Be wise equivariantise

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(1)

1) In a pencil of cubic plane curves, how many times do we see a given j-invariant? 12

2) In a $\mathbb{P}^4$ of cubic surfaces, how many times do we see a given cubic surface? 96120

3) Fix a cubic 3-fold $X \subset \mathbb{P}^4$. Among the hyperplane slices of X, how many times do we see a given cubic surface? 42120

(2)

1) $\{x_1F_1 + x_2F_2\}$ $\quad [x_1 : x_2] \in \mathbb{P}^1$

$[x_1 : x_2] \longmapsto$ iso class $V(x_1F_1 + x_2F_2)$

$\mu : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$?

$\mu^* : A^*(\overline{\mathcal{M}}_{1,1}) \to A^*(\mathbb{P}^1)$

Want $\mu^*([pt]) \in A^*(\mathbb{P}^1)$

2) $\{x_1F_1 + \dots + x_5F_5\}$

$[x_1 : \dots : x_5] \mapsto V(x_1F_1 + \dots + x_5F_5)$

$\cap$

$\mathbb{P}^4 \overset{\mu}{\dashrightarrow}$ moduli of cubic surfaces

$\mathrm{Sym}^3(\mathbb{C}^4) /\!/ GL(4)$

3) Also $\check{\mathbb{P}}^4 \dashrightarrow$

(3)

Remedy

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ \mathbb{P}^4 \arrow[rr, dashed] & & \mathrm{Sym}^3(\mathbb{C}^4) /\!/ GL(4) \end{tikzcd}$$

Good luck! (No one knows).

Alternative

$$\mathbb{P}^4 \xrightarrow{\mu} \left[\mathrm{Sym}^3(\mathbb{C}^4) / GL(4) \right]$$

$$A^*(\mathbb{P}^4) \xleftarrow{\mu^*} \boxed{A^*}$$

$$\text{Answer} \xleftarrow{\mu^*} \boxed{[\overline{pt}]}$$

with $[\overline{pt}] \in A^*$.

(4)

G algebraic group $\quad (GL(4))$

V fin dim G-rep $\quad (\mathrm{Sym}^3(\mathbb{C}^4))$

$\overline{\text{Point of } [V/G]} = \overline{G\text{-orbit in } V}$

$A^*([V/G]) = A^*_G(V)$

$[\overline{Pt}] = [\overline{G\text{-orbit}}]$

$\boxed{\text{Q: Given } v \in V, \text{ find } [\overline{Gv}] \in A^*_G(V).}$

Ans might depend on V.

Generic answer enough for our enum. questions.

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$G = GL(n)$

$A^*_G(V) = A^*_G(\cdot) = \mathbb{Z}[c_1, \dots, c_n]$

$\deg(c_i) = n$

<u>Theorem</u> (D., Patel, Tseng)

$G = GL(3)$ $\quad V = \mathrm{Sym}^3(\mathbb{C}^4)$

Then the $\overline{\text{orbit}}$ of a generic $v \in V$ has class

$$1080\,(24 c_1^4 + 12 c_1^2 c_2 - 6 c_1 c_3 + 9 c_4)$$

$\uparrow$

<u>Universal</u> Counting <u>formula</u>

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<u>Cor</u>: Let B be any proper 4 dim base and $\pi : \mathcal{X} \to B$ a family of cubic surfaces:

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & \subset \mathbb{P}V \arrow[d] \\ & B \end{tikzcd} \qquad \begin{array}{l} rk(V) = 4 \\ \mathcal{X} \in |\mathcal{O}_{\mathbb{P}V}(3)| \end{array}$$

Then* a generic cubic surface appears $1080(c_1(V)^4 + \cdots)$ times.
(this is the map $B \dashrightarrow$ moduli of cubic surfaces)

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A non-eq. version of main question.

$G = GL(n) \curvearrowright V$

$PGL(n) \curvearrowright \mathbb{P}V$.

<u>Q</u>: Given $[v] \in \mathbb{P}V$ what is degree $(\overline{PGL(n)[v]}) \subset \mathbb{P}V$?

$$V/GL(n) \leftarrow V^*/GL(n) \leftarrow V^*/\mathbb{G}_m = \mathbb{P}V$$

$$A^*_{GL(n)}(V) \longrightarrow A^*(\mathbb{P}V)$$

eqv. class $\longmapsto$ degree.

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<u>What is Known</u> ?

0) $G = \mathbb{G}_m^n$ any V.

1) $G = GL(2)$

$V = \mathrm{Sym}^n(\mathbb{C}^2)$ $\quad$ (n-pts on $\mathbb{P}^1$)

Generic orbit degree : Enriques–Fano (1890s)

All orbit degrees : Aluffi–Faber (1990s)

How does answer depend on v ?

v : $m_1 \quad m_2 \quad \cdots \quad m_\ell$

Answer depends on the multiplicities

1) continued ... $G = GL(2)$ (9)

$V = \mathrm{Sym}^n(\mathbb{C}^2)$, any $v \in V$

Eqv. class (Lee, Patel, Spink, Tseng)

any V, any $v \in V$

Eqv class (D.)

many interesting moduli spaces

$V = \mathrm{Sym}^4(\mathbb{C}^2) \oplus \mathrm{Sym}^6(\mathbb{C}^2)$: elliptic fibr.

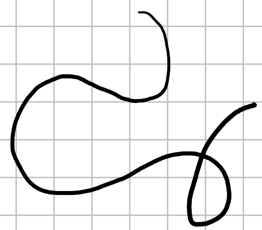
$V = \mathbb{C}^2 \otimes \mathrm{Sym}^n(\mathbb{C}^2)$: Rational self maps of deg n.

2) $G = GL(3)$ (10)

$V = \mathrm{Sym}^n(\mathbb{C}^3)$

degree : Aluffi-Faber

What does answer depend on?



Singularities

hyperflexes

Eqv. class in progress (D., Patel, Li)

(11)

3) $G = GL(4)$

$V = \mathrm{Sym}^3(\mathbb{C}^4)$ generic case

4) $G = GL(n)$

$V = \mathrm{Sym}^d(\mathbb{C}^n)$

Do not know the generic ans.

Know for hyp. arrangements (Lee, Patel, Spink, Tseng)

Answer depends on the matroid of the arrangement

(12)

How does one find the answer?

① Hard work.

Find a proper param. space for the orbit closure

$f : X \to V$ proper, image $= \overline{\text{orbit}}$

Compute $f_*[X]$

② Tricks (e.g. interpolation)

$$\text{count of cubic surfaces} = \underline{\underline{A}}\, c_1^4 + \underline{\underline{B}} \cdot c_1^2 c_2 + \underline{\underline{C}} \cdot c_1 c_3 + \underline{\underline{D}} \cdot c_2^2 + \underline{\underline{E}} \cdot c_4$$

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Find families where you know both sides.

Extra :

Recall main question : given $p : \mathrm{Spec}\, \mathbb{C} \to [V/G] = \mathcal{M}$
find $[\bar{p}] \in A^*(\mathcal{M})$.

Perfectly good question for any $\mathcal{M}$.
We want some "tautological" generators for $A^*(\mathcal{M})$
and an expression for $[\bar{p}]$ in terms of these generators.

Example : (Hannah Larsen).

$\mathcal{M} =$ Stack of vector bundles on $\mathbb{P}^1$

p = Splitting type. Larsen describes $[\bar{p}]$ in terms of certain tautological classes.