Combinatorics and dynamics of Harder-Narasimhan filtrations

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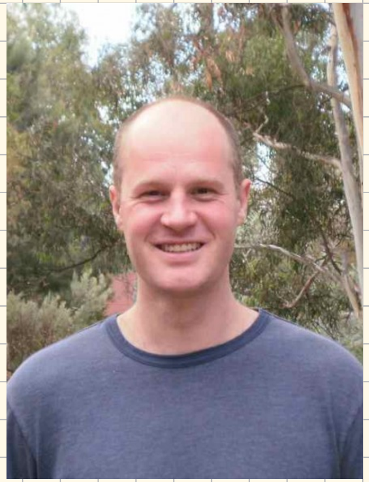
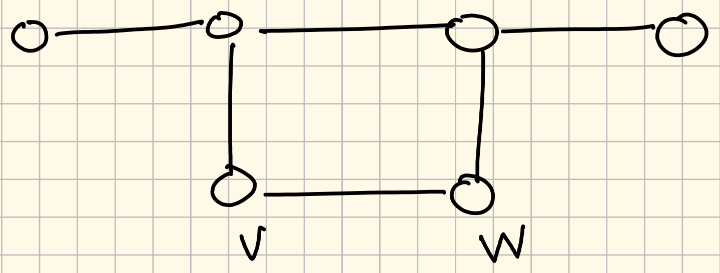


Artin-Tits braid groups

Γ 

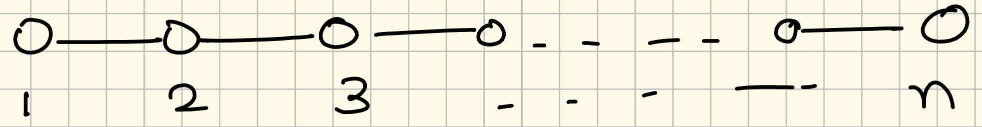
finite simple graph

$$\begin{aligned} B_\Gamma &= \text{Braid gp associated to } \Gamma \\ &= \langle \sigma_v \mid v \in \text{Vertex}(\Gamma) \rangle \;/\; \text{Relations} \end{aligned}$$

$\overset{v}{\circ}\!-\!\!-\!\!-\!\overset{w}{\circ} \quad \Rightarrow \quad \sigma_v \sigma_w \sigma_v = \sigma_w \sigma_v \sigma_w$ (Braid)

$\overset{v}{\circ} \quad \overset{w}{\circ} \quad \Rightarrow \quad \sigma_v \sigma_w = \sigma_w \sigma_v$

Example:



$=$ $(12)(23)(12)$

$= (23)(12)(23)$

$$\left\langle \sigma_i \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \leftarrow \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \leftarrow |i-j| \geq 2 \end{array} \right\rangle$$

"Usual" braid group on $(n+1)$ strands.

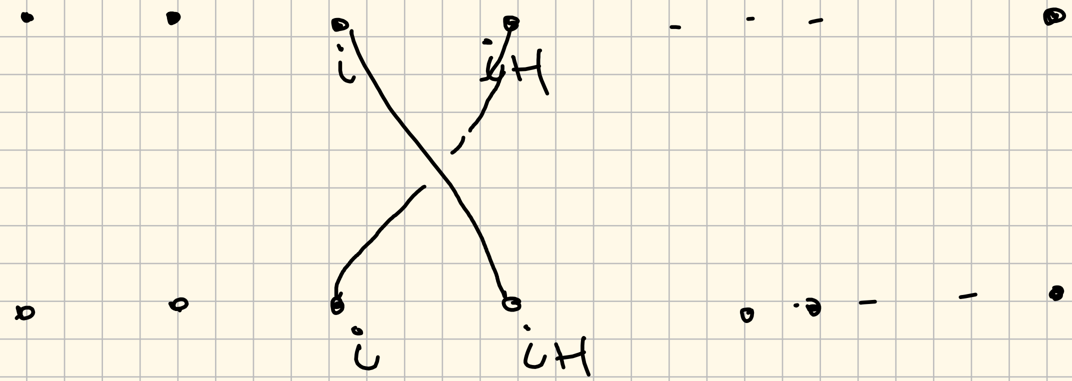
$$\downarrow$$

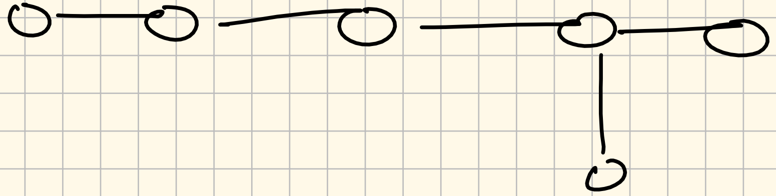
$$W_\Gamma \qquad \sigma_i^2 = 1$$

$$\cong$$

$$S_n \qquad \sigma_i = (i, i+1)$$

$\sigma_i =$





$\Gamma \rightsquigarrow B_\Gamma$

$$\langle \sigma_i \mid \text{Relations} \rangle = B_\Gamma \quad \text{Braid group}$$

$$\downarrow$$

Same gens + Relations + extra relations $\quad W_\Gamma \quad$ Coxeter group.

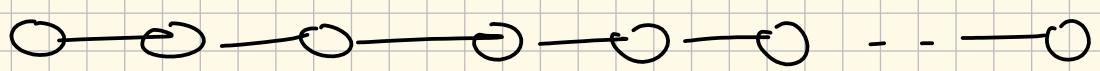
↳ $\sigma_i^2 = 1$

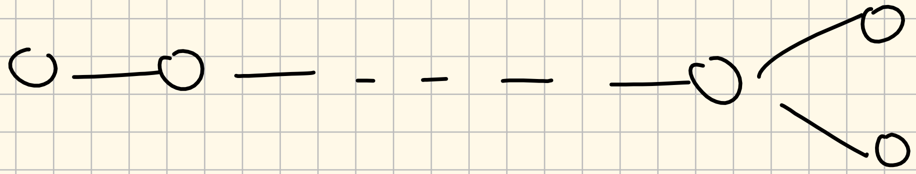
↳ Simpler

Well-understood.

<u>Ex.</u> <u>Thm</u>: W_Γ is finite <u>iff</u> Γ is an A-D-E dynkin diagram

W_Γ finite $\iff$


A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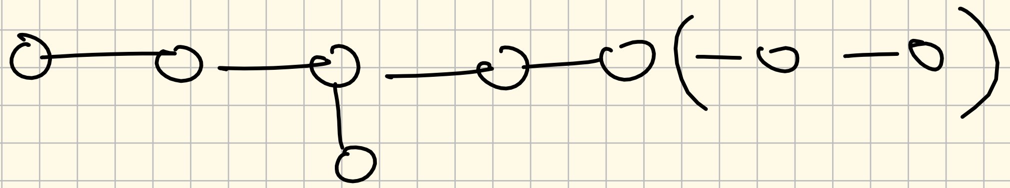

E.

B_Γ is always infinite.

$\hookrightarrow$ Not very well understood.

Ex. Rep. theory of B_Γ $\leftarrow$ Mystery

Q: Does B_Γ admit a faithful fin. dim. representation?

Does $B_\Gamma \subset GL_n$ for some n ??

Representations / actions of B_Γ

How does one study a group? $\mathbb{C}^n$ to somthing else.

$G \circlearrowright X \leftarrow$ object that you like/understand.

$\hookrightarrow$ faithful.

$G \subset \mathrm{Aut}(X) \to$ Tools to understand G.

$X = \mathbb{C}^n$, finite dim. v space.

$G \circlearrowright \mathbb{C}^n$ faithful $\rightsquigarrow G \subset GL_n$.

$G = B_\Gamma$ this proves difficult.

There is natural X on which G acts. (conj. faithful)

Spherical Objects and twists

$G \curvearrowright X$

$\downarrow$

Triang. Category

$\mathcal{C}$ = $\mathbb{C}$-linear triangulated category.

(finite type: for $X, Y \in \mathcal{C}$

the vector space $\bigoplus_n \mathrm{Hom}(X, Y[n])$ is fin. dim)

+ n-Calabi-Yau. i.e.

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X[n])^*$$

e.g. $\mathcal{C} = D^b\mathrm{Coh}(n\text{-CY manifold})$.

$\mathcal{C}$ = $\mathbb{C}$-linear <u>n</u>-Calabi-Yau category.

$X \in \mathcal{C}$ is called **spherical** if :-

$$\mathrm{Hom}(X, X[k]) = \left.\begin{cases} \mathbb{C}\ \langle \mathrm{id} \rangle & k=0 \\ 0 & \\ 0 & \\ \vdots & \\ 0 & \\ \mathbb{C} & k=n \end{cases}\right\} H^*(S^n;\mathbb{C})$$

The simplest possible End.

$\mathcal{C} \ni X \leftarrow$ spherical. $\qquad \underline{\mathrm{Hom}^*(X,Y)} = \bigoplus \mathrm{Hom}(X, Y[n])$

Then it gives an auto-equivalence

$$\boxed{\begin{aligned} \sigma_X : \mathcal{C} &\longrightarrow \mathcal{C} \\ Y &\longmapsto \underbrace{\sigma_X(Y)} \end{aligned}} \qquad X \xrightarrow{\sigma_X} X[1-n]$$

(Recall:

$$X \otimes_{\mathbb{C}} \mathrm{Hom}^*(X,Y) \xrightarrow{ev} Y$$

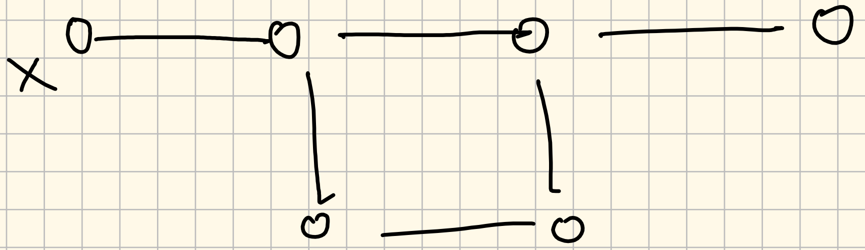
$$\sigma_X(Y) = \mathrm{Cone}(\underline{ev}) \quad)$$

From now on, $\mathcal{C}$ will be a 2-CY category

$n=2$.

$\sigma_X : X \longmapsto X[-1]$.

$Br_\Gamma \curvearrowright \mathcal{C}$

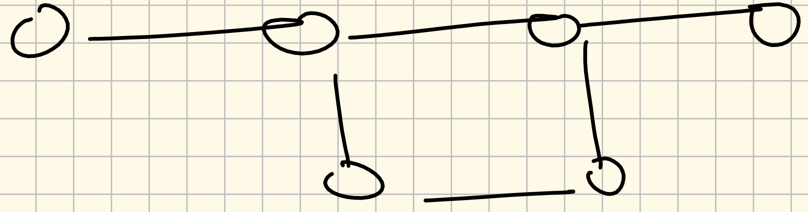


○ represents a spherical obj of $\mathcal{C}$.

$\mathrm{Hom}^*(X,Y) = 0$ dim

or 1 dim

Def: A Γ-configuration of spherical obj of $\mathcal{C}$ is a collection $\{X_v \mid v \in \mathrm{Vertex}(\Gamma)\}$ such that

$\Gamma =$ 

$$\mathrm{Hom}^*(X,Y) = 0 \quad \text{if } \underset{X}{\circ} \quad \underset{Y}{\circ}$$
$$= \mathbb{C} \quad \text{if } \underset{X}{\circ}\!-\!\!-\!\underset{Y}{\circ}$$

Prop: In this case σ_X, σ_Y satisfy the braid relations. (Khovanov, Seidel, Thomas, ...)

i.e.
$$\sigma_X \sigma_Y = \sigma_Y \sigma_X \quad \text{if } \overset{X}{\circ} \quad \overset{Y}{\circ}$$
$$\sigma_X \sigma_Y \sigma_X = \sigma_Y \sigma_X \sigma_Y \quad \text{if } \underset{X}{\circ}\!-\!\!-\!\underset{Y}{\circ}$$

Upshot: A Γ-config of sphericals in $\mathcal{C}$

$$\Downarrow$$

An action of $B_\Gamma \curvearrowright \mathcal{C}$

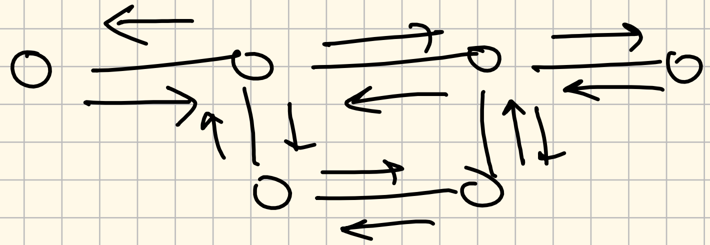
σ_v acts by σ_{X_v}

As a result, $\boxed{B_\Gamma \curvearrowright \mathcal{C} \leftarrow \text{triang. category} \quad \curvearrowright \text{ Everywhere in nature.}^{*}}$

* If you can find a Γ-config.

For every Γ, it's possible to construct $\mathcal{C} = \mathcal{C}_\Gamma$ in which we see a Γ-config of sphericals.

Here's how:



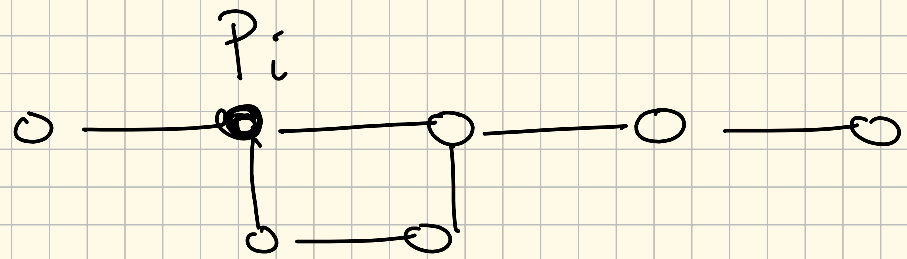
Γ

$$\Gamma \rightsquigarrow Z(\Gamma) = \text{Zig-Zag algebra of } \Gamma$$
$$= \langle \text{Path algebra of } \Gamma^{dbl} \rangle / \text{Rel}$$

Rel := • Kill all paths of length 3

• 

$(j|i|j) = (j|k|j)$



grading $=$ path length.

$Z(\Gamma)$ is a graded algebra fin-dim.

$$= Z(\Gamma)_0 \oplus Z(\Gamma)_1 \oplus Z(\Gamma)_2$$

D^b(gr-mod in $Z(\Gamma)$)

$P_i = Z(\Gamma) \cdot (i)$

$Z(\Gamma) \cdot (i) = \langle$ all paths ending at $i \rangle$

P_i: Spherical.

P_i: Graded proj. module.

Form Γ-config.

$$\mathcal{C}_\Gamma = \langle P_i \rangle \subset D^b(\text{gr-mod}).$$

$\mathcal{C}_\Gamma = \langle P_i \rangle$ P_i are spherical

& form a Γ-configuration.

Simplest / smallest $\mathcal{C}$ in which you see a Γ-config. of sphericals.

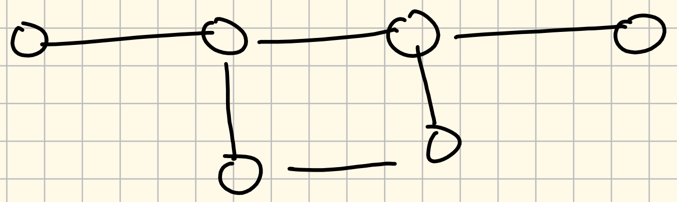
By construction

$$B_\Gamma \circlearrowright \mathcal{C}_\Gamma$$

<u>Conj</u>: This is a faithful action.

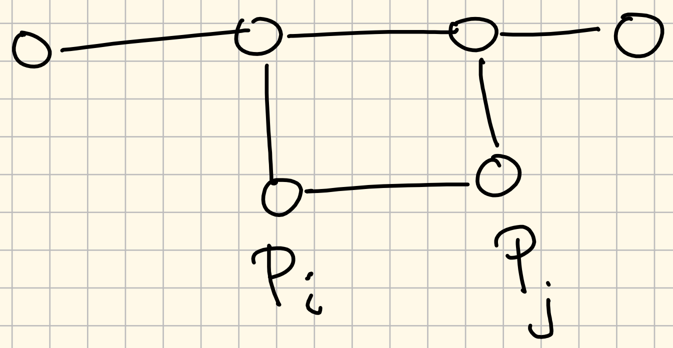
You should be able to understand B_Γ through its action on $\mathcal{C}_\Gamma$.

Γ - configuration of sphericals



Category C_T

Category $\underline{C_\Gamma}$



Example: $\Gamma = A_2$

$$\underset{1}{\circ} - \underset{2}{\circ}$$

$$C_\Gamma = \langle P_1, P_2 \rangle \circlearrowleft B_\Gamma = \langle \sigma_1, \sigma_2 \rangle$$

$$\mathrm{Hom}^*(P_i, P_i) = \overset{0}{\mathbb{C}} \oplus \overset{1}{0} \oplus \overset{2}{\mathbb{C}}$$

$$\mathrm{Hom}^*(P_i, P_j) = 0 \oplus \mathbb{C} \oplus 0 \qquad i \neq j$$

$$\mathrm{Hom}^1(P_1, P_2) \Rightarrow \exists \quad P_2 \to X \to P_1 \xrightarrow{+1} \qquad X = \sigma_1(P_2)$$

$$\mathrm{Hom}^1(P_2, P_1) \Rightarrow \exists \quad P_1 \to Y \to P_2 \xrightarrow{+1} \qquad Y = \sigma_2(P_1)$$

$$G = B_\Gamma \curvearrowright C_\Gamma = \mathcal{C}$$

How do you study this?

$X \subset \mathcal{C}_\Gamma$

$g \in G$

Look at $g(X)$

"measure the size"

or $g^2(X)$

How <u>complicated</u> does this get?

$g^3(X)$

$\vdots$

Stability Conditions

$\mathcal{C}$ = triangulated category

Stab cond = Slicing + central charge.

(Semi)-Stable objects.

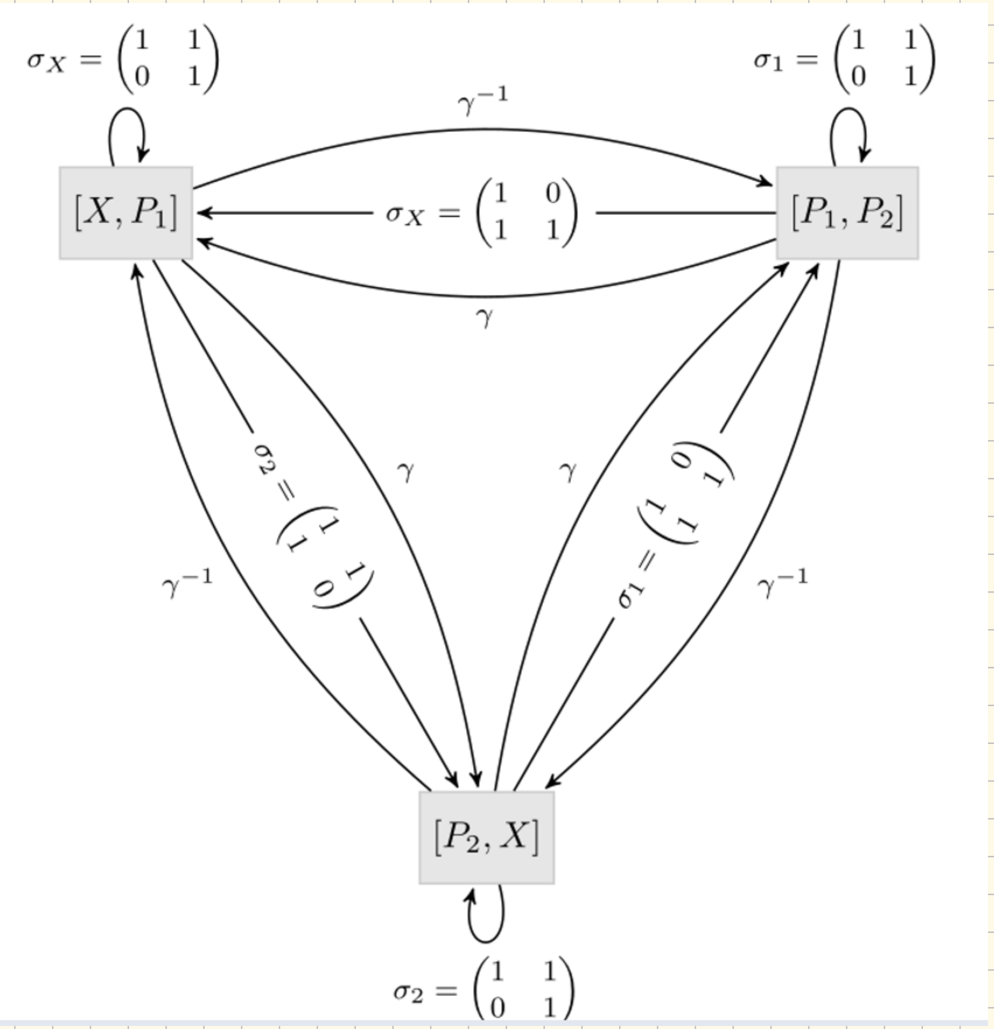
any object $X \underset{\text{unique}}{\rightsquigarrow}$ semi-stable factors.

$$\begin{tikzcd}
X \arrow[r, squiggly] \arrow[d] & g(X) \arrow[r, squiggly] \arrow[d] & g^2(X) \arrow[r, squiggly] \arrow[d] & g^3(X) \arrow[r, squiggly] & {} \\
\text{HN factors} \arrow[r, squiggly] & \text{HN factors} \arrow[r, squiggly] & \text{HN factors} & &
\end{tikzcd}$$

$\hookrightarrow$ understand how this evolves.

Evolution of HN filtrations

HN- Automaton



HN filtration of

$\underbrace{\beta(P_i)}$ for any β.

Stab. cond.

(semi)stables are P_1, P_2, X

HN filtration of a spherical involves only 2 of 3.

Any braid has an expression

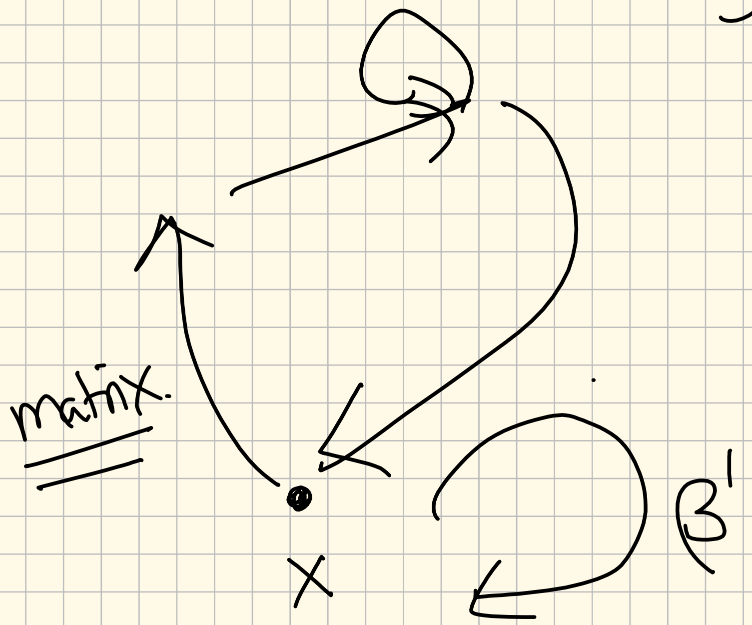
$\beta = \sigma_1 \sigma_2 \sigma_X \cdot \gamma \cdots \sigma_2$

↳ "Recognized by this Δ"

More: Every β is conjugate to β'

$\beta' =$ expression . . . . .

loop in the Δ



$(\beta')^n X$ ← easy.!

$HN(\beta' X) = \text{Matrix} \cdot HN(X).$

Cor: Entropy. (ie growth rate)

↪ Eigenvalues of this matrix.

In A_2 case (& other rank 2 categories
$\hat{A}_1$ & also some non-simply laced).

Edmund Heng

There is a linear automaton & "groupoid" that controls the growth of HN mults.

For higher rank : goal ← in progress.

Can do : Piecewise linear.

$\Gamma = A_n$: A geometric dictionary

$T = A_n$: Configurations & Stability conditions

$\Gamma = A_n$: Supports of Spherical Objects

$\Gamma = A_n$: The sphere of Spherical Objects.