Stability conditions bring triangulated categories into the realm of geometry in more than one way. We recall stability conditions and explain a few ways of thinking about them in geometric terms, particularly for the 2-Calabi–Yau (CY) categories associated to quivers defined in the lectures of Anthony Licata (see [2, §2.3]).

Let $\mathcal{C}$ be a triangulated category. A *(Bridgeland) stability condition* on $\mathcal{C}$ consists of a *central charge*, namely a $\mathbf{Z}$-linear map $Z\colon K(\mathcal{C}) \to \mathbf{C}$, and a *slicing*, namely a collection of abelian subcategories $P_\phi \subset \mathcal{C}$ indexed by real numbers ϕ, satisfying a number of conditions (see [3]). The objects of P_ϕ are called semi-stable of phase ϕ. One of the conditions in the definition ensures that every object $x \in \mathcal{C}$ admits a unique filtration, called the *Harder–Narasimhan (HN) filtration*, whose factors are semi-stable and appear in the order of descending phase. A rough analogy is to think of an object of $\mathcal{C}$ as an audio wave, the semi-stable objects as waves of pure frequency, and the decomposition of an object into semi-stable factors as the decomposition of an audio wave into pure frequencies. (This analogy, however, ignores that the HN decomposition is ordered).

A stability condition gives multiple measures of complexity of objects and morphisms. Associated to an object x is its *spread*, namely the difference between the highest and the lowest phases of the semi-stable objects in its HN filtration. The spread is a measure of homological complexity of the object.

Also associated to an object x is its *mass*, defined as follows. The mass of a semi-stable object is the absolute value of its central charge; the mass of an arbitrary object is the sum of the masses of its semi-stable factors. The mass gives another measure of complexity of the object, which is of a different flavour than the spread. The mass also allows us to define a metric on the category by declaring the length of a morphism to be the mass of its cone. With this metric, we can think of a sequence of composable morphisms in the category as a path. The sequence given by a HN filtration turns out to be a *geodesic* path. For 2-CY categories of quivers, we prove that the metric thus induced by a stability condition in fact determines the stability condition ([1, §6.1]). So, we can think of stability conditions as particular kinds of metrics on the category. It will be fantastic to understand exactly which metrics arise from stability conditions.

Thinking of stability conditions as metrics allows us to transfer ideas from metric geometry to the study of triangulated categories. One such instance is a compactification of a stability manifold inspired by Thurston's compactification of Teichmüller space ([1]).

Let $\mathcal{C}$ be the 2-CY category associated to the A_n quiver. Then the

Grothendieck group $K(\mathcal{C})$ is the root system of type A_n, which we can take to be the span of the vectors $e_i - e_j$ in $\mathbf{R}^{n+1}$. In this case, there is a beautiful geometric description of stability conditions in terms of configuration of $n+1$ points on the complex plane. Fix such a configuration $\{x_0, \ldots, x_n\} \subset \mathbf{C}$. We define the central charge by $Z(e_i - e_j) = x_i - x_j$. To define the slicing, recall that Khovanov–Seidel give a recipe to construct objects of $\mathcal{C}$ from arcs joining two marked points [4, §4]. We declare the objects represented by the straight line segments to be semi-stable. This turns out to indeed give a stability condition (see [5]). Let x be an object represented by an arc γ. In this stability condition, the HN factors of x correspond to the straight line segment pieces of γ when it is "pulled tight" around the marked points. This geometric description of the HN factors implies many non-trivial properties of the structure of HN filtrations of objects in $\mathcal{C}$. A fundamental question is to understand these properties using pure homological algebra and generalise them to a broader class of categories.

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