How to count using equivariant cohomology?

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Contents

1 Motivating question **2**

2 Why is the answer finite? **2**

3 Why cubic surfaces? **2**
3.1 Orbit closures 3
3.1.1 Question of the degree of the orbit closure 3
3.1.2 History of the degree of the orbit closure 3

4 How should we compute the answer? **4**
4.1 A natural approach using the GIT quotient 4
4.2 An even more natural approach without the GIT quotient . . 4

5 The equivariant Chow ring **5**
5.1 Question of the equivariant orbit class 5
5.2 What is the equivariant Chow ring? 5

6 The existence of a universal formula **5**

7 Finding the universal coefficients **6**
7.1 Test families 6
7.1.1 Actual families 6
7.1.2 Isotrivial families 8

8 The upshot **8**

9 The future **9**

1 Motivating question

Fix five general cubics $F_0, \dots, F_4$ in $\mathbf{C}[X, Y, Z, W]$. Consider the family of cubic surfaces defined by

$$a_0 F_0 + \cdots + a_4 F_4 = 0$$

as $[a_0 : \cdots : a_4]$ varies in $\mathbf{P}^4$. How many times does a general cubic surface appear in this family? That is, if we fix a general cubic surface S, how many $[a_0 : \cdots : a_4] \in \mathbf{P}^4$ are such that the surface defined by the equation above is isomorphic to S?

The goal of this talk is to explain the mathematics that lets us answer this question. Before I proceed, if you are dying to know the answer, I'll write it down: it is 96120.

2 Why is the answer finite?

Fact: Let $X \subset \mathbf{P}^3$ and $Y \subset \mathbf{P}^3$ be two cubic surfaces. Any isomorphism $X \cong Y$ is given by a projective linear isomorphism

$$M \colon \mathbf{P}^3 \to \mathbf{P}^3.$$

As a result, we get

$$\{\text{Cubic surfaces up to isomorphism}\} = \{\text{Cubic forms in X, Y, Z, W}\}/GL(4).$$

We do a dimension count and see that this is a moduli space of dimension

$$20 - 16 = 4.$$

So, in a four dimensional family, we expect a general isomorphism class to appear finitely many times.

3 Why cubic surfaces?

Why not cubic curves or quartic surfaces or even more generally hypersurfaces of any degree in any projective space? This is a good question, and in fact, the main question can be formulated more generally. Let us do that in a slightly different language.

3.1 Orbit closures

Fix positive integers n and d. Let $V = \mathbf{C}^n$. We look at hypersurfaces of degree d in $\mathbf{P}V$ modulo linear changes of coordinates. Fix a generic $F \in \mathrm{Sym}^d(V^*)$ and consider the orbit

$$O_F = PGL(V) \cdot F \subset \mathbf{P}\,\mathrm{Sym}^d(V^*)$$

and its closure

$$\overline{O_F} \subset \mathbf{P}\,\mathrm{Sym}^d(V^*).$$

This is a closed subvariety of dimension $n^2 - 1$. The main question is equivalent to the following.

3.1.1 Question of the degree of the orbit closure

What is the degree of $\overline{O_F}$?

More precisely, the main question is the same as the $n = 4$ and $d = 3$ case of the question above.

3.1.2 History of the degree of the orbit closure

1. Enriques and Fano answered the question for $n = 2$ (points in $\mathbf{P}^1$) in 1897. The answer is:
$$d(d-1)(d-2).$$

2. Aluffi and Faber answered the question for $n = 3$ (curves in $\mathbf{P}^2$) in 1992. The answer is:
$$d^8 - 1372d^4 + 7992d^3 - 15879d^2 + 10638d.$$
So the next non-trivial case is $n = 4$ and $d = 3$ (cubic surfaces).

3. Laura Brustenga I Moncusi, Sascha Timme, and Madeleine Weinstein numerically computed the answer to be 96120 in 2020.

I will describe a new perspective on the problem.

4 How should we compute the answer?

Let us go back to the main question, which I will restate more generally. Fix a proper moduli space of cubic surfaces, for example, the GIT quotient

$$\mathcal{M} = \text{Cubic forms}/GL.$$

Let $\mathcal{X} \to B$ be a family of cubic surfaces parametrized by a proper 4-dimensional base B. Find the degree of the rational map

$$\mu \colon B \dashrightarrow \mathcal{M}.$$

(In the main question, the base is $\mathbf{P}^4$ and the family is a linear series.)

4.1 A natural approach using the GIT quotient

If μ were a regular map, we win. The answer is simply the degree of

$$\mu^*[\text{Point}],$$

where [Point] represents the class of a point in the numerical Chow ring of $\mathcal{M}$.

If μ is not regular, we can try to find a resolution

$$B \leftarrow \widetilde{B} \xrightarrow{\widetilde{\mu}} \mathcal{M}.$$

This will give the right answer, but I do not know how to resolve even the simplest families. For example, if you know how to resolve the family in the main question, let me know!

4.2 An even more natural approach without the GIT quotient

The reason the map μ is not regular is because we are taking the GIT quotient and hence throwing away some cubic surfaces because they are "unstable". If we do not throw them away, the map μ is regular

$$\mu \colon B \to [\text{Cubic forms}/GL(4)]$$

but now the target is not a proper moduli space. But that's a minor inconvenience! In fact, that is not an inconvenience at all, and the approach works.

5 The equivariant Chow ring

To make the approach work, we first need to make sense of the Chow ring of

$$M = [X/G].$$

But Edidin and Diaz have already done this form us: this is the G-equivariant Chow ring of X:

$$A^*([X/G]) = A^*_G(X).$$

Next, we have to make sense of [Point]. We recall that a point in $[X/G]$ is the same as a G-orbit in X. We also recall that points are not necessarily closed, so we have to take the closure. Equivalentnly, we have to look at the closure of G-orbits. Now, every G-invariant closed subvariety of X has an equivariant fundamental class. This is what replaces the class of a point.

$$[\text{Point}] = [\text{Orbit closure}].$$

We should thus ask and answer the following question.

5.1 Question of the equivariant orbit class

Given $F \in \mathrm{Sym}^d(V^*)$, find the equivariant class of $\overline{O_F}$ in $A^*_{GL(V)}(\mathrm{Sym}^d(V^*))$.

5.2 What is the equivariant Chow ring?

The equivariant Chow ring is very easy to describe in our case:

$$A^*_{GL(V)}(\mathrm{Sym}^d(V^*)) = A^*_{GL(V)}(\cdot) = \mathbf{Z}[c_1, \ldots, c_n],$$

where c_i has degree i and is the i-th Chern class of the standard representation.

6 The existence of a universal formula

For the case of cubic surfaces, the equivariant orbit class lives in codimension 4, and hence is a $\mathbf{Z}$-linear combination of

$$c_1^4, c_1^2 c_2, c_1 c_3, c_2^2, c_4.$$

As a result, we get the following.

Theorem 1. *There exist integers* $a_{1^4}, a_{1^2\cdot 2}, a_{1\cdot 3}, a_{2^2}, a_4$ *such that for every (good) family of cubic surfaces*

$$\mathcal{X} \to B$$

parametrized by a proper 4-dimensional base B*, we have*

$$\text{Degree of } B \dashrightarrow \mathcal{M} = \sum a_I c_I(\mathcal{V})[B],$$

where $\mathcal{V}$ *is the rank 4 bundle of anticanonical sections*

$$\mathcal{V} = \pi_* \omega_{X/B}^{-1}.$$

These universal constants are just the coefficients of the equivariant class of the generic orbit closure in the standard basis.

7 Finding the universal coefficients

Once we know that a universal formula exists, we can find the coefficients by writing down families where we can compute both sides of the formula. Each such family gives a linear equation satisfied by the coefficients. If we can write down enough families, we get enough linear equations that we know all the coefficients!

7.1 Test families

7.1.1 Actual families

1. First family

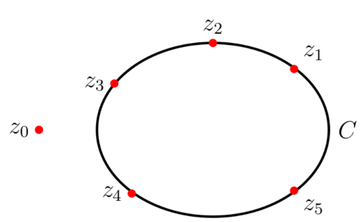

FIGURE 1. In the first test family, we blow up the plane along $\{z_0, \ldots, z_5\}$ as z_0 stays fixed and $z_1, \ldots, z_5$ vary in a general 4-dimensional series on a fixed conic C.

2. Second family

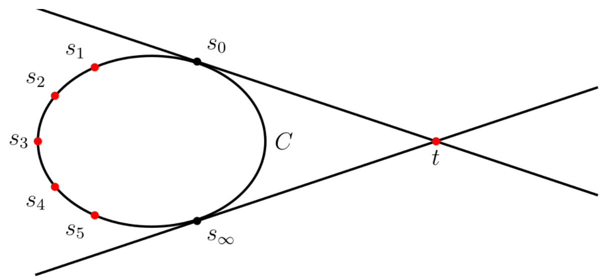

FIGURE 2. In the second test family, we blow up the plane at the points $\{s_1, \ldots, s_5, t\}$ as the seven points $\{s_1, \ldots, s_5, s_0, s_\infty\}$ vary freely on a conic C.

3. Third family

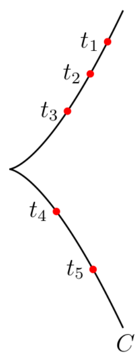

FIGURE 3. In the family B_3, we blow up the points t_i for $i = 1, \ldots, 5$ and a fixed point t_∞ at infinity (not shown) as the points $\{t_1, \ldots, t_5\}$ move freely on a fixed cuspidal cubic C with a flex at t_∞.

4. Fourth family

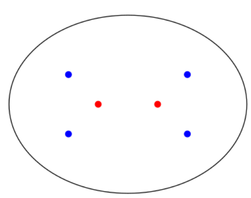

FIGURE 4. In the fourth test family, we blow up the plane at 4 fixed points (blue) and 2 variable points (red).

7.1.2 Isotrivial families

Let X be a cubic surface with an action of a group G. We can turn this into a "family"

$$\mathcal{X} = [X/G] \to [./G].$$

This gives a morphism

$$\mu\colon [./G] \to [\mathrm{Cubics}/GL].$$

Suppose that X is not in the orbit closure of a generic cubic surface. Then the pullback under μ of a generic orbit closure is zero. And hence, we get a relation

$$\sum a_I c_I(V) = 0$$

in $A^*_G(\cdot)$.

If G is a finite group, then $A^*_G(\cdot)$ is torsion, so we will only get a congruence relation on the coefficients. But if G is infinite, for example, $G = \mathbf{G}_m$, then we (may) get a non-trivial relation.

There are 4 cubic surfaces with a $\mathbf{G}_m$ action that are provably not in the orbit closure of a generic cubic surface. They give 4 additional relations.

(1) $x_0x_1x_3 = x_2^3 \quad (3A_2)$
(2) $x_3(x_0x_2 - x_1^2) = x_0x_1^2 \quad (A_3 + 2A_1)$
(3) $x_3(x_0x_2 - x_1^2) = x_0^2x_1 \quad (A_4 + A_1)$
(4) $x_3x_0^2 = x_1^3 + x_2^3 \quad (D_4)$

8 The upshot

Any 5 of the 8 families described above are sufficient to determine the universal coefficients.

Theorem 2. *For every $\mathcal{X} \to B$ (good) family of cubic surfaces parametrized by a proper 4-dimensional base B, we have*

$$\textit{Degree of } B \dashrightarrow \mathcal{M} = 1080(v_1^2v_2 - v_1v_3 + 9v_4)[B].$$

where the v_i are the Chern classes of the rank 4 bundle

$$\mathcal{V} = \pi_*\omega^{-1}_{X/B}.$$

If we apply this to the general 4-dimensional linear series, we get

$$96120.$$

If we apply this to the section of a general cubic 4-fold, we get

$$42120.$$

9 The future

In closing, I want to re-iterate the main broader question. Let G be an algebraic group and let W be a G-representation. Pick a point $w \in W$. What is the equivariant fundamental class of the orbit closure of w?

This is an important question with whose answer has rich geometrical and enumerative consequences.

1. If $W = \mathrm{Hom}(E, F)$ and $G = GL(E) \times GL(F)$ and w is a rank r-matrix, then the answer is the Porteous fomula.
2. If $W = \mathrm{Sym}^2(E)$ or $\wedge^2(E)$, and w is of rank r, then the answer is the Porteous formula due to Harris and Tu. Work of Feher, Nemethi, Rimanyi, Weber, Varchenko (etc.) generalises this.
3. For hypersurfaces, not much is known.
   - Hypersurfaces in $\mathbf{P}^1$ (Lee, Patel, Spink, Tseng)
   - Quartic plane curves (Lee, Patel, Tseng)
   - Cubic surfaces (-, Patel, Tseng)
4. For $GL(2)$ and any representation (-, Patel) in progress.