A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS

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GUIDING PRINCIPLE

Algebra / Algebraic geometry $\xleftrightarrow[\text{Mirror symmetry}]{\text{Homological}}$ Topology / Symplectic geometry

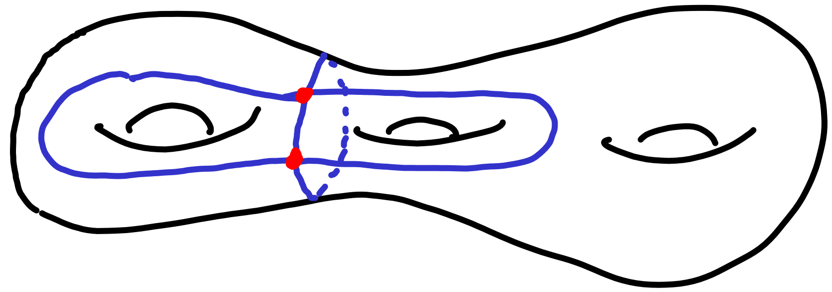
$$D^b\mathrm{Coh}(W) \cong \mathrm{Fuk}(M)$$

GUIDING PRINCIPLE

$D^b Coh(W)$

Modules / sheaves

$Fuk(M)$



GUIDING PRINCIPLE (Seidel–Thomas, Kontsevich)

Algebra/ Algebraic geometry $\longleftarrow$ Ideas $\longrightarrow$ Topology/ symplectic geometry

MAIN GOAL

$$\overline{\mathrm{Stab}(\mathcal{C})} \longleftrightarrow \overline{\mathrm{Teich}(M)}$$

(Thurston)

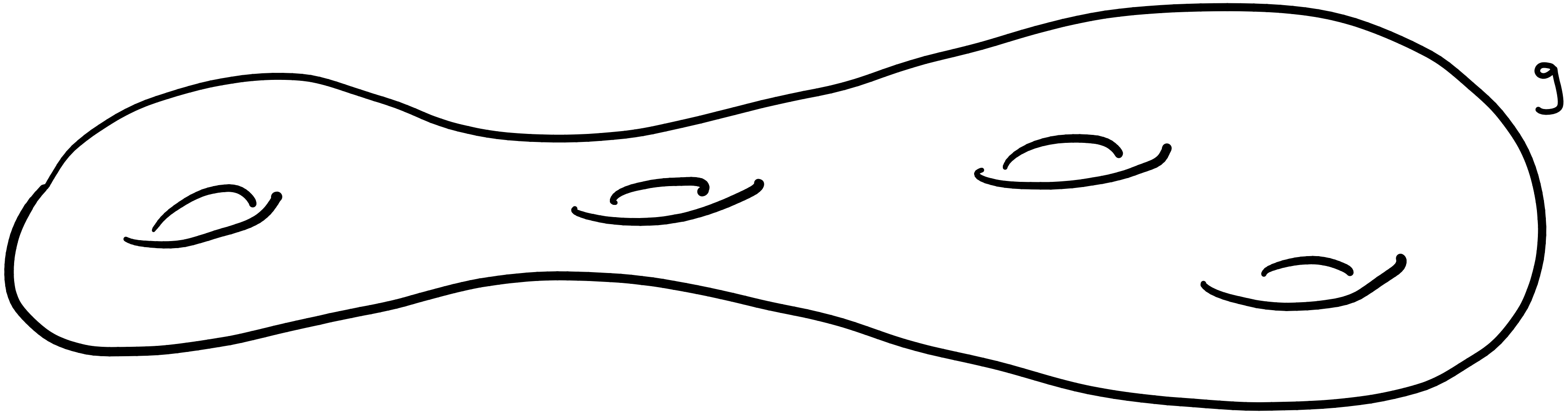
Plan : ① Thurston's construction

② Bridgeland stability conditions

③ Which $\mathcal{C}$?

④ Construction of $\overline{\mathrm{Stab}(\mathcal{C})}$

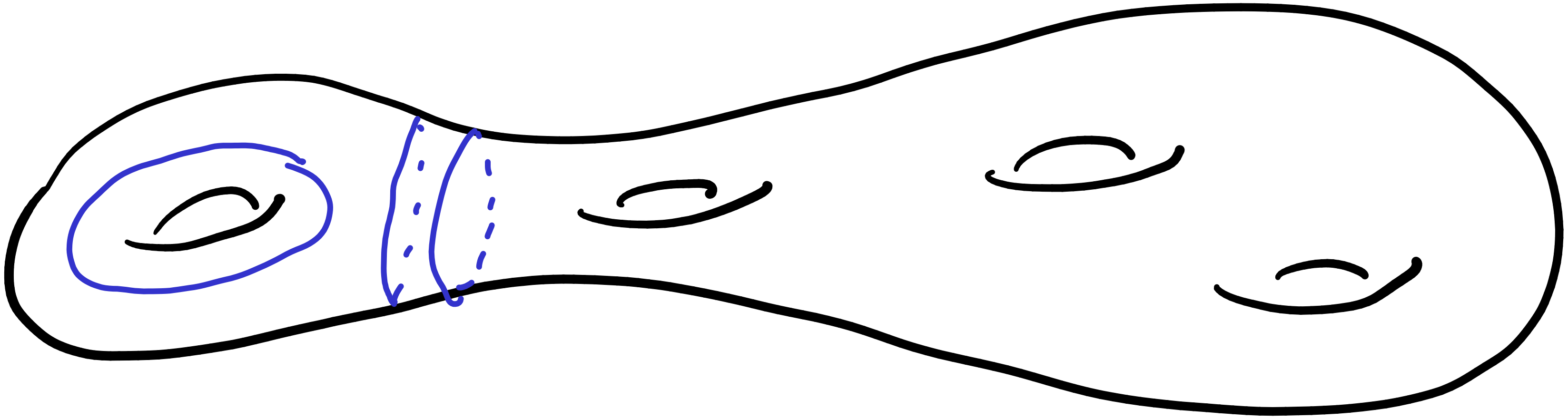
TEICH (M) AND COMPACTIFICATION

$M =$ 

$$\text{Teich}(M) = \left\{ \begin{array}{l} \text{Normalised hyperbolic} \\ \text{metrics on } M \end{array} \right\}$$

$$\cong \mathbb{R}^{6g-6}$$

TEICH(M) AND COMPACTIFICATION



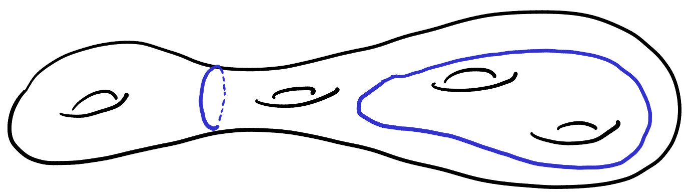
$$\underline{\mathrm{Teich}(M)} \xrightarrow{\ \ell\ } \mathbb{RP}^S$$

$$S = \left\{ \begin{array}{l} \text{simple closed} \\ \text{curves on } M \end{array} \right\} / \text{iso}$$

$$\mu \longmapsto [\, \dots : \underbrace{\mathrm{Len}_\mu(\gamma)}_{\gamma} : \dots \,]$$

DRAW CURVES

TEICH(M) AND COMPACTIFICATION



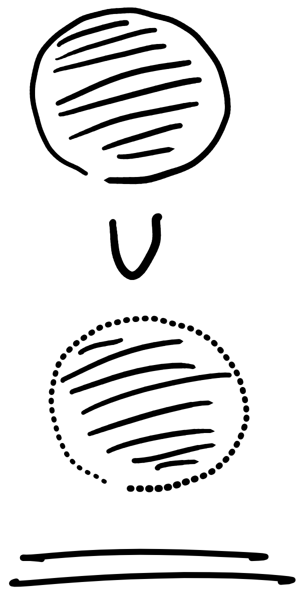
$$\underline{\underline{\mathrm{Teich}(M)}} \xrightarrow{\ \ell\ } \underline{\underline{\mathbb{RP}^{S}}}$$

Theorem (Thurston):

1) ℓ is a homeomorphism onto its image

2) The closure of the image is compact.
$\overline{\mathrm{Teich}(M)}$

3) $$\begin{array}{c} \overline{\mathrm{Teich}(M)} \\ \cup \\ \mathrm{Teich}(M) \end{array} \cong \begin{array}{c} \overline{\mathbb{D}}^{6g-6} \\ \cup \\ \mathbb{D}^{6g-6} \end{array}$$



4) S appears as a dense subset of $\partial \overline{\mathrm{Teich}(M)} = S^{6g-7}$

TEICH (M) AND COMPACTIFICATION

4) $\mathcal{S}$ appears as a dense subset of $\partial\,\mathrm{Teich}(M) = S^{6g-7}$

choose γ that minimizes

$$\mathcal{S} \xrightarrow{i} \mathbb{RP}^{\mathcal{S}}$$

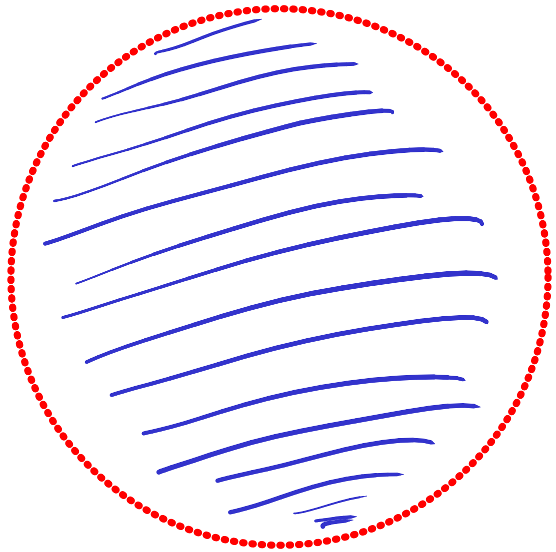
$$s \longmapsto [\ldots : \#\, s \cap \gamma : \ldots]$$

$$\gamma$$

$$\mathcal{S} \xrightarrow{i} \mathbb{RP}^{\mathcal{S}}$$

$$\|$$

$$\mathrm{Teich}(M) \xrightarrow{\iota} \mathbb{RP}^{\mathcal{S}}$$



BRIDGELAND STABILITY CONDITIONS

$\mathcal{C}$ = $\mathbb{C}$- linear triangulated Category

<u>Def</u> : A stability condition on $\mathcal{C}$ is

(Z, P)

Central charge

Slicing

Satisfying . . .

BRIDGELAND STABILITY CONDITIONS

$$\sigma = (Z, \mathcal{P})$$

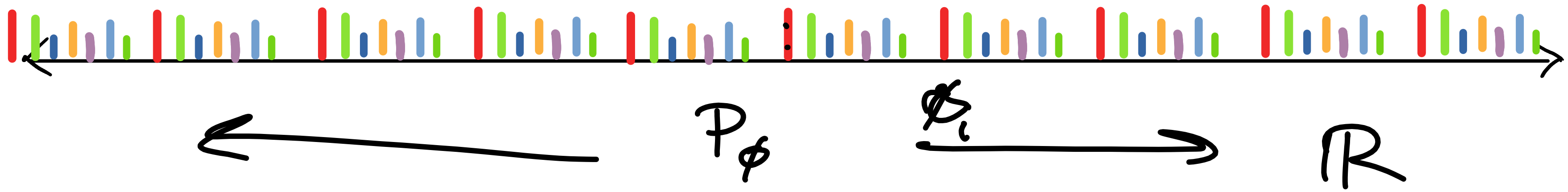
$$Z : \underline{\underline{K(\mathcal{C})}} \to \mathbb{C} \qquad \text{group hom}$$

BRIDGELAND STABILITY CONDITIONS

$$\sigma = (Z, \mathcal{P})$$

$\rightsquigarrow$ Abelian $\mathcal{P}_\phi \subset \mathcal{C}$ for $\phi \in \mathbb{R}$

Talk about hearts

$\mathcal{C} =$ 

Ψ

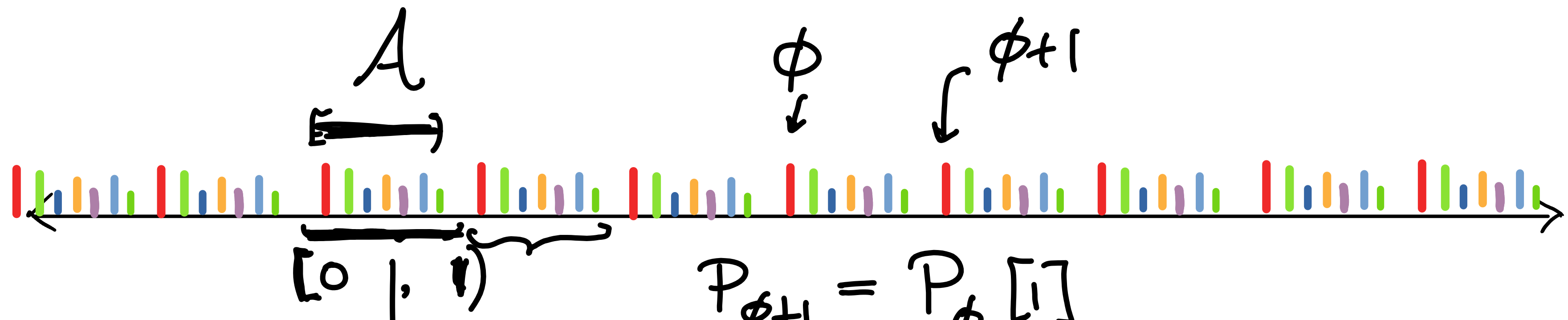
X has

$$\begin{tikzcd}
0 = X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_{i-1} \arrow[r] & X_i \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_n = X \\
& & & (Z_i) \arrow[u, dashed] & & &
\end{tikzcd}$$

Harder Narasimhan filtr.

$$\phi_1 > \phi_2 > \cdots > \phi_n \qquad Z_i \in \mathcal{P}_{\phi_i}$$

BRIDGELAND STABILITY CONDITIONS

$\mathcal{C} =$ 

$P_{\phi+1} = P_{\phi}[1]$

Heart of a bounded t-structure.

$\ldots\ldots \mathcal{A}[-1] \;\ldots\; \mathcal{A} \quad \mathcal{A}[1] \quad \mathcal{A}[2] \;\ldots\ldots$

t-structure $\longleftrightarrow$ $\mathbb{Z}$-indexed decomp of $\mathcal{C}$

Slicing $\longleftrightarrow$ $\mathbb{R}$-indexed decomp of $\mathcal{C}$

BRIDGELAND STABILITY CONDITIONS

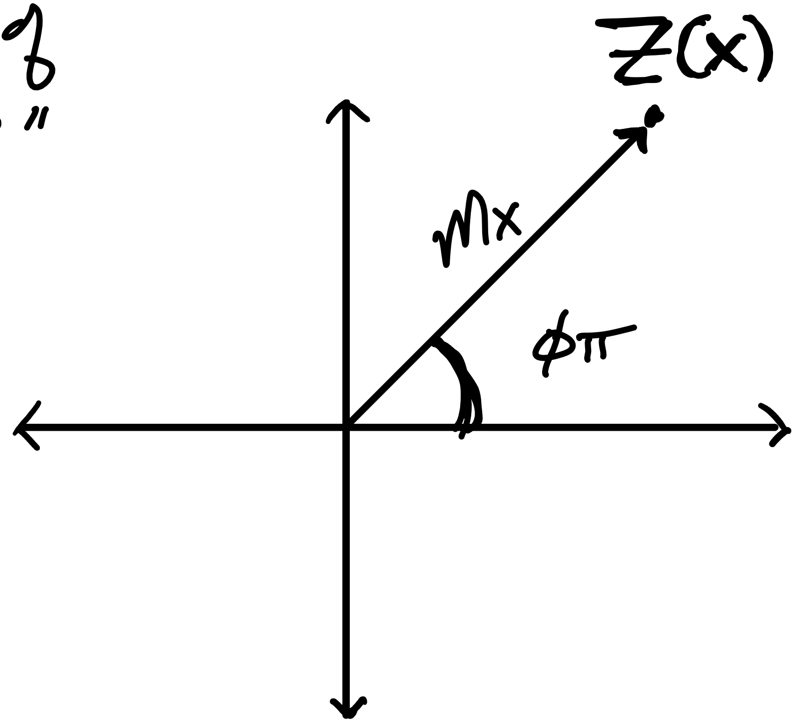
$$\sigma = (\underline{\underline{Z}}, \underline{\underline{P}})$$

Compatibility : "Semi-stable of phase ϕ"

For $X \in P_\phi$

$$Z(X) = \underline{\underline{m_X}}\, e^{i\pi\phi}$$

mass of X wrt σ



BRIDGELAND STABILITY CONDITIONS

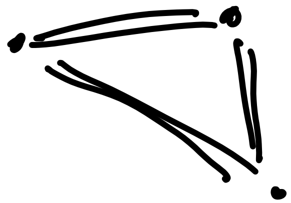
$\sigma \in \mathrm{Stab}(\mathcal{C})$

$X \in \mathcal{C}$

Analogy

metric μ curve γ $\rightsquigarrow \mathrm{Len}_\mu(\gamma)$

Def: $\underline{\underline{m_\sigma(X)}} := \sum_{HN} \underline{\underline{m_\sigma(Z_i)}}$



Satisfies "triangle inequality"

$$\mathcal{C} \ni \begin{tikzcd} \underline{X} \arrow[r] & Y \arrow[dl] \\ \underline{\underline{Z}} \arrow[u, dotted] & \end{tikzcd} \Rightarrow m(Y) \leq m(X) + m(Z)$$

BRIDGELAND STABILITY CONDITIONS

Theorem (Bridgeland): Stab($\mathcal{C}$) forms a manifold of real dimension $2 \cdot \operatorname{rank} K(\mathcal{C})$

$$\left\{ \begin{array}{c} \mathrm{Stab}(\mathcal{C}) \\ \downarrow Z \quad \text{local homeo} \\ \mathrm{Hom}(K(\mathcal{C}), \mathbb{Z}) \end{array} \right\}$$

(~~conjecturally~~ contractible ?)

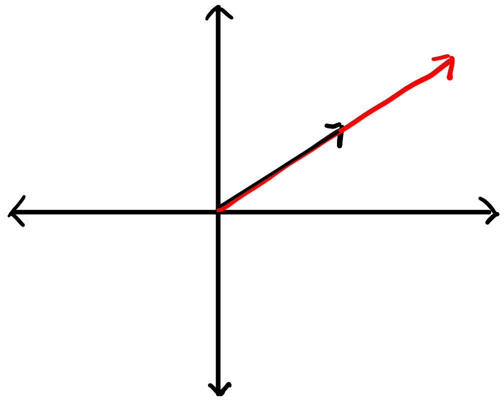
Analogy

$\mathrm{Teich}(M) \cong \mathbb{R}^{6g-6}$

BRIDGELAND STABILITY CONDITIONS

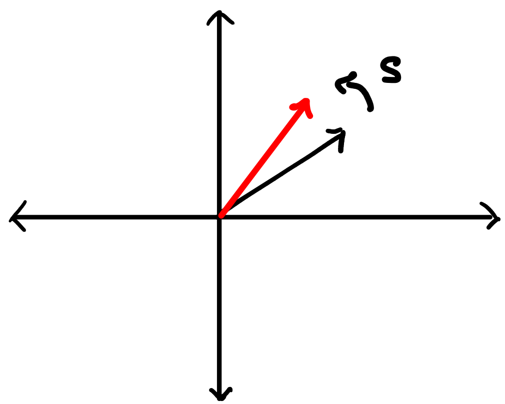
1) Scaling by $\mathbb{R}_+^\times$

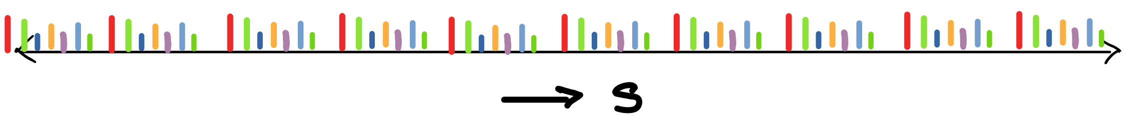
$$\underline{t} : (Z, P) \longmapsto (\underline{tZ}, P)$$



2) Rotation by $\mathbb{R}$

$$\underline{s} : (Z, P) \longmapsto (e^{i\pi s} Z, P_{\phi - s})$$





BRIDGELAND STABILITY CONDITIONS

$$\underset{\text{scaling}}{\mathbb{R}_+^{\times}} \circlearrowright \mathrm{Stab} \circlearrowleft \underset{\text{rotation}}{\mathbb{R}}$$

$$\big\downarrow$$

$$\mathrm{PStab} := \mathrm{Stab} \Big/ \begin{array}{l}\text{scaling \&}\\ \text{rotation}\end{array}$$

CATEGORIES

$\mathcal{C}$ = $\mathbb{C}$-linear triangulated **2-CY** [2]

Finite dim $\mathrm{Hom}^*(X,Y) = \bigoplus_n \mathrm{Hom}^n(X,Y)$

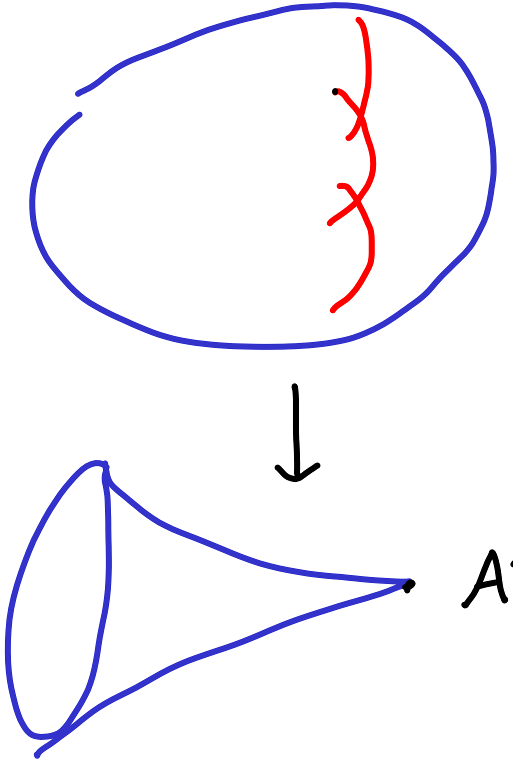
Examples: $D^b \mathrm{Coh}\, K3$

$D^b \mathrm{Coh}$ Local K3 $\leftarrow$ TBD

2-CY Quiver Category

CATEGORIES

Local K3 : $\begin{array}{c} \hat{S} \\ \downarrow \pi \\ S \end{array}$



ADE

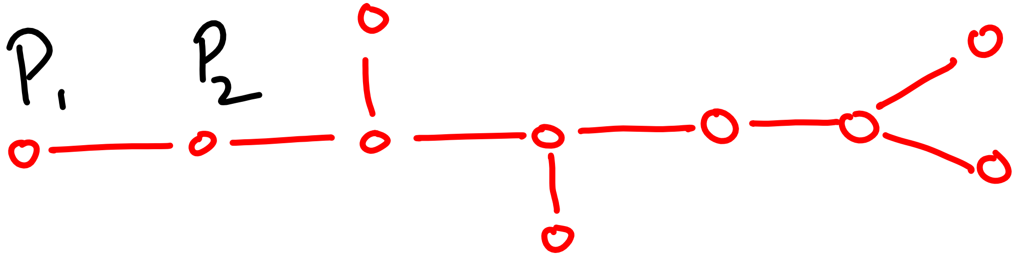
$$\mathcal{C} = \{ E \in D^b\mathrm{Coh}(\hat{S}) \mid R\pi_* E = 0 \}$$

Generated by $P_i := \mathcal{O}(-1)$ on i^{th} exc. $\mathbb{P}^1$ $\|$

satisfying hom conditions governed by the dual graph Γ of $\mathrm{Exc}(\pi)$

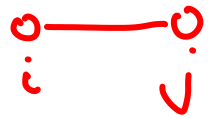
CATEGORIES

Quiver Γ



$\mathcal{C}(\Gamma) = \mathbb{C}$-linear Δ cat generated by P_i for $i \in V(\Gamma)$ satisfying

$$\hom^*(P_i, P_j) = \mathbb{C} \quad \text{if}$$



$$= 0 \quad \text{if}$$

Spherical

$$\hom^*(P_i, P_i) = H^*(S^2)$$

$\mathcal{C}(\Gamma) \circlearrowleft$ <u>Artin-Tits Braid group</u> $B(\Gamma)$

COMPACTIFYING PSTAB - MOTIVATION

$\mathcal{C} \circlearrowleft \underline{\underline{G}}$

$\rightsquigarrow$

$\mathrm{PStab}(\mathcal{C}) \circlearrowleft \underline{G}$

$\cap$

$\overline{\mathrm{PStab}}(\mathcal{C}) \circlearrowleft G$

$\rightsquigarrow$

Understand G

Analogy

$\mathrm{Teich} \circlearrowleft \underline{\underline{\mathrm{MCG}}}$

$\cap$

$\overline{\mathrm{Teich}} \circlearrowleft \mathrm{MCG}$

COMPACTIFYING PSTAB

$\mathcal{C} \supset S = \{\text{Spherical obj}\}/\text{shifts}$

$$\underline{\mathrm{PStab}} \xrightarrow{m} \underline{\mathbb{RP}}^{S}$$

$$\sigma \longmapsto [\cdots : \underset{x}{m_\sigma(x)} : \cdots]$$

Teich $\longrightarrow \mathbb{RP}^S$

$\underline{\mu} \longmapsto [\underline{\mathrm{Len}_\mu}]$

$S \xrightarrow{i} \mathbb{RP}^S$

$s \longmapsto [\# s \cap]$

$$S \xrightarrow{i} \mathbb{RP}^{S}$$

$$y \longmapsto [\cdots : \underset{x}{\underline{\underline{\mathrm{hom}}}(x,y)} : \cdots]$$

$$\sum_n \dim \mathrm{Hom}(x, y[n])$$

COMPACTIFYING PSTAB

$\underline{\text{Pstab}} \xrightarrow{m} \mathbb{RP}^S$ $\qquad\qquad$ $S \xrightarrow{i} \mathbb{RP}^S$.

1) m is a homeomorphism onto its image.

2) The closure of the image is a compact manifold with boundary ∂.

3) $\underline{S}$ embeds via i as a dense subset of ∂.

4) $\overline{\text{Pstab}} \cup \overline{\text{Pstab}}^{\,int} \cong \overline{\text{Disk}} \cup \text{Disk}$ $\qquad$ || $\qquad$ Wishlist

$\text{Pstab} = \overline{\text{Pstab}}^{\,int}$

COMPACTIFYING PSTAB

① – ④ • Theorems in rank 2 cases [Bapat, –, Licata]
(A_2 and $\hat{A}_1$ quivers) written down

• In progress in finite ($\underline{ADE}$) and ‖
affine ($\underline{\hat{A}, \hat{D}, \hat{E}}$) type

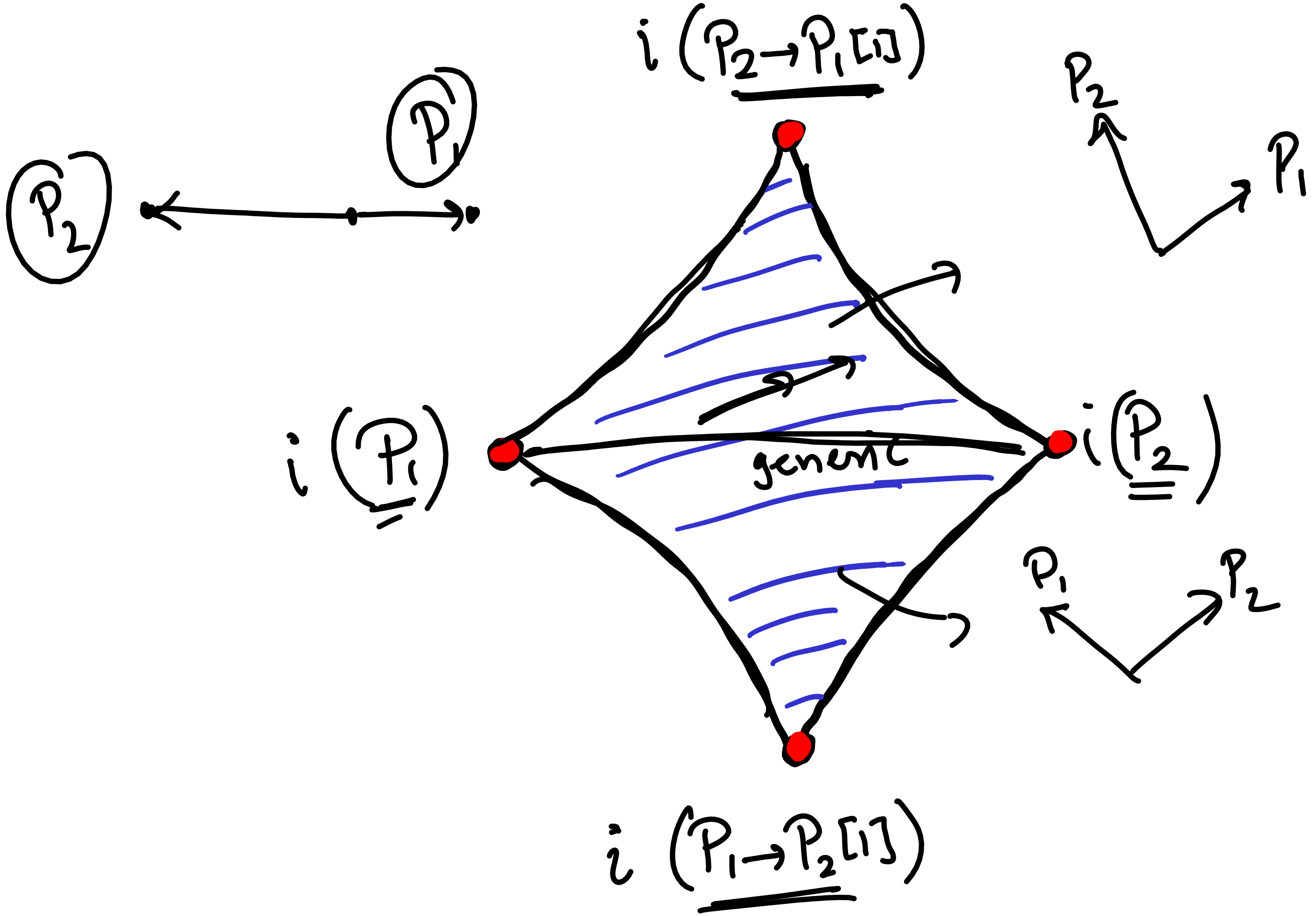
• Conjectures for arbitrary $\underline{\underline{C(\Gamma)}}$

• Dream / question more generally.

COMPACTIFYING PSTAB : A_2 PICTURE

$$\mathcal{C} = C\left(\overset{P_1}{\circ}\!-\!\!-\!\!\overset{P_2}{\circ}\right)$$

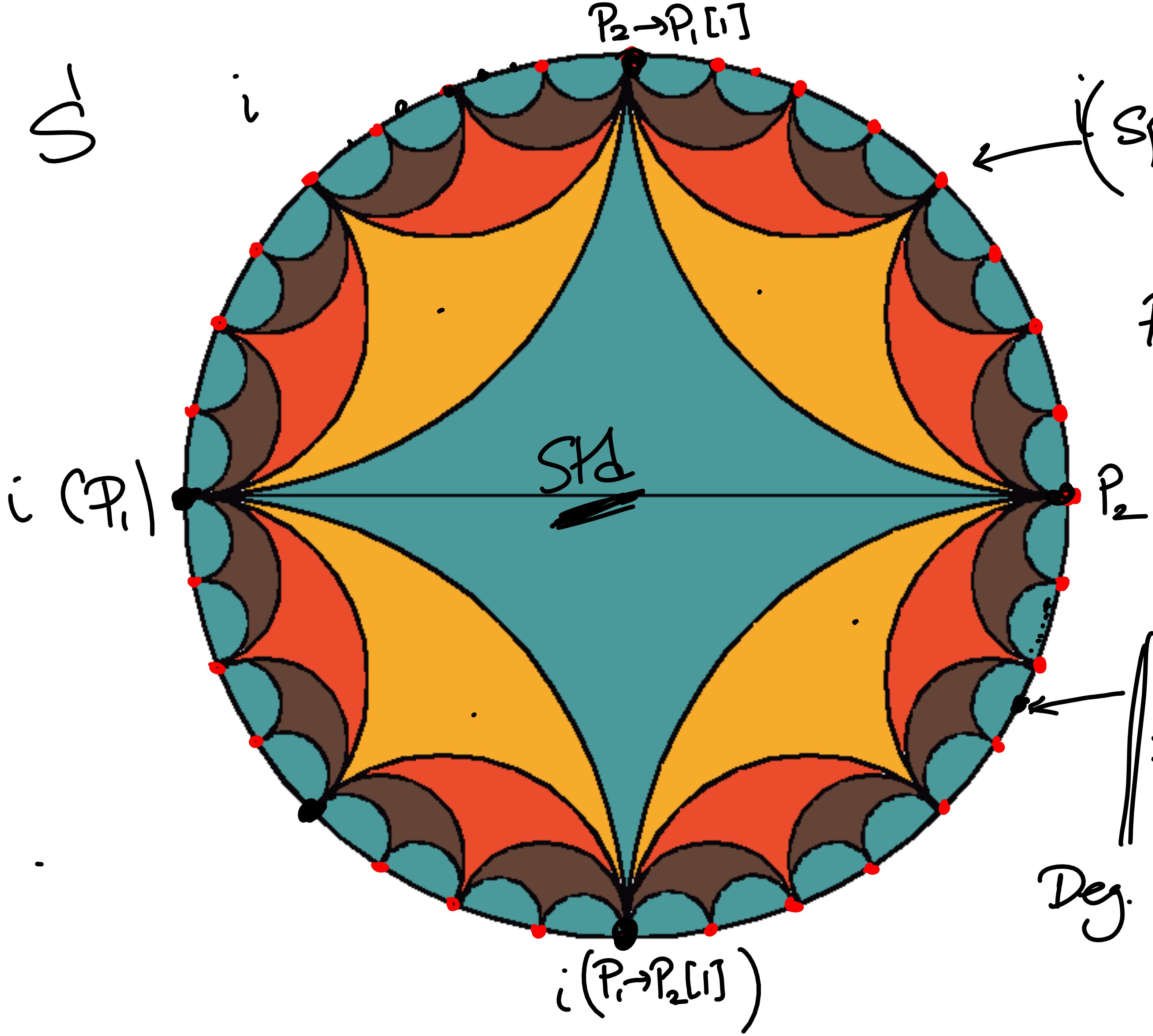
$$\overset{\text{std}}{\heartsuit} = \langle P_1, P_2 \rangle$$



Translate by B_3 $\wr$ $PSL_2(\mathbb{Z})$

COMPACTIFYING PSTAB : A_2 PICTURE

$\overset{\downarrow}{S}$ $\overset{\cdot}{i}$



i (Spherical objects)

Farey triangulation.

Modular interpretation?

Deg. stab. conds?

COMPACTIFYING PSTAB

Why is a spherical $\underline{\underline{x}} \in \overline{\text{PStab}}$?

Take $\sigma \in \text{Stab}$

$$\sigma_1 := \underline{T}\omega_x^{-1}\,\sigma$$

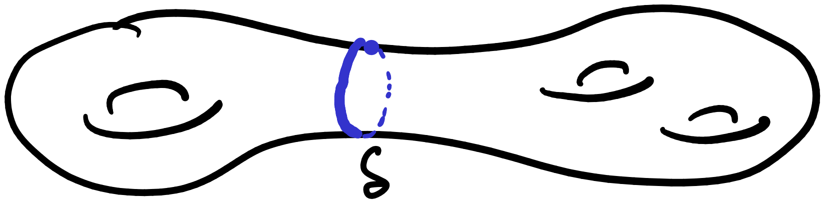
$$\underline{\underline{y}} \to \underline{\underline{T\omega_x y}} \to \underline{\underline{\mathrm{Hom}(x,y) \otimes x[1]}}$$

$$m_{\sigma_1}(y) = m_\sigma(\underline{\underline{T\omega_x y}})$$

$$\text{“}\approx\text{”}\ \underline{\underline{m_\sigma(y)}} + \underline{\underline{\mathrm{hom}(x,y) \cdot \boxed{m_\sigma(x)}}}$$

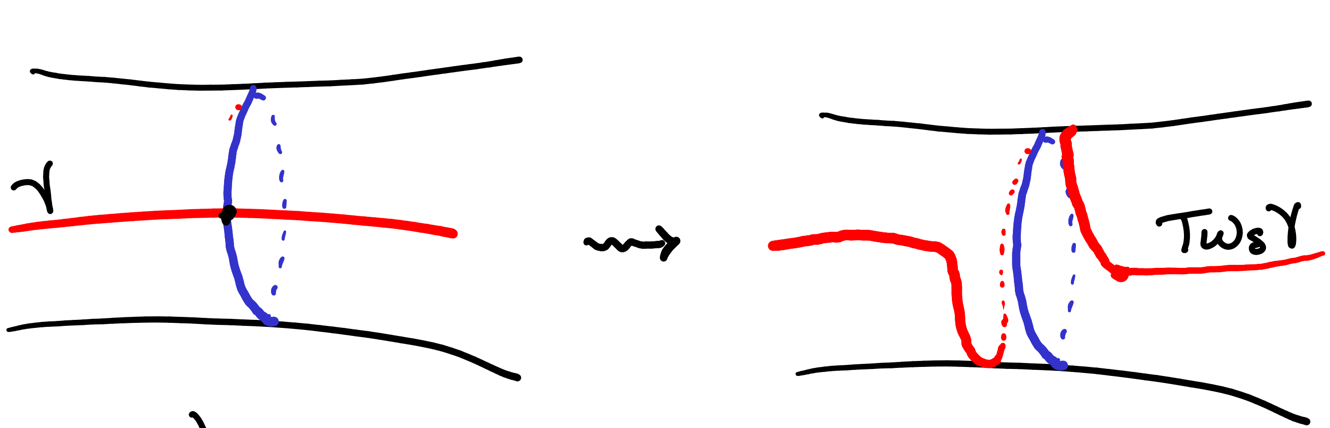
Iterate $\underline{\underline{m_{\sigma_n}(y)}} \approx a + b\ \underline{\underline{n\ \boxed{\mathrm{hom}(x,y)}}}$

Why is $\underline{\delta} \in \overline{\mathrm{Teich}}$?



μ

$\underline{\underline{\mu_1}} := Tw_\delta^{-1}(\mu)$



$$Len_{\mu_1}(\gamma) = Len_{\mu}(Tw_\delta \gamma)$$

$$\approx Len_{\mu}(\gamma) + \underline{\underline{Len(\delta)}} \cdot \# \delta \cap \gamma$$

Iterate: $Len_{\underline{\underline{\mu_n}}}(\gamma) \approx \underbrace{\underline{\underline{a}} + \underline{\underline{b}} \cdot \underline{\underline{n}} \; \underline{\underline{\# \delta \cap \gamma}}}$

THANK YOU !

$\hat{A}_1$