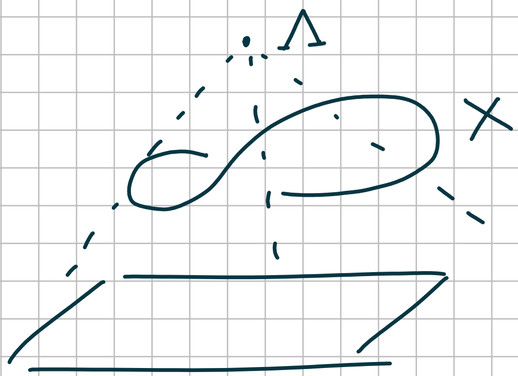
APPARENT BOUNDARIES

Joint with Anand Patel, Eduard Duryev.
A story about the extrinsic geometry of projective varieties.

Setup: Fix



$$X^r \subset \mathbb{P}^n_{\mathbb{C}}$$

↳ smooth, proj
dim r

Consider $\Lambda \subset \mathbb{P}^n$ linear subspace of codim $(r+1)$
dim $= n-r-1$

We have the projection map

$$\pi_\Lambda : X \dashrightarrow \mathbb{P}^r$$

If Λ is generic, then π_Λ is regular and finite, and we can associate to it

$$R_\Lambda \subset X$$

the ramification divisor of π_Λ.

This divisor is called the "apparent boundary" of X from the P.O.V. of Λ.

SHOW PICTURE / PROP.

Riemann-Hurwitz $\Rightarrow$

$$R_\Lambda \sim K_X + (r+1)H$$

The story is about the map:

$$\begin{array}{ccc} \rho : Gr(n-r, n+1) & \longrightarrow & |K_X + (r+1)H| \\ \cup & & \cup \\ \Lambda & \longmapsto & R_\Lambda \end{array}$$

- Flenner-Manaresi (1998, 2004)
- Cilberto
- Zak

Prop: $\dim Gr(n-r, n+1) \leq \dim |K_X + (r+1)H|$ with equality if and only if $X \subset \mathbb{P}^n$ is a variety of minimal degree.

(That is, $\deg X = n-r+1$).

Guess: $\rho : Gr \to \mathbb{P}$.

is ① generically finite for all X

② also dominant for X of minimal degree.

Flenner-Manaresi proved ① in some cases.
Cilberto proved ① for surfaces in $\mathbb{P}^4$.
Zak mentions counterexamples.

Recall: $\rho : Gr \dashrightarrow \mathbb{P}$

when does ρ extend to a regular map?

i.e. when can one define a sensible R_Λ?

A: ↳ one can if

$$\pi_\Lambda : X \dashrightarrow \mathbb{P}^r$$

is dominant. (eqv. generic fiber dim 0)

Def.: $X \subset \mathbb{P}^n$ is incompressible if for all $\Lambda \in Gr(n-r, r+1)$, the projection $\pi_\Lambda : X \dashrightarrow \mathbb{P}^r$ is dominant.

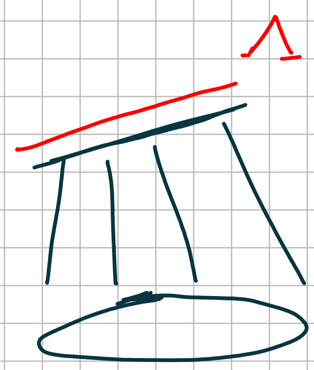
In this case $\rho : Gr \to \mathbb{P}$ is regular

$\Rightarrow \rho$ is finite.

Examples: ① Curves

② Hypersurfaces

Non example: $\mathbb{F}_1 \subset \mathbb{P}^4$
cubic surface scroll



Another class of varieties where we know ρ is generically finite.

$$\begin{array}{c} X \subset \mathbb{P}^n \\ \rightsquigarrow \\ X^* \subset \mathbb{P}^{n\vee} \\ \parallel \\ \{ H \subset \mathbb{P}^n \text{ such that } X \cap H \text{ is singular} \} \end{array}$$

Expected dim of $X^* = (n-1)$

& this holds if a generic $H \in X^*$ has the property that $X \cap H$ is singular only at a finite set of points.

<u>Thm</u> If $X^\vee$ is a hypersurface $\Rightarrow$

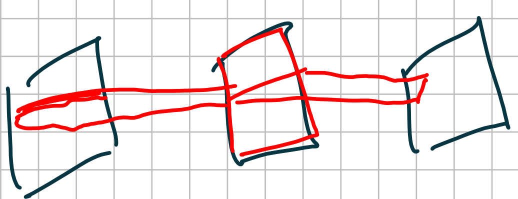
$\rho : Gr \dashrightarrow \mathbb{P}$ is generically finite.

<u>Examples</u>
- All curves and surfaces
- $X \subset \mathbb{P}^n$ embedded by sufficiently ample line bundle.

<u>Non-example</u>:

$r \geq 2$

$$\mathbb{P}^1 \times \mathbb{P}^r \hookrightarrow \mathbb{P}^{2r+1} \quad \text{segre}$$



Generic singular $X \cap H$ $\leftarrow$

Varieties of minimal degree

① Quadric hypersurfaces

② $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ Veronese

③ Scrolls

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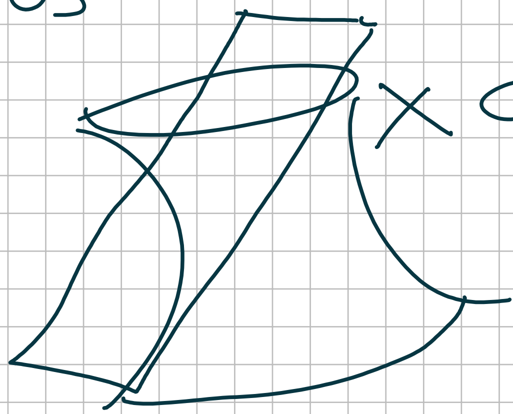
① Quadrics



$X \subset \mathbb{P}^n$

$$Gr = \mathbb{P}^n \overset{\rho}{\dashrightarrow} \mathbb{P}^{n*}$$

ρ is the duality induced by X.

$\deg(\rho) = 1$

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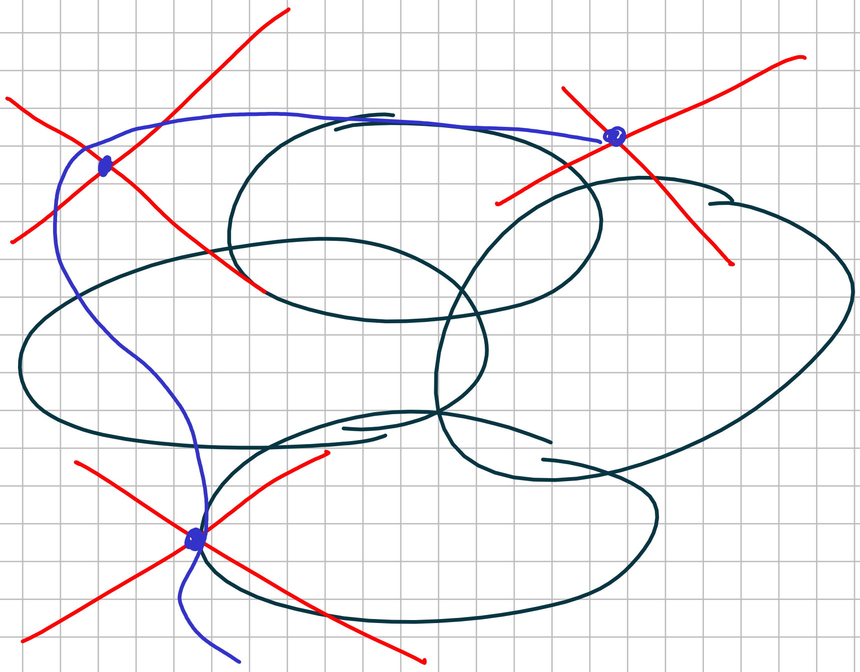
② Veronese surface

$$X = \mathbb{P}^2 \overset{|\mathcal{O}(2)|}{\hookrightarrow} \mathbb{P}^5$$

$$Gr = Gr(3, H^0(\mathcal{O}(2))) \dashrightarrow \mathbb{P}H^0(\mathcal{O}(3))$$

Net of conics $\dashrightarrow$ cubic



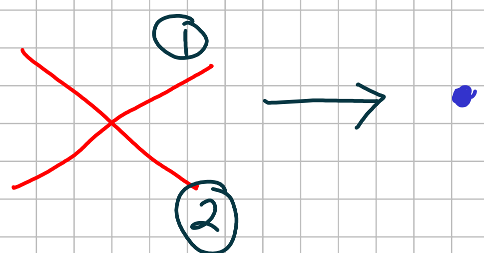
$$\begin{tikzcd} \text{Net } N \arrow[rr, bend left, "p"] & & \\ \cup & & C \\ \{\text{Singular}\} \text{ curve} \arrow[rr, "\text{singular point}"'] & & \text{Cubic Plane curve} \end{tikzcd}$$

What is $\deg(p)$?

Given C, how to reconstruct N ?

Need extra data

$$N \to C \;+\; \tilde{C} \xrightarrow{2:1} C$$



$N \leftrightarrow C$ + connected étale double cover

Given C, how many connected étale double covers? $\boxed{3}$

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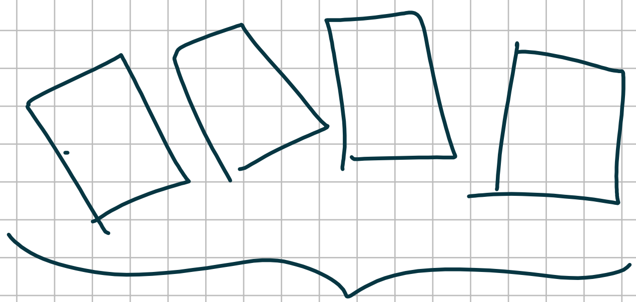
③ Scrolls

$X \subset \mathbb{P}^n$ embedded by $\mathcal{O}_{\mathbb{P}E}(1)$

$$\begin{array}{c} X \\ \parallel \\ \mathbb{P}E \end{array} , \quad E \to \mathbb{P}^1 \text{ ample vector bundle of rk } r$$



$\subset \mathbb{P}^n$

$$\underbrace{}_{X}$$

$\rho : Gr \dashrightarrow \mathbb{P}$

↳ same dim.

- ρ is not always dominant!

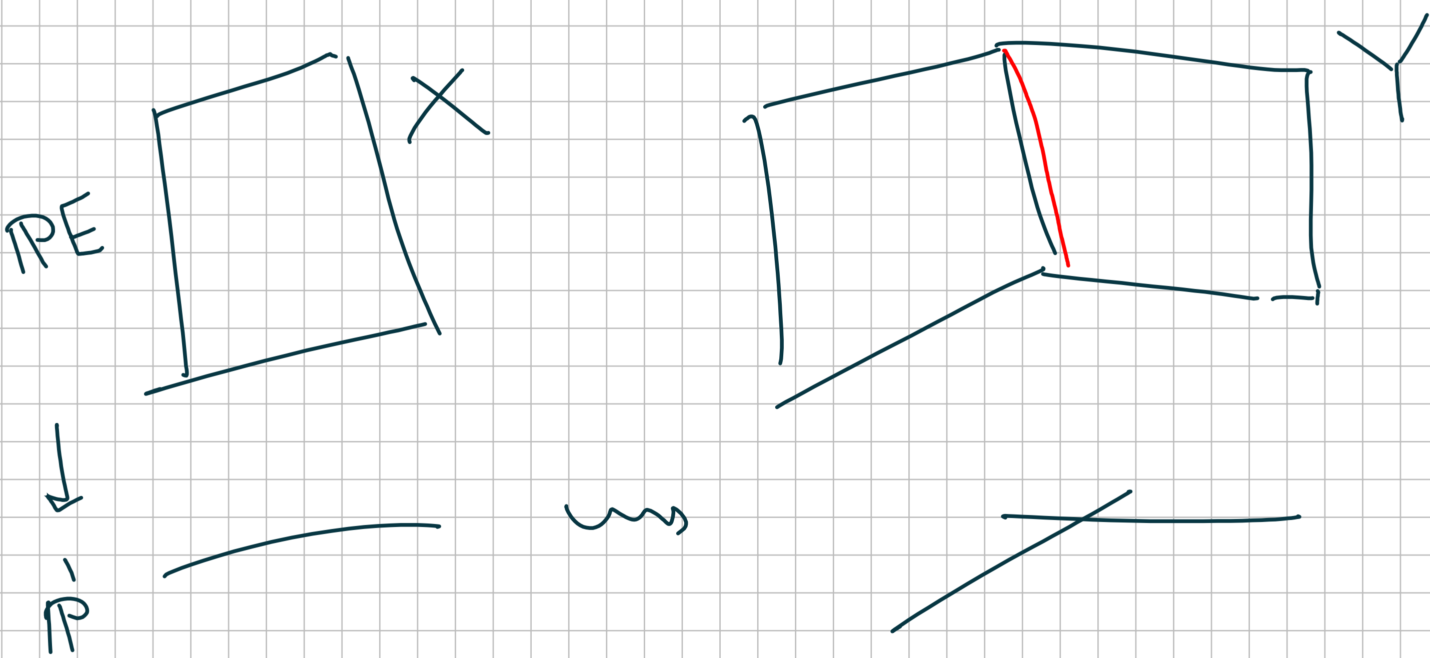
<u>Ex:</u> $E = \mathcal{O}(1)^{r-1} \oplus \mathcal{O}(2)$ for $r \geq 4$

Thm: Fix r.

Let E be the generic v.b. on $\mathbb{P}^1$ of rank r and degree $\geq (r-1)(2r-1)+1$.

Then ρ is dominant ($\Leftrightarrow$ generically finite) for $X = \mathbb{P}E$.

Sketch of pf.: Degeneration

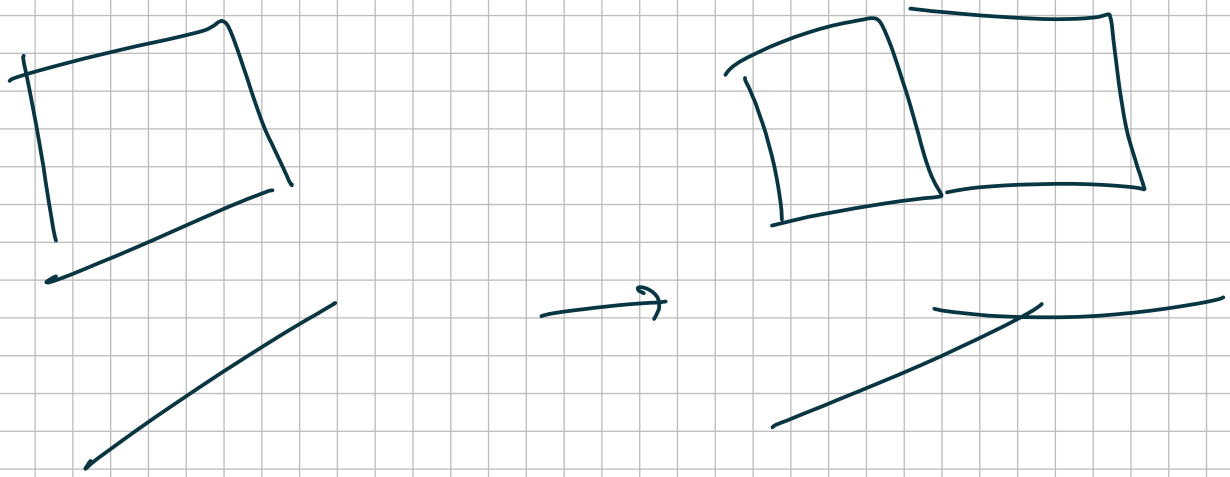


But ρ is never dominant for Y!

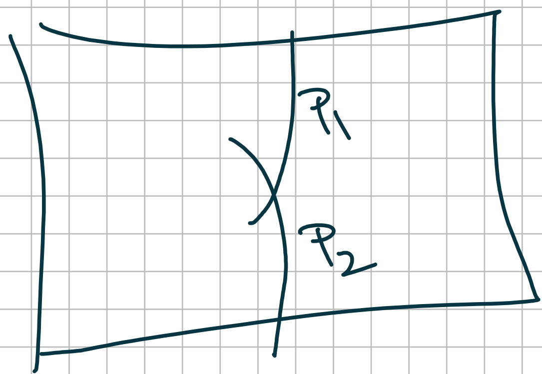
$$\rho : \left\{ \begin{matrix} \text{centers of} \\ \text{proj} \end{matrix} \right\} \dashrightarrow \left\{ \begin{matrix} \text{Ram.} \\ \text{divisor} \end{matrix} \right\}$$

Ram. div. always contains the spine!

Better degeneration



$$E \rightsquigarrow \underbrace{E_1 \vee E_2}_{\text{Not unique!}}$$



$\otimes \mathcal{O}(p_1)$

$E \rightsquigarrow E_1 \vee E_2$

$E \rightsquigarrow (E_1 \otimes \mathcal{O}(1)) \vee (E_2 \otimes \mathcal{O}(-1))$.

Section of $E \rightsquigarrow$ Section of all possible limits $E_1 \vee E_2$

Generic fiber : $Gr(n-r, H^0(E))$

(n-r) sections of E

$\rightsquigarrow$

Special fiber $Gr(\cancel{n-r}, E_1 \vee E_2)$

(n-r) sections of all possible limits.

$\hookrightarrow$ Space of limit linear series. ||

(for vector bundles).

B. Osserman, Teixidor-i-Bigas.

Better degeneration : $E \rightsquigarrow E_1 \vee E_2$

Generic fiber : $Gr \dashrightarrow \mathbb{P}$

Special fiber : Limit lin series of (n-r) sections of E_0 $\dashrightarrow$ Limit lin series of 1 section of $E_0 \otimes \det E_0 \otimes \omega$

Do some cases by hand

$(1,1,1,\dots,1)$ & $(2,2,\dots,2)$

& by degeneration as above prove:

ρ dominant for degree d

$\Rightarrow$ dominant for degree $d+(r-1)$

and for degree $d+(2r-1)$

Degrees:

$$E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$$

$\rightsquigarrow$ $N(a_1,\dots,a_r)$ = deg of ρ for $X = \mathbb{P}E$

$r=1$

a_1 \ N	1	2	3	4	5	6	7	8
	1	1	2	5	14	42	132	429

$\hookrightarrow$ Catalan numbers (Not too hard to prove)

$r=2$

$a_1 \backslash a_2$	1	2	3	4	5
1	$\boxed{1}$				
2	$\boxed{1}$	$\boxed{2}$			
3	1	$\boxed{6}$	$\boxed{22}$		
4	1	17	$\boxed{92}$	$\boxed{422}$	
5	1	43			
6	1	100			
7	1	220			

$1, 1, 2, 6, 22, 92, 422, \ldots$

↳ $\deg P$ for generic scrolls.

"A001181" = The number of Baxter permutations