The Geometric Steinitz Problem

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Discriminants

$$\mathfrak{p}_1^{e_1>1} \qquad \mathfrak{p}_1$$
$$\vdots \qquad \mathfrak{p}_2$$
$$\qquad \vdots$$

$$\begin{tikzcd}
L \arrow[r, phantom, "\supset"] & \mathcal{O}_L & \\
K \arrow[u] \arrow[r, phantom, "\supset"] & \mathcal{O}_K \arrow[u] \arrow[r, phantom, "\supset"] & \langle D \rangle
\end{tikzcd}$$

$p \mid D \qquad p \nmid D$

Example:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt{-3}) \arrow[r, phantom, "\supset"] & \mathbb{Z}(\zeta_3) & \\
\mathbb{Q} \arrow[u] \arrow[r, phantom, "\supset"] & \mathbb{Z} \arrow[u] \arrow[r, phantom, "\supset"] & (3)
\end{tikzcd}$$

$(\sqrt{-3})^2 \qquad (4+\sqrt{-3})(4-\sqrt{-3})$

$3 \qquad 19$

DISCRIMINANTS

$L/\mathbb{Q} \rightsquigarrow$ Discriminant $(D_{L/\mathbb{Q}}) \subset \mathbb{Z}$

MOTIVATING QUESTIONS

1. Which numbers are discriminants?
2. How are discriminants distributed?

DISCRIMINANTS - Which numbers?

① Quadratic -

3, 4, 5, 7, 8, 8, 11, 12, 13, 15, 17, 19, 20, ...

② Cubic -

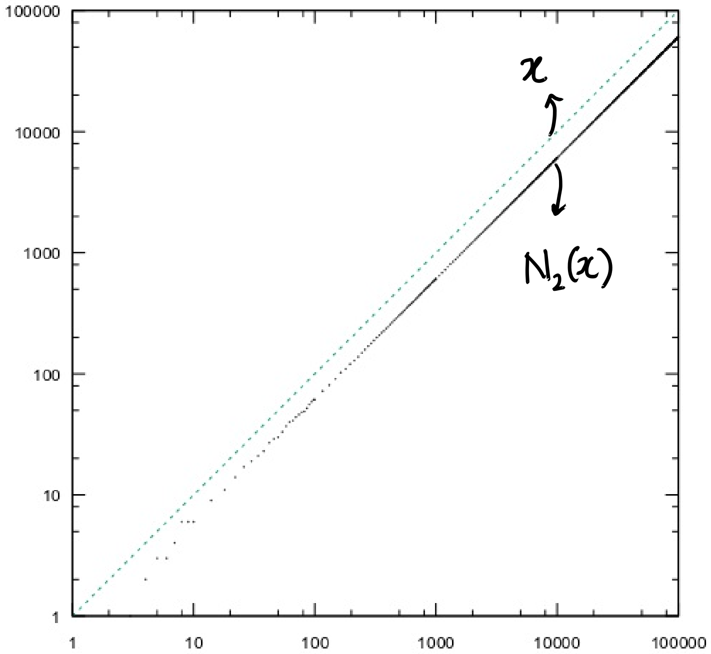
23, 31, 44, 49, 59, 76, 81, 83, 87, 104, ...

③ Quartic -

117, 125, 144, 189, 225, 229, 256, ...

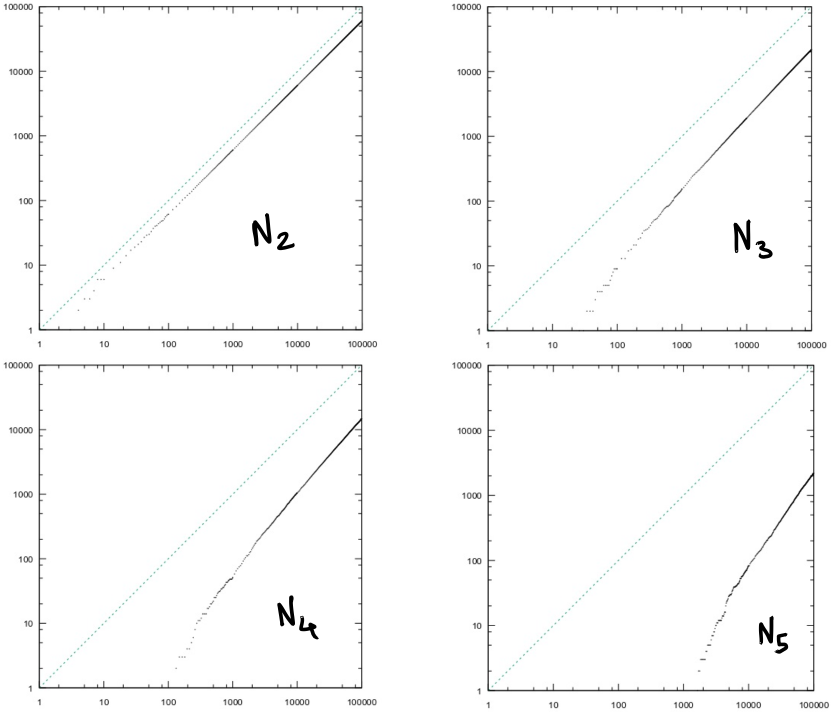
(Courtesy : PARI GP BORDEAUX)

Discriminants - Distribution.

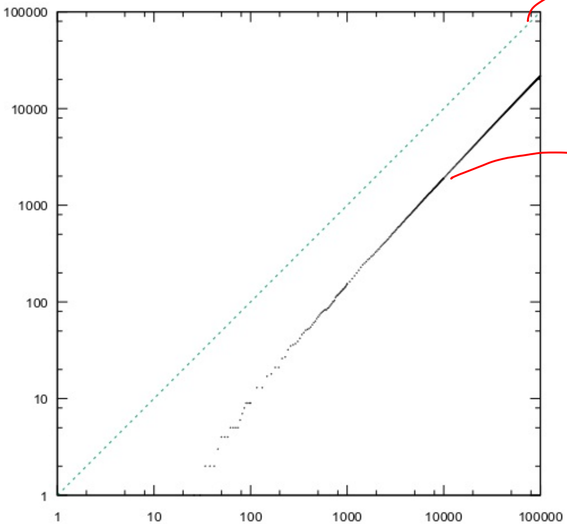


$N_2(x)$ = Number of quadratic $L/\mathbb{Q}$ with disc $< x$

Discriminants - Distribution.



DISCRIMINANTS - DISTRIBUTION.



x

$N_d(x)$

$\parallel$

Number of $L/\mathbb{Q}$ of deg d and

$|D_{L/\mathbb{Q}}| < x$

CONJECTURE : $N_d(x) \sim C_d \cdot x$

(Cohen, Malle, Venkatesh, Ellenberg)

KNOWN for $d \leq 5$ (Bhargava, Davenport-Heilbronn, Tsimermann, Shankar)

DISCRIMINANTS – WHAT ARE THEY?

HARD!

$$\begin{array}{c} L \\ \uparrow \\ K \end{array} \rightsquigarrow \quad \mathcal{D}_{L/K} \subset \mathcal{O}_K$$

$$[\mathcal{D}_{L/K}] \in Cl(K)$$

① Which ideal classes are discriminants?

② How are they distributed?

STEINITZ CLASS

$$\begin{array}{ccccc} L & \supset & \mathcal{O}_L & \cong & \mathcal{O}_K^{d-1} \oplus E \\ \uparrow & & \uparrow & & \\ K & \supset & \mathcal{O}_K & & \end{array}$$

As $\mathcal{O}_K$-modules

(Serre 1958)

$$E = E_{L/K} \in Cl(K)$$

① Which ideal classes are Steinitz ?

② How are they distributed ?

Fact : $\text{Disc} = \text{Steinitz}^2$

STEINITZ CLASS

Which ideal classes are Steinitz ?

BROADER ALGEBRA QUESTION:

A a ring
M an A-module

} When can you make M an A-algebra ?

EXAMPLE / ANALOGY

$A = \mathbb{R}$
$M = \mathbb{R}^n$

} When can M be a field, division ring, ... ?

STEINITZ CLASS

$L/K \rightsquigarrow E_{L/K} \in Cl(K)$

- $\mathcal{O}_L$ as $\mathcal{O}_K$-mod
- Sqrt of $D_{L/K}$

THEOREMS:

In degree 2, every class is Steinitz.

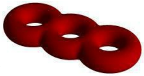
– Fröhlich (1960)

In degree 3, every class is Steinitz, and Steinitz classes of cubic (an quadratic) extensions are equidistributed.

– Kable, Wright (2006)

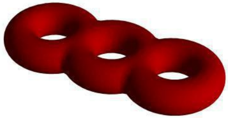
Bruche, Byott, Carter, Cobbe, Endo, Godin, Greither, Long, Massey, McCulloh, Sodaïgui

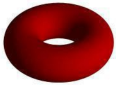
Geometric Steinitz Problem

$L = \mathbb{C}\Big($  $\Big)$

$\uparrow$

$K = \mathbb{C}\Big($  $\Big)$





$\mathcal{O}_K$ = Coord ring of a smooth affine curve X

$\mathcal{O}_L$ = Coord ring of a covering curve Y

$\cong \mathcal{O}_K^{d-1} \oplus E$ (Serre)

$E \in \mathrm{Pic}(X)$

Image courtesy: Thomas Krämer

GEOMETRIC STEINITZ PROBLEM

$$\begin{array}{l} Y \quad \text{degree } d \text{ cover} \\ \downarrow \qquad\qquad \rightsquigarrow \quad E_{Y/X} \in \mathrm{Pic}(X) \\ X \end{array}$$

THEOREM: For every degree d, every class is Steinitz.

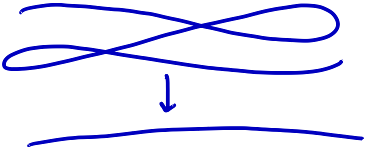
THEOREM: For covers of smooth proj. curves, every class is Steinitz up to twisting.

— D., Patel (2017)

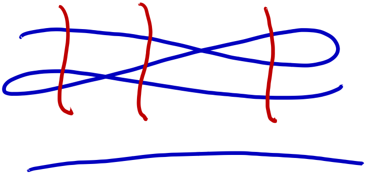
Geometric Steinitz Problem

Proof ideas.

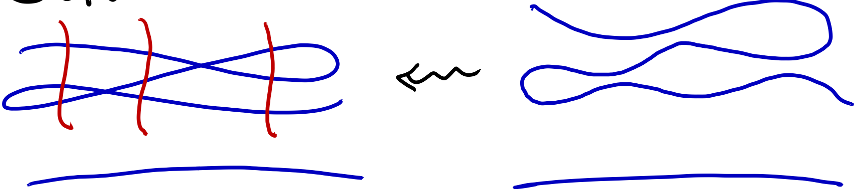
① Solve with nodal curves (orders)



② Surgery



③ Deformation.



QUESTIONS

① Geometry $\rightsquigarrow$ arithmetic ?

② Higher dimensional analogue ?

THANK YOU!