The Enumerative Geometry of projection & ramification ①

Joint with A. Patel and E. Duryev.

The goal of the talk is to introduce a natural question ~~that~~ whose answer has been known ~~to~~ in dimension 1 for centuries and whose answer in dimension 2 and above is almost entirely unknown.

The reason I want to talk about it now is because Anand P. recently (thanks to Google's computers) obtained a wealth of computational evidence in higher dimension. This has confirmed by belief that there is something interesting happening in higher dim, which is beyond our current tools. ~~for~~

Dimension 1 story

$f(x) = \dfrac{\text{polynomial}}{\text{polynomial}}$ $\deg(f) := \max(\deg(\text{num}), \deg(\text{denom})) = n.$

Consider $\{x \mid f'(x) = 0\}$. $\longleftarrow$ $(2n-2)$ critical pts.

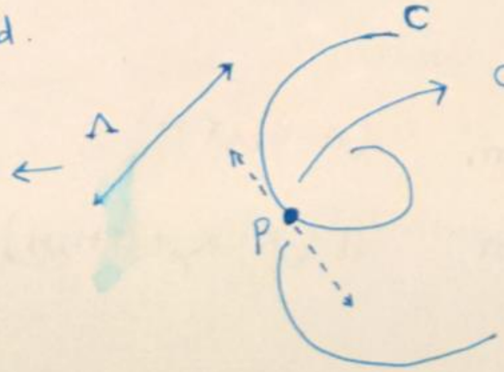
$f(x) = \dfrac{p(x)}{q(x)}$; $f'(x) = \dfrac{qp' - pq'}{q^2}$ $\leftarrow \deg (2n-2)$

Thm (Castelnuovo, Late 1800s): Given $(2n-2)$ general points, there are $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ rational $f(x)$ of deg n with these critical pts (up to mobius transf).

Pf: Homogenise :- $f : \mathbb{P}^1 \to \mathbb{P}^1$ deg n.

$$\underbrace{\begin{array}{c}\mathbb{P}^1 \hookrightarrow \mathbb{P}^n \\ [x:y] \mapsto [x^n : x^{n-1}y : \dots : y^n]\end{array}}_{\text{fixed}} \quad \text{followed by} \quad \underbrace{\text{a projection } \mathbb{P}^n \dashrightarrow \mathbb{P}^1}_{\text{Variable}}$$

Center of proj. (codim 2)



critical iff $T_pC \cap \Lambda \neq \emptyset$.

Given $(2n-2)$ pts $P_1, \dots, P_{2n-2}$ we want ②

$\# \{ \Lambda \mid \Lambda \text{ meets all } T_{P_i} C \}$ $\Lambda \in Gr(n-1, n+1)$

$\# \{ \Lambda \text{ meeting } 2n-2 \text{ lines in } \mathbb{P}^n \}$

$= [\Lambda \text{ meeting one line}]^{(2n-2)}$ in $H^*(Gr(n-1, n+1))$

Schubert Calculus: $\Rightarrow$ $=$ # Young Tableaux of shape $2 \times n$.

$\square$

Generalisation: Fix $X \subset \mathbb{P}^n$ smooth proj irred of dim r. non deg. embedded.

Choose $\Lambda \subset \mathbb{P}^n$ codim $(r+1)$ subspace (variable).

$\Lambda \rightsquigarrow \pi_\Lambda : X \to \mathbb{P}^r \rightsquigarrow \mathrm{Ram}(\pi_\Lambda)$

$= \{ x \in X \mid d\pi_\Lambda \text{ is degenerate} \}$.

$\mathrm{Ram}(\pi_\Lambda)$ is a divisor of class $K_X + (r+1)H$.

$$\begin{array}{ccc} \Lambda & \rightsquigarrow & \mathrm{Ram}(\pi_\Lambda) \\ \cap & & \cap \\ Gr(n-r, n+1) & \overset{\rho}{\dashrightarrow} & \mathbb{P} H^0(X, K_X + (r+1)H). \end{array}$$

Prop: $\dim(\text{domain } \rho) \leq \dim(\text{codom } \rho)$ with equality iff $X \subset \mathbb{P}^n$ is a variety of minimal degree.

Expect: ① Image ρ has dim $= \dim Gr(n-r, n+1)$

② If X is of min deg. then ρ is dominant & gen. finite.

Q: If X is of min deg, what is $\deg(\rho)$?

Thm ① is true if $X \subset \mathbb{P}^n$ is suff. positively embedded. (can make much more precise statements).

[~~Known partially~~ (Flenner, Manaresi, Ciliberto, –, Patel, Duryev)

② is not always true., but is true generically. (–, Patel, Duryev).

The degrees are completely mysterious, beyond $\dim X = 1$.

Varieties of minimal degree

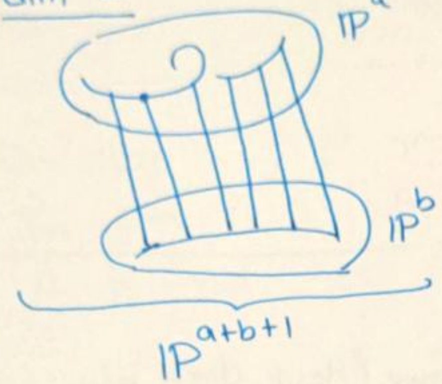
$X \subset \mathbb{P}^n$ X sm. proj var non. deg. embedded. $\dim X = r$

Then $\deg(X) \geq n-r+1$. Equality holds iff

1. $X = \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ by $\mathcal{O}(n)$ (rational normal curve)
2. $X = \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by $\mathcal{O}(2)$ (veronese $\mathbb{P}^2$)
3. $X =$ Quadric hypersurface in $\mathbb{P}^n$
4. X is a scroll.

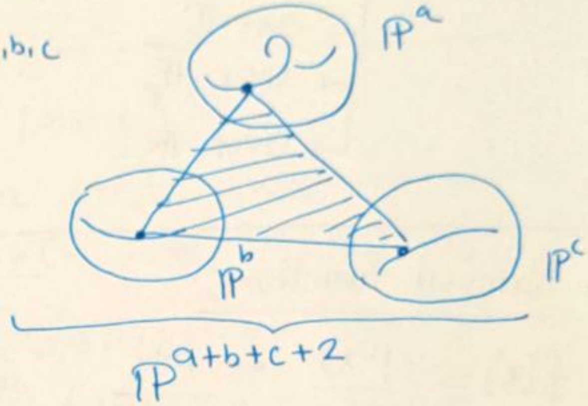
Scrolls: dim 2

$F_{a,b}$



dim 3

$F_{a,b,c}$



$F_{a_1,\dots,a_r} \subset \mathbb{P}^{\sum a_i + r - 1}$ degree $= \sum a_i$

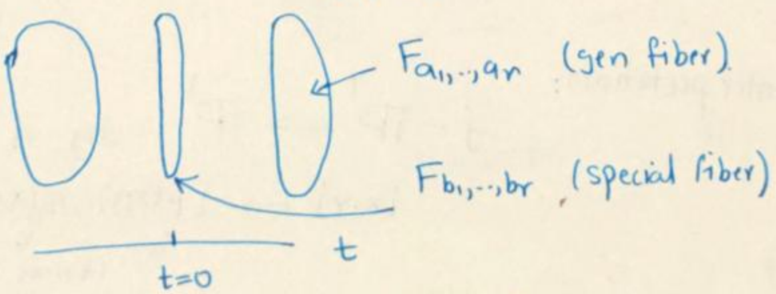
dim $= r$.

Given degree d, dim r: finite number of scrolls $\leftrightarrow$ Partition of d in r parts.

Partially ordered.

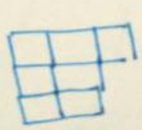
$F_{a_1,\dots,a_r} > F_{b_1,\dots,b_r}$ if $F_{a_1,\dots,a_r}$ specialises to $F_{b_1,\dots,b_r}$. $\Leftrightarrow$ Dominance order

i.e. if $\exists$



Ex  $F_{2,2} \rightsquigarrow F_{3,1}$.

Given d, r, the order has a largest elt ("most generic scroll") $=$ most balanced partition.

Ex: $d=7$, $r=3$. 

~~Thm~~

Back to ρ for varieties of min degree.

X	ρ gen. finite ?	degree of ρ
① $\mathbb{P}^1 \subset \mathbb{P}^n$	YES	C_n
② $\mathbb{P}^2 \subset \mathbb{P}^5$	YES	3
③ Quadric $\subset \mathbb{P}^n$	YES	1
④ Scrolls	It's complicated	Even more complicated.

Scrolls: In dim 2, ρ gen finite. (always).
In dim 4 & above, there exist scrolls for which ρ is not gen. finite. BUT these are very unbalanced.

Thm: In all dim, if degree $\geq (r-1)(2r-1)+1$ then ρ is gen finite for the generic scrolls (balanced).

The degree of ρ for Scrolls.

Surfaces $F_{a,b}$. degree $= a+b$

Generic scrolls: (most balanced).

$d =$	2	3	4	5	6	7	8	9	10	11	12
$\deg\rho =$	1	1	2	6	22	~~422~~ 92	~~2074~~ 422	~~10754~~ 2074	10754	58202	326240

$d=13$	14
1882960	11140560

OEIS: A001181
"Baxter permutations".

most unbalanced: $(1, d-1)$ all have $\deg\rho = 1$ (easy to prove).

Next unbalanced: $(2, d-2)$.

$d=4$	5	6	7	8	9	10
2	6	17	43	100	220	467

OEIS: A014833 $\quad 2^{d-1} - \frac{d(d-1)}{2}$

more surprises in the full table
& this is just dim $= 2$!