Towards a birational Classification of algebraic varieties — Work of Caucher Birkar.

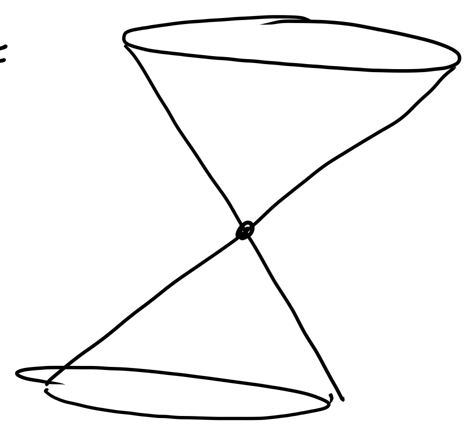
1. Algebraic Varieties.

Algebraic variety = Set of solutions of polynomial equations.

Ex. $X = \{ (x,y,z) \mid x^2+y^2 = z^2 \}$ $\dim X = 2$

Solutions in $\mathbb{Z}$ = {Pythagorean triples}.

Solutions in $\mathbb{R}$ =



cone over S^1.

Solutions in $\mathbb{C}$ = Complex cone over S^2.

Isomorphisms

An isomorphism between two complex varieties is a bijection

$$\varphi : Y \to X$$

s.t. φ & φ^{-1} are defined by polynomials.

Ex. $Y = \{ (x,y,z) \mid xy = z^2 \} \xrightarrow{\sim} X$

Via $\varphi : x \mapsto x+iy$, $y \mapsto x-iy$ $z \mapsto z$.

Problem: Describe all isomorphism classes.

$\hookrightarrow$ Hopeless.

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2. Birational algebraic geometry

A birational iso $\varphi: Y \overset{\sim}{\dashrightarrow} X$ is a bijection $\varphi: U \to V$ between a dense open $U \subset X$ & $V \subset Y$ such that φ & φ^{-1} are defined by rational functions

Ex. $\mathbb{C}^2 \overset{\sim}{\dashrightarrow} Y$ by

$$(s,t) \longmapsto \left(s, \frac{t^2}{s}, t\right)$$

$$(x,z) \longleftarrow\!\shortmid (x, y, z)$$

Problem: Describe all birational iso classes.

① Identify a distinguished element in each birat. iso class ("canonical model")

Hironaka 1970

② Describe the canonical models.

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dim 1

① There is a unique smooth & compact (proj). X in every birat class.

② $g=0$ $\quad g=1$ $\quad g=2,3,4,\ldots$

$\mathbb{P}^1$ $\quad$ 1 dim family $\quad$ $3g-3$ dim family.

3. The minimal model Program

§. The canonical class.

X an alg. variety.

$\exists$ distinguished element $K_X \in H^2(X, \mathbb{Q})$.

$K_X = c_1(\Omega_X)$ Ω_X = Holomorphic cotangent bundle.

Ex. $\dim X = 1$, X smooth compact, $H^2(X) \cong \mathbb{Q}$.
$K_X = 2g-2$.

$g=0$: $K_X < 0$
$g=1$: $K_X = 0$
$g=2$: $K_X > 0$.

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§ The trichotomy

X is
- ① Fano if K_X is anti ample ($K_X \cdot C < 0 \ \forall C$)
- ② Calabi-Yau if K_X is trivial. ($K_X \cdot C = 0 \ \forall C$)
- ③ Canonically polarised if K_X is ample ($K_X \cdot C > 0 \ \forall C$.)

① spherical - small/trivial π_1, many $\mathbb{Q}$ pts, big aut
② flat - close to abelian π_1, many but not too many,
③ hyperbolic - complicated π_1, few $\mathbb{Q}$-points, finite aut.

MMP – Up to birat. iso. every X can be broken down into these 3 archetypes.

$$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_n$$

Each step – ⓐ div. contraction
ⓑ flip.

switch

X_n = can. polarized or

X_n with Fano / CY fibers.
$\downarrow$
Y ← lower dim

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dim $X=1$ – $X = X_n$.

dim $X=2$ – only div. contr. needed.
All X_i are smooth.

(Castelnuovo-Enriques, Early 1900)

$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots$

Contract C such that $K \cdot C < 0$.

In dim ≥ 3, introduces singularities.

Identify a class of singular X – $\mathbb{Q}$ factorial terminal.

Preserved under divisorial contraction, but not small

Flip = a surgery that improves singularities
does not introduce K-neg curves. 1970

dim $=3$: Flips exist. Mori, Iitaka, Shokurov, Kollár, Kawamata, Reid.

There cannot be an inf. seq. of flips

$\Rightarrow$ MMP terminates + X_n is as expected.

Higher dim: Birkar, Cascini, Hacon, McKernan (2012)

1. Flips exist.
2. MMP terminates if X is of general type. (& flips are carefully chosen.)

(In this case X_n is canonically polarised.).

Conj.:- 1. MMP terminates in general. (Termination of flips).

2. If X_n is not canonically polarized, then it admits a CY fibration (Abundance).

Boundedness results – BAB conjecture.

Thm (Birkar). The class of Fano X of given dim with canonical $\mathbb{Q}$-factorial singularities forms a finite dim family.

even ε-log canonical for a given $\varepsilon > 0$.

\+ relative versions

\+ existence of complements (nice elements in $|-mK_X|$ for bounded m.

Flip

$$X \searrow X^-$$

"Relative Fano"

Thm $\Rightarrow$ control over flips.

Aside: Boundedness of canonically polarized X of a given dim & volume is also known (Tsuji, Hacon-McKernan, Takayama).