Vector bundles and finite covers

$f : X \to Y$ $\rightsquigarrow$ $f_* \mathcal{O}_X$

Finite flat — Vector bundle on Y.

Question: Which vector bundles arise in this way?

Endow V with the structure of an $\mathcal{O}_Y$-algebra. Then $X = \underline{\operatorname{Spec}}_Y V$. $\overset{?}{\leftsquigarrow}$ V vector bundle on Y

Question: Which vector bundles admit the structure of an $\mathcal{O}_Y$-algebra?

Suppose $V = f_* \mathcal{O}_X$.
We have

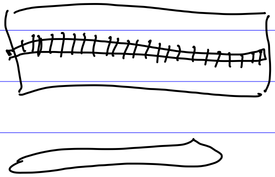
$$\begin{tikzcd} \mathcal{O}_Y \arrow[r, "``1\text{''}"'] & V \arrow[l, bend right, "\frac{1}{d}\,\mathrm{tr}"'] \end{tikzcd}$$

(char $\nmid d$)

So $$V = \mathcal{O}_Y \oplus E^\vee \qquad \text{———} \quad ①$$

Answer: Any such V admits an algebra structure.
Take $E^\vee \otimes E^\vee \to V$ to be $\underline{\underline{\text{zero}}}$.

X = Thickening $\subset E$
of zero section.



Modified Q: X, Y smooth, connected.

Then E exhibits positivity.

- $H^0(Y, E^\vee) = 0$
- For $Y = \mathbb{P}^n$, then E is ample (Lazarsfeld)
- E is weakly positive, so nef if $\dim Y = 1$.
  (Peternell-Sommese)

Not sufficient.

Example: $Y = \mathbb{P}^1$. $f : X \to Y$

Then $E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$,
where $a_1, a_2, \ldots, a_{d-1} > 0$.
called "Scrollar invariants" of X.

$d=2$: Any $a_1 > 0$ can be a scrollar invariant.

$d=3$: $a_1 \geq a_2 > 0$ are scrollar invariants iff
$$2a_2 \geq a_1 \geq a_2.$$

In general, a necessary condition for $a_1, \ldots, a_{d-1}$ to be scrollar invariants is that they are not "too far apart." (Ohbuchi, Coppens, Martens).

e.g. (Ohbuchi) $\Rightarrow$ $(d-1)\, a_{d-1} \geq a_1$ (barring some exceptions)

Asymptotic Q: Does every E arise from a finite cover up to twisting by a line bundle?

e.g. $\mathcal{O}(1) \oplus \mathcal{O}(99)$ — NO

$\mathcal{O}(1001) \oplus \mathcal{O}(1099)$ — YES!

Thm 1: Let Y be a smooth curve and E a v.b on Y. There exists N (depending on Y, E) such that for every line bundle L of degree $\geq N$, the twist $E \otimes L$ arises from a cover $f: X \to Y$ with smooth X.

Let $H_{d,g}(Y) = \left\{ f: X \to Y \;\middle|\; \begin{matrix} g(X) = g \\ \deg f = d \end{matrix} \right\}$

$$\begin{tikzcd} H_{d,g}(Y) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\ f \arrow[r, squiggly] & E_f \text{ of } \operatorname{rk}(d-1) \text{ \& } \deg b = g-1-d(g_Y-1). \end{tikzcd}$$

$$M_{d-1,b} = \{ \text{v.b. of rk } d-1 \text{ \& deg } b \text{ on } Y \}$$

Thm 2: Suppose $g(Y) \geq 2$. If g is sufficiently large, then a general $f \in H_{d,g}(Y)$ gives a stable E_f. Also, the map

$$\begin{tikzcd} H_{d,g}(Y) \arrow[r, dashed] & M_{d-1,b}(Y) \\ f \arrow[r, mapsto] & E_f \end{tikzcd}$$

is dominant.

Rem: Thm 2 proved by Kanev for $d \leq 5$ (2004, 05, 13). Using explicit structure theorems for coverings of deg ≤ 5.

NO such theorems for $d \geq 6$!

Q: Effective bounds ?

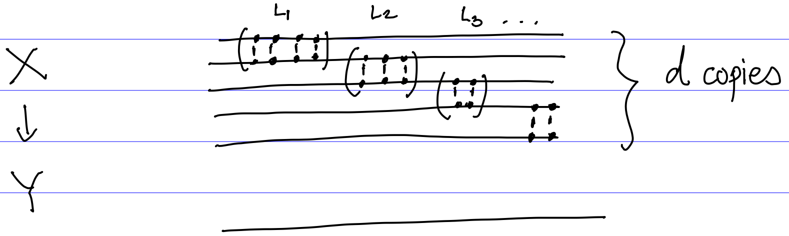
Ideas behind the proof

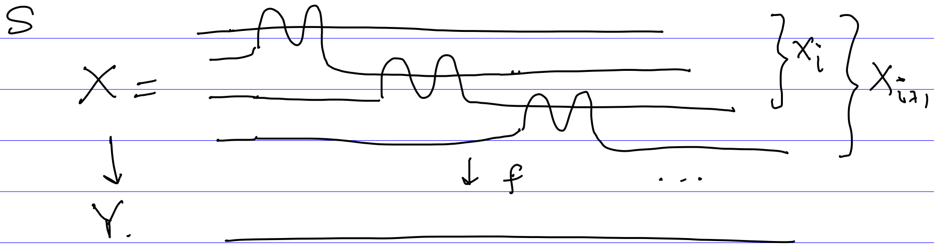
Details: Slightly weaker statement.

Thm: Every projective bundle $\mathbb{P}E$ arises from $X \to Y$ with smooth X.

Step 1:

$$E = L_1 \oplus \cdots \oplus L_{d-1}, \quad \deg L_i \gg \deg L_{i+1}$$





$$\begin{tikzcd}
0 \arrow[r] & L_i^\vee \arrow[r] & f_*\mathcal{O}_{X_{i+1}} \arrow[r] & f_*\mathcal{O}_{X_i} \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
0 \arrow[r] & L_i^\vee \arrow[r] & ? \arrow[r] & \mathcal{O} \oplus L_1^\vee \oplus \cdots \oplus L_{i-1}^\vee \arrow[r] & 0
\end{tikzcd}$$

Must be split because $\deg L_i \ll \deg L_j$ for $j < i$.

So inductively

$$f_* \mathcal{O}_X = \mathcal{O}_Y \oplus L_1^\vee \oplus \cdots \oplus L_{d-1}^\vee .$$

$$\Rightarrow \quad E_f = L_1 \oplus \cdots \oplus L_{d-1}.$$

BUT X is not smooth!.

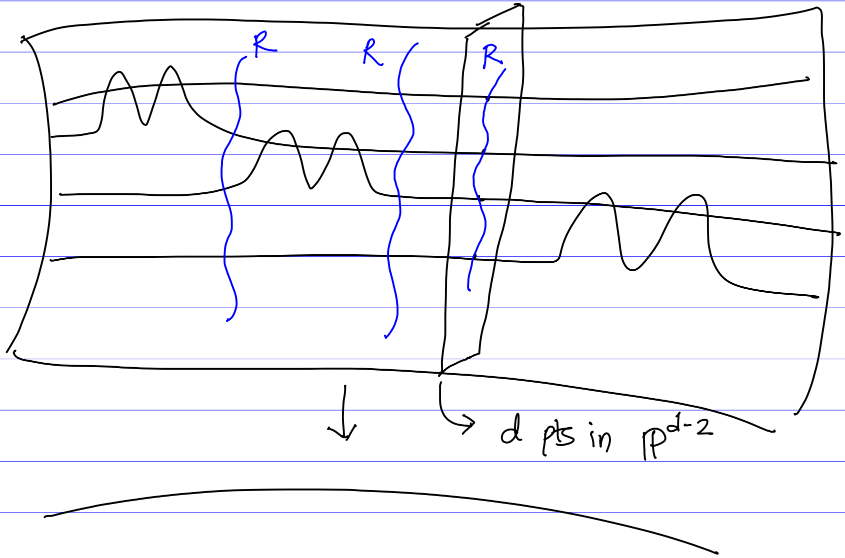
Basic fact:

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$ finite flat map of degree d

Then we have a canonical embedding i

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}E_f \arrow[d] \\ & Y \end{tikzcd}$$

So $\mathbb{P}E$



Smooth out X inside $\mathbb{P}E_f =: \mathbb{P}$

But $N_{X/\mathbb{P}}$ is typically negative.

Solution: $X' = X \cup \{\text{Rational normal curves}\}$.

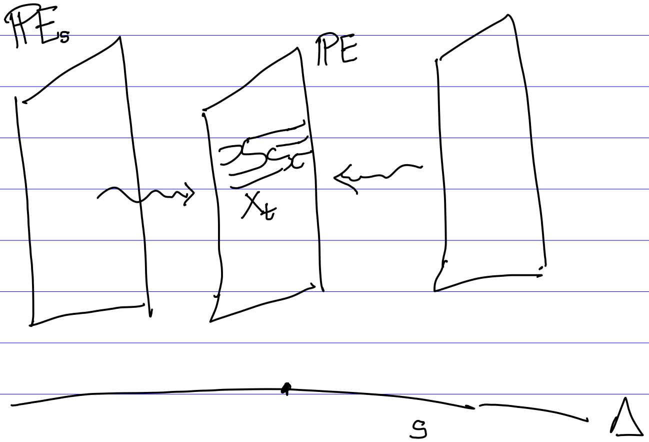
Prop: For generic choices of Rational normal curves, $N_{X'/\mathbb{P}}$ becomes sufficienty positive. i.e

① $H^0(N_{X'/\mathbb{P}}) \twoheadrightarrow H^0(K_s)$ $\forall\, s \in \operatorname{sing} X'$

② $H^1(N_{X'/\mathbb{P}}) = 0$.

Conseq: 1) X' is a limit of smooth $X_t \subset \mathbb{P}E$ (X_t give the same scroll $\mathbb{P}E$).

2) $BrCov \xrightarrow{\tau} ProjBun$ is smooth at $[X_t \to Y]$.
so any scroll that isotrivially deg. to $\mathbb{P}E$ arises from br. covers.



2) $\Rightarrow$ $Hilb/\Delta$ is smooth at $[X_t]$
so X_t can be deformed into the gen. fiber.

Step3: Any v.b. isotriv. specializes to
$L_1 \oplus \cdots \oplus L_{d-1}$
$\deg L_i >> \deg L_{i+1}$ (Exercise).

$\square$

Higher dimensions.

Let Y be a smooth proj var & L an ample line bundle on Y

⊛ Given a v.b. E on Y, $E \otimes L^n$ arises from a finite cover for sufficiently large n.

Set $d = \mathrm{rk}\, E + 1$.

⊛ is false if $\dim Y \gg d$.
Consider the multiplication

$$E^\vee \otimes E^\vee \to \mathcal{O} \oplus E^\vee$$

Must be 0 for some $y \in Y$.
$\Rightarrow$ Fiber of $X \to Y$ over y is a fat point
Contradicts X is smooth (even Gorenstein).

Lazarsfeld $\Rightarrow$ $\ast$ is false for $Y = \mathbb{P}^r$, $r \geq d+1$.
For $\dim Y \geq d+1$, there are nontrivial restrictions on topology of X.

Q: Is $\ast$ true for $\dim Y \leq d$?