The Geometry and Combinatorics of Harder-Narasimhan Filtrations

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(joint work with Asilata Bapat
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Groups acting on Categories

$$G \circlearrowright \mathcal{C}$$

Why?

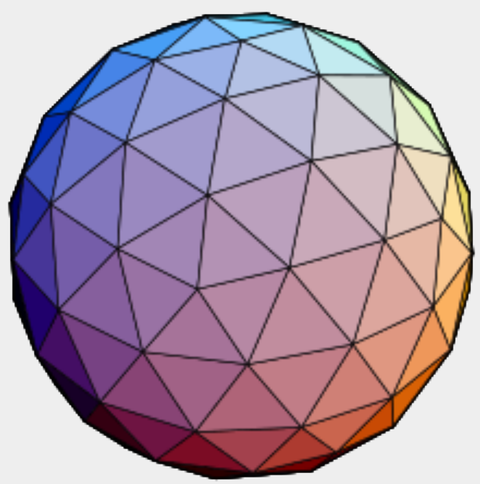
- You like the category.
- You like the group.

Main Picture (Bapat, D, Licata)

$B_n \circlearrowright \mathcal{C}_n$ = 2-CY Category of

A_n

$B_n \circlearrowright$



Piecewise $\mathbb{Q}$-linear Sphere of dim $2n-3$

Spherical Objects

$A \in \mathcal{C}$, $\mathbb{K}$-linear, k-Calabi-Yau, triangulated

$$E = \bigoplus_i \mathrm{Hom}(A, A[i])$$

$$\underbrace{\begin{array}{ccccccccc} i & = & 0 & 1 & 2 & \cdots & k-1 & k \\ E_i & = & \mathbb{K} & 0 & 0 & \cdots & 0 & \mathbb{K} \end{array}}_{H^*(S^k,\, \mathbb{K})}$$

Then A is called "spherical."

Spherical Objects $\longleftrightarrow$ Roots in a lattice

$$\Lambda := K_0 \mathcal{C}$$

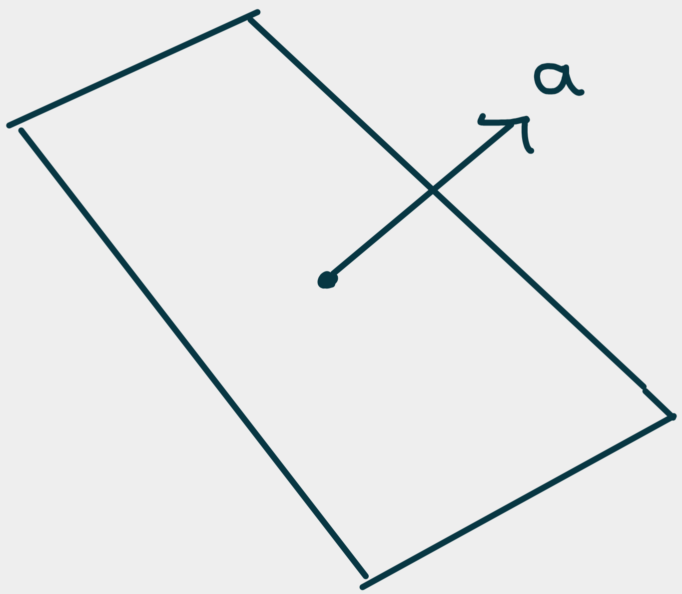
$$= \text{Grothendieck group of } \mathcal{C}$$

$$\langle A, B \rangle := \sum (-1)^i \dim \mathrm{Hom}(A, B[i])$$

$(k \text{ even}) \Rightarrow$ Symmetric even bilinear form

A spherical $\Rightarrow \langle A, A \rangle = 2$

Spherical Objects $\longleftrightarrow$ Roots in a lattice



$S_a : \Lambda \longrightarrow \Lambda$

Reflection in $a^\perp$

$\bullet\ A$

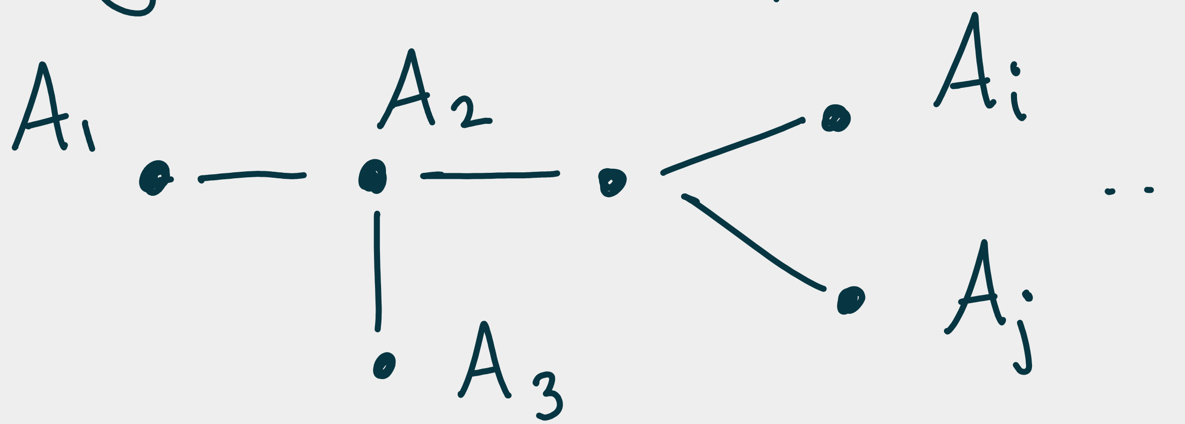
Spherical

$\sigma_A : \mathcal{C} \longrightarrow \mathcal{C}$

Twist in A.

Braid group actions

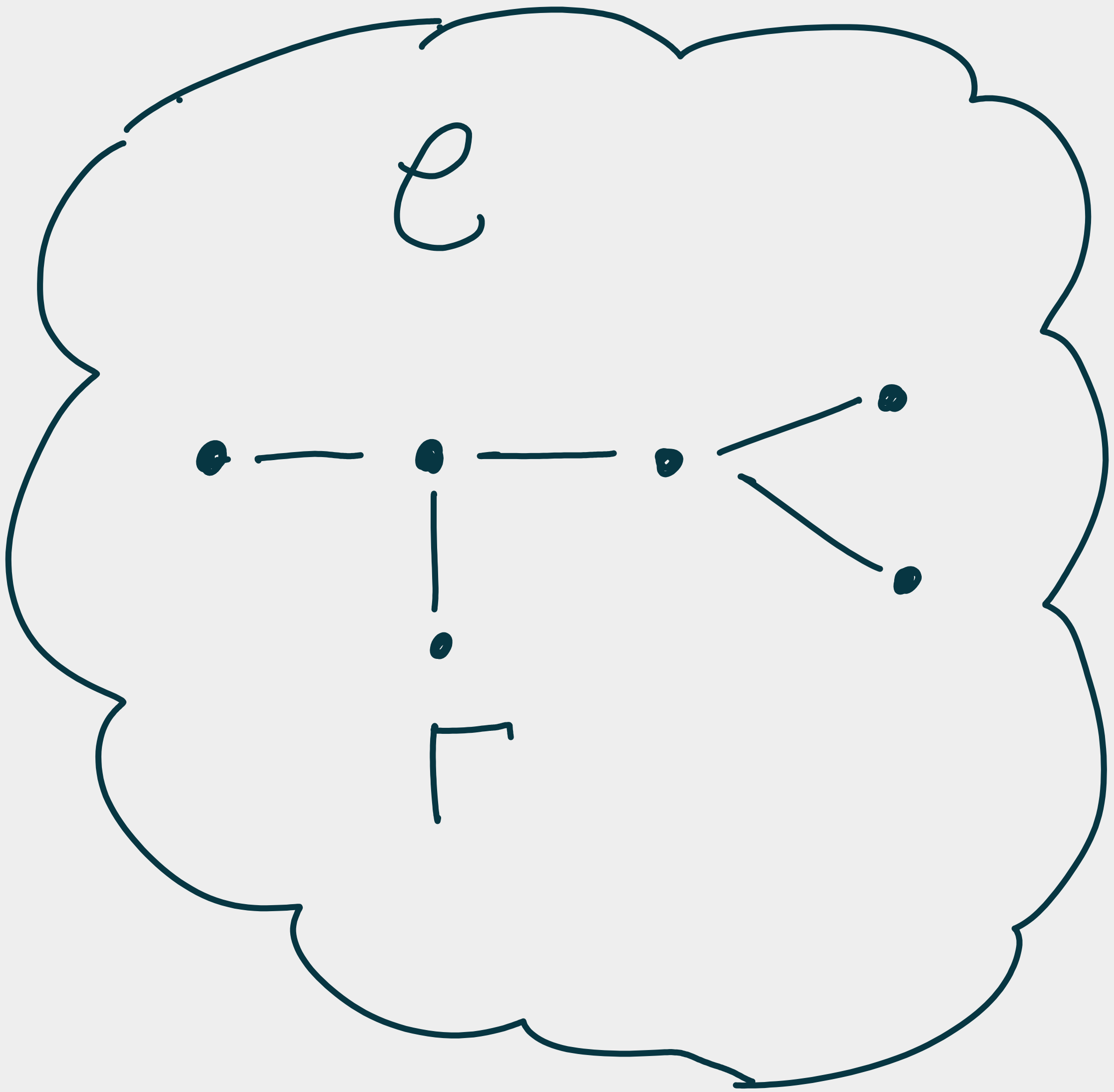
Γ - Configuration of sphericals



$$\dim \mathrm{Hom}^*(A_i, A_j) = 1 \quad \text{if} \quad i - j$$
$$= 0 \quad \text{if} \quad i \not- j$$

Then σ_{A_i} satisfy braid relations

Braid group actions



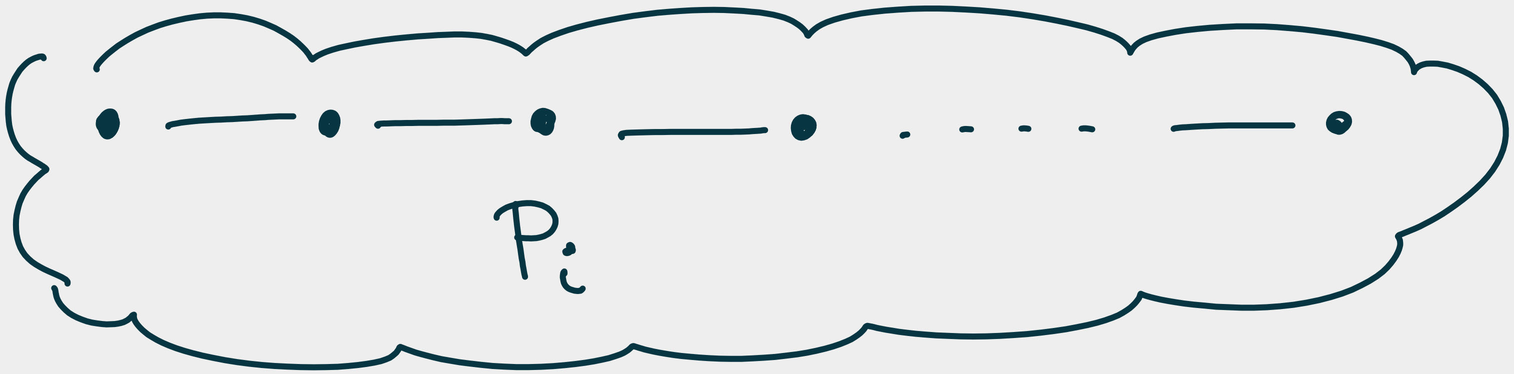
$$\Rightarrow \quad \begin{tikzcd} B_\Gamma \arrow[d] \arrow[r, phantom, "\curvearrowright"] & \mathcal{C} \arrow[d] \\ W_\Gamma \arrow[r, phantom, "\curvearrowright"] & K_0\mathcal{C} \end{tikzcd}$$

The Category $\mathcal{C}_\Gamma$

$$\Gamma : \quad \bullet - \bullet - \underset{P_i}{\bullet} - \bullet \cdots - \bullet$$

$\mathcal{C}_\Gamma$ is 2-CY, generated by P_i

The Category $\mathcal{C}_\Gamma$



$$\Downarrow$$

$$\begin{tikzcd}
B_\Gamma \arrow[d] & \circlearrowright & \mathcal{C}_\Gamma \arrow[d] \\
W_\Gamma & \circlearrowright & K_0\mathcal{C}_\Gamma
\end{tikzcd}$$

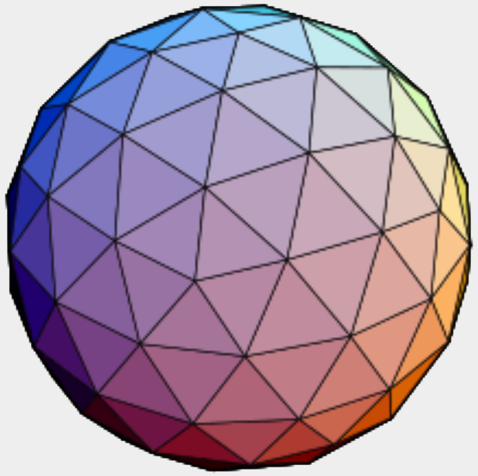
$\leftarrow$ Geometric / Burau representation

Main Picture

$$B_n \circlearrowright \mathcal{C}_n = \mathcal{C}_{A_n}$$

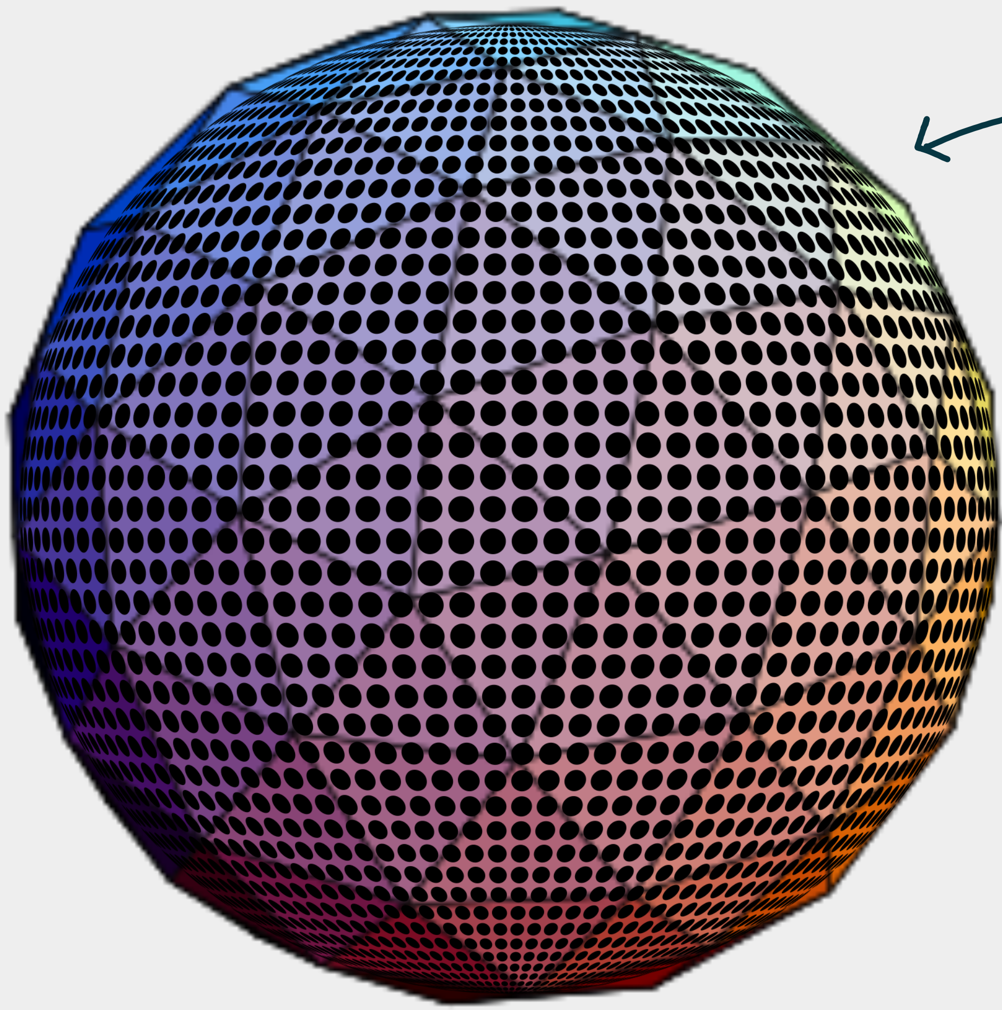
$$\rightsquigarrow$$

$$B_n \circlearrowright$$



Piecewise $\mathbb{Q}$-linear Sphere of dim $2n-3$

The Sphere of Spherical Objects



(Dense) Set of $\mathbb{Q}$ - points

$\updownarrow$

Spherical Objects

Main Picture

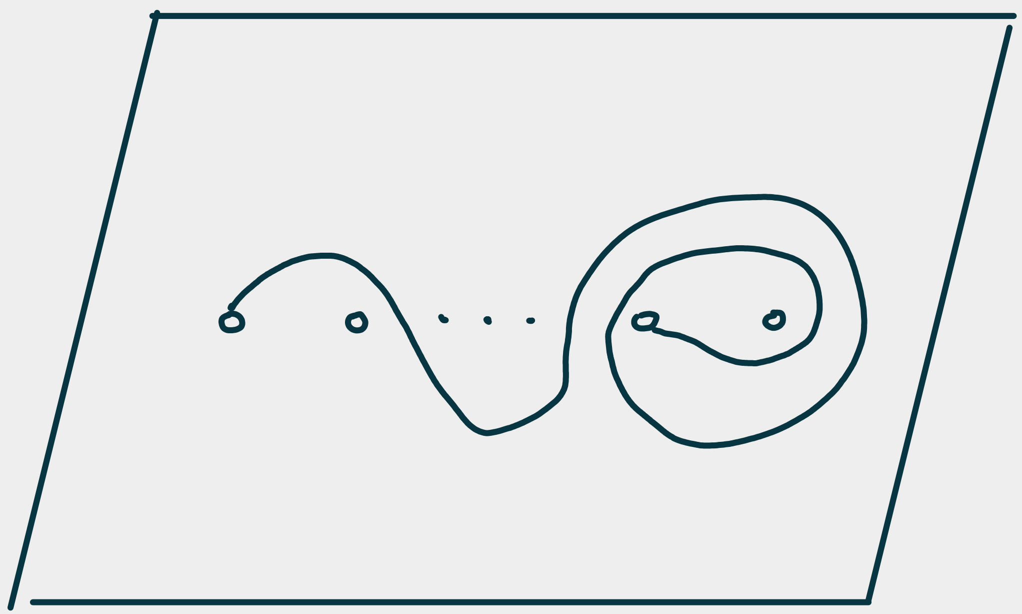
1. Spherical objects $\longleftrightarrow$ $\mathbb{Q}$-points of a PL-manifold

2. $B \quad G \quad \mathcal{C}$ $\rightsquigarrow$ PL-action on the manifold.

Q: How general is this picture?

$\mathcal{C}_n$ and the punctured plane

$$\mathcal{C}_n = \mathcal{C}_{A_n}$$

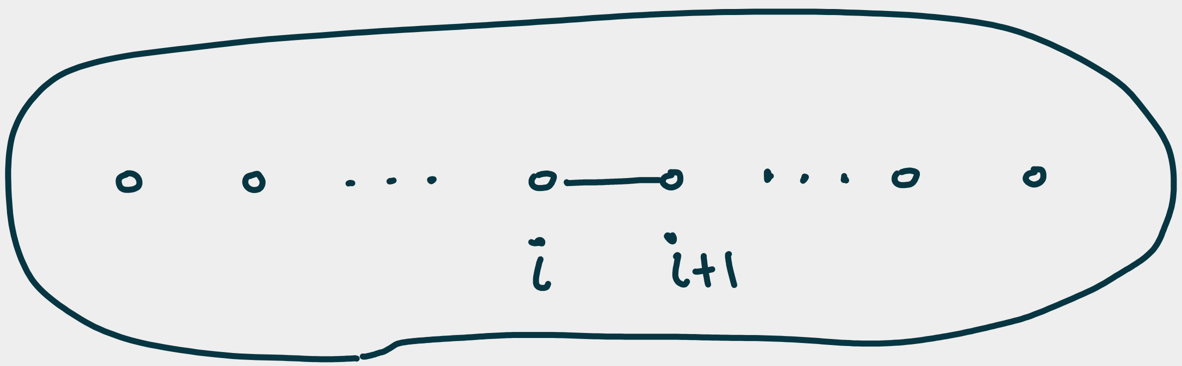


Curve in $\mathbb{R}^2 - (n+1)$ pts

$\Big\downarrow$ (Khovanov-Seidel)

Spherical Object of $\mathcal{C}_n$

$\mathcal{C}_n$ and the punctured plane



$\rightsquigarrow \quad P_i \in \mathcal{C}_n$

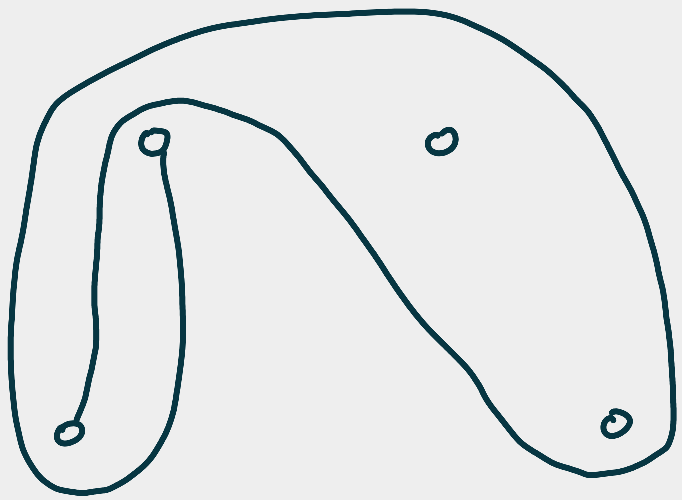
Curves $\rightsquigarrow$ Objects

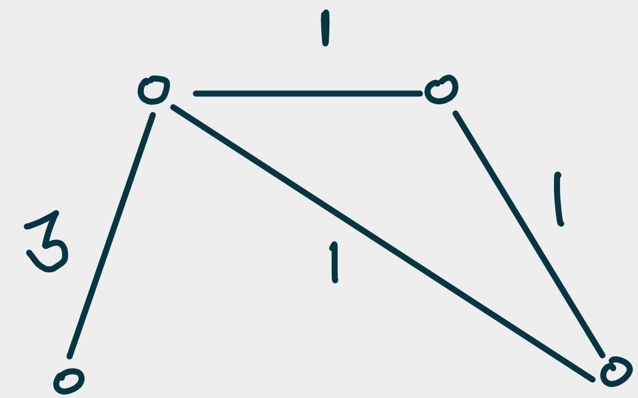
$\circlearrowleft \qquad\qquad \circlearrowleft$

$B_n \qquad\qquad B_n$

$\mathcal{C}_n$ and the punctured plane

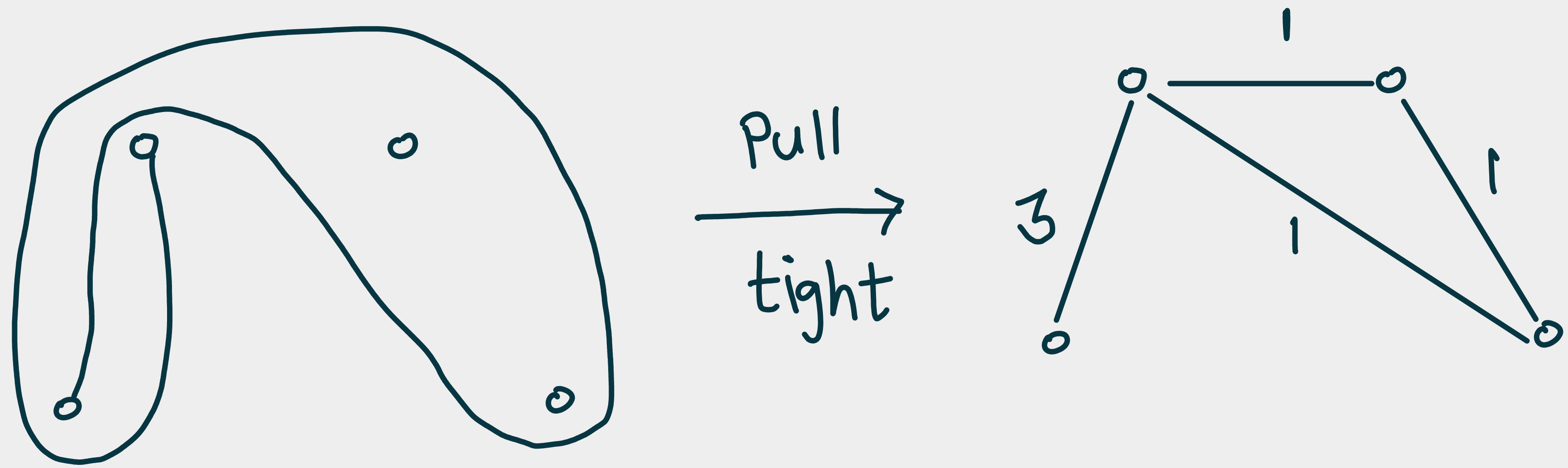
Fix a configuration Σ



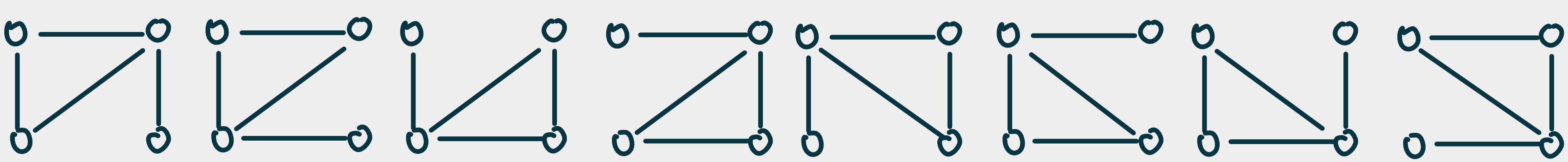


$$\text{Curve} \xrightarrow[\text{tight}]{\text{Pull}} \text{Multiset of edges}$$

C_n and the punctured plane

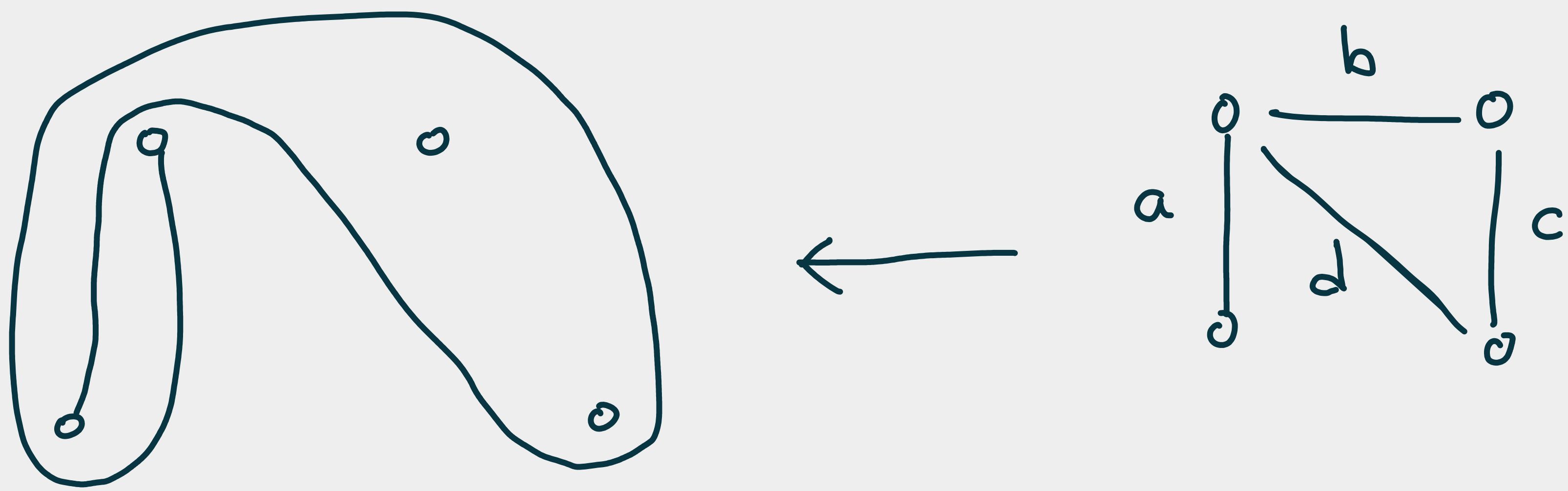


Possible supports = Triangulation − external edge



$\mathcal{C}_n$ and the punctured plane

Conversely



(multi)-Curve $\longleftarrow$ $\mathbb{Z}_{\geq 0}$-weighted triangulation$^{\circ}$

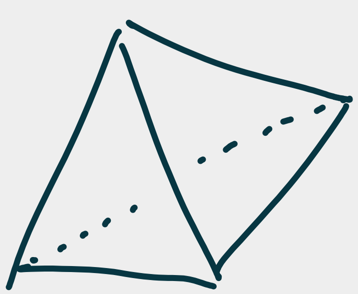
($\oplus$ objects)

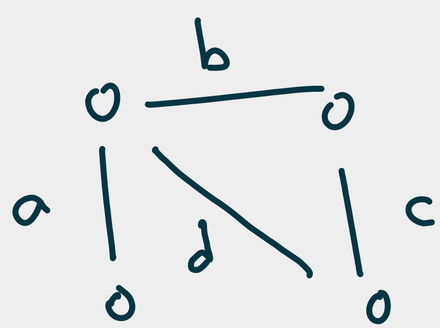
$\mathcal{C}_n$ and the punctured plane

Spherical Objects of $\mathcal{C}_n$ $\longleftrightarrow$ Curves on the $(n+1)$ punctured plane $\longleftrightarrow$ $\mathbb{Z}_{\geq 0}$ weighted triangulations° up to scaling

$\updownarrow$

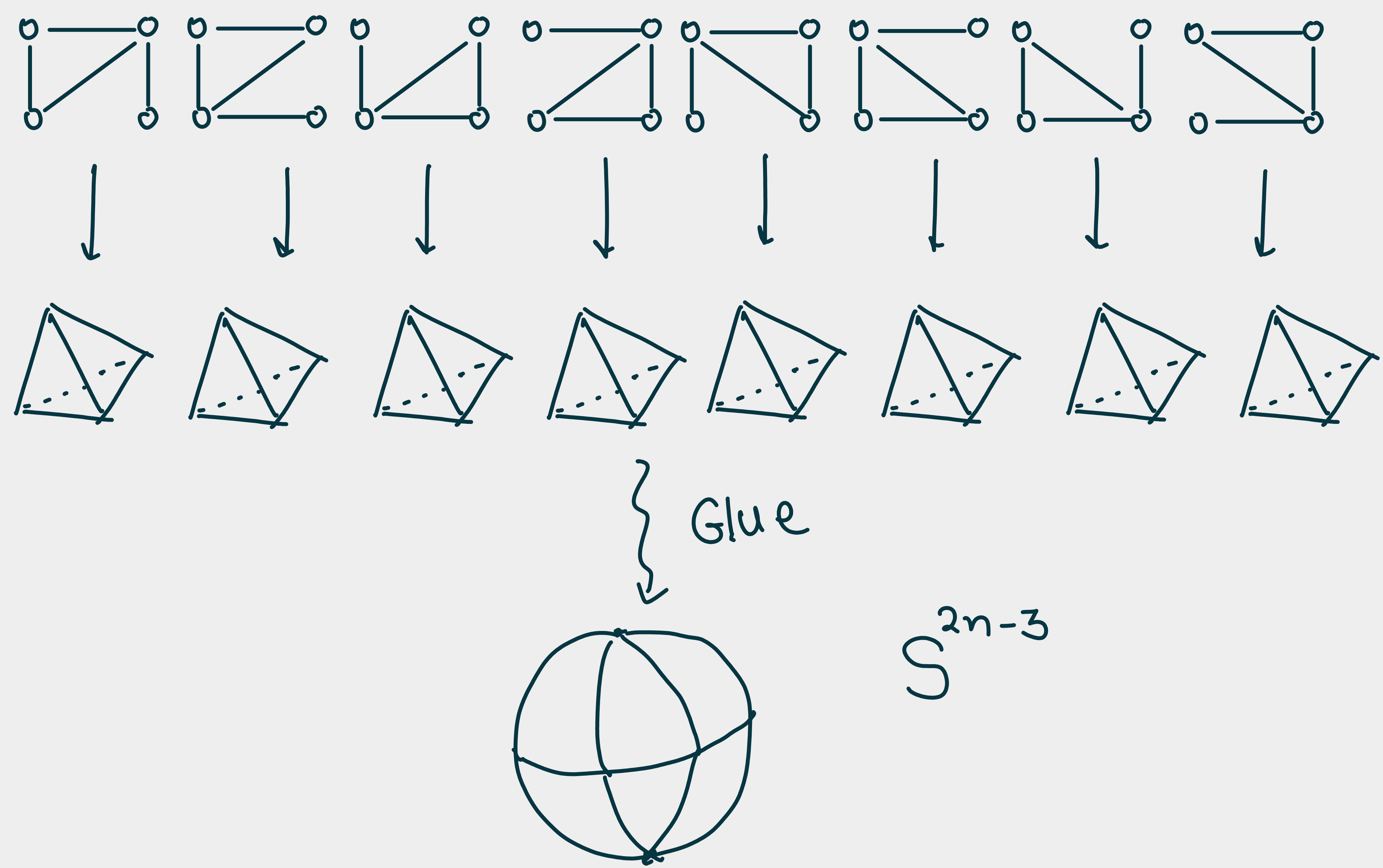
$\mathbb{Q}$-pts of





$a, b, c, d \in \mathbb{Z}_{\geq 0}$ mod scaling

$\mathcal{C}_n$ and the punctured plane



S^{2n-3}

A categorical interpretation

Choice of a point configuration

in $\mathbb{R}^2 = \mathbb{C}$

$\rightsquigarrow$

Choice of a Bridgeland Stability Condition on $\mathcal{C}_n$

Bridgeland Stability Conditions

A stability condition σ on $\mathcal{C}$ includes

⊛ A collection of 'semistable' objects satisfying Harder-Narasimhan property:

Every $X \in \mathcal{C}$ has a unique* filtration

$$0 \to X_1 \to X_2 \to \cdots \to X_n = X$$

with semistable

$$\begin{tikzcd} X_{i-1} \arrow[rr] & & X_i \arrow[dl] \\ & A_i \arrow[ul, dotted] & \end{tikzcd}$$

(+ more ...)

Bridgeland Stability Conditions

Stability Condition on $\mathcal{C}$ $\rightsquigarrow$ Metric on $\mathcal{C}$

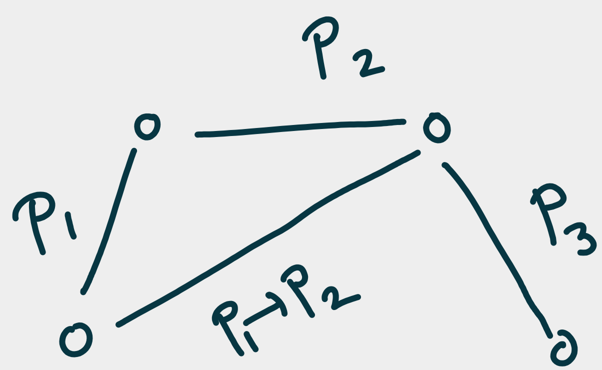
$$0 \xrightarrow{A_1} \cdot \xrightarrow{A_2} \cdot \xrightarrow{A_3} \cdot \xrightarrow{\cdots} \cdot \xrightarrow{A_n} X$$

HN factors = Steps on a geodesic path

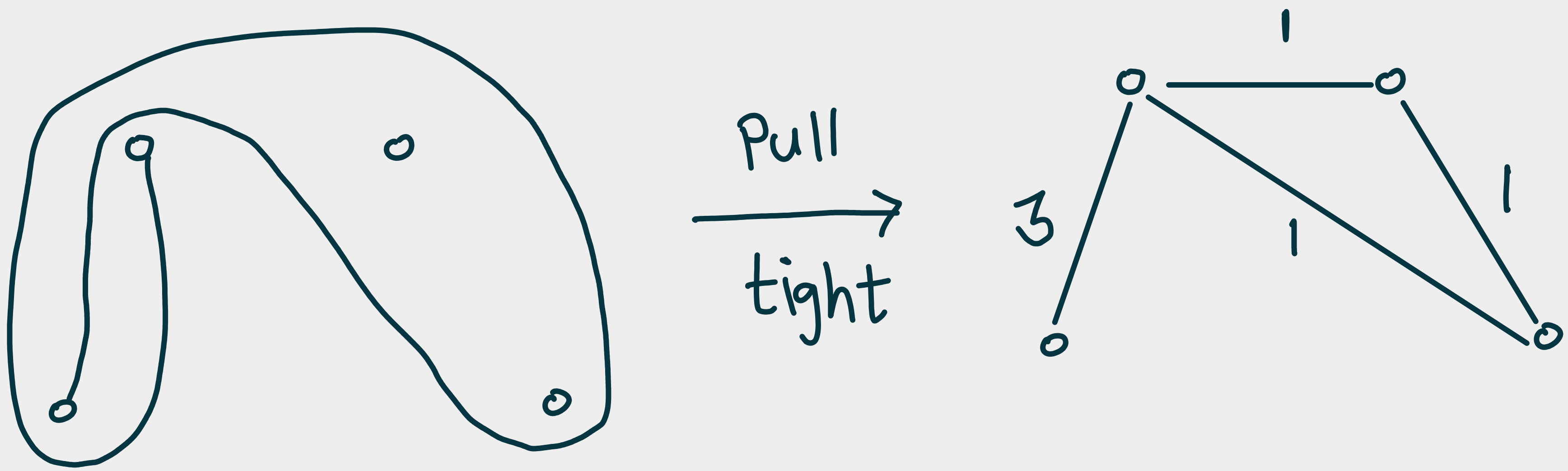
A categorical interpretation

$\rightsquigarrow$ σ where semi-stable
$\|$
Represented by a straight line

Example :



A categorical interpretation



$$X \xrightarrow[\text{decomposition}]{\text{HN}} 3P_1 + P_2 + P_3 + (P_1 \to P_2)$$

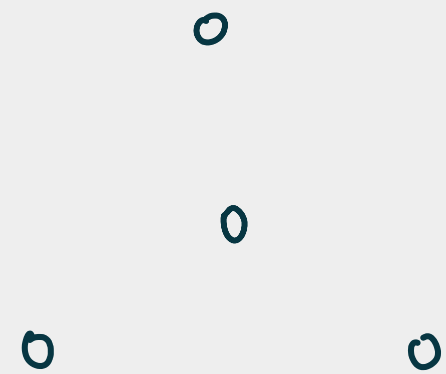
A categorical interpretation

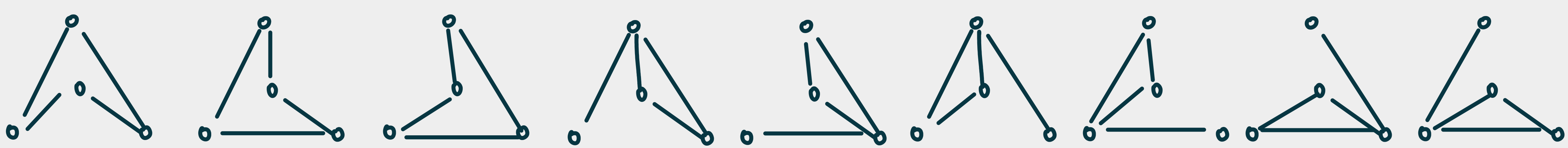
$$
\begin{tikzcd}
\text{Spherical Object} \arrow[r, rightsquigarrow, "HN"] \arrow[d, rightsquigarrow] & \text{Multi-set of Stable Spherical objects} \arrow[d, rightsquigarrow] \\
\text{Point on a manifold} & \text{Co-ordinates} \arrow[l, rightsquigarrow]
\end{tikzcd}
$$

(Type A ✓, DE, $\mathcal{C}_\Gamma$, more generally?)

Wall Crossing

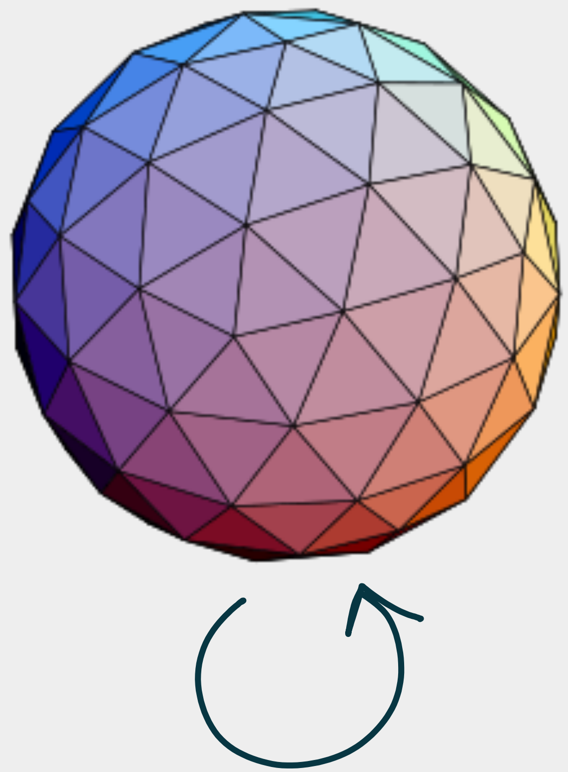
Change σ





Get a different Simplicial Structure
(PL-isomorphic to previous)

Linearising the action

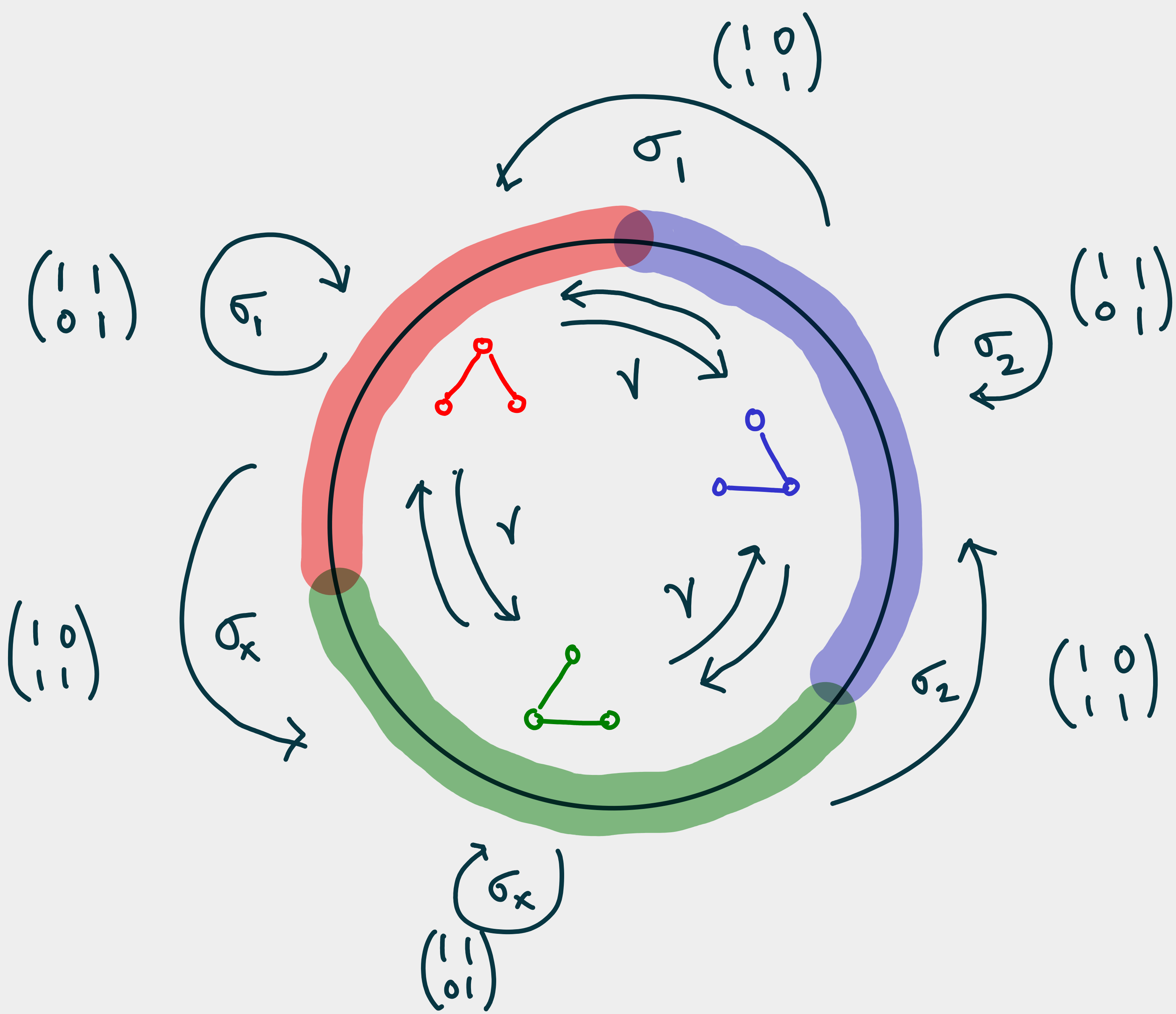


B_n

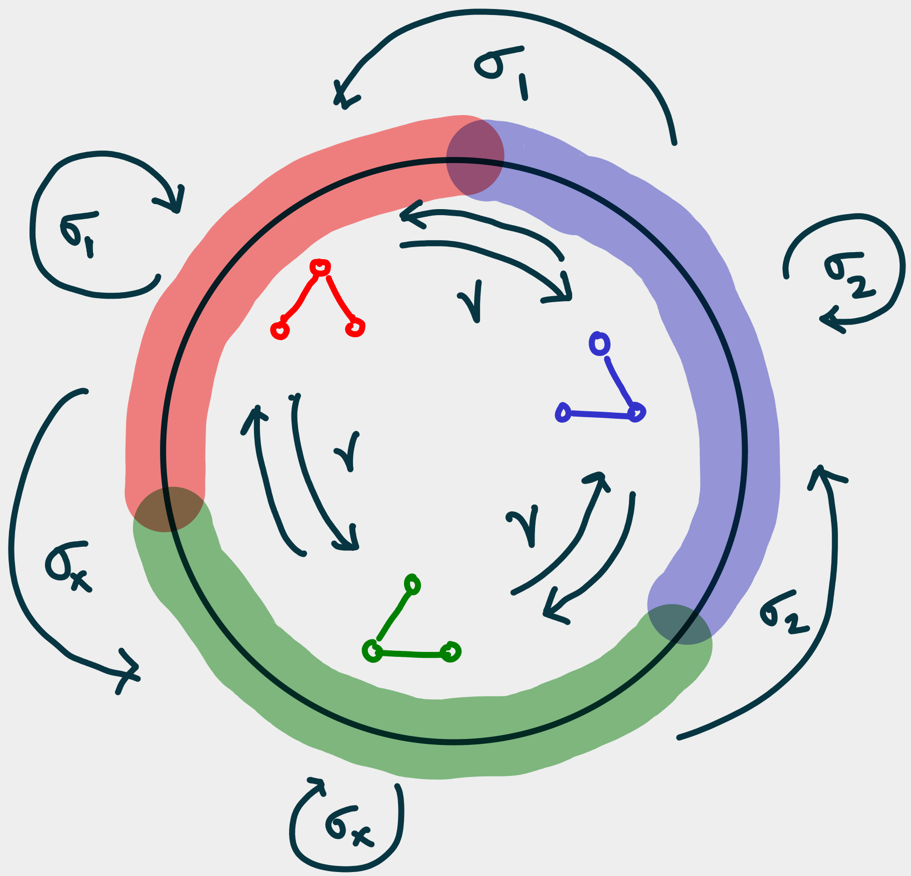
Dynamics ?

Linearising the action - HN automata

$n=2$ (3 strand braid group)



Linearising the action - HN automata



① Every braid has an acceptable word

② Every braid has a power conjugate to a loop (Edmund Heng)

⇓

☺ Dynamical properties

THANK YOU!