Rationality in families of threefolds

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Abstract We prove that in a family of projective threefolds defined over an algebraically closed field, the locus of rational fibers is a countable union of closed subsets of the locus of separably rationally connected fibers. When the ground field has characteristic zero, this implies that the locus of rational fibers in a smooth family of projective threefolds is the union of at most countably many closed subfamilies.

1 Introduction

Let $f : X \to T$ be a projective equidimensional morphism onto a connected reduced scheme T of finite type over an algebraically closed field k. Assume that the fibers $X_t := f^{-1}(t)$ are varieties for all $t \in T$, and let n denote the relative dimension of f. We will refer to f as a family of projective varieties. We are interested in understanding the algebraic structure of the *rational locus*

$$\text{Rat}(f) := \{t \in T \mid X_t \text{ is a rational variety}\}$$

of the family.

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It follows by general facts that Rat(f) is a countable union of locally closed subsets of T. Once singularities are allowed, it is easy to pick up examples of families of rational varieties that specialize to nonrational ones. In characteristic zero, however, the following question regarding smooth families has been around for some time.

Question 1.1 Assuming that $f \colon X \to T$ is a smooth family of projective varieties over an algebraically closed field of characteristic zero, is Rat(f) equal to a countable union of closed subsets of T?

The answer is easy and well-known in dimension one, and follows from Castelnuovo's rationality criterion in dimension two. It is expected that in higher dimensions Rat(f) can be a proper subset, possibly with infinitely many components; this should occur for instance in smooth families of cubic fourfolds.

In this paper we study the three-dimensional case. We do not put conditions on the characteristic of the ground field k; for this reason we consider the *separably rationally connected locus*

$$\text{SRC}(f) := \{t \in T \mid X_t \text{ is separably rationally connected}\}$$

of the family. There is an inclusion Rat(f) $\subset$ SRC(f), and equality holds for projective families of relative dimension $n \leq 2$. We prove the following result.

Theorem 1.2 *For every family $f \colon X \to T$ of projective varieties of dimension three over an algebraically closed field,* Rat(f) *is a countable union of closed subsets of* SRC(f).

In the case where the ground field has characteristic zero, this yields a positive answer to Question 1.1 in dimension three.

Corollary 1.3 *For a smooth family $f \colon X \to T$ of projective threefolds over an algebraically closed field of characteristic zero,* Rat(f) *is a countable union of closed subsets of* T.

The proof of these results is based on two basic properties: the countability and properness of the irreducible components of Hilbert schemes, and the fact that divisorial valuations are geometric. In characteristic zero, one can alternatively use the Weak Factorization Theorem in place of the property on valuations. A key step of the proof is a result regarding one-parameter degenerations of rational varieties, which is stated and proven below in Theorem 3.1. A special case of this result, where the ground field is $k = \mathbb{C}$ and the degeneration is given by a smooth family of complex threefolds, was also obtained using Hodge theoretic methods by Timmerscheidt [15].

In view of Theorem 1.2, one could ask, as a plausible extension of Question 1.1 to arbitrary characteristics, whether given any family $f \colon X \to T$ of projective varieties over an algebraically closed field, Rat(f) is always a countable union of closed subsets of SRC(f). The examples discussed below in Example 3.4 suggest however that this may be false in higher dimensions, even in characteristic zero; in fact, in view of these examples, it seems likely that the hypothesis on the dimension in the theorem is optimal. It is important to remark that the families in these examples are not smooth, so they do not bring enough evidence to disbelieve Question 1.1. Rather, they suggest that if a positive answer is expected to this question, then the smoothness of the family should play a key role in the proof.

We do not know the full history of Question 1.1. It is an old problem to find an example of a family of nonrational projective varieties that specializes to a smooth rational variety. The question whether in smooth families the locus of rational fibers can always be expressed as a countable union of closed subsets has been considered in conversations between Paolo

Francia and Alessandro Verra; it is likely that the same question has been raised by other mathematicians as well. The special case of cubic hypersurfaces in $\mathbb{P}^5$ is representative: rationality questions about cubic fourfolds have attracted the attention of the mathematical community for a long time, starting with the work of Morin [14] and Fano [5] if not earlier. The construction of countably many families of rational cubic fourfolds due to Hassett [8,9], in particular, fits naturally in the context of Question 1.1 and has prompted more people to consider the rationality problem from this point of view.

2 General properties

We work over an algebraically closed field k. All schemes are assumed to be of finite type over k. With the term *variety* we mean an integral scheme. A morphism of varieties $f: X \to Y$ is *separable* if it is dominant and the field extension $K(X) \supset K(Y)$ is separably generated. A variety X is *rational* if its function field is purely transcendental over k, or, equivalently, if X is birational to $\mathbb{P}^n$ where $n = \dim X$. A variety X is *separably rationally connected* if there is a variety V and a morphism $u: \mathbb{P}^1 \times V \to X$ such that

$$u^{(2)}: \mathbb{P}^1 \times \mathbb{P}^1 \times V \to X \times X$$

is separable, or equivalently, is dominant and smooth at the generic point (cf. [10, Definition IV.3.2]).

The following property is a direct consequence of the definition (cf. [10, Proposition IV.3.3.1]).

Proposition 2.1 *If X and X' are two proper varieties that are birationally equivalent, then X is separably rationally connected if and only if X' is.*

It is straightforward from the definitions that a proper rational variety is separably rationally connected, and the converse holds in dimension two (cf. [10, Exercise IV.3.3.5]).

Proposition 2.2 *Let X be a proper surface. If X is separably rationally connected, then it is rational.*

Proof By Lipman [13], there exists a resolution of singularities of X. Since both rationality and separably rational connectedness are birational properties, we may thus assume without loss of generality that X is smooth. Then, by Kollár [10, Theorem IV.3.7], there is a morphism $g: \mathbb{P}^1 \to X$ such that f^*T_X is ample. This implies that every section of $(\wedge^q \Omega_X)^{\otimes m}$, for any $q, m \geq 1$, vanishes along $g(\mathbb{P}^1)$. As these curves cover a dense set in X, we conclude that all sections of $(\wedge^q \Omega_X)^{\otimes m}$ are zero. Therefore X is rational by Castelnuovo's criterion.   □

Suppose now that $f: X \to T$ is a family of projective varieties parameterized by a connected, reduced scheme T of finite type over k. The rational locus $\mathrm{Rat}(f)$ of the family has the following algebraic structure.

Proposition 2.3 $\mathrm{Rat}(f)$ *is a countable union of locally closed subsets of T.*

We learned the following proof, which simplifies our original arguments, from Claire Voisin.

Proof Let $P := T \times \mathbb{P}^n$, where n is the relative dimension of f. First observe that every closed subscheme $Z \subset X \times_T P$ determines a birational map $X_t \dashrightarrow P_t \cong \mathbb{P}^n$ for every t such that Z_t is irreducible and both projections $Z_t \to X_t$ and $Z_t \to P_t$ are birational; conversely, all birational maps from fibers of f to $\mathbb{P}^n$ arise in this way.

Consider the relative Hilbert scheme $H := \mathrm{Hilb}(X \times_T P/T)$ of $X \times_T P$, and let $U \to H$ be the universal family: U is a closed subscheme of $X \times_T P \times_T H$, flat over H. For every irreducible component H_j of H, consider the set of points $h \in H_j$ such that U_h is irreducible and, if $t \in T$ is the image of h, then the projections $U_h \to X_t$ and $U_h \to P_t$ are birational. By applying Lemma 2.4 to $U_j \to X \times_T H_j \to H_j$ and $U_j \to P \times_T H_j \to H_j$ where $U_j := U \times_H H_j$, we see that this set is constructible in H_j. By Chevalley's theorem, its image in T is also constructible, and as such can be written as a finite union of locally closed subsets. The union of all these sets, as H_j varies among the irreducible components of the Hilbert scheme, is $\mathrm{Rat}(f)$. The statement then follows by the fact that the Hilbert scheme has countably many irreducible components.   □

Lemma 2.4 *Let $U \to V \to H$ be morphisms of schemes of finite type over k, with $U \to H$ flat and $V \to H$ projective. Then the set $h \in H$ such that U_h is irreducible and $U_h \to V_h$ is birational is constructive.*

Proof Assuming without loss of generality that H is irreducible, these properties hold at the generic point of H if and only if they hold over a nonempty open set of H. The statement then follows by Noetherian induction.   □

Remark 2.5 An analogous property is satisfied by the locus of unirational varieties: the argument easily adjusts to this case by relaxing the condition on $U_h \to X_t$ from being birational to being dominant. A related result concerning the behavior of uniruledness in families is proven in [10, Therem IV.1.8], where it is shown that the locus of uniruled varieties in an equidimensional proper family is a countable union of closed subsets of the base.

Regarding the general structure of $\mathrm{SRC}(f)$, several interesting cases are covered by the following proposition.

Proposition 2.6 *Let $f : X \to T$ as above.*

(a) *In any setting where embedded resolution of singularities exists, $\mathrm{SRC}(f)$ is a constructible subset of T.*
(b) *If f is smooth, then $\mathrm{SRC}(f)$ is open in T.*
(c) *If f is smooth and k has characteristic zero, then $\mathrm{SRC}(f)$ is open and closed in T (and thus is either empty or equal to T).*

Proof The assertions in (b) and (c) are proven in [10, Theorem IV.3.11]. Regarding (a), first note that f is separable as it has reduced fibers (cf. [7, Theorem II.8.6A and Proposition II.8.10]) , and so is the restriction of f over any locally closed subset of T. Let $Y \to X$ be a resolution of singularities, and consider the composition map $g : Y \to T$. Since g is separable, there is a nonempty open set T° in the regular locus of T over which the induced map $g^\circ : g^{-1}(T^\circ) \to T^\circ$ is smooth (the proof of [7, Corollary III.10.7] goes through without assumptions on the characteristic of the ground field as long as one assumes that the morphism is separable). By (b), $\mathrm{SRC}(g^\circ)$ is an open subset of T°. Note on the other hand that $\mathrm{SRC}(f) \cap T^\circ = \mathrm{SRC}(g^\circ)$ by Proposition 2.1, since every fiber of g° is birational to the corresponding fiber of f. Thus the assertion follows by Noetherian property, by considering a suitable stratification of T.   □

Remark 2.7 A different definition of separably rational connectedness has been considered in works of de Jong, Graber, and Starr (cf. [4,6]), where a projective variety X is said to be separably rationally connected if there exists a morphism $g : \mathbb{P}^1 \to X_{\mathrm{reg}}$ such that $g^* T_{X_{\mathrm{reg}}}$ is ample. It is elementary to see that this property is open in families. It follows by the

deformation theory of rational curves (see for instance the argument in the proof of [10, Theorem IV.3.5]) that a projective variety that is separably rationally connected in the sense of de Jong, Graber and Starr is also separably rationally connected in the sense defined in the previous section, and the two notions coincide whenever X is smooth by Kollár [10, Theorem IV.3.7]. It is however not clear to us whether being separably rationally connected in the sense of de Jong, Graber and Starr is a birational property among projective varieties. In particular, we do not know whether a rational projective variety is necessarily separably rationally connected in this sense.

It is easy to construct examples of families of rational projective varieties degenerating to singular varieties that are not rational, and vice versa. We do not know any example of a (connected) smooth family of projective varieties containing both rational and nonrational members. It is expected in general that one needs to consider countable unions in Proposition 2.3 and in Question 1.1.

Example 2.8 Complex cubic fourfolds in $\mathbb{P}^5$ form a particularly interesting class of varieties from the point of view of rationality. The quest for rational examples goes back at least to Morin [14], who gave an incorrect argument that would have implied that the general cubic in $X \subset \mathbb{P}^5$ is rational. In the same paper, however, Morin correctly proves the rationality of general Pfaffian cubic fourfolds: these span a codimension one family of smooth rational cubics fourfolds which was further studied by Fano [5], Tregub [16], and Beauville and Donagi [2]. A crucial step in the study of cubic fourfolds is Voisin's proof of a Torelli Theorem for these varieties [17]. More examples of rational cubic fourfolds were found by Zarhin [19], and later Hassett [8,9] constructed a countable series of distinct families of smooth rational cubic fourfolds: these are parameterized by divisors on the family of cubics containing a plane, which has codimension one in the whole space of cubics. It is expected on the other hand that not only the general cubic in $\mathbb{P}^5$, but also the very general element among those containing a plane is not rational. An explicit conjecture has been formalized in the language of derived categories by Kuznetsov [12]. Knowing this conjecture would give an example of a family where the rational locus is, strictly speaking, a countable union of closed subfamilies.

3 The three dimensional case

In dimension three, we have the following property regarding one-parameter degenerations of rational projective varieties.

Theorem 3.1 *Let $f : X \to T$ be a projective morphism from a variety X onto a smooth curve T defined over an uncountable algebraically closed field k. Let $0 \in T$ be a closed point. Assume that X_t is a rational variety for every $t \neq 0$. Then every reduced, irreducible component D of X_0 that is separably rationally connected is rational.*

Proof Using the same notation as in the proof of Proposition 2.3, there are countably many irreducible locally closed subsets

$$S_i \subset H := \mathrm{Hilb}(X \times_T (T \times \mathbb{P}^3)/T), \quad i \in \mathbb{N},$$

such that, if U is the universal family of H then, for every $h \in S_i$, U_h gives a birational correspondence between a fiber X_t of X and $\mathbb{P}^3$, and the rational locus of f is given by the

union

$$\mathrm{Rat}(f) = \bigcup_i T_i$$

of the images $T_i \subset T$ of the sets S_i. Note that each T_i is an irreducible constructible subset of T, and thus is either a point or an open subset.

Since we are assuming that $\mathrm{Rat}(f) = T \smallsetminus \{0\}$ and the ground field is uncountable, there is at least one index i_0 such that T_{i_0} is a dense open subset of T. Let $\overline{S}_{i_0}$ be the closure of S_{i_0} in H. By the properness of the Hilbert scheme over T, $\overline{S}_{i_0}$ maps onto T. Let $C \subset \overline{S}_{i_0}$ be a general complete intersection curve; we assume in particular that C is irreducible, that it intersects S_{i_0}, and that the map $C \to T$ is surjective. Note that $C \cap S_{i_0}$ is open and dense in C. Let then $T' \to C$ be the normalization, let $g : T' \to T$ be the composition map, and fix a point $0' \in g^{-1}(0)$. The fiber product $X' := T' \times_T X$ is a variety, and by base change we obtain a projective morphism

$$f' : X' \to T'$$

with fiber $X'_{0'} \cong X_0$ over $0'$.

Let $H' := \mathrm{Hilb}(X' \times_{T'} (T' \times \mathbb{P}^3)/T')$, with universal family U'. By base change, we have a commutative diagram

$$
\begin{array}{ccccc}
U_{T'} & \longrightarrow & U' & \longrightarrow & U \quad , \\
\downarrow & & \downarrow & & \downarrow \\
T' & \longrightarrow & H' & \longrightarrow & H \\
& \diagdown & \downarrow & & \downarrow \\
& & T' & \longrightarrow & T
\end{array}
$$

where

$$U_{T'} := U \times_H T' = U' \times_{H'} T'$$

is the pullback of the universal family to T'. By construction, the image of the induced map $U_{T'} \to X' \times_{T'} (T' \times \mathbb{P}^3)$ is a closed subscheme

$$Z' \subset X' \times_{T'} (T' \times \mathbb{P}^3)$$

such that for every $s \in g^{-1}(T_{i_0})$ the fiber Z'_s is irreducible and both projections $Z'_s \to X'_s$ and $Z'_s \to \{s\} \times \mathbb{P}^3$ are birational. Since $g^{-1}(T_{i_0})$ is an open dense subset of T', it follows that the support of Z' is the graph of a birational map

$$\phi : X' \dashrightarrow T' \times \mathbb{P}^3$$

defined over T'.

Let D' be the irreducible component of $X'_{0'}$ mapping to D via the isomorphism $X'_{0'} \cong X_0$. Since the fiber $X'_{0'}$ is a Cartier divisor on X' that is reduced at the generic point $\eta_{D'}$ of D', $\eta_{D'}$ is contained in the regular locus of X'. Thus the vanishing order at $\eta_{D'}$ defines a divisorial valuation on the function field of X'. Let ν be the induced valuation on the function field of $T' \times \mathbb{P}^3$. Note that the center C_0 of ν in $T' \times \mathbb{P}^3$ is contained in the fiber $\{0'\} \times \mathbb{P}^3$.

Consider the sequence of blow-ups

$$\cdots \to Y_i \to Y_{i-1} \to \cdots \to Y_1 \to Y_0 := T' \times \mathbb{P}^3$$

where each $g_i \colon Y_i \to Y_{i-1}$ is the blow-up of Y_{i-1} along the the center C_{i-1} of ν. Note that, for every i, C_i is contained in the exceptional divisor of the blow-up g_i, and $g_i(C_i) = C_{i-1}$.
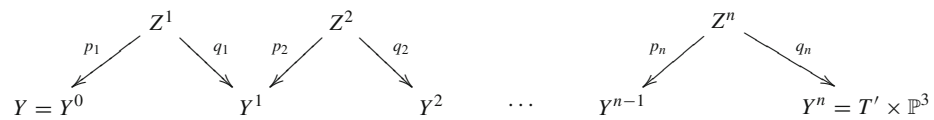
By induction on i, both Y_{i-1} and C_{i-1} are smooth at the generic point of C_{i-1}, and therefore there is a dense open set $Y_{i-1}^\circ \subset Y_{i-1}$, contained in the regular locus of Y_{i-1}, such that $C_{i-1}^\circ := C_{i-1} \cap Y_{i-1}^\circ$ is smooth and the induced map $g_i^{-1}(Y_{i-1}^\circ) \to Y_{i-1}^\circ$ is the blow-up of the normal bundle $\mathcal{N}_{i-1}$ of C_{i-1}° in Y_{i-1}°. In particular, the restriction of the exceptional locus of g_i over C_{i-1}° is isomorphic to the projective bundle $\mathbb{P}_{C_{i-1}^\circ}(\mathcal{N}_{i-1})$.

It follows by a theorem of Zariski (cf. [11, Lemma 2.45]) that there is an integer $m \geq 0$ such that the center C_m of ν has codimension one in Y_m and ν is given by the order of vanishing at the generic point of C_m. In particular, C_m is birational to D' since both their function fields are equal to the residue field of the valuation (geometrically, C_m is the proper transform of D' under the birational map $X' \dashrightarrow Y_m$). We can pick m to be the least integer with these properties.

If $m = 0$, then the center of ν in $T' \times \mathbb{P}^3$ is the whole fiber $\{0'\} \times \mathbb{P}^3$. This means that ϕ induces a birational map from D' to $\{0'\} \times \mathbb{P}^3$, and therefore D' is rational.

Suppose then that $m \geq 1$. In this case the projection $C_m \to C_{m-1}$ is a surjective map from a threefold to a variety of dimension at most two. Note that C_m is separably rationally connected, since it is birational to X_0 which is separably rationally connected by hypothesis, and being separably rationally connected is a birational property (see Proposition 2.1). Since the map $C_m \to C_{m-1}$ is smooth over C_{m-1}°, it follows that C_{m-1} is separably rationally connected too. The assumption on the relative dimension of f implies that $\dim C_{m-1} \leq 2$. If C_{m-1} has dimension at most one then it is clearly rational, and the same conclusion holds if C_{m-1} is a surface by Proposition 2.2. Note, on the other hand, that C_m contains $g^{-1}(C_{m-1}^\circ)$ as a dense open set, and the latter is isomorphic to $\mathbb{P}_{C_{m-1}^\circ}(\mathcal{N}_{m-1})$. We conclude that C_m is rational. Therefore D is rational.   □

Remark 3.2 If the ground field k has characteristic zero then one can use an alternative argument, based on the Weak Factorization Theorem [1,18], to prove Theorem 3.1. The argument goes as follows. Let $\phi \colon X' \dashrightarrow T' \times \mathbb{P}^3$ and D' be as in the proof of the theorem, and suppose that ϕ contracts D' (so that it does not induce directly a birational map from D' to $\{0'\} \times \mathbb{P}^3$). Let $Y \to X'$ be a resolution of singularities. By the Weak Factorization Theorem applied to the induced birational map $Y \dashrightarrow T' \times \mathbb{P}^3$, we can find a sequence of blow-ups p_i and blow-downs q_j with smooth irreducible centers

$$
\begin{array}{ccccccccc}
 & Z^1 & & & Z^2 & & & & Z^n \\
 & p_1 \nearrow & \searrow q_1 & p_2 \nearrow & & \searrow q_2 & & p_n \nearrow & & \searrow q_n \\
Y = Y^0 & & Y^1 & & Y^2 & \cdots & Y^{n-1} & & Y^n = T' \times \mathbb{P}^3
\end{array}
$$

(we allow isomorphisms among the maps p_i and q_j). Since ϕ contracts D', there is a model Z^i, for some $1 \leq i \leq n$, where the proper transform D^i of D' is the exceptional divisor of $q_i \colon Z^i \to Y^i$. Since D^i is rationally connected, so is its image $W_i := q_i(D^i)$, which is therefore rational. This implies that D^i is rational, as it is isomorphic to the projectivization of the normal bundle of W_i in Y^i. Therefore D is rational.

Remark 3.3 When the ground field is $k = \mathbb{C}$ and the family $f \colon X \to T$ is smooth, Theorem 3.1 also follows by Timmerscheidt [15, Theorem 1].

Proof of Theorem 1.2 The statement of the theorem is trivial if the ground field k if finite or countable, since in this case any subset of T can be expressed as a countable union of closed subsets. Thus we can assume that k is uncountable.

By Proposition 2.3, $\mathrm{Rat}(f)$ is a countable union of locally closed subsets of $R_i \subset T$. Suppose that $\mathrm{Rat}(f)$ cannot be written as a countable union of closed subsets of $\mathrm{SRC}(f)$. Then we can find a point $p \in \mathrm{SRC}(f) \setminus \mathrm{Rat}(f)$ that belongs to the closure $\overline{R}_i$ of R_i in T for some i. Let $S \subset \overline{R}_i$ be a curve passing through p and with generic point in R_i. Let $\tilde{S} \to S$ be the normalization of S and fix a point $0 \in \tilde{S}$ in the pre-image of p. Let then $\tilde{T} \subset \tilde{S}$ be an open neighborhood of 0 such that $\tilde{T} \setminus \{0\}$ maps into R_i. By taking the base change

$$\tilde{f} \colon \tilde{X} := X \times_T \tilde{T} \to \tilde{T},$$

we reduce to the setting of Theorem 3.1, which implies that $\tilde{X}_0$ is rational. Since $\tilde{X}_0 \cong X_p$, this contradicts the fact that $p \notin \mathrm{Rat}(f)$. $\qquad\square$

Proof of Corollary 1.3 In the hypothesis of Question 1.1, assume that f has relative dimension 3. Suppose that $\mathrm{Rat}(f) \neq \emptyset$. Then $\mathrm{SRC}(f)$ is nonempty, and thus it is equal to T by Proposition 2.6. Therefore the corollary reduces to a special case of Theorem 3.1. $\qquad\square$

Example 3.4 Consider the projection $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^1$. Fix a point $0 \in \mathbb{P}^1$, and let W be a smooth hypersurface of degree n in the fiber $\{0\} \times \mathbb{P}^n$. By de Fernex [3], W is nonrational if $n \geq 4$ and the ground field has characteristic zero. Although W might be *stably rational* (which would mean that $W \times \mathbb{P}^m$ is rational for some m), it is quite possible that $W \times \mathbb{P}^1$ is nonrational. In fact, it is conceivable (and possibly expected) that W is not even unirational if it is general, and this would certainly imply that $W \times \mathbb{P}^1$ is not rational. Now, let $\mathcal{L} = O_{\mathbb{P}^1}(2) \boxtimes O_{\mathbb{P}^n}(n)$ and $\mathcal{I}_W$ be the ideal sheaf of W in $\mathbb{P}^1 \times \mathbb{P}^n$. The sheaf $\mathcal{L} \otimes \mathcal{I}_W$ is globally generated, and thus the linear system $|\mathcal{L} \otimes \mathcal{I}_W|$ defines a rational map

$$\psi \colon \mathbb{P}^1 \times \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}H^0(\mathcal{L} \otimes \mathcal{I}_W)$$

which is resolved by the blow-up $Y = \mathrm{Bl}_W(\mathbb{P}^1 \times \mathbb{P}^n)$ of $\mathcal{I}_W$. Here X denotes the closure of the image of the map. The map ψ is defined over $\mathbb{P}^1$, and thus there is a morphism $f \colon X \to \mathbb{P}^1$. Furthermore, ψ induced an isomorphism away from the fibers over 0, so that $X_t \cong \mathbb{P}^n$ for $t \neq 0$. On the other hand the fiber X_0 is birational to $W \times \mathbb{P}^1$. Indeed, the induced morphism $\psi' \colon Y \to X$ contracts the proper transform of $\{0\} \times \mathbb{P}^n$ to a point and maps the exceptional divisor of the blow-up birationally to the fiber X_0, which is thus isomorphic to the projective cone in $\mathbb{P}^{n+1}$ over W. In particular, X_0 is not rational if $W \times \mathbb{P}^1$ is not rational; note however that X_0 is always separably rationally connected (for every W in characteristic zero, and for general W in positive characteristics by Zhu [20]). This example suggests that the analogous statement of Theorem 3.1 in higher dimensions may be false, possibly starting from dimension four.

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