^pH^0$. A **perverse sheaf** on X with respect to p is a constructible sheaf in the heart of this t-structure.

Of course, we must show that this actually is a t-structure. That is quite easy; we just use the gluing theorem and induction on the number of strata.

Theorem 6. *The perverse t-structure on $D^b(X)$ is a t-structure.*

Remark 7. By far, the most common setting for perverse sheaves is one in which all strata are even-dimensional, and the perversity function used is $p(S) = -\frac{1}{2}\dim S$. The advantages of this will become clear when we discuss Verdier duality for perverse sheaves.

In any situation in which the perversity function is not specified, it should be assumed that this is the perversity function being used.

One reason the theory of perverse sheaves is so useful is that there is a new kind of extension functor with very good properties.

Theorem 8. *In the context of Theorem 2, let $\mathcal{T}$, $\mathcal{T}_U$, and $\mathcal{T}_Z$ be the hearts of the t-structures on D, D_U, and D_Z, respectively. Let $\mathcal{F}$ be an object of $\mathcal{T}_U$. There is a unique object $\mathcal{G}$ of $\mathcal{T}$, up to isomorphism, such that*

$$
j^{-1}\mathcal{G} \simeq \mathcal{F}, \qquad i^{-1}\mathcal{G} \in D_Z^{\leq -1}, \qquad i^!\mathcal{G} \in D_Z^{\geq 1}.
$$

*This object is denoted by $\mathcal{G} = j_{!*}\mathcal{F}$ and is called the **middle extension** or **Goresky–MacPherson extension** of $\mathcal{F}$.*

In addition, let us define two t-structures $(D^{\pm,\leq 0}, D^{\pm,\geq 0})$ on D obtained by gluing $(D_U^{\leq 0}, D_U^{\geq 0})$ and one of $(D_Z^{\leq \pm 1}, D_Z^{\geq \pm 1})$, and let $\tau^{\pm}_{\leq 0}$, $\tau^{\pm}_{\geq 0}$ be the corresponding truncation functors. Then the middle extension can be characterized as

$$
j_{!*}\mathcal{F} \simeq \tau^{-}_{\leq 0} Rj_*\mathcal{F} \simeq \tau^{+}_{\geq 0} Rj_!\mathcal{F}.
$$

Definition 9. Let X be a stratified space with set of strata $\mathcal{S}$ and perversity function $p : \mathcal{S} \to \mathbb{Z}$. Let S be a stratum of X, and let $\mathcal{E}$ be a local system (locally constant ordinary sheaf) on S. Let $j_S : S \hookrightarrow \overline{S}$ be the inclusion of S into its closure, and let $i_{\overline{S}} : \overline{S} \hookrightarrow X$ be the inclusion of $\overline{S}$ into X. $\mathcal{E}[p(S)]$ is a perverse sheaf on S, so $j_{S!*}(\mathcal{E}[p(S)])$ is a perverse sheaf on $\overline{S}$, and the sheaf

$$\mathrm{IC}^p(\overline{S}, \mathcal{E}) = i_{\overline{S}*} j_{S!*}(\mathcal{E}[p(S)])$$

is a perverse sheaf on X. (If the perversity function to be used is unambiguous, the "p" may be omitted from the notation.) A perverse sheaf obtained in this way is called an **intersection cohomology complex**. In the case $\mathcal{E} = \underline{\mathbb{C}}_S$, $\mathrm{IC}(\overline{S}, \underline{\mathbb{C}}_S)$ is also denoted $\mathrm{IC}(\overline{S})$ or $\mathrm{IC}_{\overline{S}}$.

But wait! It is clear that intersection cohomology complexes are objects in the heart of the perverse t-structure, but are they constructible?

This question will be resolved in the next set of notes. At the same time, we will deal with the three issues that were put off until "later" in this set of notes.

Goresky–MacPherson Stratifications and Perversities

April 12, 2007

Intersection cohomology complexes are in some sense the most important perverse sheaves—essentially all perverse sheaves that come up in practice are direct sums of intersection cohomology complexes. But as pointed out in the last set of notes, it is not clear that intersection cohomology complexes are actually constructible. In fact, we need to modify the definition of "stratification" to make sure that they are.

Definition 1. Let X be a topological space equipped with a stratification $\mathcal{S}$, and let $n = \max\{\dim S \mid S \in \mathcal{S}\}$. $\mathcal{S}$ is called a **topological stratification** (or a **Goresky–MacPherson stratification**) if it satisfies the following additional condition: for any point x in a stratum of dimension k with $k < n$, there is a neighborhood U together with a homeomorphism

$$U \simeq \mathbb{R}^k \times C$$

where

$$C = (Y \times [0, \infty)/(Y \times \{0\})$$

(the "cone" on Y), and where Y is a topologically stratified space of dimension $n-k-1$. Moreover, the set of strata $\mathcal{T}$ of Y should be in bijection with the strata of X that are larger than S in the partial order on strata: specifically, if $S' \in \mathcal{S}$ is such that $S \subset \overline{S'}$ but $S \neq S'$, then there must exist a stratum T of Y such that the homeomorphism $U \simeq \mathbb{R}^k \times C$ restricts to a homeomorphism

$$S' \cap U \simeq \mathbb{R}^k \times (T \times (0, \infty)),$$

and every $T \in \mathcal{T}$ must arise in this way from some S'.

Theorem 2. *If $\mathcal{S}$ is a topological stratification of X, then for any stratum S and any local system $\mathcal{E}$ on S, the sheaf $Ri_{S*}\mathcal{E}$ (where $i_S : S \hookrightarrow X$ is inclusion) is constructible.*

Now we can go back and revisit the theorems on gluing of t-structures and on middle extension. Those theorems were stated in terms of categories D, D_U, and D_Z, together with six pull-back and push-forward functors. On a topologically stratified space, the relevant pull-back and push-forward functors all take constructible sheaves to constructible sheaves, so we can invoke those theorems with $D = D^b_c(X)$, $D_U = D^b_c(U)$, and $D^b_c(Z)$. In particular, we can define a "perverse t-structure" on $D^b_c(X)$ (not just on the larger category $D^b(X)$), and we can revise the definition of "perverse sheaves" as follows.

Definition 3. The category of **perverse sheaves** on X (with respect to a topological stratification $\mathcal{S}$ and a perversity function $p : \mathcal{S} \to \mathbb{Z}$), denoted $M(X)$, is the heart of the **perverse *t*-structure** on $D^b_c(X)$ given by

$$\begin{aligned} {}^pD^b_c(X)^{\leq 0} &= \{\mathcal{F} \mid i_S^{-1}\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\leq p(S)} \text{ for all } S \in \mathcal{S}\} \\ {}^pD^b_c(X)^{\geq 0} &= \{\mathcal{F} \mid i_S^{!}\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\geq p(S)} \text{ for all } S \in \mathcal{S}\} \end{aligned}$$

Then, revisiting the middle extension theorem gives the following result.

Corollary 4. *On a topologically stratified space, the intersection cohomology complexes $\mathrm{IC}(\overline{S}, \mathcal{E})$ are constructible.*

Recall that a **simple** or **irreducible object** A in an abelian category is an object with the property that any monomorphism $A' \to A$ is either 0 or an isomorphism. Equivalently, any epimorphism $A \to A''$ must be either 0 or an isomorphism.

We call a local system "simple" if it has no nontrival ordinary subsheaf that is also a local system. A simple local system on a stratum of a stratified space is a simple object in the category of ordinary constructible sheaves, but not in the category of all ordinary sheaves.

Theorem 5. *The simple objects in the category $M(X)$ are precisely the intersection cohomology complexes $\mathrm{IC}(\overline{S}, \mathcal{E})$ where a $\mathcal{E}$ is a simple local system on S.*

So far, perversity functions have been arbitrary integer-valued functions on the set of strata. In the literature, however, one usually imposes a few mild conditions on perversities, and these conditions have a number of useful consequences.

Definition 6. A perversity function $p : \mathcal{S} \to \mathbb{Z}$ is said to be **of Goresky–MacPherson type** if the following conditions hold:

(1) $p(S)$ depends only on the dimension of S: that is, there is a function $\tilde{p} : \mathbb{N} \to \mathbb{Z}$ such that $p(S) = \tilde{p}(\dim S)$.
(2) $\tilde{p}$ is weakly decreasing.
(3) $\tilde{p}^* : \mathbb{N} \to \mathbb{Z}$, defined by $\tilde{p}^*(k) = -k - \tilde{p}(k)$, is also weakly decreasing.

The last two conditions are equivalent to requiring that

$$0 \leq \tilde{p}(l) - \tilde{p}(k) \leq k - l$$

whenever $0 \leq l \leq k$.

Often, Goresky–MacPherson perversities are normalized by also requiring $\tilde{p}(0) = 0$. With this normalization, it is clear that the Goresky–MacPherson perversity with the smallest possibles values (the "bottom perversity") is

$$\tilde{p}(n) = -n.$$

Conversely, the "top perversity" is the constant perversity

$$\tilde{p}(n) = 0.$$

(As we have noted before, perverse sheaves with respect to a constant perversity function are the same as ordinary constructible sheaves.) In between are the "upper middle" and "lower middle" perversities

$$\tilde{p}(n) = \lceil -n/2 \rceil \qquad \text{and} \qquad \tilde{p}(n) = \lfloor -n/2 \rfloor.$$

On a space with only even-dimensional strata, these two perversities coincide with each other and with what we have previously called the "middle perversity."