Limits of plane quintics via covers of stacky curves

Q: Let

$Q_d = \{$ Smooth plane curves of deg $d \}$

Describe $\overline{Q}_d \subset \overline{M}_g$

$g = \binom{d-1}{2}$

d	g	$\overline{Q}_d$
1	0	–
2	0	–
3	1	Everything
4	3	Everything
5	6	?

Hassett : KSBA compact of $(\mathbb{P}^2, C)$

$d = 4 \Rightarrow$ iso to $\overline{M}_3$

$d \geq 5$ many comp.

Hacking : $(\mathbb{P}^2, \varepsilon C)$ small ε

All $d \Rightarrow$ Nice space

No map to $\overline{M}_g$

Our idea . a la Abr. Vistoli.

 $\leftarrow$  $\leftrightarrow$ Map to $\widetilde{M}_{0,d-1}$

① Compactify space of maps nicely

② Take image in $\overline{M}_g$.

Spaces of Maps to X

• X proj scheme

$M_g(X) = \{ \varphi : C_g \to X \}$ (sm)

$\overline{M}_g(X) = \{ \varphi : C_g \to X \}$ (nodal)

• X a proper DM stack

$M_g(X) = \{ \varphi : C_g \to X \}$ (sm)

$\overline{M}_g(X) = \{ \varphi : C_g \to X \}$ (orbi-nodal)

 $\leftarrow (\mathbb{C}[x,y]/xy) / \mu_n$

$\zeta(x,y) = (\zeta x, \zeta^{-1} y)$

• Problems

→ Has many components

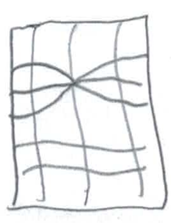
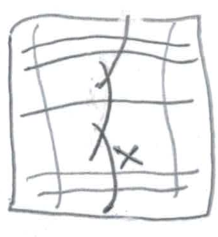
→ Hard to identify just the closure of the good one.

X a curve : Admissible Covers

$H_g^d(X) = \{ \varphi : C_g \xrightarrow{d} X$ simp. br $\}$

$\overline{H}_g^d(X) = \{ \varphi : C \to Z$ simp. br

$Z \to X$ "bubbling" $\}$

 $\leftarrow$ 

Thm: X a smooth stacky curve.

$\overline{H}^d_g(X)$ = Space of adm covers of X

$$\{\varphi : C \to Z \text{ simp br}, \; Z \to X \text{ bubbling}\}$$

is a smooth, proper DM stack with

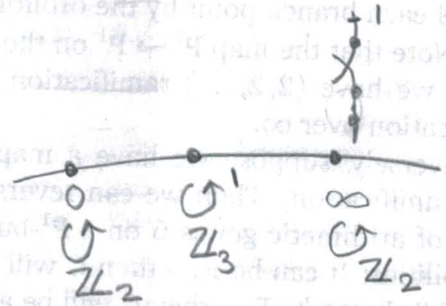
$\overline{H}^d_g(X) \setminus H^d_g(X)$ normal crossings

Admits a morphism to $\overline{M}_g(X)$.

$$X = \overline{M}_{0,4}/S_4 =: \widetilde{M}_{0,4}$$

Gen stab $K = \{\mathrm{id}, (12)(34), \dots\}$

$$X' = \overline{M}_{0,4}/S_3 =: \widetilde{M}_{0,1+3}$$



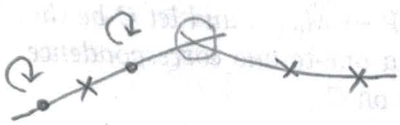
$\overline{H}^d_g(X)$:

- X orbifold

- Generic

$\times$ = br pt

- Codim 1



or

- Etc

Quintics

$\longrightarrow P \to X$

$P = \mathbb{P}^1(\sqrt{br})$

$4\sigma + 5F \subset \mathbb{F}_1$

18 br pts

$$\left.\begin{array}{l} \text{① } 4\sigma+5F \text{ on } \mathbb{F}_1 \\ \text{② } (4,3) \text{ on } \mathbb{F}_0 \\ \text{③ } \sigma \sqcup 3\sigma+6F \text{ on } \mathbb{F}_2 \end{array}\right\} \longleftarrow P \to X$$

- X has gen stab

$X \to X'$ ← orbifold

$$\overline{H}^d_g(X) = \{\varphi : C \to Z', \; Z = Z' \times_{X'} X, \; Z' \to X' \text{ as above}\}$$

$$\downarrow$$

$$\overline{H}^d_g(X')$$

$\overline{Q}$ = Adm cov. comp of $P \to \widetilde{M}_{0,4}$

$$= \overline{Q}^{odd} \sqcup \overline{Q}^{even} \sqcup \overline{Q}^{zero}$$

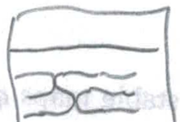
$$\downarrow$$

$\overline{T}$ = Adm cov. comp of $P \to \widetilde{M}_{0,1+3}$ (irred)

3 components

$v \in \overline{T}$ gen pt.

$C_v \subset \mathbb{F}_2$ trig curve



$2F$ a theta char

Thm: we have a bij

$$\left\{ \text{2-torsion line bundles on } C_v \right\} \updownarrow \left\{ \text{Pts of } \overline{Q} \text{ over } v \right\}$$

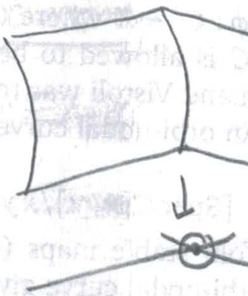
$Q^{\text{zero}} \leftrightarrow 0$

odd $\leftrightarrow$ odd Θchar

even $\leftrightarrow$ even Θ-char

Boundary of plane quintics

- Explicitly analyze the boundary
- Curves in degenerate scrolls

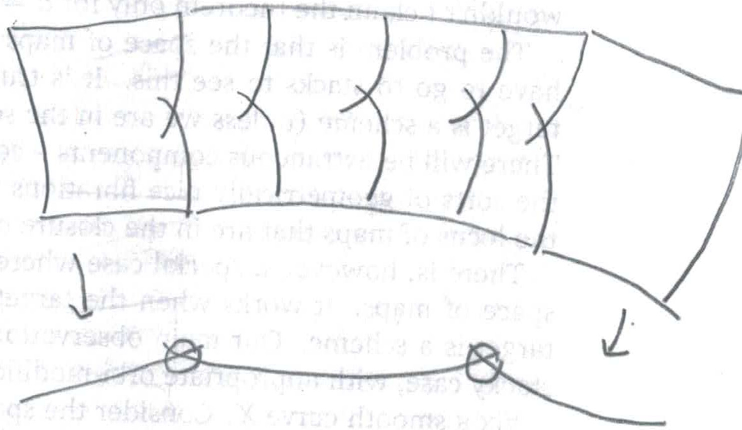


example

$$\begin{array}{ccc}
\boxed{\text{maps to } \overline{M}_{0,4}} & \longrightarrow & \boxed{\text{Maps to } \overline{M}_{0,1+3}} \\
\| & & \| \\
\boxed{\text{Tetr. curves in } \mathbb{F}_{odd}} & \searrow & \\
 & & \boxed{\text{Trig curves of highest Maroni}} \\
\boxed{\text{Tetr curves in } \mathbb{F}_{even}} & \nearrow &
\end{array}$$

Odd / even Θ-char

Recillas corresp



example