AN OVERVIEW OF MORIHIKO SAITO'S THEORY OF MIXED HODGE MODULES

CHRISTIAN SCHNELL

Abstract. After explaining the definition of pure and mixed Hodge modules on complex manifolds, we describe some of Saito's most important results and their proofs, and then discuss two simple applications of the theory.

A. Introduction

1. Nature of Saito's theory. Hodge theory on complex manifolds bears a surprising likeness to ℓ-adic cohomology theory on algebraic varieties defined over finite fields. This was pointed out by Deligne [Del71], who also proposed a "heuristic dictionary" for translating results from one language into the other, and used it to predict, among other things, the existence of limit mixed Hodge structures and the notion of admissibility for variations of mixed Hodge structure. But the most important example of a successful translation is without doubt Morihiko Saito's theory of mixed Hodge modules: it is the analogue, in Hodge theory, of the mixed ℓ-adic complexes introduced by Beilinson, Bernstein, and Deligne [BBD82]. One of the main accomplishments of Saito's theory is a Hodge-theoretic proof for the decomposition theorem; the original argument in [BBD82, §6.2] was famously based on reduction to positive characteristic.

Contained in two long papers [Sai88, Sai90b], Saito's theory is a vast generalization of classical Hodge theory, built on the foundations laid by many people during the 1970s and 1980s: the theory of perverse sheaves, $\mathscr{D}$-module theory, and the study of variations of mixed Hodge structure and their degenerations. Roughly speaking, classical Hodge theory can deal with two situations: the cohomology groups of a single complex algebraic variety, to which it assigns a mixed Hodge structure; and the cohomology groups of a family of smooth projective varieties, to which it assigns a variation of Hodge structure. The same formalism, based on the theory of harmonic forms and the Kähler identities, also applies to cohomology groups of the form $H^k(B, \mathcal{H})$, where B is a smooth projective variety and $\mathcal{H}$ a polarizable variation of Hodge structure. Saito's theory, on the other hand, can deal with arbitrary families of algebraic varieties, and with cohomology groups of the form $H^k(B, \mathcal{H})$, where B is non-compact or singular, and where $\mathcal{H}$ is an admissible variation of mixed Hodge structure. It also provides a nice formalism, in terms of filtered $\mathscr{D}$-modules, that treats all those situations in a consistent way.

Conceptually, there are two ways of thinking about mixed Hodge modules:

(1) Mixed Hodge modules are *perverse sheaves with mixed Hodge structure*. They obey the same six-functor formalism as perverse sheaves; whenever an

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operation on perverse sheaves produces a $\mathbb{Q}$-vector space, the corresponding operation on mixed Hodge modules produces a mixed Hodge structure.

(2) Mixed Hodge modules are *a special class of filtered $\mathscr{D}$-modules* with good properties. Whereas arbitrary filtered $\mathscr{D}$-modules do not behave well under various operations, the ones in Saito's theory do. The same is true for the coherent sheaves that make up the filtration on the $\mathscr{D}$-module; many applications of the theory rely on this fact.

The first point of view is more common among people working in Hodge theory; the second one among people in other areas who are using mixed Hodge modules to solve their problems. Saito's theory has been applied with great success in representation theory [BW04, SV11], singularity theory [Sai91c, BMS06, BFNP09, DMS11], algebraic geometry [Sai91b, Sai01, PS14], and nowadays also in Donaldson-Thomas theory [BBD$^+$12, BBS13]. I should also mention that the whole theory has recently been generalized to arbitrary semisimple representations of the fundamental group on algebraic varieties [Sab12].

2. About this article. The purpose of this article is to explain the definition of pure and mixed Hodge modules on complex manifolds; to describe some of the most important results and their proofs; and to discuss two simple applications of the theory. Along the way, the reader will also find some remarks about mixed Hodge modules on analytic spaces and on algebraic varieties, and about the construction of various functors. While it is easy to learn the formalism of mixed Hodge modules, especially for someone who is already familiar with perverse sheaves, it helps to know something about the internals of the theory, too. For that reason, more than half of the text is devoted to explaining Saito's definition and the proofs of two crucial theorems; in return, I had to neglect other important parts of the story.

The article has its origins in two lectures that I gave during the workshop on geometric methods in representation theory in Sanya. In editing the notes, I also tried to incorporate what I learned during a week-long workshop about mixed Hodge modules and their applications, held in August 2013 at the Clay Mathematics Institute in Oxford. Both in my lectures and in the text, I decided to present Saito's theory with a focus on filtered $\mathscr{D}$-modules and their properties; in my experience, this aspect of the theory is the one that is most useful in applications.

For those looking for additional information about mixed Hodge modules, there are two very readable surveys by Saito himself: a short introduction that focuses on "how to use mixed Hodge modules" [Sai89b], and a longer article with more details [Sai94]. The technical aspects of the theory are discussed in a set of lecture notes by Shimizu [RIM92], and a brief axiomatic treatment can be found in the book by Peters and Steenbrink [PS08, Ch. 14]. There are also unpublished lecture notes by Sabbah [Sab07], who gives a beautiful account of the theory on curves.

3. Left and right D-modules. An issue that always comes up in connection with Saito's theory is whether one should use left or right $\mathscr{D}$-modules. This is partly a matter of taste, because the two notions are equivalent on complex manifolds, but each choice brings its own advantages and disadvantages. Saito himself dealt with this problem by using left $\mathscr{D}$-modules in the introduction to [Sai88, Sai90b], and right $\mathscr{D}$-modules in the definitions and proofs. After much deliberation, I decided to follow Saito and to write this article entirely in terms of *right $\mathscr{D}$-modules*. Since this is sure to upset some readers, let me explain why.

One reason is that the direct image and duality functors are naturally defined for right $\mathscr{D}$-modules. Both functors play important roles in the theory: the duality functor is needed to define polarizations, and the direct image functor is needed for example to define mixed Hodge modules on singular spaces. The idea is that $\mathscr{D}$-modules on a singular space X are the same thing as $\mathscr{D}$-modules on some ambient complex manifold with support in X; to make this definition independent of the choice of embedding, it is better to use right $\mathscr{D}$-modules. Another reason is that it makes sense to distinguish, conceptually, between variations of Hodge structure and Hodge modules: for example, it is easy to define inverse images for variations of Hodge structure, but not for Hodge modules. Replacing a flat bundle by the corresponding right $\mathscr{D}$-module further emphasizes this distinction.

4. Brief summary. Let X be a complex algebraic variety. Although Saito's theory works more generally on complex manifolds and analytic spaces, we shall begin with the case of algebraic varieties, because it is technically easier and some of the results are stronger. Saito constructs the following two categories:[1]

$$\begin{aligned} \mathrm{HM}(X, w) &= \text{polarizable Hodge modules of weight } w \\ \mathrm{MHM}(X) &= \text{graded-polarizable mixed Hodge modules} \end{aligned}$$

Both are abelian categories, with an exact and faithful functor

$$\mathrm{rat}\colon \mathrm{MHM}(X) \to \mathrm{Perv}_{\mathbb{Q}}(X)$$

to the category of perverse sheaves (with coefficients in $\mathbb{Q}$). All the familiar operations on perverse sheaves – such as direct and inverse images along arbitrary morphisms, Verdier duality, nearby and vanishing cycles – are lifted to mixed Hodge modules in a way that is compatible with the functor “rat”. This means that whenever some operation on perverse sheaves produces a $\mathbb{Q}$-vector space, the corresponding operation on mixed Hodge modules endows it with a mixed Hodge structure. It also means that every semisimple perverse sheaf of geometric origin is a direct summand of $\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{rat}\, M$ for some polarizable Hodge module M. After going to the derived category $\mathrm{D}^b \mathrm{MHM}(X)$, one has a formalism of weights, similar to the one for mixed complexes in [BBD82, §5.1.5].

The relation between (mixed) Hodge modules and variations of (mixed) Hodge structure is similar to the relation between perverse sheaves and local systems. Every polarizable variation of Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of an irreducible subvariety $Z \subseteq X$ extends uniquely to a polarizable Hodge module $M \in \mathrm{HM}(X, w)$ with strict support Z, which means that $\mathrm{rat}\, M$ is the intersection complex of the underlying local system of $\mathbb{Q}$-vector spaces. Conversely, every polarizable Hodge module on X is isomorphic to a direct sum of objects of the above type, and this so-called *decomposition by strict support* is unique. The category $\mathrm{HM}(X, w)$ is semi-simple; the perverse sheaves that one gets by applying the functor “rat” are also semi-simple.

The category $\mathrm{MHM}(X)$ of graded-polarizable mixed Hodge modules is no longer semi-simple, but every object $M \in \mathrm{MHM}(X)$ comes with a finite increasing filtration $W_\bullet M$, called the *weight filtration*, such that

$$\mathrm{gr}_\ell^W M = W_\ell M / W_{\ell-1} M \in \mathrm{HM}(X, \ell).$$

[1] To be precise, $\mathrm{HM}(X, w)$ should be $\mathrm{HM}^p(X, w)$, and $\mathrm{MHM}(X)$ should be $\mathrm{MHM}_{\mathrm{alg}}(X)$.

Every mixed Hodge module is therefore an extension of several pure Hodge modules; the extensions among these are not arbitrary, but are controlled by an "admissibility" condition similar to the one appearing in the theory of variations of mixed Hodge structure by Steenbrink and Zucker [SZ85] and Kashiwara [Kas86]. In the same way that a perverse sheaf can be obtained by gluing together local systems on a stratification [Ver85, Beĭ87], one can think of a mixed Hodge module as being obtained by gluing together admissible variations of mixed Hodge structure.

5. Examples. In order to get some intuition, let us look at a few typical examples of mixed Hodge modules. The exact definitions will be given in §12 and §20 below; for the time being, we shall describe mixed Hodge modules by giving the underlying perverse sheaf, the underlying filtered $\mathscr{D}$-module, and the weight filtration. In more detail, suppose that $M \in \mathrm{MHM}(X)$ is a mixed Hodge module on a smooth algebraic variety X, considered as a complex manifold. The perverse sheaf is $\operatorname{rat} M$; its complexification is isomorphic to the de Rham complex of a regular holonomic right $\mathscr{D}_X$-module $\mathcal{M}$, and the additional piece of data is an increasing filtration $F_\bullet \mathcal{M}$ by coherent $\mathscr{O}_X$-modules, compatible with the order filtration on $\mathscr{D}_X$.

Example 5.1. The most basic example of a polarizable Hodge module on a nonsingular algebraic variety X of dimension n is the canonical bundle ω_X. It is naturally a right $\mathscr{D}$-module, and together with the filtration

$$F_{-n-1}\,\omega_X = 0 \quad \text{and} \quad F_{-n}\,\omega_X = \omega_X$$

and the perverse sheaf $\mathbb{Q}_X[n]$, it is an object of the category $\mathrm{HM}(X,n)$, usually denoted by $\mathbb{Q}_X^H[n]$. One can obtain many additional examples by applying various functors, such as the direct image by a morphism $f\colon X \to Y$. When f is proper, the filtration on $\mathcal{H}^i f_* \mathbb{Q}_X^H[n]$ starts with the coherent $\mathscr{O}_Y$-module $R^i f_* \omega_X$; this fact is behind several results in algebraic geometry, such as Kollár's theorems about higher direct images of dualizing sheaves (see §25).

Example 5.2. More generally, one can consider an irreducible subvariety $Z \subseteq X$, and a polarizable variation of Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of the smooth locus of Z. As mentioned above, it extends uniquely to a polarizable Hodge module $M \in \mathrm{HM}(X,w)$ with strict support Z. In this case, the perverse sheaf $\operatorname{rat} M$ is simply the intersection complex of the underlying local system of $\mathbb{Q}$-vector spaces. The filtration $F_\bullet \mathcal{M}$ on the corresponding $\mathscr{D}$-module is also uniquely determined by the Hodge filtration, but hard to make explicit (except in special cases, for instance when the variation is defined on the complement of a normal crossing divisor).

Example 5.3. One case where one can describe the filtration in geometric terms is the family of hyperplane sections of a smooth projective variety [Sch12]. Suppose that X is a smooth projective variety of dimension n, and that $Y \subseteq X$ is a nonsingular hyperplane section (in a fixed projective embedding). The entire cohomology of Y is determined by that of X, with the exception of the so-called *variable part*

$$H_0^{n-1}(Y,\mathbb{Q}) = \ker\Big(H^{n-1}(Y,\mathbb{Q}) \to H^{n+1}\big(X,\mathbb{Q}(1)\big)\Big).$$

Now let B be the projective space parametrizing hyperplane sections of X, and let $f\colon \mathscr{Y} \to B$ be the universal family. Let $B_0 \subseteq B$ be the open subset corresponding to nonsingular hyperplane sections, and let $\mathcal{V}$ be the variation of Hodge structure on the variable part of their cohomology. As in the previous example, it determines a Hodge module M on B, and one can show that M is isomorphic to a direct summand of $\mathcal{H}^0 f_* \mathbb{Q}_{\mathscr{Y}}^H[d+n-1]$, where $d = \dim B$. The filtration $F_\bullet \mathcal{M}$ on the underlying $\mathscr{D}$-module can be described using residues of meromorphic forms. For a nonsingular hyperplane section $Y \subseteq X$, a classical theorem by Carlson and Griffiths [CG80] says that the *residue mapping*

$$\operatorname{Res}_Y \colon H^0\bigl(X, \Omega_X^n(pY)\bigr) \to F^{n-p} H_0^{n-1}(Y, \mathbb{C})$$

is surjective, provided the line bundle $\mathscr{O}_X(Y)$ is sufficiently ample. Given an open subset $U \subseteq B$ and a meromorphic form

$$\omega \in H^0\Bigl(U \times X, \Omega_{B\times X}^{d+n}\bigl((n+d+p)\mathscr{Y}\bigr)\Bigr),$$

one can therefore take the residue along smooth fibers to obtain a holomorphic section of $\omega_B \otimes j_* F^{-d-p}\mathcal{V}$, where $j \colon B_0 \hookrightarrow B$ denotes the inclusion. Then $F_p\mathcal{M}$ is exactly the subsheaf of $\omega_B \otimes j_* F^{-d-p}\mathcal{V}$ given by all sections of this kind.

Example 5.4. One can obtain mixed Hodge modules by taking the direct image along an open embedding $j \colon U \hookrightarrow X$. Suppose that X is a nonsingular algebraic variety of dimension n, and that $U = X \setminus D$ is the complement of a divisor. Then $j_*\mathbb{Q}_U^H[n]$ is a mixed Hodge module on X; the underlying perverse sheaf is $\mathbf{R}j_*\mathbb{Q}[n]$, and the underlying regular holonomic $\mathscr{D}$-module is $\omega_X(*D)$, the sheaf of meromorphic n-forms on X that are holomorphic on U. When the divisor D is nonsingular, one can also say explicitly what the Hodge filtration and the weight filtration are. The Hodge filtration satisfies

$$F_{-n-1}\,\omega_X(*D) = 0 \quad \text{and} \quad F_p\,\omega_X(*D) = \omega_X\bigl((p+n+1)D\bigr) \quad \text{for } p \geq -n,$$

and is therefore essentially the filtration by pole order; the weight filtration satisfies

$$W_n j_*\mathbb{Q}_U^H[n] = \mathbb{Q}_X^H[n] \quad \text{and} \quad W_{n+1} j_*\mathbb{Q}_U^H[n] = j_*\mathbb{Q}_U^H[n],$$

and the quotient $\operatorname{gr}_{n+1}^W j_*\mathbb{Q}_U^H[n]$ is isomorphic to the direct image of $\mathbb{Q}_D^H(-1)[n-1]$, with a Tate twist to change the weight from $n-1$ to $n+1$. When the divisor D is singular, both filtrations still exist, but are hard to describe concretely.

Example 5.5. An interesting special case of the previous construction has been studied by Cautis and Kamnitzer [Cau13]. Let $G(k,n)$ denote the Grassmannian parametrizing k-dimensional subspaces of $\mathbb{C}^n$, and consider the divisor

$$D = \bigl\{\, (V,W) \in G(k,n) \times G(n-k,n) \bigm| V \cap W \neq 0 \,\bigr\}$$

inside the product $X = G(k,n) \times G(n-k,n)$; without loss of generality, $k \leq n/2$. We would like to describe the direct image $j_*\mathbb{Q}_U^H[d] \in \operatorname{MHM}(X)$, where $d = 2k(n-k)$ is the dimension of X. The open set U is one of the orbits of the natural $\operatorname{GL}(n)$-action on X; the divisor D is the union of the other orbits

$$Z_r = \bigl\{\, (V,W) \in G(k,n) \times G(n-k,n) \bigm| \dim V \cap W \geq r \,\bigr\}$$

for $1 \leq r \leq k$, and therefore singular once $k \geq 2$. Using the group action, one can show that for $r = 0, \dots, k$, the graded quotient

$$\operatorname{gr}_{d+r}^W j_*\mathbb{Q}_U^H[d] \in \operatorname{HM}(X, d+r)$$

of the weight filtration is the Hodge module corresponding to the constant variation of Hodge structure $\mathbb{Q}\bigl(-\tfrac{1}{2}r(r+1)\bigr)$ on the orbit Z_r; it has the correct weight because

$$\dim Z_r + r(r+1) = d - r^2 + r(r+1) = d + r.$$

One can also compute the graded module $\mathrm{gr}_\bullet^F \omega_X(*D)$ and show that the characteristic variety is the union of the conormal varieties of the orbit closures.

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B. Pure Hodge modules

After a brief review of nearby and vanishing cycle functors, we explain the definition of (polarizable) Hodge modules on complex manifolds. A Hodge module is basically a certain type of $\mathscr{D}$-module with a filtration and a rational structure. The goal is understand the need for the various technical conditions that Saito imposes on such objects in order to define the category $\mathrm{HM}^p(X, w)$ of polarizable Hodge modules of weight w on a complex manifold X.

7. Basic objects. Saito's theory is an extension of classical Hodge theory, and the objects that he uses are a natural generalization of variations of Hodge structure. Recall that a *polarized variation of Hodge structure of weight w* on a complex manifold X consists of the following four things:

(1) A local system V of finite-dimensional $\mathbb{Q}$-vector spaces.
(2) A holomorphic vector bundle $\mathcal{V}$ with integrable connection $\nabla\colon \mathcal{V} \to \Omega_X^1 \otimes \mathcal{V}$.
(3) A Hodge filtration $F^\bullet \mathcal{V}$ by holomorphic subbundles.
(4) A bilinear form $Q\colon V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-w)$, where $\mathbb{Q}(-w) = (2\pi i)^{-w}\mathbb{Q} \subseteq \mathbb{C}$.

They are related by the condition that the local system of ∇-flat holomorphic sections of $\mathcal{V}$ is isomorphic to $V \otimes_{\mathbb{Q}} \mathbb{C}$; in particular, $\mathcal{V} \simeq V \otimes_{\mathbb{Q}} \mathscr{O}_X$. At every point $x \in X$, the filtration $F^\bullet \mathcal{V}_x$ and the rational structure V_x should define a Hodge structure of weight w on the $\mathbb{C}$-vector space $\mathcal{V}_x$, and this Hodge structure should be polarized by the bilinear form Q_x. Globally, the Hodge filtration is required to satisfy the Griffiths transversality relation

$$\nabla\bigl(F^p \mathcal{V}\bigr) \subseteq \Omega_X^1 \otimes F^{p-1}\mathcal{V}.$$

By the holomorphic Poincaré lemma, the holomorphic de Rham complex

$$\mathcal{V} \to \Omega_X^1 \otimes \mathcal{V} \to \cdots \to \Omega_X^{\dim X} \otimes \mathcal{V}$$

is a resolution of the locally constant sheaf $V \otimes_{\mathbb{Q}} \mathbb{C}$; this gives another way to describe the relationship between V and $(\mathcal{V}, \nabla)$. Lastly, recall that a variation of Hodge structure is called *polarizable* if it admits at least one polarization.

Saito's idea is to generalize variations of Hodge structure by allowing perverse sheaves instead of local systems; because of the Riemann-Hilbert correspondence, it then becomes necessary to use regular holonomic $\mathscr{D}$-modules instead of vector bundles with integrable connection. This leads to the following definition.

Definition 7.1. A *filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure* is a triple $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$, consisting of the following objects:

(1) A constructible complex of $\mathbb{Q}$-vector spaces K.

(2) A regular holonomic right $\mathscr{D}_X$-module $\mathcal{M}$ with an isomorphism

$$\mathrm{DR}(\mathcal{M}) \simeq \mathbb{C} \otimes_{\mathbb{Q}} K.$$

By the Riemann-Hilbert correspondence, this makes K a perverse sheaf.

(3) A good filtration $F_\bullet \mathcal{M}$ by $\mathscr{O}_X$-coherent subsheaves of $\mathcal{M}$, such that

$$F_p\mathcal{M} \cdot F_k\mathscr{D} \subseteq F_{p+k}\mathcal{M}$$

and such that $\mathrm{gr}_\bullet^F \mathcal{M}$ is coherent over $\mathrm{gr}_\bullet^F \mathscr{D}_X \simeq \mathrm{Sym}^\bullet \mathscr{T}_X$.

Here $\mathscr{T}_X$ is the tangent sheaf of the complex manifold X. The *de Rham complex* of a right $\mathscr{D}$-module $\mathcal{M}$ is the following complex:

$$\mathrm{DR}(\mathcal{M}) = \Big[\mathcal{M} \otimes \textstyle\bigwedge^n \mathscr{T}_X \to \cdots \to \mathcal{M} \otimes \mathscr{T}_X \to \mathcal{M}\Big][n]$$

It is concentrated in degrees $-n, \dots, -1, 0$, where $n = \dim X$; and naturally filtered by the family of subcomplexes

$$F_p \,\mathrm{DR}(\mathcal{M}) = \Big[F_{p-n}\mathcal{M} \otimes \textstyle\bigwedge^n \mathscr{T}_X \to \cdots \to F_{p-1}\mathcal{M} \otimes \mathscr{T}_X \to F_p\mathcal{M}\Big][n].$$

As I mentioned earlier, one can think of M as a perverse sheaf with an additional filtration on the corresponding regular holonomic $\mathscr{D}$-module; the functor "rat" to the category of perverse sheaves is of course defined by setting $\mathrm{rat}\, M = K$.

Example 7.2. To every variation of Hodge structure, one can associate a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure by the following procedure. The perverse sheaf is $K = V[n]$, and the regular holonomic $\mathscr{D}$-module is

$$\mathcal{M} = \omega_X \otimes_{\mathscr{O}_X} \mathcal{V},$$

with right $\mathscr{D}$-module structure given by the rule $(\omega \otimes m) \cdot \xi = (\omega \cdot \xi) \otimes m - \omega \otimes \nabla_\xi m$ for local holomorphic sections $\omega \in \omega_X$, $m \in \mathcal{M}$, and $\xi \in \mathscr{T}_X$. It is filtered by the coherent subsheaves

$$F_p\mathcal{M} = \omega_X \otimes_{\mathscr{O}_X} F^{-p-n}\mathcal{V};$$

we will see later that the shift in the filtration gives M the weight $w + n$. The definitions are set up in such a way that the de Rham complex $\mathrm{DR}(\mathcal{M})$ is isomorphic to the holomorphic de Rham complex of $\mathcal{V}$, shifted n steps to the left; because of the holomorphic Poincaré lemma, this means that $\mathrm{DR}(\mathcal{M}) \simeq \mathbb{C} \otimes_{\mathbb{Q}} K$.

Example 7.3. A typical example is the constant variation of Hodge structure $\mathbb{Q}_X$ of weight 0. The corresponding triple is $\big(\omega_X, F_\bullet\omega_X, \mathbb{Q}_X[n]\big)$; here ω_X is naturally a right $\mathscr{D}$-module, and the filtration is such that $\mathrm{gr}_p^F \omega_X = 0$ for $p \neq -n$.

Example 7.4. The *Tate twist* of M by an integer k is the new triple

$$M(k) = \Big(\mathcal{M}, F_{\bullet-k}\mathcal{M}, K \otimes_{\mathbb{Q}} \mathbb{Q}(k)\Big),$$

where $\mathbb{Q}(k) = (2\pi i)^k \mathbb{Q} \subseteq \mathbb{C}$. For variations of Hodge structure, this definition specializes to the usual notion.

Example 7.5. When X is compact, the cohomology groups $H^i\big(X, \mathrm{DR}(\mathcal{M})\big)$ are finite-dimensional $\mathbb{C}$-vector spaces; they come with a filtration induced by $F_\bullet \,\mathrm{DR}(\mathcal{M})$, and a $\mathbb{Q}$-structure induced by the isomorphism with $H^i(X, K) \otimes_{\mathbb{Q}} \mathbb{C}$. We will see later that they are polarizable Hodge structures when X is projective and M is a polarizable Hodge module.

The class of all filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure is of course much too large for the purposes of Hodge theory; the problem is to find a subclass of "Hodge modules" that is small enough to have good properties, but still large enough to contain all polarizable variations of Hodge structure. We would also like the resulting theory to be similar to the theory of perverse sheaves: for example, there should be a way of extending a polarizable variation of Hodge structure on a Zariski-open subset of an irreducible subvariety $Z \subseteq X$ to a polarizable Hodge module on X, similar to taking the intersection complex of a local system.

We shall define polarizable Hodge modules by imposing several additional conditions on $(\mathcal{M}, F_\bullet \mathcal{M})$ and K; these conditions should be strong enough that, at the end of the day, every polarizable Hodge module M of weight w is of the form

$$M \simeq \bigoplus_{Z \subseteq X} M_Z,$$

where M_Z is obtained from a polarizable variation of Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of the irreducible subvariety Z. This is a reasonable demand for two reasons: (1) The perverse sheaves appearing in the decomposition theorem are direct sums of intersection complexes. (2) The underlying local system of a polarizable variation of Hodge structure is semisimple [Del87].

In other words, we know from the beginning what objects we want in the theory; the problem is to find a good description by conditions of a local nature. Moreover, the conditions have to be such that we can check them after applying various operations. Saito's solution is, roughly speaking, to consider only those objects whose "restriction" to every point is a polarized Hodge structure; this idea does not directly make sense at points where $\mathcal{M}$ is singular, but can be implemented with the help of the nearby and vanishing cycle functors (which are a replacement for the naive operation of restricting to a divisor).

Note. As Schmid pointed out after my lectures, the $\mathbb{Q}$-structure is not essential. In fact, Saito's theory works just as well when K is a constructible complex of $\mathbb{R}$-vector spaces; with a little bit of extra effort, one can even get by with coefficients in $\mathbb{C}$. This point is discussed in more detail in [DS13, Section 3.2].

8. Review of nearby and vanishing cycles. Before we can understand Saito's definition, we need to become sufficiently familiar with nearby and vanishing cycle functors, both for perverse sheaves and for regular holonomic $\mathscr{D}$-modules. This topic is somewhat technical – but it is really at the heart of Saito's theory, and so we shall spend some time reviewing it. I decided to state many elementary properties in the form of exercises; working them out carefully is strongly recommended.

To begin with, suppose that X is a complex manifold, and that $f \colon X \to \Delta$ is a holomorphic function that is submersive over the punctured unit disk $\Delta^* = \Delta \setminus \{0\}$. Remembering that the function $e \colon \mathbb{H} \to \Delta$, $e(z) = e^{2\pi i z}$, makes the upper half-plane into the universal covering space of Δ^*, we get the following commutative diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] & X_0 \arrow[l, "i"'] \arrow[d] \\ \mathbb{H} \arrow[r, "e"] & \Delta & \{0\} \arrow[l] \end{tikzcd}$$

Here $\tilde{X}$ is the fiber product of X and $\mathbb{H}$ over Δ. For a constructible complex of $\mathbb{C}$-vector spaces K on X, we define the complex of *nearby cycles*

$$\psi_f K = i^{-1}\mathbf{R}k_*\big(k^{-1}K\big);$$

roughly speaking, this means that we pull back K to the "generic fiber" of f, and then retract onto the "special fiber". Accordingly, $\psi_f K$ contains more information about the behavior of K near the divisor X_0 than the naive restriction $i^{-1}K$. There is an obvious morphism $i^{-1}K \to \psi_f K$, and we define the *vanishing cycles*

$$\phi_f K = \operatorname{Cone}\Big(i^{-1}K \to \psi_f K\Big).$$

By construction, there is a canonical morphism $\mathrm{can}\colon \psi_f K \to \phi_f K$; it is also possible to construct a morphism $\mathrm{var}\colon \phi_f K \to \psi_f K(-1)$ going in the opposite direction.[2] Both $\psi_f K$ and $\phi_f K$ are constructible complexes of $\mathbb{C}$-vector spaces on X_0. Gabber showed that when K is a perverse sheaf, the shifted complexes

$$^p\psi_f K = \psi_f K[-1] \quad \text{and} \quad {}^p\phi_f K = \phi_f K[-1]$$

are again perverse sheaves (see [Bry86, p. 14] for more information). By construction, the complex $^p\psi_f K$ has a monodromy operator T, induced by the automorphism $z \mapsto z+1$ of the upper half-plane $\mathbb{H}$. Since perverse sheaves form an abelian category, we can decompose into generalized eigenspaces

$$^p\psi_{f,\lambda} K = \ker(T - \lambda\,\mathrm{id})^m, \qquad m \gg 0.$$

In summary, we have a decomposition

$$^p\psi_f K = \bigoplus_{\lambda \in \mathbb{C}^\times} {}^p\psi_{f,\lambda} K$$

in the category of perverse sheaves on X_0. There is a similar decomposition for $^p\phi_f K$; by construction, $^p\psi_{f,\lambda} K \simeq {}^p\phi_{f,\lambda} K$ for every complex number $\lambda \neq 1$. On the unipotent part $^p\psi_{f,1} K$, the composition $\mathrm{var} \circ \mathrm{can}$ is equal to the nilpotent operator $N = (2\pi i)^{-1} \log T$; the same goes for $\mathrm{can} \circ \mathrm{var}$ on $^p\phi_{f,1} K$.

As the Riemann-Hilbert correspondence would suggest, there is also a notion of nearby and vanishing cycles for regular holonomic $\mathscr{D}$-modules. It is due to Malgrange and Kashiwara, and involves the use of an additional filtration called the *Kashiwara-Malgrange filtration*. Such a filtration only exists when the divisor X_0 is smooth; in general, one uses the graph embedding to reduce to this situation.

Let us first consider the case where we have a smooth function $t\colon X \to \mathbb{C}$ and a global vector field ∂_t such that $[\partial_t, t] = 1$. Then the *Kashiwara-Malgrange filtration* on a right $\mathscr{D}$-module $\mathcal{M}$ is an increasing filtration $\mathcal{V}_\bullet \mathcal{M}$, indexed by $\mathbb{Z}$, such that

(1) each $\mathcal{V}_k\mathcal{M}$ is coherent over $V_0\mathscr{D}_X = \big\{ P \in \mathscr{D}_X \bigm| P \cdot I_{X_0} \subseteq I_{X_0} \big\}$,
(2) $\mathcal{V}_k\mathcal{M} \cdot t \subseteq \mathcal{V}_{k-1}\mathcal{M}$ and $\mathcal{V}_k\mathcal{M} \cdot \partial_t \subseteq \mathcal{V}_{k+1}\mathcal{M}$,
(3) $\mathcal{V}_k\mathcal{M} \cdot t = \mathcal{V}_{k-1}\mathcal{M}$ for $k \ll 0$,
(4) all eigenvalues of $t\partial_t$ on $\mathrm{gr}_k^{\mathcal{V}} \mathcal{M} = \mathcal{V}_k\mathcal{M}/\mathcal{V}_{k-1}\mathcal{M}$ have real part in $(k-1, k]$.

Kashiwara proved that a filtration with the above properties exists and is unique provided that $\mathcal{M}$ is holonomic [Kas83]. One can also show that when $\mathcal{M}$ is (regular) holonomic, all the graded quotients $\mathrm{gr}_k^{\mathcal{V}} \mathcal{M}$ are again (regular) holonomic $\mathscr{D}$-modules on the complex submanifold $t^{-1}(0) \subseteq X$.

Exercise 8.1. Prove the following results about the Kashiwara-Malgrange filtration:

[2]For historical reasons, Saito denotes this morphism by the symbol "Var".

(a) $t\colon \mathcal{V}_k\mathcal{M} \to \mathcal{V}_{k-1}\mathcal{M}$ is an isomorphism for $k < 0$.
(b) $\partial_t\colon \operatorname{gr}_k^{\mathcal{V}} \mathcal{M} \to \operatorname{gr}_{k+1}^{\mathcal{V}} \mathcal{M}$ is an isomorphism for $k \neq -1$.
(c) $\mathcal{M}$ is generated, as a $\mathscr{D}_X$-module, by the subsheaf $\mathcal{V}_0\mathcal{M}$.

Exercise 8.2. Calculate the Kashiwara-Malgrange filtration when $\operatorname{Supp} \mathcal{M} \subseteq t^{-1}(0)$.

Exercise 8.3. Prove the uniqueness of the Kashiwara-Malgrange filtration by showing that there can be at most one filtration $\mathcal{V}_\bullet\mathcal{M}$ with the above properties.

Exercise 8.4. Suppose that $\mathcal{M}$ is the right $\mathscr{D}$-module defined by a vector bundle with integrable connection. Calculate the Kashiwara-Malgrange filtration, and show that $\operatorname{gr}_{-1}^{\mathcal{V}} \mathcal{M}$ is isomorphic to the restriction of $\mathcal{M}$ to the submanifold $t^{-1}(0)$.

Now suppose that $f\colon X \to \mathbb{C}$ is an arbitrary non-constant holomorphic function. To reduce to the smooth case, we consider the graph embedding

$$(\mathrm{id}, f)\colon X \hookrightarrow X \times \mathbb{C}, \qquad x \mapsto \bigl(x, f(x)\bigr).$$

Let t be the coordinate on $\mathbb{C}$, and set $\partial_t = \partial/\partial t$; note that $t \circ (\mathrm{id}, f) = f$. Instead of the original $\mathscr{D}$-module $\mathcal{M}$, we consider the direct image

$$\mathcal{M}_f = (\mathrm{id}, f)_+\mathcal{M} = \mathcal{M}[\partial_t]$$

on $X \times \mathbb{C}$; here the action by ∂_t is the obvious one, and the action by t is defined by using the relation $[\partial_t, t] = 1$. Since $\mathcal{M}$ and $\mathcal{M}_f$ uniquely determine each other, it makes sense to consider the Kashiwara-Malgrange filtration on $\mathcal{M}_f$ with respect to the smooth function t. When $\mathcal{M}$ is regular holonomic, the Kashiwara-Malgrange filtration for $\mathcal{M}_f$ exists, and the graded quotients $\operatorname{gr}_k^{\mathcal{V}} \mathcal{M}_f$ are again regular holonomic $\mathscr{D}$-modules on $X = t^{-1}(0)$ whose support is contained in the original divisor $X_0 = f^{-1}(0)$.

The regular holonomic $\mathscr{D}$-modules $\operatorname{gr}_k^{\mathcal{V}} \mathcal{M}_f$ are related to the nearby and vanishing cycles for the perverse sheaf $\mathrm{DR}(\mathcal{M})$ in the following manner. For any complex number $\alpha \in \mathbb{C}$, let us define

$$\mathcal{M}_{f,\alpha} = \ker(t\partial_t - \alpha \,\mathrm{id})^m, \qquad m \gg 0,$$

as the generalized eigenspace with eigenvalue α for the action of $t\partial_t$ on $\operatorname{gr}_k^{\mathcal{V}} \mathcal{M}_f$, where k is the unique integer with $k - 1 < \operatorname{Re}\alpha \leq k$. Then for $\lambda = e^{2\pi i\alpha}$, one has

$$\begin{aligned} \mathrm{DR}\bigl(\mathcal{M}_{f,\alpha}\bigr) &\simeq {}^p\psi_{f,\lambda}\bigl(\mathrm{DR}(\mathcal{M})\bigr) \quad \text{for } -1 \leq \operatorname{Re}\alpha < 0, \\ \mathrm{DR}\bigl(\mathcal{M}_{f,\alpha}\bigr) &\simeq {}^p\phi_{f,\lambda}\bigl(\mathrm{DR}(\mathcal{M})\bigr) \quad \text{for } -1 < \operatorname{Re}\alpha \leq 0, \end{aligned} \tag{8.5}$$

and the operator T coming from the monodromy action corresponds to

$$e^{2\pi i t\partial_t} = e^{2\pi i\alpha} \cdot e^{2\pi i(t\partial_t - \alpha)} = \lambda \cdot e^{2\pi i(t\partial_t - \alpha)}$$

under this isomorphism. In the same way, the two morphisms

$$\mathrm{can}\colon {}^p\psi_{f,1}K \to {}^p\phi_{f,1}K \quad \text{and} \quad \mathrm{var}\colon {}^p\phi_{f,1}K \to {}^p\phi_{f,1}K(-1)$$

correspond, on the level of $\mathscr{D}$-modules, to

$$\partial_t\colon \mathcal{M}_{f,-1} \to \mathcal{M}_{f,0} \quad \text{and} \quad t\colon \mathcal{M}_{f,0} \to \mathcal{M}_{f,-1};$$

the composition $N = \mathrm{var} \circ \mathrm{can}$ is therefore represented by $\partial_t t = t\partial_t + 1$ (since we are using right $\mathscr{D}$-modules).

Exercise 8.6. Let $(\mathcal{V}, \nabla)$ be a flat bundle on the punctured disk Δ^*, and let $\mathcal{M}$ be the right $\mathscr{D}$-module corresponding to the Deligne's canonical meromorphic extension. Show that the Kashiwara-Malgrange filtration with respect to the coordinate t on Δ contains exactly the same information as Deligne's canonical lattices for $(\mathcal{V}, \nabla)$.

9. Nearby and vanishing cycles in Saito's theory. For reasons that will become clear in a moment, Saito does not directly use the Kashiwara-Malgrange filtration; instead, he uses a refined notion that works better for filtered $\mathscr{D}$-modules. Let $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$ be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure on a complex manifold X. Given a non-constant holomorphic function $f\colon X \to \mathbb{C}$, we would like to define the nearby cycles $\psi_f M$ and the vanishing cycles $\phi_f M$ in the category of filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure.

As before, we use the graph embedding $(\mathrm{id}, f)\colon X \hookrightarrow X \times \mathbb{C}$ and replace the original filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ by its direct image

$$\begin{aligned} \mathcal{M}_f &= (\mathrm{id}, f)_+\mathcal{M} = \mathcal{M}[\partial_t], \\ F_\bullet\mathcal{M}_f &= F_\bullet(\mathrm{id}, f)_+\mathcal{M} = \bigoplus_{i=0}^{\infty} F_{\bullet-i}\mathcal{M} \otimes \partial_t^i. \end{aligned} \tag{9.1}$$

Let $\mathcal{V}_\bullet\mathcal{M}_f$ denote the Kashiwara-Malgrange filtration with respect to $t = 0$.

Following Saito, we shall only consider objects with quasi-unipotent local monodromy; this assumption comes from the theory of variations of Hodge structure [Sch73]. In our situation, it means that all eigenvalues of the monodromy operator T on ${}^p\psi_f K$ are roots of unity.[3] Then all eigenvalues of $t\partial_t$ on $\mathrm{gr}_0^{\mathcal{V}} \mathcal{M}_f$ are rational numbers, and one can introduce a refined filtration $V_\bullet\mathcal{M}_f$ indexed by $\mathbb{Q}$. Given $\alpha \in \mathbb{Q}$, we let $k = \lceil \alpha \rceil$, and define $V_\alpha\mathcal{M}_f \subseteq \mathcal{V}_k\mathcal{M}_f$ as the preimage of

$$\bigoplus_{k-1<\beta\leq\alpha} \mathcal{M}_{f,\beta} \subseteq \mathrm{gr}_k^{\mathcal{V}} \mathcal{M}_f$$

under the projection $\mathcal{V}_k\mathcal{M}_f \to \mathrm{gr}_k^{\mathcal{V}} \mathcal{M}_f$; we define $V_{<\alpha}\mathcal{M}_f$ in the same way, but taking the direct sum over $k-1 < \beta < \alpha$. The resulting filtration $V_\bullet\mathcal{M}_f$ is called the *(rational) V-filtration*, to distinguish it from the Kashiwara-Malgrange filtration.

Exercise 9.2. Prove the following assertions about the V-filtration:

(a) The operator $t\partial_t - \alpha\,\mathrm{id}$ acts nilpotently on $\mathrm{gr}_\alpha^V \mathcal{M} = V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$.
(b) $t\colon V_\alpha\mathcal{M} \to V_{\alpha-1}\mathcal{M}$ is an isomorphism for $\alpha < 0$.
(c) $\partial_t\colon \mathrm{gr}_\alpha^V \mathcal{M} \to \mathrm{gr}_{\alpha+1}^V \mathcal{M}$ is an isomorphism for $\alpha \neq -1$.
(d) $V_{<0}\mathcal{M}$ only depends on the restriction of $\mathcal{M}$ to $X \setminus X_0$.

Note. More generally, one can consider the case where the eigenvalues of T on ${}^p\psi_f K$ have absolute value 1; the V-filtration is then naturally indexed by $\mathbb{R}$.

The correct choice of filtration on the nearby and vanishing cycles is a little bit subtle. Let me explain what the issue is. The formulas in (8.5), together with the

[3] Saito makes this assumption in [Sai88, Sai90b], but not in certain other papers [Sai90a].

obvious isomorphisms $\operatorname{gr}_\alpha^V \mathcal{M}_f \simeq \mathcal{M}_{f,\alpha}$, yield

$$\mathrm{DR}\left(\bigoplus_{-1\leq\alpha<0} \operatorname{gr}_\alpha^V \mathcal{M}_f\right) \simeq \mathbb{C}\otimes_{\mathbb{Q}} {}^p\psi_f K,$$
$$\mathrm{DR}\left(\bigoplus_{-1<\alpha\leq0} \operatorname{gr}_\alpha^V \mathcal{M}_f\right) \simeq \mathbb{C}\otimes_{\mathbb{Q}} {}^p\phi_f K,$$

and the two $\mathscr{D}$-modules on the left-hand side are regular holonomic. To make them into filtered $\mathscr{D}$-modules, we should use a filtration that is compatible with the decompositions above. There are two reasons for this: (1) In the case of a polarizable variation of Hodge structure on a punctured disk, the correct choice of filtration on the nearby cycles is the limit Hodge filtration; according to Schmid's nilpotent orbit theorem [Sch73], this filtration is compatible with the decomposition into generalized eigenspaces. (2) On a more formal level, it turns out that certain properties of M can be described very naturally in terms of the filtration on the individual $\mathscr{D}$-modules $\operatorname{gr}_\alpha^V \mathcal{M}_f$; we shall revisit this point below.

The upshot is that we should endow each $\mathscr{D}$-module $\operatorname{gr}_\alpha^V \mathcal{M}_f$ with the filtration induced by $F_\bullet\mathcal{M}_f$; concretely, for $p\in\mathbb{Z}$, we set

$$F_p \operatorname{gr}_\alpha^V \mathcal{M}_f = \frac{F_p\mathcal{M}_f \cap V_\alpha\mathcal{M}_f}{F_p\mathcal{M}_f \cap V_{<\alpha}\mathcal{M}_f}.$$

Note that this leads to a different filtration on the $\mathscr{D}$-module

$$\operatorname{gr}_0^{\mathcal{V}} \mathcal{M}_f \simeq \bigoplus_{-1<\alpha\leq0} \operatorname{gr}_\alpha^V \mathcal{M}$$

than if we simply took the filtration induced by $F_\bullet\mathcal{M}_f$; while this choice might seem more natural, it would be wrong from the point of view of Hodge theory.

For any non-constant holomorphic function $f\colon X\to\mathbb{C}$, we can now set

$$(9.3)\qquad \begin{aligned} \psi_f M &= \bigoplus_{-1\leq\alpha<0}\Bigl(\operatorname{gr}_\alpha^V \mathcal{M}_f, F_{\bullet-1}\operatorname{gr}_\alpha^V \mathcal{M}_f, {}^p\psi_{f,e^{2\pi i\alpha}}K\Bigr)\\ \psi_{f,1} M &= \Bigl(\operatorname{gr}_{-1}^V \mathcal{M}_f, F_{\bullet-1}\operatorname{gr}_{-1}^V \mathcal{M}_f, {}^p\psi_{f,1}K\Bigr)\\ \phi_{f,1} M &= \Bigl(\operatorname{gr}_0^V \mathcal{M}_f, F_\bullet\operatorname{gr}_0^V \mathcal{M}_f, {}^p\phi_{f,1}K\Bigr)\end{aligned}$$

Except for $\lambda=1$, the individual perverse sheaves ${}^p\psi_{f,\lambda}K$ are generally not defined over $\mathbb{Q}$; in order to have a $\mathbb{Q}$-structure on $\psi_f M$, we are forced to keep them together. Provided that the induced filtration $F_\bullet \operatorname{gr}_\alpha^V \mathcal{M}_f$ is good for every $\alpha\in[-1,0]$, all three objects are filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure on X; by construction, their support is contained in the divisor $f^{-1}(0)$. The logarithm of the unipotent part of the monodromy

$$N = \frac{1}{2\pi i}\log T_u\colon \psi_f M \to \psi_f M(-1)$$

is a nilpotent endomorphism (up to a Tate twist).

10. Decomposition by strict support. Now we can start thinking about the definition of Hodge modules. Recall that the category $\mathrm{HM}^p(X, w)$ should be such that every polarizable Hodge module decomposes into a finite sum

$$M \simeq \bigoplus_{Z \subseteq X} M_Z,$$

where M_Z is supported on an irreducible subvariety Z, and in fact comes from a variation of Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of the smooth locus of Z. On the level of perverse sheaves, this means that K_Z should be an intersection complex, and should therefore not have nontrivial subobjects or quotient objects that are supported on proper subvarieties of Z. The following definition captures this property in the case of $\mathscr{D}$-modules.

Definition 10.1. Let $Z \subseteq X$ be an irreducible subvariety. We say that a $\mathscr{D}$-module $\mathcal{M}$ has *strict support* Z if the support of every nonzero subobject or quotient object of $\mathcal{M}$ is equal to Z.

Note that a regular holonomic $\mathscr{D}$-module has strict support Z if and only if the corresponding perverse sheaf $\mathrm{DR}(\mathcal{M})$ is the intersection complex of a local system on a dense Zariski-open subset of Z. It turns out that this property can also be detected with the help of the V-filtration.

Exercise 10.2. Let $f\colon X \to \mathbb{C}$ be a non-constant holomorphic function, and let $\mathcal{M}$ be a regular holonomic $\mathscr{D}$-module on X.

(a) Show that $\mathcal{M}$ has no nonzero subobject supported on $f^{-1}(0)$ if and only if $t\colon \operatorname{gr}_0^V \mathcal{M}_f \to \operatorname{gr}_{-1}^V \mathcal{M}_f$ is injective.
(b) Show that $\mathcal{M}$ has no nonzero quotient object supported on $f^{-1}(0)$ if and only if $\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f$ is surjective.

More generally, one can use the V-filtration to test whether or not $\mathcal{M}$ decomposes into a direct sum of $\mathscr{D}$-modules with strict support.

Exercise 10.3. Let $f\colon X \to \mathbb{C}$ be a non-constant holomorphic function, and let $\mathcal{M}$ be a regular holonomic $\mathscr{D}$-module on X. Show that

$$\operatorname{gr}_0^V \mathcal{M}_f = \ker\bigl(t\colon \operatorname{gr}_0^V \mathcal{M}_f \to \operatorname{gr}_{-1}^V \mathcal{M}_f\bigr) \oplus \operatorname{im}\bigl(\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f\bigr) \tag{10.4}$$

if and only if $\mathcal{M} \simeq \mathcal{M}' \oplus \mathcal{M}''$, where $\operatorname{Supp} \mathcal{M}' \subseteq f^{-1}(0)$, and $\mathcal{M}''$ does not have nonzero subobjects or quotient objects whose support is contained in $f^{-1}(0)$.

Using the result of the preceding exercise, one can easily prove the following local criterion for the existence of a decomposition by strict support. Note that such a decomposition is necessarily unique, because there are no nontrivial morphisms between $\mathscr{D}$-modules with different strict support.

Proposition 10.5. *Let $\mathcal{M}$ be a regular holonomic $\mathscr{D}$-module on X. Then $\mathcal{M}$ decomposes into a direct sum of $\mathscr{D}$-modules with strict support if and only if* (10.4) *is true for every f.*

This is the first indication that nearby and vanishing cycle functors are useful in describing global properties of $\mathscr{D}$-modules in terms of local conditions.

11. Compatibility with the filtration. Let $(\mathcal{M}, F_\bullet\mathcal{M}, K)$ be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure on a complex manifold X. Saito realized that the nearby and vanishing cycle functors are also good for imposing restrictions on the filtration $F_\bullet\mathcal{M}$. To see why this might be the case, let us suppose for a moment that $\mathcal{M}$ has strict support $Z \subseteq X$. Let $f\colon X \to \mathbb{C}$ be a holomorphic function whose restriction to Z is not constant; the most interesting case is when X_0 contains the singular locus of $\mathcal{M}$, or in other words, when K is the intersection complex of a local system on $Z \setminus Z \cap X_0$.

To simplify the notation, suppose that we are actually dealing with a smooth function $t\colon X \to \mathbb{C}$, and that there is a global vector field ∂_t with $[\partial_t, t] = 1$. Clearly, we can always put ourselves into this situation by considering the $\mathscr{D}$-module $\mathcal{M}_f$ on the product $X \times \mathbb{C}$: it will have strict support $i_f(Z)$, and the two filtrations $F_\bullet\mathcal{M}$ and $F_\bullet\mathcal{M}_f$ determine each other by (9.1). We know from Exercise 10.2 that

$$t\colon \operatorname{gr}_0^V \mathcal{M} \to \operatorname{gr}_{-1}^V \mathcal{M} \quad \text{is injective,}$$
$$\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M} \to \operatorname{gr}_0^V \mathcal{M} \quad \text{is surjective.}$$

Because of the properties of the V-filtration, one then has

$$\mathcal{M} = \sum_{i=0}^{\infty} \bigl(V_{<0}\mathcal{M}\bigr)\partial_t^i,$$

and so $V_{<0}\mathcal{M}$ generates $\mathcal{M}$ as a right $\mathscr{D}$-module; recall from Exercise 9.2 that this is the part of the V-filtration that only depends on the restriction of $\mathcal{M}$ to $X \setminus X_0$. We would like to make sure that the filtration $F_\bullet\mathcal{M}$ is also determined by its restriction to $X \setminus X_0$. First, we need a condition that allows us to recover the coherent sheaves

$$F_p V_{<0}\mathcal{M} = V_{<0}\mathcal{M} \cap F_p\mathcal{M}.$$

from information on $X \setminus X_0$.

Exercise 11.1. Denote by $j\colon X \setminus X_0 \hookrightarrow X$ the inclusion. Show that one has

$$F_p V_{<0}\mathcal{M} = V_{<0}\mathcal{M} \cap j_* j^* F_p\mathcal{M}$$

if and only if $t\colon F_p V_\alpha \mathcal{M} \to F_p V_{\alpha-1}\mathcal{M}$ is surjective for every $\alpha < 0$.

Next, we observe that $F_p\mathcal{M}$ contains $\bigl(F_{p-i}V_{<0}\mathcal{M}\bigr)\partial_t^i$ for every $i \geq 0$. The following exercise gives a criterion for when these subsheaves generate $F_p\mathcal{M}$.

Exercise 11.2. Suppose $\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M} \to \operatorname{gr}_0^V \mathcal{M}$ is surjective. Show that one has

$$F_p\mathcal{M} = \sum_{i=0}^{\infty} \bigl(F_{p-i}V_{<0}\mathcal{M}\bigr)\partial_t^i$$

if and only if $\partial_t\colon F_p \operatorname{gr}_\alpha^V \mathcal{M} \to F_{p+1} \operatorname{gr}_{\alpha+1}^V \mathcal{M}$ is surjective for every $\alpha \geq -1$.

This criterion is one reason for considering the induced filtration on each $\operatorname{gr}_\alpha^V \mathcal{M}$. Almost exactly the same condition also turns out to be useful for describing the filtration in the case where the support of $\mathcal{M}$ is contained in the divisor X_0.

Exercise 11.3. Suppose that $\operatorname{Supp}\mathcal{M} \subseteq X_0$, and define a filtered $\mathscr{D}$-module on X_0 by setting $\mathcal{M}_0 = \ker(t\colon \mathcal{M} \to \mathcal{M})$ and $F_p\mathcal{M}_0 = F_p\mathcal{M} \cap \mathcal{M}_0$. Show that

$$\mathcal{M} \simeq \mathcal{M}_0[\partial_t] \quad \text{and} \quad F_p\mathcal{M} \simeq \bigoplus_{i=0}^{\infty} F_{p-i}\mathcal{M}_0 \otimes \partial_t^i$$

if and only if $\partial_t \colon F_p \operatorname{gr}_\alpha^V \mathcal{M} \to F_{p+1} \operatorname{gr}_{\alpha+1}^V \mathcal{M}$ is surjective for every $\alpha > -1$.

We now return to the case of an arbitrary filtered regular holonomic $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$. Let $f \colon X \to \mathbb{C}$ be a non-constant holomorphic function. The three exercises above motivate the following definition.

Definition 11.4. We say that $(\mathcal{M}, F_\bullet \mathcal{M})$ is *quasi-unipotent along* $f = 0$ if all eigenvalues of the monodromy operator on ${}^p\psi_f K$ are roots of unity, and if the V-filtration $V_\bullet \mathcal{M}_f$ satisfies the following two additional conditions:

(1) $t \colon F_p V_\alpha \mathcal{M}_f \to F_p V_{\alpha-1} \mathcal{M}_f$ is surjective for $\alpha < 0$.
(2) $\partial_t \colon F_p \operatorname{gr}_\alpha^V \mathcal{M}_f \to F_{p+1} \operatorname{gr}_{\alpha+1}^V \mathcal{M}_f$ is surjective for $\alpha > -1$.

We say that $(\mathcal{M}, F_\bullet \mathcal{M})$ is *regular along* $f = 0$ if $F_\bullet \operatorname{gr}_\alpha^V \mathcal{M}_f$ is a good filtration for every $-1 \leq \alpha \leq 0$.

The properties of the V-filtration guarantee that the two morphisms in the definition are isomorphisms in the given range. Note that we do not include $\alpha = -1$ on the second line, because we want a notion that also makes sense when $\partial_t \colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f$ is not surjective. The regularity condition ensures that $\psi_f M$ and $\phi_{f,1} M$ are filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure.

To connect the definition with the discussion above, suppose that $(\mathcal{M}, F_\bullet \mathcal{M})$ has strict support Z, and that it is quasi-unipotent and regular along $f = 0$ for a holomorphic function $f \colon X \to \mathbb{C}$ whose restriction to Z is not constant. Then

$$F_p \mathcal{M}_f = \sum_{i=0}^{\infty} \bigl(V_{<0} \mathcal{M}_f \cap j_* j^* F_{p-i} \mathcal{M}_f \bigr) \partial_t^i, \tag{11.5}$$

provided that $\partial_t \colon F_p \operatorname{gr}_{-1}^V \mathcal{M}_f \to F_{p+1} \operatorname{gr}_0^V \mathcal{M}_f$ is also surjective. This last condition will be automatically satisfied for Hodge modules: the recursive definition (in §12) implies that the morphism

$$\partial_t \colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f$$

is strictly compatible with the filtrations (up to a shift by 1), and (11.5) therefore follows from the surjectivity of ∂_t. What this means is that the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ is uniquely determined by its restriction to $Z \setminus Z \cap X_0$.

Another good feature of the conditions in Definition 11.4 is that they say something interesting even when the support of $\mathcal{M}$ is contained in the divisor $f^{-1}(0)$.

Exercise 11.6. Suppose that $\operatorname{Supp} \mathcal{M} \subseteq f^{-1}(0)$. Show that $(\mathcal{M}, F_\bullet \mathcal{M})$ is quasi-unipotent and regular along $f = 0$ if and only if the filtration satisfies

$$(F_p \mathcal{M}) \cdot f \subseteq F_{p-1} \mathcal{M}$$

for every $p \in \mathbb{Z}$.

We can also upgrade the criterion in Proposition 10.5 to filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure. In the statement, note that the filtration on the image of $\operatorname{can} \colon \psi_{f,1} M \to \phi_{f,1} M$ is induced by that on $\psi_{f,1} M$.

Theorem 11.7. *Let M be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure, and suppose that $(\mathcal{M}, F_\bullet \mathcal{M})$ is quasi-unipotent and regular along $f = 0$ for every locally defined holomorphic function f. Then M admits a decomposition*

$$M \simeq \bigoplus_{Z \subseteq X} M_Z$$

by strict support, in which each M_Z is again a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure, if and only if one has

$$\phi_{f,1}M = \ker\bigl(\mathrm{var}\colon \phi_{f,1}M \to \psi_{f,1}M(-1)\bigr) \oplus \mathrm{im}\bigl(\mathrm{can}\colon \psi_{f,1}M \to \phi_{f,1}M\bigr)$$

for every locally defined holomorphic function f.

12. Definition of pure Hodge modules. After these technical preliminaries, we are now ready to define the category of pure Hodge modules. The basic objects are filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure; we have to decide when a given $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$ should be called a Hodge module. Saito uses a recursive procedure to define a family of auxiliary categories

$$\mathrm{HM}_{\leq d}(X, w) = \left\{ \begin{matrix} \text{Hodge modules of weight } w \text{ on } X \\ \text{whose support has dimension } \leq d \end{matrix} \right\},$$

indexed by $d \geq 0$. This procedure has the advantage that results can then be proved by induction on the dimension of the support.[4] Since the definition involves the nearby and vanishing cycle functors, we impose the following condition:

(12.1) The pair $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $f = 0$ for every locally defined holomorphic function $f\colon U \to \mathbb{C}$.

Both $\psi_f M$ and $\phi_{f,1}M$ are then well-defined filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure on U, whose support is contained in $f^{-1}(0)$. Since we are only interested in objects that admit a decomposition by strict support, we require:

(12.2) M admits a decomposition by strict support, in the category of regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure.

Recall that this can also be tested with the help of the nearby and vanishing cycle functors, using the criterion in Theorem 11.7. Now the problem is reduced to defining, for each irreducible closed subvariety $Z \subseteq X$, a suitable category

$$\mathrm{HM}_Z(X, w) = \left\{ \begin{matrix} \text{Hodge modules of weight } w \text{ on } X \\ \text{with strict support equal to } Z \end{matrix} \right\};$$

we can then take for $\mathrm{HM}_{\leq d}(X, w)$ the direct sum of all $\mathrm{HM}_Z(X, w)$ with $\dim Z \leq d$.

The first case to consider is obviously when $Z = \{x\}$ is a point. Here the perverse sheaf K is the direct image of a $\mathbb{Q}$-vector space by the morphism $i\colon \{x\} \hookrightarrow X$, and this suggests defining

$$\mathrm{HM}_{\{x\}}(X, w) = \bigl\{\, i_* H \bigm| H \text{ is a } \mathbb{Q}\text{-Hodge structure of weight } w \,\bigr\}. \tag{12.3}$$

Now comes the most important part of the definition: for arbitrary $Z \subseteq X$, we say that $M \in \mathrm{HM}_Z(X, w)$ if and only if, for every locally defined holomorphic function $f\colon U \to \mathbb{C}$ that does not vanish identically on $Z \cap U$, one has

$$\mathrm{gr}_\ell^{W(N)}(\psi_f M) \in \mathrm{HM}_{\leq d-1}(X, w - 1 + \ell). \tag{12.4}$$

Here $W(N)$ is the monodromy filtration of the nilpotent operator $N = (2\pi i)^{-1} \log T_u$ on the nearby cycles $\psi_f M$. One can deduce from this last condition that

$$\mathrm{gr}_\ell^{W(N)}(\phi_{f,1} M) \in \mathrm{HM}_{\leq d-1}(X, w + \ell),$$

by using the isomorphism in Theorem 11.7. Of course, the motivation for using the monodromy filtration comes from Schmid's SL(2)-orbit theorem [Sch73]: in

[4] Saito says that he found the definition by axiomatizing certain arguments that he used to prove Theorem 16.1 in the case $M = \mathbb{Q}_X^H[\dim X]$.

the case of a polarizable variation of Hodge structure on the punctured disk, the nearby cycles carry a mixed Hodge structure whose Hodge filtration is the limit Hodge filtration and whose weight filtration is the monodromy filtration of N, up to a shift by the weight of the variation.

Definition 12.5. The category of *Hodge modules of weight* w on X has objects

$$\mathrm{HM}(X, w) = \bigcup_{d \geq 0} \mathrm{HM}_{\leq d}(X, w) = \bigoplus_{Z \subseteq X} \mathrm{HM}_Z(X, w);$$

its morphisms are the morphisms of regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure.

One can show that every morphism between two Hodge modules is strictly compatible with the filtrations, and that $\mathrm{HM}(X, w)$ is therefore an abelian category. Moreover, one can show that there are no nontrivial morphisms from one Hodge module to another Hodge module of strictly smaller weight.

An important point is that all the conditions in the definition are local; Saito's insight is that they are strong enough to have global consequences, such as the decomposition theorem. Of course, the recursive nature of the definition makes it hard to prove that a given M is a Hodge module: in fact, it takes considerable work to establish that even a very basic object like

$$\bigl(\omega_X, F_\bullet \omega_X, \mathbb{Q}_X[n]\bigr)$$

is a Hodge module of weight $n = \dim X$. It is also not clear that Hodge modules are stable under various operations such as direct or inverse images – about the only thing that is obvious from the definition is that

$$\mathrm{HM}(pt, w) = \bigl\{\mathbb{Q}\text{-Hodge structures of weight } w\bigr\}.$$

On the positive side, Hodge modules are by definition stable under the application of nearby and vanishing cycle functors: once we know that something is a Hodge module, we immediately get a large collection of other Hodge modules by taking nearby and vanishing cycles.

Example 12.6. Suppose M is a Hodge module on a smooth curve X. Then for every local coordinate t, the nearby cycles $\psi_t M$ carry a mixed Hodge structure.

13. Polarizations. In order to define polarizable Hodge modules, we also need to introduce the concept of a polarization. Let $M = (\mathcal{M}, F_\bullet \mathcal{M}, K)$ be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure on an n-dimensional complex manifold X; of course, we will be mostly interested in the case $M \in \mathrm{HM}(X, w)$.

Recall that in the case of a variation of Hodge structure V of weight $w - n$, a polarization is a bilinear form $V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-w + n)$ whose restriction to every point $x \in X$ polarizes the Hodge structure V_x; equivalently, it is a $\mathbb{Q}$-linear mapping $V(w - n) \to V^*$ with the same property. The natural analogue for perverse sheaves is to consider morphisms of the form

$$K(w) \to \mathbf{D}K,$$

where $\mathbf{D}K = \mathbf{R}\mathcal{H}om\bigl(K, \mathbb{Q}_X(n)\bigr)[2n]$ is the Verdier dual of the perverse sheaf K, with a Tate twist to conform to Saito's conventions for weights. This suggests that a polarization should be a morphism $M(w) \to \mathbf{D}M$ ... except that there is no duality functor for arbitrary filtered $\mathscr{D}$-modules. The holonomic dual

$$R^n \mathcal{H}om_{\mathscr{D}_X}\bigl(\mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X\bigr)$$

of $\mathcal{M}$ is a regular holonomic $\mathscr{D}$-module, whose de Rham complex is isomorphic to $\mathbf{D}\big(\mathrm{DR}(\mathcal{M})\big)$; but this operation does not interact well with the filtration $F_\bullet\mathcal{M}$. In fact, a necessary and sufficient condition is that $(\mathcal{M}, F_\bullet\mathcal{M})$ is Cohen-Macaulay, meaning that $\mathrm{gr}_\bullet^F\mathcal{M}$ should be a Cohen-Macaulay module over $\mathrm{gr}_\bullet^F\mathscr{D}_X$. Fortunately, Saito has shown that Hodge modules always have this property, and that every $M \in \mathrm{HM}(X, w)$ has a well-defined dual $\mathbf{D}M \in \mathrm{HM}(X, -w)$; see §29.

With this issue out of the way, here is the definition. A *polarization* on a Hodge module $M \in \mathrm{HM}(X, w)$ is a morphism $K(w) \to \mathbf{D}K$ with the following properties:

(1) It is nondegenerate and compatible with the filtration, meaning that it extends to an isomorphism $M(w) \simeq \mathbf{D}M$ in the category of Hodge modules.

(2) For every summand M_Z in the decomposition of M by strict support, and for every locally defined holomorphic function $f \colon U \to \mathbb{C}$ that is not identically zero on $U \cap Z$, the induced morphism

$$^p\psi_f K_Z(w) \to \mathbf{D}(^p\psi_f K_Z)$$

is a polarization of Hodge-Lefschetz type (= on primitive parts for N).

(3) If $\dim \operatorname{Supp} M_Z = 0$, then $K_Z(w) \to \mathbf{D}K_Z$ is induced by a polarization of Hodge structures in the usual sense.

The second condition uses the compatibility of the duality functor with nearby cycles, and the fact that $K(w) \to \mathbf{D}K$ is automatically compatible with the decomposition by strict support (because there are nonontrivial morphisms between perverse sheaves with different strict support). Since $^p\psi_f K_Z$ is by construction supported in a subset of smaller dimension, the definition is again recursive.

Example 13.1. Let M be the filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure associated with a variation of Hodge structure of weight w. Provided one is sufficiently careful with signs, a polarization $V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-w)$ determines a morphism $M(w + \dim X) \to \mathbf{D}M$; we will see later that it is a polarization in the above sense.

Note that we do not directly require any “positivity” of the morphism giving the polarization; instead, we are asking that once we apply sufficiently many nearby cycle functors to end up with a vector space, certain induced Hermitian forms on this vector space are positive definite. Because of the indirect nature of this definition, the problem of signs is quite nontrivial, and one has to be extremely careful in choosing the signs in various isomorphisms. For that reason, Saito actually defines polarizations not as morphisms $K(w) \to \mathbf{D}K$, but as pairings $K \otimes_{\mathbb{Q}} K \to \mathbb{Q}(-w)[2n]$; this is equivalent to our definition, but makes it easier to keep track of signs.

Definition 13.2. We say that a Hodge module is *polarizable* if it admits at least one polarization, and we denote by

$$\mathrm{HM}^p(X, w) \subseteq \mathrm{HM}(X, w) \quad \text{and} \quad \mathrm{HM}_Z^p(X, w) \subseteq \mathrm{HM}_Z(X, w)$$

the full subcategories of polarizable Hodge modules.

14. Kashiwara's equivalence and singular spaces. To close this chapter, let me say a few words about the definition of Hodge modules on analytic spaces; a systematic treatment of $\mathscr{D}$-modules on analytic spaces can be found in [Sai91a]. The idea is similar to how one defines holomorphic functions on analytic spaces: embed a given analytic space X into a complex manifold, and consider objects on the ambient manifold that are supported on X. Of course, such embeddings may only exist locally, and so in the most general case, one has to cover X by embeddable open subsets and impose conditions on the pairwise intersections.

To simplify the discussion, we shall only consider those analytic spaces X that are globally embeddable into a complex manifold; this class includes, for example, all quasi-projective algebraic varieties. By definition, a right $\mathscr{D}$-module on X is a right $\mathscr{D}$-module on the ambient manifold whose support is contained in X. This definition is independent of the choice of embedding because of the following fundamental result by Kashiwara [Kas95].

Proposition 14.1 (Kashiwara's Equivalence)**.** *Let X be a complex manifold, and $i\colon Z \hookrightarrow X$ the inclusion of a closed submanifold. Then the direct image functor*

$$i_+\colon M_{coh}(\mathscr{D}_Z) \to M_{coh}(\mathscr{D}_X)$$

gives an equivalence between the category of coherent right $\mathscr{D}_Z$-modules and the category of coherent right $\mathscr{D}_X$-modules whose support is contained in Z.

Exercise 14.2. Suppose that $Z = t^{-1}(0)$ for a holomorphic function $t\colon X \to \mathbb{C}$. Prove Kashiwara's equivalence in this case, by using the decomposition given by the kernels of the operators $t\partial_t - i$ for $i \geq 0$.

Since the direct image functor for left $\mathscr{D}$-modules involves an additional twist by the relative canonical bundle $\omega_Z \otimes \omega_X^{-1}$, both Kashiwara's equivalence and the definition of $\mathscr{D}$-modules on singular spaces are more natural for right $\mathscr{D}$-modules.

Definition 14.3. Let X be an analytic space that can be globally embedded into a complex manifold Y. Then we define

$$\mathrm{HM}(X,w) \subseteq \mathrm{HM}(Y,w) \quad \text{and} \quad \mathrm{HM}^p(X,w) \subseteq \mathrm{HM}^p(Y,w)$$

by taking all (polarizable) Hodge modules on Y whose support is contained in X.

One subtle point is that Kashiwara's equivalence is not true for filtered $\mathscr{D}$-modules, because a coherent sheaf with support in X is not the same thing as a coherent sheaf on X. To prove that the category $\mathrm{HM}(X,w)$ is independent of the choice of embedding, one has to use the fact that the filtered $\mathscr{D}$-modules in Saito's theory are always quasi-unipotent and regular.

Exercise 14.4. In the notation of Proposition 14.1, suppose that $\operatorname{Supp}\mathcal{M} \subseteq Z$, and that $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $f = 0$ for every locally defined holomorphic function in the ideal of Z. Show that

$$(\mathcal{M}, F_\bullet\mathcal{M}) \simeq i_+(\mathcal{M}_Z, F_\bullet\mathcal{M}_Z)$$

for a filtered $\mathscr{D}$-module $(\mathcal{M}_Z, F_\bullet\mathcal{M}_Z)$ on Z, unique up to isomorphism.

Now let $M \in \mathrm{HM}(X,w)$. The underlying regular holonomic right $\mathscr{D}$-module $\mathcal{M}$, as well as the coherent sheaves $F_\bullet\mathcal{M}$ in the Hodge filtration, are really defined on the ambient complex manifold Y; they are not objects on X, although the support of $\mathcal{M}$ is contained in X. On the other hand, the result in Exercise 11.6 can be used to show that each $\mathrm{gr}_k^F \mathcal{M}$ is a coherent sheaf on X.

Exercise 14.5. Let $M \in \mathrm{HM}(X,w)$. Deduce from Exercise 14.4 that the graded quotients $\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M})$ of the de Rham complex are well-defined complexes of coherent sheaves on X, independent of the choice of embedding.

C. Two important theorems

In this chapter, we describe two of Saito's most important results: the structure theorem and the direct image theorem. The structure theorem relates polarizable Hodge modules and polarizable variations of Hodge structure; the direct image theorem says that polarizable Hodge modules are stable under direct images by projective morphisms. Taken together, they justify the somewhat complicated definition of the category $\mathrm{HM}^p(X,w)$.

15. Structure theorem. One of the main results in Saito's second paper [Sai90b] is that polarizable Hodge modules on X with strict support Z are the same thing as generically defined polarized variations of Hodge structure on Z.

Theorem 15.1 (Structure Theorem)**.** *Let X be a complex manifold, and $Z \subseteq X$ an irreducible closed analytic subvariety.*

(1) Every polarizable variation of $\mathbb{Q}$-Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of Z extends uniquely to an object of $\mathrm{HM}_Z^p(X,w)$.
(2) Every object of $\mathrm{HM}_Z^p(X,w)$ is obtained in this way.

Together with the condition (12.2) in the definition of Hodge modules, this result implies that every polarizable Hodge $M \in \mathrm{HM}(X,w)$ is of the form

$$M = \bigoplus_{Z \subseteq X} M_Z,$$

where M_Z is obtained from a polarizable variation of Hodge structure of weight $w - \dim Z$ on a Zariski-open subset of the smooth locus of Z; conversely, every object of this type is a polarizable Hodge module. We have therefore achieved our goal, which was to describe this class in terms of local conditions.

Example 15.2. On every complex manifold X,

$$\mathbb{Q}_X^H[n] = \bigl(\omega_X, F_\bullet \omega_X, \mathbb{Q}_X[n]\bigr) \in \mathrm{HM}_X^p(X,n)$$

is a polarizable Hodge module of weight $n = \dim X$.

Example 15.3. More generally, we can consider the constant variation of Hodge structure on the smooth locus of an irreducible analytic subvariety $Z \subseteq X$. By Theorem 15.1, it determines a polarizable Hodge module of weight $\dim Z$ on X; the underlying perverse sheaf is the intersection complex of Z.

Example 15.4. Another consequence of the structure theorem is that the inverse image of a polarizable Hodge module under a smooth morphism $f \colon Y \to X$ is again a polarizable Hodge module (with a shift in weight by $\dim Y - \dim X$); see §30. This statement looks innocent enough, but trying to prove it directly from the definition is hopeless, because there are too many additional functions on Y.

The proof of the second assertion in Theorem 15.1 is not hard. In fact, given $M \in \mathrm{HM}_Z^p(X,w)$, there is a Zariski-open subset of the smooth locus of Z where $\mathrm{rat}\, M$ is a local system (up to a shift), and one has all the data necessary to define a polarizable variation of Hodge structure of weight $w - \dim Z$. One then uses the definition to argue that the variation of Hodge structure uniquely determines the original M: for example, (11.5) shows how to recover $F_\bullet \mathcal{M}$. The real content is therefore in the first assertion; we shall say more about the proof in §18 below.

16. Direct image theorem. Given the local nature of the definition, it is also not clear that the category of Hodge modules is preserved by the direct image functor. The main result in Saito's first paper [Sai88] is that this is true for direct images by projective morphisms. Along the way, Saito also proved the decomposition theorem for those perverse sheaves that underlie polarizable Hodge modules.

Let $f\colon X \to Y$ be a projective morphism between complex manifolds, and let

$$M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$$

be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure. In a suitable derived category, one can define the direct image $f_* M$: the underlying constructible complex is $\mathbf{R}f_* K$, and the underlying complex of filtered $\mathscr{D}$-modules is $f_+(\mathcal{M}, F_\bullet\mathcal{M})$; see §27 for more details. The content of the direct image theorem is, roughly speaking, that $M \in \mathrm{HM}^p(X, w)$ implies $\mathcal{H}^i f_* M \in \mathrm{HM}^p(Y, w+i)$. A tricky point is that the cohomology sheaves $\mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$ are in general not filtered $\mathscr{D}$-modules, but live in a larger abelian category (see §26 below). Unless the complex $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is what is called "strict", it is therefore not possible to define $\mathcal{H}^i f_* M$ as a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure.

Before we can give the precise statement of the direct image theorem, we need to introduce the Lefschetz operator; it plays an even greater role here than in classical Hodge theory. Let $\ell \in H^2\bigl(X, \mathbb{Z}(1)\bigr)$ be the first Chern class of a relatively ample line bundle on X. It gives rise to a morphism $\ell\colon K \to K(1)[2]$ in the derived category of constructible complexes on X; using the fact that ℓ can also be represented by a closed $(1,1)$-form, one can lift this morphism to a morphism $\ell\colon M \to M(1)[2]$. Now we apply the direct image functor; provided that $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict, we obtain a collection of morphisms

$$\ell\colon \mathcal{H}^i f_* M \to \mathcal{H}^{i+2} f_* M(1),$$

which together constitute the *Lefschetz operator*.

Theorem 16.1 (Direct Image Theorem)**.** *Let $f\colon X \to Y$ be a projective morphism between two complex manifolds, and let $M \in \mathrm{HM}^p(X, w)$. Then one has:*

(1) The complex $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict, and $\mathcal{H}^i f_* M \in \mathrm{HM}^p(Y, w+i)$.
(2) For every $i \geq 0$, the morphism

$$\ell^i\colon \mathcal{H}^{-i} f_* M \to \mathcal{H}^i f_* M(i)$$

is an isomorphism of Hodge modules.
(3) Any polarization on M induces a polarization on $\bigoplus_i \mathcal{H}^i f_* M$ in the Hodge-Lefschetz sense (= on primitive parts for ℓ).

Example 16.2. In the case where $f\colon X \to pt$ is a morphism from a smooth projective variety to a point, the direct image theorem says that the i-th cohomology group $H^i(X, K)$ of the perverse sheaf K carries a polarized Hodge structure of weight $w+i$, and that the hard Lefschetz theorem holds.

The most notable consequence of Theorem 16.1 is the decomposition theorem for perverse sheaves underlying polarizable Hodge modules. This result, which contains [BBD82, Théorème 6.2.5] as a special case, is one of the major accomplishments of Saito's theory.[5]

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[5]De Cataldo and Migliorini [dCM05] later found a more elementary Hodge-theoretic proof for the decomposition theorem in the special case $K = \mathbb{Q}_X[\dim X]$.

Corollary 16.3. *Let $f\colon X \to Y$ be a projective morphism between complex manifolds, and K a perverse sheaf that underlies a polarizable Hodge module. Then*

$$\mathbf{R}f_*K \simeq \bigoplus_{i\in\mathbb{Z}} \bigl({}^pR^i f_*K\bigr)[-i]$$

in the derived category of perverse sheaves on Y.

Proof. We have a morphism $\ell\colon \mathbf{R}f_*K \to \mathbf{R}f_*K(1)[2]$ with the property that

$$\ell^i\colon {}^pR^{-i}f_*K \to {}^pR^i f_*K(i)$$

is an isomorphism for every $i \geq 0$. We can now apply a result by Deligne [Del68] to obtain the desired splitting; note that it typically depends on ℓ. □

The decomposition theorem also holds for the underlying filtered $\mathscr{D}$-modules, because of the strictness of the complex $f_+(\mathcal{M}, F_\bullet\mathcal{M})$.

Corollary 16.4. *Under the same assumptions as above, let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered $\mathscr{D}$-module that underlies a polarizable Hodge module. Then*

$$f_+(\mathcal{M}, F_\bullet\mathcal{M}) \simeq \bigoplus_{i\in\mathbb{Z}} \bigl(\mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})\bigr)[-i]$$

in the derived category of filtered $\mathscr{D}$-modules on Y.

17. Proof of the direct image theorem. Both Theorem 15.1 and Theorem 16.1 are ultimately consequences of results about polarized variations of Hodge structure. In the case of the structure theorem, the main input are the results of Cattani, Kaplan, and Schmid [Sch73, CKS86, CKS87] and Kashiware and Kawai [KK87] about degenerating polarized variations of Hodge on the complement of a normal crossing divisor. In the case of the direct image theorem, the main input are the results of Zucker [Zuc79] about the cohomology of a polarized variation of Hodge structure on a punctured Riemann surface. Note that the classical theory only deals with objects that are mildly singular; Saito's theory can thus be seen as a mechanism that reduces problems about polarizable variations of Hodge structure with arbitrary singularities to ones with mild singularities.

Let me begin by explaining the proof of the direct image theorem [Sai88, §5.3]. Given the recursive nature of the definitions, it is clear that Theorem 16.1 has to be proved by induction on $\dim Z$. The argument has three parts:

(1) Establish the theorem for $\dim X = 1$.
(2) Prove the theorem in the case $\dim f(Z) \geq 1$.
(3) Prove the theorem in the case $\dim f(Z) = 0$.

In the third part, we shall allow ourselves to use the structure theorem; one can avoid this by using some ad-hoc arguments, but it simplifies the presentation. This may look like a circular argument, because the proof of the structure theorem relies on the direct image theorem – but in fact it is not: the direct image theorem is only used for resolutions of singularities, which fall into the case $\dim f(Z) \geq 1$. Although I decided to keep the two proofs separate, one can actually make everything work out by proving both theorems together by induction on $\dim Z$.

Part 1. Saito first proves Theorem 16.1 for the mapping from a compact Riemann surface to a point; in fact, only the case $X = \mathbb{P}^1$ is needed. Let $M \in \mathrm{HM}_Z^p(X, w)$ be a polarized Hodge module with strict support $Z \subseteq X$. Then Z is either a point, in which case M comes from a polarized Hodge structure of weight w; or Z is equal to X, in which case M comes from a polarized variation of Hodge structure of weight $w - 1$ on the complement of finitely many points $x_1, \dots, x_m$. The first case is trivial; in the second case, the direct image theorem follows from Zucker's work. In [Zuc79], which predates the theory of Hodge modules, Zucker proves that the L^2-cohomology groups of the variation of Hodge structure carry a polarized Hodge structure. Although he does not explicitly mention $\mathscr{D}$-modules, he also proves that the filtered L^2-complex of the variation of Hodge structure is quasi-isomorphic to the filtered de Rham complex of $(\mathcal{M}, F_\bullet\mathcal{M})$. The key point is that the asymptotic behavior of the Hodge norm near each x_i is controlled by the V-filtration on $\mathcal{M}$ and by the weight filtration on the nearby cycles; this is a consequence of Schmid's results [Sch73]. Zucker's construction therefore leads to the same filtration and to the same pairing as in Saito's theory, and hence it implies the direct image theorem. The interested reader can find a detailed account of the proof in [Sab07, Chapter 3].

Part 2. The next step is to prove the direct image theorem in the case $\dim f(Z) \geq 1$. By induction, we can assume that we already know the result for all polarized Hodge modules with strict support of dimension less than $\dim Z$. The key point in the case $\dim f(Z) \geq 1$ is that applying a nearby or vanishing cycle functor on Y will reduce the dimension of Z, and can therefore be handled by induction.

To show that each $\mathcal{H}^i f_* M$ is a Hodge module of weight $w + i$, we have to verify the conditions in the definition; the same goes for showing that the polarization on M induces a polarization on the primitive part $P_\ell \mathcal{H}^{-i} f_* M = \ker \ell^{i+1}$. This involves applying the functors ψ_g and $\phi_{g,1}$, where g is a locally defined holomorphic function on Y. Some care is needed because $\mathcal{H}^i f_* M$ is a priori not a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure, but only lives in a larger abelian category (see §27).

Set $h = g \circ f$, and consider first the case where $f(Z) \not\subseteq g^{-1}(0)$, or equivalently, $Z \not\subseteq h^{-1}(0)$. In the abelian category mentioned above, one has a spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q} f_* \big(\mathrm{gr}_{-p}^W \psi_h M\big) \Longrightarrow \mathcal{H}^{p+q} f_* \psi_h M.$$

Each $\mathrm{gr}_{-p}^W \psi_h M$ is a polarized Hodge module of weight $w - 1 - p$ on $h^{-1}(0) \cap Z$; by induction, $E_1^{p,q}$ is therefore a polarized Hodge module of weight $w - 1 + q$. Note that we only have a polarization on the primitive part of $\mathrm{gr}_{-p}^W \psi_h M$ with respect to N, and that Theorem 16.1 only produces a polarization on the primitive part with respect to ℓ; this makes it necessary to study the simultaneous action of ℓ and N. In any case, it follows that the entire spectral sequence is taking place in the category of Hodge modules, and so it degenerates at E_2 for weight reasons. Now some general results about the compatibility of the direct image functor with nearby and vanishing cycles imply that

$$\psi_g \mathcal{H}^i f_* M \simeq \mathcal{H}^i f_* \psi_h M \quad \text{and} \quad \phi_{g,1} \mathcal{H}^i f_* M \simeq \mathcal{H}^i f_* \phi_{h,1} M. \tag{17.1}$$

This already shows that the objects on the left-hand side are Hodge modules on Y. Another key point is that the filtration induced by the spectral sequence is the monodromy filtration for the action of N on $\psi_g \mathcal{H}^i f_* M$; this is proved by showing that the Lefschetz property for (ℓ, N) on the E_1-page of the spectral sequence continues to hold on the E_2-page.

At this point, we can finally prove that the complex $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict, and hence that each $\mathcal{H}^i f_* M$ is a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure. By controlling $\psi_g M$ and $\phi_{g,1} M$ in the case when $f(Z) \not\subseteq g^{-1}(0)$, we actually know the whole associated graded of $\mathcal{M}$ with respect to the V-filtration. In sufficiently negative degrees, the V-filtration is equivalent to the g-adic filtration, and so we obtain information about $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ in a small analytic neighborhood of $g^{-1}(0)$. Since $\dim f(Z) \geq 1$, such neighborhoods cover $f(Z)$; this shows that $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict on all of Y. Similar reasoning is used to prove the Lefschetz isomorphism.

Now we can start checking the conditions in the definition. Since $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $h = 0$, the arguments leading to (17.1) show that $\mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $g = 0$, too. This proves (12.1) in the case $f(Z) \not\subseteq g^{-1}(0)$; the case $f(Z) \subseteq g^{-1}(0)$ is more or less trivial. The recursive condition in (12.4), as well as the assertions about the polarization, then follow from the analysis of the spectral sequence above.

The only remaining condition is (12.3), namely that $\mathcal{H}^i f_* M$ admits a decomposition by strict support. Somewhat surprisingly, the proof of this fact uses the polarization in an essential way. The idea is that the criterion in Theorem 11.7 reduces the problem to an identity among Hodge modules, which can be checked pointwise after decomposing by strict support; in the end, it becomes a linear algebra problem about certain families of polarized Hodge structures. (The same result is used again during the proof of the structure theorem.)

Part 3. It remains to prove Theorem 16.1 in the case $\dim f(Z) = 0$. We may clearly assume that X is projective space and Y is a point; now the idea is to use a pencil of hyperplane sections to get into a situation where one can apply the inductive hypothesis. Let $\pi \colon \tilde{X} \to X$ be the blowup of X at a generic linear subspace of codimension 2; as shown in the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] \arrow[d, "p"] \arrow[dr, "\tilde{f}"] & X \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r] & pt \end{tikzcd}$$

we also get a new morphism $p \colon \tilde{X} \to \mathbb{P}^1$. To apply the inductive hypothesis, we need to construct a polarizable Hodge module $\tilde{M}$ on $\tilde{X}$; this can easily be done with the help of the structure theorem. Let $\tilde{Z} \subseteq \pi^{-1}(Z)$ be the irreducible component birational to Z. From Theorem 15.1, we get an equivalence of categories

$$\mathrm{HM}^p_Z(X, w) \simeq \mathrm{HM}^p_{\tilde{Z}}(\tilde{X}, w),$$

and so M determines a Hodge module $\tilde{M} \in \mathrm{HM}^p_{\tilde{Z}}(\tilde{X}, w)$ with a polarization. We already know that the direct image theorem is true for $\tilde{M}$ and π, and so $\mathcal{H}^0 \pi_* \tilde{M}$ is a polarizable Hodge module of weight w; it is clear from the construction that the summand with strict support X is isomorphic to M (with the original polarization).

Now we apply the direct image theorem to $\tilde{M}$ and $p \colon \tilde{X} \to \mathbb{P}^1$, and then again to the Hodge modules $\mathcal{H}^i p_* \tilde{M}$ on $\mathbb{P}^1$; the result is that the complex $\tilde{f}_+(\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}})$ is strict, and that $H^i(\tilde{X}, \tilde{M}) = \mathcal{H}^i \tilde{f}_* \tilde{M}$ is a Hodge structure of weight $w + i$. Both properties are inherited by direct summands, and so it follows that $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is also strict, and that $H^i(X, M) = \mathcal{H}^i f_* M$ is also a Hodge structure of weight $w + i$.

The remainder of the argument consists in proving the Lefschetz isomorphism and the assertion about the polarization. Here Saito's method is to use the weak Lefschetz theorem as much as possible, rather than trying to compare Lefschetz operators and polarizations for M and $\tilde{M}$ directly; the point is that $\ell \simeq i_* \circ i^*$, where $i\colon Y \hookrightarrow X$ is the inclusion of a general hyperplane. This takes care of all cases except for proving that the polarization on M induces a polarization of the Hodge structure $P_\ell H^0(X, M) = \ker \ell$. In this remaining case, one has

$$P_\ell H^0(X, M) \hookrightarrow H^0\big(\mathbb{P}^1, P_{\tilde{\ell}} \mathcal{H}^0 p_* \tilde{M}\big),$$

with a corresponding identity for the polarizations; since we control the polarization on $P_{\tilde{\ell}} \mathcal{H}^0 p_* \tilde{M}$ by induction, an appeal to the theorem on $\mathbb{P}^1$ finishes the proof.

18. Proof of the structure theorem. Now let me explain how Saito proves the difficult half Theorem 15.1, namely that a generically defined polarized variation of Hodge structure on Z of weight $w - \dim Z$ extends uniquely to an object of $\mathrm{HM}^p_Z(X, w)$. Since it is easy to deduce from the definition that there can be at most one extension, we shall concentrate on the problem of constructing it.

We first consider a simplified local version of the problem to which we can apply the theory of degenerating variations of Hodge structure. Let $X = \Delta^n$ be a product of disks, with coordinates $t_1, \dots, t_n$, and let $(\mathcal{V}, F^\bullet\mathcal{V}, V, Q)$ be a polarized variation of Hodge structure on the complement of the divisor $t_1 \cdots t_n = 0$. We shall do the following:

(1) Extend the variation of Hodge structure to a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure M, and construct a candidate for a polarization.
(2) Show that M is of normal crossing type, and that many of its properties are therefore determined by combinatorial data.
(3) Verify the conditions in the special case of a monomial function g.

Afterwards, we can use resolution of singularities and the direct image theorem to show that this special case is enough to solve the problem in general.

Part 1. For the time being, we assume that we have a polarized variation of Hodge structure of weight $w - n$, defined on the complement of the divisor $t_1 \cdots t_n = 0$ in $X = \Delta^n$, with quasi-unipotent local monodromy. We have to construct an extension $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$ to a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure. Define K as the intersection complex of the local system V, and $\mathcal{M}$ as the corresponding regular holonomic $\mathscr{D}$-module; both have strict support X. Since $(\mathcal{M}, F_\bullet\mathcal{M})$ is supposed to be quasi-unipotent and regular along $t_1 \cdots t_n = 0$, we are forced to define the filtration $F_\bullet\mathcal{M}$ as in (11.5). More concretely,

$$\mathcal{M} = \Big(\omega_X \otimes \tilde{\mathcal{V}}^{(-1,0]}\Big) \cdot \mathscr{D}_X \subseteq \omega_X \otimes \tilde{\mathcal{V}}$$

is the $\mathscr{D}$-submodule generated by the lattice $\tilde{\mathcal{V}}^{(-1,0]}$ in Deligne's meromorphic extension $\tilde{\mathcal{V}}$ of the flat bundle $(\mathcal{V}, \nabla)$, and the filtration is given by the formula

$$F_p\mathcal{M} = \sum_{i=0}^{\infty} \Big(\omega_X \otimes F^{i-p-n}\tilde{\mathcal{V}}^{(-1,0]}\Big) \cdot F_i\mathscr{D}_X.$$

From the polarization on V, one can also construct a morphism $K(w) \to \mathbf{D}K$ that is a candidate for being a polarization on M.

Part 2. To prove that M is a polarized Hodge module, we use induction on $n \geq 0$. Here it is useful to work inside a larger class of objects that is preserved by various operations. From the nilpotent orbit theorem [Sch73], one can deduce that M has the following properties:

(a) The perverse sheaf K is quasi-unipotent, and the natural stratification of Δ^n is adapted to K.
(b) The $n+1$ filtrations $F_\bullet \mathcal{M}, V_\bullet^1 \mathcal{M}, \dots, V_\bullet^n \mathcal{M}$ are compatible, where $V_\bullet^i \mathcal{M}$ denotes the V-filtration with respect to the function t_i.
(c) Set $\partial_i = \partial/\partial t_i$ for every $i = 1, \dots, n$; then one has

$$F_p V_\alpha^i \mathcal{M} \cdot t_i = F_p V_{\alpha-1}^i \mathcal{M} \quad \text{for } \alpha < 0,$$
$$F_p \operatorname{gr}_\alpha^{V^i} \mathcal{M} \cdot \partial_i = F_{p+1} \operatorname{gr}_{\alpha+1}^{V^i} \mathcal{M} \quad \text{for } \alpha > -1,$$

which is a special case of the conditions in Definition 11.4.

In general, we say that an object with these properties is of *normal crossing type.*

This definition is closely related to the combinatorial description of regular holonomic $\mathscr{D}$-modules [GGM85]. If the first condition is satisfied, then $\mathcal{M}$ is determined by the following *combinatorial data*: the collection of vector spaces

$$\operatorname{gr}_\alpha^V \mathcal{M} = \operatorname{gr}_{\alpha_1}^{V^1} \cdots \operatorname{gr}_{\alpha_n}^{V^n} \mathcal{M}, \quad \text{for } \alpha \in (\mathbb{Q} \cap [-1, 0])^n,$$

and the linear mappings induced by $t_1, \dots, t_n$ and $\partial_1, \dots, \partial_n$ and $N_1, \dots, N_n$. As in Definition 11.4, Saito introduces the other two conditions in order to deal with the filtration $F_\bullet \mathcal{M}$ (which is not itself combinatorial). For objects of normal crossing type, the combinatorial data controls the properties of morphisms: for example, if $\varphi \colon M_1 \to M_2$ is a morphism between objects of normal crossing type, then $\ker \varphi$ and $\operatorname{im} \varphi$ are again of normal crossing type, provided that the induced morphisms on combinatorial data are strictly compatible with the Hodge filtration.

Saito also shows that if M is of normal crossing type, then the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ is Cohen-Macaulay, and quasi-unipotent and regular along any monomial $g = t_1^{m_1} \cdots t_n^{m_n}$; moreover, the vanishing cycles $\psi_g M$ are again of normal crossing type. Part of the argument consists in finding formulas for the combinatorial data of $\psi_g M$, and for the nilpotent operator N, in terms of the combinatorial data of M itself.

Part 3. Now we can verify all the conditions in the case of a monomial $g = t_1^{m_1} \cdots t_n^{m_n}$. Since we got M from a polarized variation of Hodge structure, the combinatorial data lives in the category of mixed Hodge structures: for nearby cycles, this follows from the SL(2)-orbit theorem [CKS86], and for vanishing cycles from the "vanishing cycle theorem" [KK87] respectively "descent lemma" [CKS87]. In particular, $t_1, \dots, t_n$ and $\partial_1, \dots, \partial_n$ are morphisms of mixed Hodge structure, and therefore strictly compatible with the Hodge filtration.

Since M is of normal crossing type, $(\mathcal{M}, F_\bullet \mathcal{M})$ is quasi-unipotent and regular along $g = 0$, and $\psi_g M$ is again an object of normal crossing type. From the combinatorial data for M, one can write down that for $\psi_g M$, and show that it also lives in the category of mixed Hodge structures. Because N is a morphism of mixed Hodge structures, and therefore strictly compatible with the Hodge filtration, it follows that the monodromy filtration $W(N)_\bullet \psi_g M$, as well as the primitive decomposition for N, are also of normal crossing type. In particular, the primitive part $P_N \operatorname{gr}_\ell^{W(N)} \psi_g M$ is again an object of normal crossing type.

The next step is to prove that $P_N \operatorname{gr}_\ell^{W(N)} \psi_g M$ admits a decomposition by strict support. It is enough to check the condition in Theorem 11.7 for the functions $t_1, \dots, t_n$; because we are dealing with objects of normal crossing type, the problem further reduces to a statement about polarized Hodge structures (which has already been used during the proof of the direct image theorem). Note that every summand in the decomposition by strict support is again of normal crossing type.

Now let M' be any summand in the decomposition; after permuting the coordinates, its support will be of the form $\Delta^m \times \{0\} \subseteq \Delta^n$. To conclude the argument, we have to show that M' is again an extension of a polarized variation of Hodge structure: by induction, this will guarantee that M' is a Hodge module of weight $w-1+\ell$, and that the induced morphism $K'(w-1+\ell) \to \mathbf{D}K'$ is a polarization. Because M' is an object of normal crossing type, $(\mathcal{M}', F_\bullet \mathcal{M}')$ is quasi-unipotent and regular along $t_1 \cdots t_m = 0$, and so M' must be the extension of a variation of Hodge structure $(\mathcal{V}', F^\bullet \mathcal{V}', V')$; the problem is to show that the induced bilinear form Q' on V' is a polarization. Here Saito appeals to the SL(2)-orbit theorem and its consequences [CKS86]: by looking at the combinatorial data, one sees that $\psi_{t_1} \cdots \psi_{t_m} M'$ is a nilpotent orbit; this implies that Q' polarizes the variation of Hodge structure in a small neighborhood of the origin in Δ^m.

Proof of Theorem 15.1. Suppose first that V is defined on the complement of a normal crossing divisor D in a complex manifold X. The same construction as above produces an extension M to a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure and strict support X. Now we have to check that all the conditions are satisfied for an arbitrary locally defined holomorphic function g. Resolution of singularities, together with some technical results from the proof of the direct image theorem, reduces this to the case where X is a product of disks and g is a monomial; since we have already proved the result in that case, we conclude that M is a polarized Hodge module.

To deal with the general case, we again use resolution of singularities and the direct image theorem. Choose an embedded resolution of singularities $f \colon \tilde{Z} \to X$ that is an isomorphism over the subset of Z where the variation of Hodge structure is defined, and that makes the complement into a normal crossing divisor. We already know that we can extend the variation of Hodge structure to an object $\tilde{M} \in \mathrm{HM}^p_{\tilde{Z}}(\tilde{Z}, w)$; since we can choose f to be projective, we can then apply Theorem 16.1 to conclude that $\mathcal{H}^0 f_* \tilde{M} \in \mathrm{HM}^p(X, w)$. Now the direct summand with strict support Z is the desired extension. $\square$

D. Mixed Hodge modules

The cohomology groups of algebraic varieties that are not smooth and projective still carry mixed Hodge structures. To include those cases into his theory, Saito was led to consider mixed objects. In fact, the definition of pure Hodge modules already hints at the existence of a more general theory: one of the conditions says that $\operatorname{gr}_\ell^{W(N)}(\psi_f M)$ is a Hodge module of weight $w-1+\ell$ for every $\ell \in \mathbb{Z}$; this suggests that the nearby cycles $\psi_f M$, together with the monodromy filtration for N shifted by $w-1$ steps, should actually be a "mixed" Hodge module.

19. Weakly mixed Hodge modules. To define mixed Hodge modules, Saito first introduces an auxiliary category $\mathrm{MHW}(X)$ of *weakly mixed Hodge modules.*

Its objects are pairs $(M, W_\bullet M)$, where M is a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure, and $W_\bullet M$ is a finite increasing filtration with the property that

$$\mathrm{gr}_\ell^W M \in \mathrm{HM}(X, \ell).$$

A weakly mixed Hodge module is called *graded-polarizable* if the individual Hodge modules $\mathrm{gr}_\ell^W M$ are polarizable; we denote this category by the symbol $\mathrm{MHW}^p(X)$.

Certain results from the pure case carry over to this setting without additional effort. For example, suppose that $(M, W_\bullet M) \in \mathrm{MHW}^p(X)$, and that we have a projective morphism $f\colon X \to Y$. Then the spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q} f_*\big(\mathrm{gr}_{-p}^W M\big) \Longrightarrow \mathcal{H}^{p+q} f_* M$$

degenerates at E_2, and each $\mathcal{H}^i f_* M$, together with the filtration induced by the spectral sequence, again belongs to $\mathrm{MHW}^p(Y)$. This is an easy consequence of Theorem 16.1 and the fact that there are no nontrivial morphisms between polarizable Hodge modules of different weights. One can also show that every morphism in the category $\mathrm{MHW}(X)$ is strictly compatible with the filtrations $F_\bullet \mathcal{M}$ and $W_\bullet \mathcal{M}$ (in the strong sense).

In order to have a satisfactory theory of mixed Hodge modules, however, we will need to impose restrictions on how the individual pure Hodge modules can be put together. In fact, the same problem already appears in the study of variations of mixed Hodge structure. The work of Cattani, Kaplan, and Schmid [CKS86] shows that every polarizable variation of Hodge structure degenerates in a controlled way; but for variations of mixed Hodge structure, this is no longer the case. To get a theory similar to the pure case, one has to restrict to *admissible* variations of mixed Hodge structure; this notion was introduced by Steenbrink and Zucker [SZ85], and further developed by Kashiwara [Kas86].

Example 19.1. When we assemble a mixed Hodge structure from $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$, the result is determined by a non-zero complex number, because

$$\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(0), \mathbb{Z}(1)\big) \simeq \mathbb{C}^\times.$$

A variation of mixed Hodge structure of this type, say on the punctured disk, is therefore the same thing as a holomorphic function $f\colon \Delta^* \to \mathbb{C}^\times$; note that f may have an essential singularity at the origin. In this simple example, the admissibility condition amounts to allowing only meromorphic functions f.

Note that admissibility is always defined relative to a partial compactification; two partial compactifications that are not bimeromorphically equivalent may lead to different notions of admissibility. Only in the case of algebraic varieties, where any two compactifications are automatically birationally equivalent, can we speak of admissibility without specifying the compactification.

20. Definition of mixed Hodge modules. In this section, we present a simplified definition of the category of mixed Hodge modules that Saito has developed over the years.[6] The point is that there is some redundancy in the original definition [Sai90b, §2.d], which can be eliminated by a systematic use of the stability of $\mathrm{MHM}^p(X)$ under taking subquotients.

We shall only define mixed Hodge modules on complex manifolds; more general analytic spaces are again dealt with by local embeddings into complex manifolds.

[6] I learned the details of this new definition from a lecture by Mochizuki.

Let $(M, W_\bullet M)$ be a weakly mixed Hodge module on a complex manifold X. We have to decide when M should be called a mixed Hodge module; the idea is to adapt the recursive definition from the pure case, by also imposing a condition similar to admissibility in terms of nearby and vanishing cycle functors.

Suppose then that $f\colon X \to \mathbb{C}$ is a non-constant holomorphic function on X. By induction on the length of the weight filtration, one can easily show that the underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ is quasi-unipotent and regular along $f = 0$; recall that this means the existence of a rational V-filtration $V_\bullet \mathcal{M}_f$ that interacts well with the Hodge filtration $F_\bullet \mathcal{M}_f$. Together with the weight filtration, we now have three filtrations on $\mathcal{M}_f$, and so we require that

(20.1) the three filtrations $F_\bullet \mathcal{M}_f$, $V_\bullet \mathcal{M}_f$, and $W_\bullet \mathcal{M}_f$ are compatible.

Roughly speaking, compatibility means that when we compute the associated graded, the order of the three filtrations does not matter; this is automatic in the case of two filtrations, but not usually true in the case of three or more.[7]

Assuming the condition above, the nearby and vanishing cycles of each $W_i M$ are defined, and so we can introduce the naive limit filtrations

$$L_i(\psi_f M) = \psi_f\big(W_{i+1} M\big) \quad \text{and} \quad L_i(\phi_{f,1} M) = \phi_{f,1}\big(W_i M\big).$$

Now recall that both $\psi_f M$ and $\phi_{f,1} M$ are equipped with a nilpotent endomorphism $N = (2\pi i)^{-1} \log T_u$; since N preserves the weight filtration, it also preserves the two limit filtrations. As in the admissibility condition of Steenbrink and Zucker, we ask for the existence of the so-called *relative monodromy filtration*:

(20.2) the relative monodromy filtrations $W_\bullet(\psi_f M) = W_\bullet\big(N, L_\bullet(\psi_f M)\big)$ and $W_\bullet(\phi_{f,1} M) = W_\bullet\big(N, L_\bullet(\phi_{f,1} M)\big)$ for the action of N exist.

This is the natural replacement for the monodromy filtration of N in the pure case; but while the existence of the monodromy filtration is automatic, the existence of the relative monodromy filtration is a nontrivial requirement. We say that $(M, W_\bullet M)$ is *admissible along* $f = 0$ if both of the conditions in (20.1) and (20.2) are satisfied; one should think of this as saying that the restriction of $(M, W_\bullet M)$ to the open subset $X \setminus f^{-1}(0)$ is admissible relative to X.

Exercise 20.3. Consider a weakly mixed Hodge module $(M, W_\bullet M)$ on the disk Δ, whose restriction to Δ^* is a variation of mixed Hodge structure with unipotent monodromy. Show that $(M, W_\bullet M)$ is admissible along $t = 0$ if and only if the variation is admissible in the sense of Steenbrink and Zucker [SZ85, §3.13].

Now we can define the category of mixed Hodge modules; as in the pure case, the definition is recursive. A weakly mixed Hodge module $(M, W_\bullet M) \in \mathrm{MHW}(X)$ is called a *mixed Hodge module* if, for every locally defined holomorphic function $f\colon U \to \mathbb{C}$, the pair $(M, W_\bullet M)$ is admissible along $f = 0$, and

(20.4) $\big(\psi_f M, W_\bullet(\psi_f M)\big)$ and $\big(\phi_{f,1} M, W_\bullet(\phi_{f,1} M)\big)$ are mixed Hodge modules

whenever $f^{-1}(0)$ does not contain any irreducible components of $U \cap \operatorname{Supp} \mathcal{M}$. This definition makes sense because $\psi_f M$ and $\phi_{f,1} M$ are then supported in a subset of strictly smaller dimension. Note that the shifts by $w - 1$ and w from the pure case are now built into the definition of the relative monodromy filtration.

[7] This problem has been studied at length in [Sai90b, §1].

Definition 20.5. We denote by

$$\mathrm{MHM}(X) \subseteq \mathrm{MHW}(X) \quad \text{and} \quad \mathrm{MHM}^p(X) = \mathrm{MHM}(X) \cap \mathrm{MHW}^p(X)$$

the full subcategories of all (graded-polarizable) mixed Hodge modules.

A morphism between two Hodge modules is simply a morphism between the underlying filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure that is compatible with the weight filtrations. As in the case of $\mathrm{MHW}(X)$, one can show that every morphism is strictly compatible with the filtrations $F_\bullet\mathcal{M}$ and $W_\bullet\mathcal{M}$ (in the strong sense).

21. Properties of mixed Hodge modules. In this section, we collect a few important properties of mixed Hodge modules. First, $\mathrm{MHM}(X)$ and $\mathrm{MHM}^p(X)$ are abelian categories; here the main point is that every subquotient of a mixed Hodge module in the category $\mathrm{MHW}(X)$ is again a mixed Hodge module. Almost by definition, (graded-polarizable) mixed Hodge modules are stable under the application of nearby and vanishing cycle functors. Extending the other standard functors from perverse sheaves to mixed Hodge modules is more involved. Given a projective morphism $f\colon X \to Y$, one has a collection of cohomological functors

$$\mathcal{H}^i f_*\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(Y);$$

see the discussion near the beginning of §19. As explained in §30, one can also define the cohomological inverse image functors

$$\mathcal{H}^i f^*\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(Y) \quad \text{and} \quad \mathcal{H}^i f^!\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(Y)$$

for an arbitrary morphism $f\colon Y \to X$. Lastly, one can define a duality functor

$$\mathbf{D}\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(X)^{opp},$$

compatible with Verdier duality for the underlying perverse sheaves (see §29).

There is also a version of Theorem 15.1, relating mixed Hodge modules and admissible variations of mixed Hodge structure [Sai90b, Theorem 3.27]. Suppose that $(M, W_\bullet M)$ is a graded-polarizable mixed Hodge module on a complex manifold X, and let $Z \subseteq X$ be an irreducible component of the support of M. After restricting to a suitable Zariski-open subset of Z, we obtain a graded-polarizable variation of mixed Hodge structure; it is not hard to deduce from the definition that it must be admissible relative to Z. The converse is also true, but much harder to prove.

Theorem 21.1. *Let X be a complex manifold, and $Z \subseteq X$ an irreducible closed analytic subvariety of X. A graded-polarizable variation of mixed Hodge structure on a Zariski-open subset of Z can be extended to a mixed Hodge module on X if and only if it is admissible relative to Z.*

Another important result is the description of mixed Hodge modules by a "gluing" procedure, similar to the case of perverse sheaves [Ver85, Beĭ87]. Suppose that X is a complex manifold, and $Z \subseteq X$ a closed analytic subvariety; we denote by $j\colon X \setminus Z \hookrightarrow X$ the inclusion of the open complement. Saito shows that a mixed Hodge module $M \in \mathrm{MHM}^p(X)$ can be reconstructed from the following information: its restriction to $X \setminus Z$; a mixed Hodge module on Z; and some gluing data. Let me explain how this works in the special case where $Z = f^{-1}(0)$ is the zero locus of a holomorphic function $f\colon X \to \mathbb{C}$. Here one can show that $M \in \mathrm{MHM}^p(X)$ is uniquely determined by the quadruple

$$\bigl(j^{-1}M, \phi_{f,1}M, \mathrm{can}, \mathrm{var}\bigr).$$

Recall that can: $\psi_{f,1}M \to \phi_{f,1}M$ and var: $\phi_{f,1}M \to \psi_{f,1}M(-1)$ are morphisms of mixed Hodge modules, and that the unipotent nearby cycles $\psi_{f,1}M$ only depend on $j^{-1}M$. This suggests considering all quadruples of the form

$$\bigl(M', M'', u, v\bigr),$$

where $M' \in \mathrm{MHM}^p(X \setminus Z)$ is *extendable* to X; where $M'' \in \mathrm{MHM}^p(Z)$; and where

$$u\colon \psi_{f,1}M' \to M'' \quad \text{and} \quad v\colon M'' \to \psi_{f,1}M'(-1)$$

are morphisms of mixed Hodge modules with the property that $v \circ u = N$. We denote the category of such quadruples (with the obvious morphisms) by the symbol $\mathrm{MHM}^p_{ex}(X \setminus Z, Z)$. One has the following result [Sai90b, Proposition 2.28].

Theorem 21.2. *The functor*

$$\mathrm{MHM}^p(X) \to \mathrm{MHM}^p_{ex}(X \setminus Z, Z), \quad M \mapsto \bigl(j^{-1}M, \phi_{f,1}M, \mathrm{can}, \mathrm{var}\bigr),$$

is an equivalence of categories.

The proof is essentially the same as in the case of perverse sheaves [Beĭ87], and makes use of Beilinson's maximal extension functor. There is a similar theorem for more general $Z \subseteq X$, in terms of Verdier specialization. This result can be used to define the two functors

$$j_*j^{-1}\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(X) \quad \text{and} \quad j_!j^{-1}\colon \mathrm{MHM}^p(X) \to \mathrm{MHM}^p(X),$$

in a way that is compatible with the corresponding functors for perverse sheaves. As the notation suggests, $j_*j^{-1}M$ and $j_!j^{-1}M$ only depend on the restriction of M to $X \setminus Z$, and can thus be considered as functors from extendable mixed Hodge modules on $X \setminus Z$ to mixed Hodge modules on X.

Example 21.3. In the special case considered above, $j_*j^{-1}M$ is the mixed Hodge module corresponding to the quadruple $\bigl(j^{-1}M, \psi_{f,1}M(-1), N, \mathrm{id}\bigr)$, and $j_!j^{-1}M$ the one corresponding to $\bigl(j^{-1}M, \psi_{f,1}M, \mathrm{id}, N\bigr)$.

Note that an arbitrary graded-polarizable mixed Hodge module on $X \setminus Z$ may not be extendable to X: for graded-polarizable variations of mixed Hodge structure, for example, extendability is equivalent to admissibility (by Theorem 21.1). This is why one cannot define j_* and $j_!$ as functors from $\mathrm{MHM}^p(X \setminus Z)$ to $\mathrm{MHM}^p(X)$.

22. Algebraic mixed Hodge modules. In the case where X is a complex algebraic variety, it is more natural to consider only algebraic mixed Hodge modules on X. Choose a *compactification* of X, meaning a proper algebraic variety $\bar{X}$ containing X as a dense Zariski-open subset. Then according to Serre's G.A.G.A. theorem, a coherent sheaf on the associated analytic space X^{an} is algebraic if and only if it is the restriction of a coherent sheaf from $\bar{X}^{an}$; this condition is independent of the choice of $\bar{X}$. A similar result holds for coherent filtered $\mathscr{D}$-modules. This suggests the following definition.

Definition 22.1. Let X be a complex algebraic variety. The category $\mathrm{MHM}_{\mathrm{alg}}(X)$ of *algebraic mixed Hodge modules* is defined as the image of $\mathrm{MHM}^p(\bar{X}^{an})$ under the natural restriction functor $\mathrm{MHM}^p(\bar{X}^{an}) \to \mathrm{MHM}^p(X^{an})$.

One can show that the resulting category is independent of the choice of $\bar{X}$; the reason is that any two complete algebraic varieties containing X as a dense Zariski-open subset are birationally equivalent. When M is an algebraic mixed

Hodge module, the perverse sheaf $\operatorname{rat} M$ on X^{an} is constructible with respect to an algebraic stratification of X, and the coherent sheaves $F_\bullet \mathcal{M}$ are algebraic; this justifies the name. It is important to remember that algebraic mixed Hodge modules are, by definition, polarizable and extendable.

Exercise 22.2. Use the structure theorem to prove that every polarizable variation of Hodge structure on an algebraic variety X is an object of $\mathrm{MHM}_{\mathrm{alg}}(X)$.

A drawback of the above definition is that it involves choosing a complete variety containing X. Fortunately, Saito [Sai13] has recently published an elegant intrinsic description of the category $\mathrm{MHM}_{\mathrm{alg}}(X)$. We shall state it as a theorem here; in an alternative treatment of the theory, it could perhaps become the definition.

Theorem 22.3. *A weakly mixed Hodge module $(M, W_\bullet M) \in \mathrm{MHW}(X^{an})$ belongs to the category $\mathrm{MHM}_{\mathrm{alg}}(X)$ if and only if X can be covered by Zariski-open subsets U with the following four properties:*

(1) There exists $f \in \mathscr{O}_X(U)$ such that $U \setminus f^{-1}(0)$ is smooth and dense in U.
(2) The restriction of $(M, W_\bullet M)$ to the open subset $U^{an} \setminus f^{-1}(0)$ is a graded-polarizable admissible variation of mixed Hodge structure.
(3) The pair $(M, W_\bullet M)$ is admissible along $f = 0$.
(4) The pair $\big(\psi_f M, W_\bullet(\psi_f M)\big)$ belongs to $\mathrm{MHM}_{\mathrm{alg}}\big(f^{-1}(0)\big)$.

Note that if one takes the conditions in the theorem as the definition $\mathrm{MHM}_{\mathrm{alg}}(X)$, then every graded-polarizable admissible variation of mixed Hodge structure on X is automatically an algebraic mixed Hodge module (because $f^{-1}(0)$ is allowed to be empty). In fact, the point behind Theorem 22.3 is precisely that every graded-polarizable admissible variation of mixed Hodge structure on X^{an} can be extended to an object of $\mathrm{MHM}^p(\bar{X}^{an})$.

23. Derived categories and weights. As mentioned in the introduction, Saito's theory is more satisfactory in the case of algebraic varieties. The reason is that admissibility is independent of the compactification (because any two compactifications are birationally equivalent); and that one has direct image functors for arbitrary morphisms (because every morphism can be factored into the composition of an open embedding and a proper morphism).

We denote by $\mathrm{D}^b \mathrm{MHM}_{\mathrm{alg}}(X)$ the bounded derived category of the abelian category $\mathrm{MHM}_{\mathrm{alg}}(X)$; its objects are bounded complexes of algebraic mixed Hodge modules on X. By construction, we have an exact functor

$$\operatorname{rat}\colon \mathrm{D}^b \mathrm{MHM}_{\mathrm{alg}}(X) \to \mathrm{D}^b_c(\mathbb{Q}_{X^{an}})$$

to the bounded derived category of algebraically constructible complexes. The functor associates to a mixed Hodge module the underlying perverse sheaf; it is faithful, essentially because a morphism between two mixed Hodge modules is zero if and only if the corresponding morphism on perverse sheaves is zero.

The advantage of working with algebraic varieties is that $\mathrm{D}^b \mathrm{MHM}_{\mathrm{alg}}(X)$ satisfies the same six-functor formalism as in the case of constructible complexes [BBD82]. Setting up this formalism is, however, not a trivial task. The easiest case is that of the duality functor (see §29): because

$$\mathbf{D}\colon \mathrm{MHM}_{\mathrm{alg}}(X) \to \mathrm{MHM}_{\mathrm{alg}}(X)^{opp}$$

is exact, it extends without ado to the derived category. As explained in §30, we have a collection of cohomological inverse image functors

$$\mathcal{H}^j f^* \colon \operatorname{MHM}_{\mathrm{alg}}(Y) \to \operatorname{MHM}_{\mathrm{alg}}(X) \quad \text{and} \quad \mathcal{H}^j f^! \colon \operatorname{MHM}_{\mathrm{alg}}(Y) \to \operatorname{MHM}_{\mathrm{alg}}(X)$$

for an arbitrary morphism $f \colon X \to Y$ between algebraic varieties; the functors f^* and $f^!$ on the derived category are defined by working with Čech complexes for suitable affine open coverings. One also has a collection of cohomological direct image functors

$$\mathcal{H}^j f_* \colon \operatorname{MHM}_{\mathrm{alg}}(X) \to \operatorname{MHM}_{\mathrm{alg}}(Y) \quad \text{and} \quad \mathcal{H}^j f_! \colon \operatorname{MHM}_{\mathrm{alg}}(X) \to \operatorname{MHM}_{\mathrm{alg}}(Y).$$

In the case where f is proper, this follows from Theorem 16.1 and Chow's lemma; in the general case, one factors f into an open embedding j followed by a proper morphism, and uses the fact that the two functors j_* and $j_!$ are well-defined for algebraic mixed Hodge modules. The functors f_* and $f_!$ are then again constructed with the help of affine open coverings, following the method introduced in [Beĭ87]. After all the functors have been constructed, one then has to prove that they satisfy all the standard compatibility and adjointness relations.

Example 23.1. Once the whole theory is in place, one can use it to put mixed Hodge structures on the cohomology groups of algebraic varieties. Let $f \colon X \to pt$ denote the morphism to a point; then one has a complex of algebraic mixed Hodge modules

$$\mathbb{Q}_X^H = f^* \mathbb{Q}(0) \in \mathrm{D}^b \operatorname{MHM}_{\mathrm{alg}}(X).$$

When X is smooth and n-dimensional, $\mathbb{Q}_X^H[n]$ is concentrated in degree 0, and is isomorphic to the pure Hodge module we were denoting by $\mathbb{Q}_X^H[n]$ earlier. Now

$$H^i(X, \mathbb{Q}_X^H) = H^i f_* \mathbb{Q}_X^H \quad \text{and} \quad H_c^i(X, \mathbb{Q}_X^H) = H^i f_! \mathbb{Q}_X^H$$

are graded-polarizable mixed Hodge structures on the $\mathbb{Q}$-vector spaces $H^i(X, \mathbb{Q})$ and $H_c^i(X, \mathbb{Q})$; with considerable effort, Saito managed to show that these mixed Hodge structures are the same as the ones defined by Deligne [Sai00].

Let me end this chapter by briefly discussing the weight formalism on the derived category of algebraic mixed Hodge modules; it works in exactly the same way as in the case of mixed complexes [BBD82, §5.1.5].

Definition 23.2. We say that a complex $M \in \mathrm{D}^b \operatorname{MHM}_{\mathrm{alg}}(X)$ is

(a) *mixed of weight* $\leq w$ if $\operatorname{gr}_i^W \mathcal{H}^j(M) = 0$ for $i > j + w$;
(b) *mixed of weight* $\geq w$ if $\operatorname{gr}_i^W \mathcal{H}^j(M) = 0$ for $i < j + w$;
(c) *pure of weight* w if $\operatorname{gr}_i^W \mathcal{H}^j(M) = 0$ for $i \neq j + w$.

Saito shows that when M is mixed of weight $\leq w$, both $f_! M$ and $f^* M$ are again mixed of weight $\leq w$; when M is mixed of weight $\geq w$, both $f_* M$ and $f^! M$ are again mixed of weight $\geq w$. It is also easy to see that M is mixed of weight $\leq w$ if and only if $\mathbf{D} M$ is mixed of weight $\geq -w$. In particular, pure complexes are stable under direct images by proper morphisms, and under the duality functor. If M_1 is mixed of weight $\leq w_1$, and M_2 is mixed of weight $\geq w_2$, then $\operatorname{Ext}^i(M_1, M_2) = 0$ for $i > w_1 - w_2$. This implies formally that every pure complex splits into the direct sum of its cohomology modules. The resulting non-canonical isomorphism

$$M \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(M)[-j] \in \mathrm{D}^b \operatorname{MHM}_{\mathrm{alg}}(X)$$

gives another perspective on the decomposition theorem.

E. Two applications

The two results in this chapter are among the first applications of Saito's theory; the interested reader can find more details in an article by Masahiko Saito [RIM92].

24. Saito's vanishing theorem. In [Sai90b, §2.g], Saito proved the following vanishing theorem for mixed Hodge modules. It contains many familiar results in algebraic geometry as special cases: for example, if X is an n-dimensional smooth projective variety, and if we take $M = \mathbb{Q}_X^H[n]$, then

$$\operatorname{gr}_{-p}^F \operatorname{DR}(\omega_X) \simeq \Omega_X^p[n-p],$$

and Saito's result specializes to the Kodaira-Nakano vanishing theorem.

Theorem 24.1 (Vanishing Theorem)**.** *Suppose that $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a mixed Hodge module M on a projective algebraic variety X. Then one has*

$$H^i\Big(X, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes L\Big) = 0 \quad \text{if } i > 0,$$
$$H^i\Big(X, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes L^{-1}\Big) = 0 \quad \text{if } i < 0,$$

where L is an arbitrary ample line bundle on X.

Recall from the discussion in §14 that $(\mathcal{M}, F_\bullet\mathcal{M})$ lives on some ambient projective space; nevertheless, each $\operatorname{gr}_k^F \operatorname{DR}(\mathcal{M})$ is a well-defined complex of coherent sheaves on X, independent of the choice of embedding (by Exercise 14.5).

The proof of Theorem 24.1 is similar to Ramanujam's method for deducing the Kodaira vanishing theorem from the weak Lefschetz theorem [Ram72]. Choose a sufficiently large integer $d \geq 2$ that makes L^d very ample, and let $i\colon X \hookrightarrow \mathbb{P}^N$ denote the corresponding embedding into projective space; we can assume that $M \in \operatorname{MHM}(\mathbb{P}^N)$, with $\operatorname{Supp} M \subseteq X$. Let $H \subseteq \mathbb{P}^N$ be a generic hyperplane; then $H \cap X$ is the divisor of a section of L^d, and therefore determines a branched covering $\pi\colon Y \to X$ of degree d to which one can pull back M. Then one can show that the $\mathscr{D}$-module underlying $\pi_*\pi^*M$ contains a summand isomorphic to $\mathcal{M}(*H) \otimes L^{-1}$; this implies the vanishing of

$$H^i\Big(X, \operatorname{gr}_k^F \operatorname{DR}\big(\mathcal{M}(*H)\big) \otimes L^{-1}\Big)$$

for $i \neq 0$. In fact, the formalism of mixed Hodge modules reduces this to Artin's vanishing theorem: on the affine variety $Y \setminus \pi^{-1}(H \cap X)$, the cohomology of a self-dual perverse sheaf vanishes in all degrees $\neq 0$. One can now deduce the desired vanishing for $\operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes L^{-1}$ from the exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}(*H) \to i_+\Big(\mathcal{M}\big|_H\Big) \to 0$$

by induction on the dimension. A recent article by Popa [Pop14] explains the details of the proof, as well as some applications of Saito's theorem to algebraic geometry. An alternative proof of Theorem 24.1 can be found in [Sch14].

25. Kollár's conjecture. In [Kol86a, Kol86b], Kollár proved several striking results about the higher direct images of dualizing sheaves: when $f\colon X \to Y$ is a surjective morphism from a smooth projective variety X to an arbitrary projective variety Y, the sheaves $R^i f_*\omega_X$ are torsion-free and satisfy a vanishing theorem, and the complex $\mathbf{R}f_*\omega_X$ splits in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$. He also conjectured that the same results should hold in much greater generality [Kol86b, Section 5].

More precisely, he predicted that, for every polarizable variation of Hodge structure V on a Zariski-open subset of a projective variety X, there should be a coherent sheaf $S(X, V)$ on X with the following properties:[8]

(1) If X is smooth and V is defined on all of X, then $S(X, V) = \omega_X \otimes F^p\mathcal{V}$, where $F^{p+1}\mathcal{V} = 0$ and $F^p\mathcal{V} \neq 0$.
(2) If L is an ample line bundle on X, then $H^j\big(X, S(X, V) \otimes L\big) = 0$ for $j > 0$.
(3) If $f\colon X \to Y$ is a surjective morphism to another projective variety Y, then the sheaves $R^i f_* S(X, V)$ are torsion-free.
(4) In the same situation, one has a decomposition

$$\mathbf{R}f_* S(X, V) \simeq \bigoplus_{i=0}^{r} \big(R^i f_* S(X, V)\big)[-i],$$

where $r = \dim X - \dim Y$.

In fact, Kollár's conjecture follows quite easily from the theory of mixed Hodge modules. Saito announced the proof with the laconic remark: "This implies some conjecture by Kollár, combined with the results in §3." [Sai90b, p. 276]. Fortunately, he later published a more detailed proof [Sai91b].

The idea is to extend the polarizable variation of Hodge structure V to a polarizable Hodge module $M = (\mathcal{M}, F_\bullet\mathcal{M}, K) \in \mathrm{HM}^p_X(X, w)$ with strict support X, by appealing to Theorem 15.1, and then to define

$$S(X, V) = F_{p(M)}\mathcal{M},$$

where $p(M) = \min\big\{\, p \in \mathbb{Z} \;\big|\; F_p\mathcal{M} \neq 0 \,\big\}$. Recall from Exercise 14.5 that $F_{p(M)}\mathcal{M}$ is a well-defined coherent sheaf on X, even though $(\mathcal{M}, F_\bullet\mathcal{M})$ only lives on some ambient projective space. With this definition of $S(X, V)$, Kollár's predictions become consequences of general results about polarizable Hodge modules: the formula in (1) follows from Theorem 15.1; the vanishing in (2) follows from Theorem 24.1; the splitting of the complex $\mathbf{R}f_* S(X, V)$ in (4) follows from Corollary 16.4; etc. Saito also found a very pretty argument for showing that the higher direct image sheaves $R^i f_* S(X, V)$ are torsion-free when $f\colon X \to Y$ is surjective [Sai91b, Proposition 2.6]. It is based on the following observation.

Proposition 25.1. *Let $f\colon X \to Y$ be a projective morphism between complex manifolds, and let M be a polarizable Hodge module with strict support $Z \subseteq X$. If M' is any summand of $\mathcal{H}^i f_* M$, then $p(M') > p(M)$ unless* $\operatorname{Supp} M' = f(Z)$.

Proof. Here is a brief outline of the proof. Suppose that $\operatorname{Supp} M' \neq f(Z)$; then it suffices to show that $F_{p(M)}\mathcal{M}' = 0$. Using the compatibility of the nearby cycle functor with direct images, one reduces the problem to showing that

$$F_{p(M)} \operatorname{gr}_0^V \mathcal{M}_h = 0,$$

where $V_\bullet\mathcal{M}_h$ denotes the V-filtration with respect to $h = 0$, and where $h = g \circ f$ for a locally defined holomorphic function g that vanishes along $\operatorname{Supp} M'$ but not along $f(Z)$. But because $F_{p(M)-1}\mathcal{M}_h = 0$, this follows from the fact that

$$\partial_t\colon F_{p-1} \operatorname{gr}_{-1}^V \mathcal{M}_h \to F_p \operatorname{gr}_0^V \mathcal{M}_h$$

is surjective for every $p \in \mathbb{Z}$ (see the discussion after Definition 11.4). □

[8] This is a simplified version; the actual conjecture is more precise.

F. Various functors and strictness

In this final chapter, we are concerned with various functors on mixed Hodge modules, in particular, the direct image, duality, and inverse image functors. They are relatively easy to define for perverse sheaves and regular holonomic $\mathscr{D}$-modules, but the presence of the filtration leads to complications. We shall also discuss some of the remarkable properties of those filtered $\mathscr{D}$-modules $(\mathcal{M}, F_\bullet \mathcal{M})$ that underlie mixed Hodge modules – they are due to the strong restrictions on the filtration $F_\bullet \mathcal{M}$ that are built into the definition.

26. Derived category and strictness. In this section, we briefly discuss the definition of the derived category of filtered $\mathscr{D}$-modules, and the important notion of strictness. Let X be a complex manifold, and let $\mathrm{MF}(\mathscr{D}_X)$ denote the category of filtered $\mathscr{D}$-modules; its objects are pairs $(\mathcal{M}, F_\bullet \mathcal{M})$, where $\mathcal{M}$ is a right $\mathscr{D}_X$-module, and $F_\bullet \mathcal{M}$ is a compatible filtration by $\mathscr{O}_X$-submodules. Note that $\mathrm{MF}(\mathscr{D}_X)$ is not an abelian category. It is, however, an exact category: a sequence of the form

$$0 \longrightarrow (\mathcal{M}', F_\bullet \mathcal{M}') \xrightarrow{u} (\mathcal{M}, F_\bullet \mathcal{M}) \xrightarrow{v} (\mathcal{M}'', F_\bullet \mathcal{M}'') \longrightarrow 0,$$

is considered to be short exact if and only if $v \circ u = 0$ and

$$0 \longrightarrow \mathrm{gr}_\bullet^F \mathcal{M}' \xrightarrow{\mathrm{gr}\, u} \mathrm{gr}_\bullet^F \mathcal{M} \xrightarrow{\mathrm{gr}\, v} \mathrm{gr}_\bullet^F \mathcal{M}'' \longrightarrow 0,$$

is a short exact sequence of graded $\mathrm{gr}_\bullet^F \mathscr{D}_X$-modules. Acyclic complexes are defined in a similar way; after localizing at the subcategory of all acyclic complexes, one obtains the derived category $\mathrm{DF}(\mathscr{D}_X)$. This category turns out to have a natural t-structure, whose heart is an abelian category containing $\mathrm{MF}(\mathscr{D}_X)$; more details about this construction can be found in [Lau83].

In fact, the category $\mathrm{DF}(\mathscr{D}_X)$ and the t-structure on it can be described in more concrete terms, starting from the general principle that graded objects form an abelian category, whereas filtered objects do not. Consider the *Rees algebra* of $(\mathscr{D}_X, F_\bullet \mathscr{D}_X)$, defined as

$$\mathscr{R}_X = R_F \mathscr{D}_X = \bigoplus_{k=0}^{\infty} F_k \mathscr{D}_X \cdot z^k;$$

here z is an auxiliary variable. It is a graded $\mathscr{O}_X$-algebra with

$$\mathscr{R}_X/(z-1)\mathscr{R}_X \simeq \mathscr{D}_X \quad \text{and} \quad \mathscr{R}_X/z\mathscr{R}_X \simeq \mathrm{gr}_\bullet^F \mathscr{D}_X,$$

and therefore interpolates between the non-commutative $\mathscr{O}_X$-algebra $\mathscr{D}_X$ and its associated graded $\mathrm{gr}_\bullet^F \mathscr{D}_X \simeq \mathrm{Sym}^\bullet \mathscr{T}_X$. By the same method, we can associate with every filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ a graded $\mathscr{R}_X$-module

$$R_F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} \cdot z^k \subseteq \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X[z, z^{-1}],$$

which is coherent over $\mathscr{R}_X$ exactly when the filtration $F_\bullet \mathcal{M}$ is good.

Exercise 26.1. Prove that $R_F \mathcal{M}/(z-1) R_F \mathcal{M} \simeq \mathcal{M}$ and $R_F \mathcal{M}/z R_F \mathcal{M} \simeq \mathrm{gr}_\bullet^F \mathcal{M}$.

This construction defines a functor $R_F \colon \mathrm{MF}(\mathscr{D}_X) \to \mathrm{MG}(\mathscr{R}_X)$ from the category of filtered right $\mathscr{D}$-modules to the category of graded right $\mathscr{R}$-modules.

Exercise 26.2. Prove that $R_F\colon \mathrm{MF}(\mathscr{D}_X) \to \mathrm{MG}(\mathscr{R}_X)$ is faithful and identifies $\mathrm{MF}(\mathscr{D}_X)$ with the subcategory of all graded $\mathscr{R}$-modules without z-torsion.

One can show that R_F induces an equivalence of categories

$$R_F\colon \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X) \to \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_X), \tag{26.3}$$

under which the t-structure on $\mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X)$ corresponds to the standard t-structure on $\mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_X)$. From this equivalence, we get a collection of cohomology functors

$$\mathcal{H}^i\colon \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X) \to \mathrm{MG}_{coh}(\mathscr{R}_X);$$

the thing to keep in mind is that $\mathcal{H}^i$ of a complex of filtered $\mathscr{D}$-modules is generally *not* a filtered $\mathscr{D}$-module, only a graded $\mathscr{R}$-module. By a similar procedure, one can define the bounded derived category of filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure; in that case, the cohomology functors $\mathcal{H}^i$ go to the abelian category of graded regular holonomic $\mathscr{R}$-modules with $\mathbb{Q}$-structure.

Definition 26.4. A graded $\mathscr{R}$-module is called *strict* if it has no z-torsion. More generally, a complex of filtered $\mathscr{D}$-modules is called *strict* if, for every $i \in \mathbb{Z}$, the graded $\mathscr{R}$-module obtained by applying $\mathcal{H}^i$ is strict.

In other words, a graded $\mathscr{R}$-module comes from a filtered $\mathscr{D}$-module if and only if it is strict. Strictness of a complex is equivalent to all differentials in the complex being strictly compatible with the filtrations.

Exercise 26.5. Show that a complex of filtered $\mathscr{D}$-modules

$$(\mathcal{M}', F_\bullet\mathcal{M}') \xrightarrow{u} (\mathcal{M}, F_\bullet\mathcal{M}) \xrightarrow{v} (\mathcal{M}'', F_\bullet\mathcal{M}'')$$

is strict at $(\mathcal{M}, F_\bullet\mathcal{M})$ if and only if u is strictly compatible with the filtrations.

The definition of various derived functors (such as the direct image functor or the duality functor) requires an abelian category, and therefore produces not filtered $\mathscr{D}$-modules but graded $\mathscr{R}$-modules. For Saito's theory, it is important to know that they are strict in the case of mixed Hodge modules.

27. Direct image functor. Our first concern is the definition of the direct image functor in the case where $f\colon X \to Y$ is a proper morphism between two complex manifolds; by using local charts, one can extend the definition to the case where X and Y are analytic spaces. We would like to have an exact functor

$$f_+\colon \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X) \to \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_Y)$$

that is compatible with the direct image functor $\mathbf{R}f_*$ for constructible complexes. Saito's construction is based on his theory of filtered differential complexes [Sai88, §2.2]; what we shall do here is briefly sketch another one based on Koszul duality.

Recall first how the elementary construction of f_+ works in the case of a single filtered right $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. Using the factorization

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "f"'] & X \times Y \arrow[r, "p_2"] & Y \end{tikzcd}$$

through the graph of f, one only has to define the direct image for closed embeddings and for smooth projections. In the case of a closed embedding $i\colon X \hookrightarrow Y$, the direct image is again a filtered $\mathscr{D}$-module, given by

$$i_+(\mathcal{M}, F_\bullet \mathcal{M}) = i_*\bigl(\mathcal{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\bigr),$$

where $(\mathscr{D}_{X\to Y}, F_\bullet \mathscr{D}_{X\to Y}) = \mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}(\mathscr{D}_Y, F_\bullet \mathscr{D}_Y)$. Locally, the embedding is defined by holomorphic functions $t_1, \dots, t_r$; if we set $\partial_i = \partial/\partial t_i$, then

$$i_+\mathcal{M} \simeq \mathcal{M}[\partial_1, \dots, \partial_r],$$

with filtration given by

$$F_k\bigl(i_+\mathcal{M}\bigr) = \sum_{a\in\mathbb{N}^r} F_{k-(a_1+\dots+a_r)}\mathcal{M} \otimes \partial_1^{a_1} \cdots \partial_r^{a_r}.$$

Strictness is clearly not an issue in this situation. In the case of a smooth projection $p_2\colon X \times Y \to Y$, the direct image is a complex of filtered $\mathscr{D}$-modules, given by

$$\mathbf{R}p_{2*}\, \mathrm{DR}_{X\times Y/Y}(\mathcal{M}, F_\bullet\mathcal{M}).$$

If we use the canonical Godement resolution to define $\mathbf{R}p_{2*}$, the filtration on the complex is given by the subcomplexes

$$\mathbf{R}p_{2*} F_k\, \mathrm{DR}_{X\times Y/Y}(\mathcal{M}, F_\bullet\mathcal{M});$$

there is no reason why the resulting filtered complex should be strict.

Example 27.1. For a morphism $f\colon X \to pt$ to a point, strictness of $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is equivalent to the E_1-degeneration of the Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^{p+q}\bigl(X, \mathrm{gr}^F_{-p}\, \mathrm{DR}(\mathcal{M})\bigr) \Longrightarrow H^{p+q}\bigl(X, \mathrm{DR}(\mathcal{M})\bigr).$$

This degeneration is of course an important issue in classical Hodge theory, too.

A one-step construction of the direct image functor uses the equivalence of categories in (26.3) and Koszul duality [BGS96, Theorem 2.12.6]; it works more or less in the same way as Saito's theory of induced $\mathscr{D}$-modules [Sai89a]. This construction also explains why direct images are more naturally defined using right $\mathscr{D}$-modules.

The key point is that the graded $\mathscr{O}_X$-algebra $\mathscr{R}_X$ is a *Koszul algebra*, meaning that a certain Koszul-type complex constructed from it is exact. It therefore has a Koszul dual $\mathscr{K}_X$, which is a certain graded $\mathscr{O}_X$-algebra constructed from the holomorphic de Rham complex $(\Omega_X^\bullet, d)$; concretely, $\mathscr{K}_X = \Omega_X^\bullet[d]$, with relations $d^2 = 0$ and $[d, \alpha] + (-1)^{\deg \alpha} d\alpha = 0$. Now *Koszul duality* gives an equivalence of categories of right modules

$$K_X\colon \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_X) \to \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{K}_X).$$

What makes this useful is that one has a morphism $\mathscr{K}_Y \to f_*\mathscr{K}_X$, and so $\mathbf{R}f_*$ of a complex of graded right $\mathscr{K}_X$-modules is naturally a complex of graded right $\mathscr{K}_Y$-modules. Since $\mathscr{K}_X$ has finite rank as on $\mathscr{O}_X$-module, and f is proper, it is also obvious that this operation preserves boundedness and coherence. If we put everything together, we obtain an exact functor

$$f_+\colon \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_X) \to \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_Y)$$

by taking the composition $K_Y^{-1} \circ \mathbf{R}f_* \circ K_X$.

Example 27.2. Consider the case of a filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. By definition,

$$f_+(\mathcal{M}, F_\bullet\mathcal{M}) = f_+(R_F\mathcal{M}) \in \mathrm{D}^b_{coh}\mathrm{G}(\mathscr{R}_Y),$$

and without additional assumptions on the filtration $F_\bullet\mathcal{M}$, the graded $\mathscr{R}_Y$-modules $\mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$ may fail to be strict.

28. Strictness of direct images. As part of Theorem 16.1, Saito proved that when $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a polarizable Hodge module on a complex manifold X, and $f\colon X \to Y$ is a projective morphism to another complex manifold Y, then the complex $f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict; the result is easily extended to mixed Hodge modules and to proper morphisms between algebraic varieties. In particular, every cohomology module $\mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is again a filtered $\mathscr{D}$-module.

Saito's theorem is a powerful generalization of the fact that, on a smooth projective variety X, the Hodge-de Rham spectral sequence degenerates at E_1. It has numerous important consequences; here we only have room to mention one, namely *Laumon's formula* for the associated graded of the filtered $\mathscr{D}$-module underlying the Hodge module $\mathcal{H}^i f_* M$. Laumon [Lau85] described the "associated graded" of the complex $f_+(\mathcal{M}, F_\bullet\mathcal{M}) \in \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_Y)$, which really means the derived tensor product with $\mathscr{R}_X/z\mathscr{R}_X$. This is not at all the same thing as the associated graded of the underlying $\mathscr{D}$-modules $\mathcal{H}^i f_+\mathcal{M}$, except when the complex is strict. In the case where M is a mixed Hodge module and $f\colon X \to Y$ is a projective morphism, this leads to the isomorphism

$$\mathrm{gr}^F_\bullet\big(\mathcal{H}^i f_+\mathcal{M}\big) \simeq R^i f_*\Big(\mathrm{gr}^F_\bullet\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y\Big),$$

where $\mathscr{A}_X = \mathrm{gr}^F_\bullet\mathscr{D}_X$. To say this in more geometric terms, let $\mathscr{G}(X, M)$ denote the coherent sheaf on the cotangent bundle T^*X determined by $\mathrm{gr}^F_\bullet\mathcal{M}$. Then

$$\mathscr{G}(Y, \mathcal{H}^i f_* M) \simeq R^i p_{1*}\Big(\mathbf{L}(df)^*\mathscr{G}(X, M)\Big),$$

where the notation is as in the following diagram:

$$\begin{tikzcd} T^*Y \times_Y X \arrow[r, "df"] \arrow[d, "p_1"] & T^*X \\ T^*Y \end{tikzcd}$$

Laumon's formula explains, in the case of mixed Hodge modules, how taking the associated graded interacts with the direct image functor. Saito has proved a similar statement for the associated graded of the de Rham complex [Sai88, §2.3.7].

Theorem 28.1. *Let $f\colon X \to Y$ be a projective morphism between complex manifolds. If $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a mixed Hodge module on X, one has*

$$\mathbf{R}f_*\,\mathrm{gr}^F_p\,\mathrm{DR}(\mathcal{M}) \simeq \mathrm{gr}^F_p\,\mathrm{DR}(f_+\mathcal{M}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^F_p\,\mathrm{DR}\big(\mathcal{H}^i f_+\mathcal{M}\big)[-i]$$

for every $p \in \mathbb{Z}$.

29. Duality functor. We also have to say a few words about the definition of the duality functor for mixed Hodge modules. The goal is to have an exact functor

$$\mathbf{D} \colon \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X) \to \mathrm{D}^b_{coh}\mathrm{F}(\mathscr{D}_X)^{opp}$$

that is compatible with the Verdier dual $\mathbf{D}K$ for constructible complexes. It is again easiest to define such a functor in terms of graded right $\mathscr{R}$-modules. The point is that the tensor product $\omega_X \otimes_{\mathscr{O}_X} \mathscr{R}_X$ has two commuting structures of right $\mathscr{R}_X$-module; when we apply

$$\mathbf{R}\mathcal{H}om_{\mathscr{R}_X}\Big(-, \omega_X \otimes_{\mathscr{O}_X} \mathscr{R}_X\Big)[\dim X]$$

to a complex of graded right $\mathscr{R}_X$-modules, we therefore obtain another complex of graded right $\mathscr{R}_X$-modules. Note that if we tensor by $\mathscr{R}_X/(z-1)\mathscr{R}_X$, this operation specializes to the usual duality functor

$$\mathbf{R}\mathcal{H}om_{\mathscr{D}_X}\Big(-, \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X\Big)[\dim X]$$

for $\mathscr{D}$-modules, and is therefore nicely compatible with Verdier duality.

Example 29.1. In the case of a single filtered regular holonomic $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, the complex of graded $\mathscr{R}_X$-modules

$$\mathbf{R}\mathcal{H}om_{\mathscr{R}_X}\Big(R_F\mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \mathscr{R}_X\Big)[\dim X] \tag{29.2}$$

can have cohomology in negative degrees and therefore fail to be strict. In general, all one can say is that the cohomology in negative degrees must be z-torsion, because the complex reduces to the holonomic dual of $\mathcal{M}$ after tensoring by $\mathscr{R}_X/(z-1)\mathscr{R}_X$.

Fortunately, the problem above does not arise for Hodge modules; this is the content of the following theorem by Saito [Sai88, Lemme 5.1.13].

Theorem 29.3. *If $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a Hodge module $M \in \mathrm{HM}(X, w)$, then the dual complex is strict and again underlies a Hodge module $\mathbf{D}M \in \mathrm{HM}(X, -w)$.*

It is proved by induction on the dimension of $\operatorname{Supp} M$; the key point is the compatibility of the duality functor with nearby and vanishing cycles. The result extends without much trouble to the case of mixed Hodge modules: given a mixed Hodge module $(M, W_\bullet M) \in \mathrm{MHM}(X)$, the pair

$$\big(\mathbf{D}M, \mathbf{D}W_{-\bullet}M\big)$$

is again a mixed Hodge module.

In fact, the strictness of the complex in (29.2) is equivalent to $\mathrm{gr}^F_\bullet \mathcal{M}$ being *Cohen-Macaulay* as an $\mathscr{A}_X$-module, where $\mathscr{A}_X = \mathrm{gr}^F_\bullet \mathscr{D}_X$; therefore Theorem 29.3 is saying that $\mathrm{gr}^F_\bullet \mathcal{M}$ is a Cohen-Macaulay module whenever $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a mixed Hodge module. This has the following useful consequence: if $(\mathcal{M}', F_\bullet\mathcal{M}')$ denotes the filtered $\mathscr{D}$-module underlying $M' = \mathbf{D}M$, then

$$\mathrm{gr}^F_\bullet \mathcal{M}' \simeq \mathbf{R}\mathcal{H}om_{\mathscr{A}_X}\Big(\mathrm{gr}^F_\bullet \mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \mathscr{A}_X\Big)[n],$$

where $n = \dim X$. It also implies that the coherent sheaf $\mathscr{G}(X, M)$ on the cotangent bundle T^*X is Cohen-Macaulay. This fact can be used to get information about the coherent sheaves $R^i\mathcal{H}om_{\mathscr{O}_X}(\mathrm{gr}^F_p\mathcal{M}, \mathscr{O}_X)$ from the geometry of the characteristic variety $\mathrm{Ch}(\mathcal{M})$ of the $\mathscr{D}$-module [Sch11]. For example, suppose that the fiber of the projection $\mathrm{Ch}(\mathcal{M}) \to X$ over a point $x \in X$ has dimension $\leq d$; then

$$R^i\mathcal{H}om_{\mathscr{O}_X}(\mathrm{gr}^F_p\mathcal{M}, \mathscr{O}_X) = 0 \quad \text{for every } i \geq d+1.$$

30. Inverse image functors. The inverse image functors for mixed Hodge modules lift the two operations $f^{-1}K$ and $f^!K$ for perverse sheaves. We shall only discuss the case of a morphism $f\colon Y \to X$ between two complex manifolds; some additional work is needed to deal with morphisms between analytic spaces. Using the factorization

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[rr, bend right, "f"'] & Y \times X \arrow[r, "p_2"] & X \end{tikzcd}$$

through the graph of f, it is enough to define inverse images for closed embeddings and for smooth projections.

Let us first consider the case of a smooth morphism $f\colon Y \to X$. For variations of Hodge structure, the inverse image is obtained by pulling back the vector bundle and the connection; to get a sensible definition for Hodge modules, we therefore have to tensor by the relative canonical bundle $\omega_{Y/X} = \omega_Y \otimes f^*\omega_X^{-1}$. There also has to be a shift by the relative dimension $r = \dim Y - \dim X$ of the morphism, because of Saito's convention that a variation of Hodge structure of weight w defines a Hodge module of weight $w + \dim X$.

With this in mind, let $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$ be a filtered regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure on X. We first define an auxiliary object

$$\tilde{M} = \big(\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}}, \tilde{K}\big),$$

where $\tilde{K} = f^{-1}K(-r)$ is again a perverse sheaf on Y, and where

$$\tilde{\mathcal{M}} = \omega_{Y/X} \otimes_{\mathscr{O}_Y} f^*\mathcal{M} \quad \text{and} \quad F_p\tilde{\mathcal{M}} = \omega_{Y/X} \otimes_{\mathscr{O}_Y} f^*F_{p+r}\mathcal{M}$$

is again a filtered $\mathscr{D}$-module on Y, through the natural morphism $\mathscr{D}_Y \to f^*\mathscr{D}_X$. It requires a small calculation to show that $\mathrm{DR}(\tilde{\mathcal{M}}) \simeq \mathbb{C} \otimes_{\mathbb{Q}} \tilde{K}$; this is where the factor $\omega_{Y/X}$ comes in. Now the crucial result is the following.

Theorem 30.1. *Suppose that $M \in \mathrm{HM}^p(X, w)$. Then $\tilde{M} \in \mathrm{HM}^p(Y, w + r)$.*

Unfortunately, one cannot prove directly that $\tilde{M}$ satisfies the conditions for being a Hodge module; instead, the proof has to go through the equivalence of categories in Theorem 15.1. Since M admits a decomposition by strict support, we may assume that $M \in \mathrm{HM}^p_Z(X, w)$, and therefore comes from a generically defined polarizable variation of Hodge structure on Z of weight $w - \dim Z$; its pullback is a generically defined polarizable variation of Hodge structure on $f^{-1}(Z)$ of the same weight, and therefore extends uniquely to an object of

$$\mathrm{HM}^p_{f^{-1}(Z)}(Y, w + r).$$

One then checks that this extension is isomorphic to $\tilde{M}$.

With this surprisingly deep result in hand, it is straightforward to construct the two inverse image functors in general. Let $(M, W_\bullet) \in \mathrm{MHW}(X)$ be a weakly mixed Hodge module on X. For a smooth morphism $f\colon Y \to X$, we define

$$\mathcal{H}^{-r} f^*(M, W_\bullet M) = \Big(\tilde{M}, W_{\bullet+r}\tilde{M}\Big)$$
$$\mathcal{H}^{r} f^!(M, W_\bullet M) = \Big(\tilde{M}(r), W_{\bullet-r}\tilde{M}(r)\Big)$$

as objects of $\mathrm{MHW}(Y)$; the need for shifting the weight filtration is explained by Theorem 30.1. One can show that both functors take (graded-polarizable) mixed Hodge modules to (graded-polarizable) mixed Hodge modules.

Next, we turn our attention to the case of a closed embedding $i\colon Y \hookrightarrow X$. Here the idea is to construct cohomological functors

$$\mathcal{H}^j i^*\colon \operatorname{MHM}(X) \to \operatorname{MHM}(Y) \quad \text{and} \quad \mathcal{H}^j i^!\colon \operatorname{MHM}(X) \to \operatorname{MHM}(Y) \tag{30.2}$$

in terms of nearby and vanishing cycles. This procedure has the advantage of directly producing mixed Hodge modules; note that it leads in general to a different filtration than the one used by Laumon for arbitrary filtered $\mathscr{D}$-modules [Lau83].

To make things simpler, let me explain the construction when $\dim Y = \dim X - 1$. Locally, the submanifold Y is then defined by a single holomorphic function t, so let us first treat the case where $Y = t^{-1}(0)$. Fix a mixed Hodge module $M \in \operatorname{MHM}(X)$ and set $K = \operatorname{rat} M$. By construction of the vanishing cycle functor (see §8), we have a distinguished triangle

$$i^{-1}K \to \psi_{t,1}K \to \phi_{t,1}K \to i^{-1}K[1]$$

in the derived category of constructible complexes; it is therefore reasonable to define $\mathcal{H}^j i^* M \in \operatorname{MHM}(Y)$, for $j \in \{-1, 0\}$, as the cohomology modules of the complex of mixed Hodge modules

$$\Big[\psi_{t,1}M \xrightarrow{\operatorname{can}} \phi_{t,1}M\Big][1].$$

To describe this operation on the level of filtered $\mathscr{D}$-modules, let $V_\bullet\mathcal{M}$ denote the V-filtration relative to the divisor $t = 0$; then the corresponding complex of filtered $\mathscr{D}$-modules is

$$\Big[\big(\operatorname{gr}_{-1}^V \mathcal{M}, F_{\bullet-1}\operatorname{gr}_{-1}^V \mathcal{M}\big) \xrightarrow{\partial_t} \big(\operatorname{gr}_0^V \mathcal{M}, F_\bullet \operatorname{gr}_0^V \mathcal{M}\big)\Big][1], \tag{30.3}$$

where the filtration is the one induced by $F_\bullet\mathcal{M}$.

To deal with the general case, we observe that the V-filtration is independent of the choice of local equation for Y, and that both $\operatorname{gr}_{-1}^V \mathcal{M}$ and $\operatorname{gr}_0^V \mathcal{M}$ are well-defined sheaves of $\mathscr{O}_Y$-modules. The same is true for the action of $t\partial_t$; the only thing that actually depends on t is the $\mathscr{D}_Y$-module structure. In fact, both sheaves carry an action by $\operatorname{gr}_0^V \mathscr{D}_X$, but this $\mathscr{O}_Y$-algebra is only locally isomorphic to $\mathscr{D}_Y[t\partial_t]$. On the other hand, one can show that the quotient of $\operatorname{gr}_0^V \mathscr{D}_X$ by $t\partial_t$ is canonically isomorphic to $\mathscr{D}_Y$; this gives both

$$\operatorname{coker}\Big(\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M} \to \operatorname{gr}_0^V \mathcal{M}\Big) \quad \text{and} \quad \ker\Big(\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M} \to \operatorname{gr}_0^V \mathcal{M}\Big) \otimes_{\mathscr{O}_Y} \mathscr{N}_{Y|X}$$

a canonical $\mathscr{D}_Y$-module structure that does not depend on the choice of t. Thus we obtain two filtered regular holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure and weight filtration; because the conditions in the definition are local, both are mixed Hodge modules on Y.

The same procedure can be used to define $\mathcal{H}^j i^! M \in \operatorname{MHM}(Y)$, for $j \in \{0, 1\}$. When $Y = t^{-1}(0)$, one has a similar distinguished triangle for $i^! K$, which suggests to look at the complex of mixed Hodge modules

$$\Big[\phi_{t,1}M \xrightarrow{\operatorname{var}} \psi_{t,1}M(-1)\Big].$$

The corresponding complex of filtered $\mathscr{D}$-modules is

$$\Big[\big(\operatorname{gr}_0^V \mathcal{M}, F_\bullet \operatorname{gr}_0^V \mathcal{M}\big) \xrightarrow{t} \big(\operatorname{gr}_{-1}^V \mathcal{M}, F_\bullet \operatorname{gr}_{-1}^V \mathcal{M}\big)\Big]. \tag{30.4}$$

As before, one proves that the cohomology of this complex leads to two well-defined mixed Hodge modules on Y.

Exercise 30.5. Let $i^!\mathcal{M}$ denote the complex of $\mathscr{D}$-modules in (30.4). Use the fact that $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent and regular along $t=0$ to construct a morphism

$$F_p(i^!\mathcal{M}) \to \mathbf{L}i^*(F_p\mathcal{M})[-1]$$

in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. Show that this morphism is an isomorphism when $\mathcal{M}$ comes from a vector bundle with integrable connection.

For an arbitrary morphism $f\colon Y \to X$, we use the factorization $f = p_2 \circ i$ given by the graph of f, and define

$$\mathcal{H}^j f^*(M, W_\bullet M) = \mathcal{H}^{j+\dim Y} i^* \Big(\mathcal{H}^{-\dim Y} p_2^*(M, W_\bullet M)\Big)$$
$$\mathcal{H}^j f^!(M, W_\bullet M) = \mathcal{H}^{j-\dim Y} i^! \Big(\mathcal{H}^{\dim Y} p_2^!(M, W_\bullet M)\Big).$$

One can check that the resulting functors take $\mathrm{MHM}^p(X)$ into $\mathrm{MHM}^p(Y)$, and that they are (up to canonical isomorphism) compatible with composition.

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651
E-mail address: `cschnell@math.sunysb.edu`