Log surfaces of almost K3 type and curves of genus 4

joint with Changho Han
(and Valery Alexeev, Phil Engel)

Broader Context -

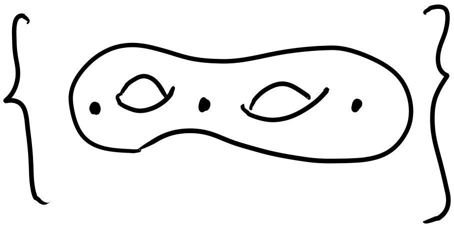
Problem - Understand compact moduli of varieties of (log) general type.

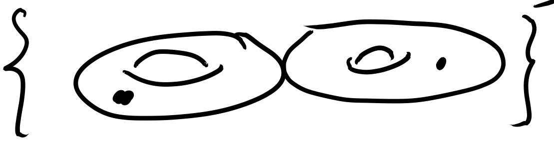
Fix $\varepsilon \in \mathbb{Q}_{>0}$

$$\left\{ \begin{array}{c} \text{Smooth } (X, D) \\ K_X + \varepsilon D \\ \text{ample} \end{array} \right\} \cup \left\{ \begin{array}{c} \text{Degenerate} \\ (X, D) \end{array} \right\}$$

Curves:

$$\left\{\begin{array}{l}\text{Smooth } (X,D) \\ K_X+\varepsilon D \text{ ample}\end{array}\right\} \cup \left\{\begin{array}{l}(X,D) \\ X \text{ nodal} \\ D \subset X^{sm} \\ \text{mult}_p(\varepsilon D) \leq 1 \\ \omega_X(\varepsilon D) \text{ ample}\end{array}\right\}$$





$\rightsquigarrow$ Projective coarse moduli well-understood. | Deligne, Mumford, Knudson, Hassett

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Higher dim:

$$\left\{\begin{array}{l}(X,D) \text{ sm.} \\ K_X+\varepsilon D \text{ ample}\end{array}\right\} \cup \left\{ (X,D) \ \dots \right\}$$

① $(X, \varepsilon D)$ has semi-log-canonical (SLC) singularities

② $K_X + \varepsilon D$ ample.

$\rightsquigarrow$ Projective coarse moduli.

(KSBA)

| Kóllar - Shepherd Barron
Alexeev
Birkar - Cascini - Hacon - McKernan
Xu, Kovácz - Patakfalvi
...

- Do not know much about the geometry (sing., tangent spaces, boundary comp...)
- Probably hopeless – satisfy Murphy's Law.

<u>BUT</u>. Important special cases are better behaved.

<u>Ex.</u> $K_X \sim 0$

- X an Abelian Variety
  $D \subset X$ the theta divisor.
  (Alexeev 2002, Olsson 2008).

- X a K3 surfaces of deg 2
  $D \in |L|$ Laza 2012
  $D =$ Ram. div. of $X \to \mathbb{P}^2$
  (Alexeev-Engel-Thompson 2018).

- Hacking : (2004). Fix a positive integer d.

KSBA compact. of $\{(S, (\frac{3}{d}+\varepsilon)D)\}$
where $S \cong \mathbb{P}^2$, D CS curve of deg d,
& ε very small.

- Smooth DM stack (if $3 \nmid d$).
- Fairly explicit description of the boundary
- $d=4$ recovers Schuberts compactification of M_3.

Salient feature -

$(S, (\frac{3}{d}+\varepsilon)D)$ is log gen. type

$K_S + (\frac{3}{d}+\varepsilon)D$ ample

but barely so.

"Almost K3"

Main def: Fix a positive rational $r = \frac{m}{n}$

An almost K3 stable log surface is a pair (S, D).

- S : Connected, reduced, Coh. Mac., Proj. surface
- D : Effective weil divisur on S

such that

① For all sufficiently small $\varepsilon > 0$, $(S, (r+\varepsilon) D)$ is stable.

 ↳ SLC + $K_S + (r+\varepsilon)D$ ample.

② $n K_S + m D \sim 0$

Rmk ① Both K_S & D are $\mathbb{Q}$-Cartier

② S smooth $\Rightarrow$ S del Pezzo

Goal: Understand moduli of almost K3 log surfaces.

Today: A highly interesting special case

$S \cong \mathbb{P}^1 \times \mathbb{P}^1$ (generically).

$D \subset S$ of type $(3,3)$.

Aside :- Families

$\pi : S \to B$ flat proper Cohen-Macaulay
rel dim 2

$D \subset S$ a relative Weil divisor

such that (i). $\omega_\pi^{[i]}$ and $\mathcal{O}(D)^{[i]}$
commute with base change $\forall\, i \in \mathbb{Z}$.

(ii) All geometric fibers are almost K3
stable log surfaces.

Main theorem:

Let $\mathcal{X}$ be the moduli stack of smoothable
almost K3 stable log surfaces (S, D)
with $K_S^2 = 8$ & $\rho_a(D) = 4$. Then $\mathcal{X}$
is proper with proj coarse space.

1. irreducible
2. smooth.
3. the boundary ($:= \mathcal{X} \setminus \mathcal{X}^\circ$
$\mathcal{X}^\circ = \{ (S, D) \mid S \,\&\, D \text{ smooth} \}$) is
the union of 4 irreducible divisors.

⑥ $\mathcal{X}^\circ \longrightarrow M_4$ is an isomorphism onto the complement of the Gieseker-Petri locus.

$$\begin{tikzcd} \mathcal{X}^\circ \arrow[r] \arrow[d, hook] & M_4 \arrow[d, equal] \\ \mathcal{X}^c \arrow[r] & M_4 \end{tikzcd}$$

$(S,D) \longmapsto D$

$\mathcal{X}^c \longrightarrow M_4$ is the blow up of the hyperell. locus.

$\mathcal{X}^c = \{(S,D) \mid D \text{ smooth}\}$

<u>Interior</u>: $\mathcal{X}^\circ \cong M_4$

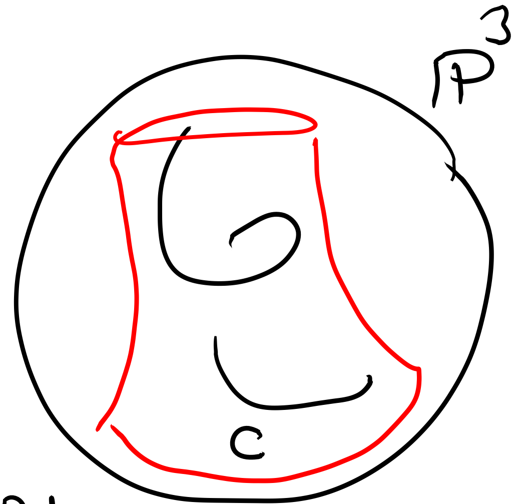
D smooth non-hyp. of genus 4

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^3 \\ S \arrow[ur, hook] & \end{tikzcd}$$

$D \hookrightarrow \mathbb{P}^3$ canonical

S unique quadric

(smooth if D is Gieseker-Petri general.

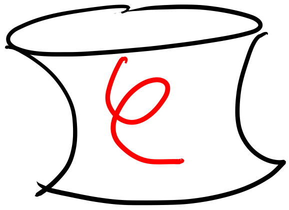


so $D \rightsquigarrow (S,D)$

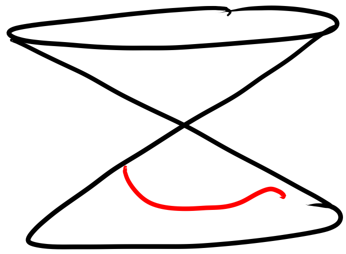
Boundary:

① (S, D) $\quad S \cong \mathbb{P}^1 \times \mathbb{P}^1$

D singular of type $(3,3)$



② $S \cong$ Quadric cone, $D \subset S$ general in $|-\frac{3}{2}K_S|$



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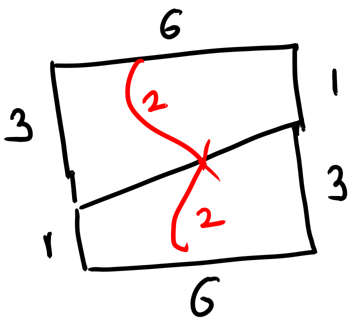
③ $S =$ Smoothing of $\mathbb{P}(1,2,9)$ at the A_1-sing.

$D \subset S$ general in $|-\frac{3}{2}K_S|$ $\quad (\Rightarrow D$ hyperelliptic$)$

④ $S =$ Toric deg. of $\mathbb{P}^1 \times \mathbb{P}^1$ given by



$D \in |-\frac{3}{2}K_S|$ generic.

Proof :- (Dream) :- Understand all $\mathbb{Q}$-Gor. degenerations of $\mathbb{P}^1 \times \mathbb{P}^1$ & use this to get the boundary (Manetti, Hacking-Prokhorov for $\mathbb{P}^2$).

(Reality) :- Tour de force.

Relationship with other moduli spaces

① $\mathcal{X} \dashrightarrow M_4$

② K3 surfaces.

$(S, D) \rightsquigarrow T :=$ cyclic triple cov. of S along D.

$T \downarrow$ K3 with a lattice polarization coming from the pull-backs of the two rulings of S + μ_3-action.

Lattice Pol. Hodge-Struct. with μ_3-symm. $\overset{\sim}{\leftsquigarrow}$

$\uparrow$

Type I period domain D

$$
\begin{tikzcd}
 & & \overline{(D/\Gamma)}^{\text{toroidal}} \arrow[dll, bend right] \arrow[dd] \\
\mathcal{X} \arrow[d, dashed, leftrightarrow] \arrow[drr] & & \\
M_4 \arrow[r, dashed, leftrightarrow, "(\text{Kondo})"'] & D/\Gamma \arrow[r, hook] & \overline{(D/\Gamma)}^{\text{B.B.}}
\end{tikzcd}
$$

Key to our understanding of $\mathcal{X}$.

$$
\begin{tikzcd}
 & \overline{H}_{3,4}\left(\frac{1}{6}+\varepsilon\right) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, dashed] \\
\left\{ \begin{array}{l} C \text{ genus } 4 \\ \downarrow 3 \\ \mathbb{P}^1 \end{array} \right\} = & H_{3,4} \arrow[r, "2:1"] & M_4
\end{tikzcd}
$$