Quadrature and algebraic geometry

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Numerical integration using quadrature

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$$\int f \; d\mu$$

for functions f on X.

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Solution (One approach)

Find $x_1, \ldots, x_n \in X$ and $w_1, \ldots, w_n \in \mathbf{R}$ or $\mathbf{C}$ so that

$$\int f \; d\mu \approx \sum_{i=1}^{n} w_i f(x_i).$$

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$$\int f \; d\mu = \sum_{i=1}^{n} w_i f(x_i)$$

for f in a suitable subspace V of the space of functions on X.

One dimensional case

Take $X = \mathbf{C}^1$ and $V = \{\text{Polynomials of degree up to } m\}$.

Find $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ such that

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

for all $f \in V$.

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for all $f \in V$.

1. What is the minimum possible n?
2. For this n, how do we find x_i and w_i?

Quadrature equations: one dimensional case

Choose a basis of V

$$V = \langle 1, x, \dots, x^m \rangle.$$

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Write the quadrature equation for the basis elements:

$$w_1 \cdot x_1^i + w_2 \cdot x_2^i + \cdots + w_n \cdot x_n^i = \int_{-1}^{1} x^i dx,$$

for $i = 1, \ldots, m$.

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where $a_i = \int x^i dx$ are constants.

A geometric view: one dimensional case

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^m & x_1^m & \dots & x_n^m \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m. \end{pmatrix}$$

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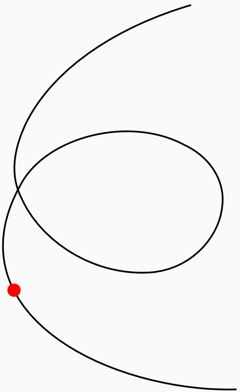
Consider $X \subset \mathbf{P}^m$ given by the Zariski closure of the set of points with homogeneous coordinates $[1 : x : \dots : x^m]$.

X is called the *rational normal curve of degree m*.

A geometric view: one dimensional case

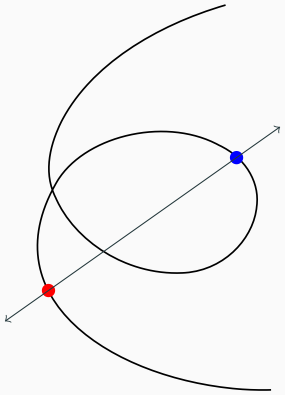
$$\begin{pmatrix} 1 \\ x_0 \\ \vdots \\ x_0^m \end{pmatrix} \in$$



Rational normal curve of degree m.

A geometric view: one dimensional case

$$\begin{pmatrix} 1 & 1 \\ x_0 & x_1 \\ \vdots & \vdots \\ x_0^m & x_1^m \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \equiv$$



Rational normal curve of degree m.

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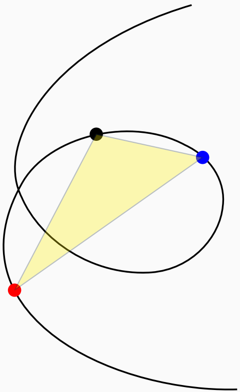

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Secant varieties

Definition

Let X be a variety in projective space. The i-th secant variety $\sigma_i(X)$ is the Zariski closure of the union of all sub-spaces spanned by i points of X.

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The system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^m & x_1^m & \dots & x_n^m \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m. \end{pmatrix}$$

has a solution only if $[a_0 : \cdots : a_m] \in \sigma_{n+1}(X)$.

General case

X an algebraic variety.

V a finite dimensional space of algebraic functions on X.

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Choose a basis $\langle f_0, \ldots, f_m \rangle$ for V. Then we get:

$$w_1 \cdot f_i(x_1) + w_2 \cdot f_i(x_2) + \cdots + w_n \cdot f_i(x_n) = a_i,$$

where $a_i = \int f \ d\mu$ are constants.

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A geometric view: general case

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say: "Find $x_0, \dots, x_n \in X$ such that the span of $f(x_0), \dots, f(x_n)$ contains $[a_0 : \cdots : a_m]$."

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Let $\overline{X}$ be the closure of $f(X) \subset \mathbf{P}^m$. Quadrature equations have a solution only if $[a_0 : \cdots : a_m] \in \sigma_{n+1}\left(\overline{X}\right)$.

Examples in higher dimension

1. If $V = \langle 1, x, y, xy \rangle$, then

$$\overline{X} = \overline{\{[1 : x : y : xy] \mid x \in \mathbf{C}, y \in \mathbf{C}\}} \subset \mathbf{P}^3$$

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2. If $V = \{\text{All polynomials in } x, y \text{ of total degree at most } n\}$, then $\overline{X}$ is the Veronese surface $v_n(\mathbf{P}^2) \subset \mathbf{P}^N$ where $N = \binom{n+2}{2} - 1$.

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3. If $V = \{$All polynomials in x, y of x and y degree at most $n\}$, then $\overline{X}$ is the Segre–Veronese surface $v_n(\mathbf{P}^1) \times v_n(\mathbf{P}^1) \subset \mathbf{P}^N$ where $N = (n+1)^2 - 1$.

Recap

Given X and V, an $(m+1)$ dimensional space of algebraic functions on X, we get $\overline{X} \subset \mathbf{P}^m$.

Quadrature equations for X and V are related to the secant varieties $\sigma_{n+1}(\overline{X})$.

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1. Determine if $[a_0 : \cdots : a_m] \in \sigma_{n+1}(\overline{X})$.
2. If yes, find $(n+1)$ points $x_0, \ldots, x_n \in X$ whose span contains $[a_0 : \cdots : a_m]$.

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2. If yes, find $(n+1)$ points $x_0, \ldots, x_n \in X$ whose span contains $[a_0 : \cdots : a_m]$.
3. Find numerically stable and feasible algorithms to answer the above questions.

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Let $X \subset \mathbf{P}^m$ be the rational normal curve of degree m, with $m = 2k + 1$.

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Solution (First question (Sylvester, 1852))

The point $[a_0 : \cdots : a_m]$ *lies in* $\sigma_{n+1}(X)$ *if and only if the Catalecticant matrix*

$$\begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k+1} \\ a_2 & a_3 & \dots & a_{k+2} \\ \vdots & & \ddots & \vdots \\ a_{k+1} & a_{m+2} & \dots & a_{2k+1} \end{pmatrix}$$

has rank at most $(n+1)$*. In particular, if the* a_i *are generic, then* $[a_0 : \cdots : a_{2k+1}] \in \sigma_{k+1}(X)$*.*

Dimension one: answers

Solution (Second question (Sylvester, 1852))

Let $(\lambda_0, \ldots, \lambda_{k+1})$ *be in the (right) kernel of the catalecticant. Then the* $(k+1)$ *points* $x_0, \ldots, x_k$ *whose span contains* $[a_0 : \cdots : a_{2k+1}]$ *satisfy*

$$\lambda_{k+1}X^{k+1} + \lambda_k X^k + \cdots + \lambda_0 = 0.$$

For quadrature, this polynomial is called the Legendre polynomial.

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Example ($m = 5$)

In this case, $[a_0 : \cdots : a_5] = [2 : 0 : \frac{2}{3} : 0 : \frac{2}{5} : 0]$.

$$\text{Catalecticant} = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \\ 0 & \frac{2}{5} & 0. \end{pmatrix} \rightsquigarrow \text{Legendre polynomial} = \frac{5}{2}X^3 - \frac{3}{2}X.$$

Better answers for numerical methods?

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{2k+1} & x_1^{2k+1} & \dots & x_k^{2k+1} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2k+1} \end{pmatrix},$$

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- Pick a_i, eliminate w_i, and all but one $x_i \rightsquigarrow$ Legendre polynomial.

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- Pick a_i, eliminate $w_i \rightsquigarrow$ An ideal

$$\langle h_0(x_0, \dots, x_k), h_1(x_1, \dots, x_k), \dots, h_{k-1}(x_{k-1}, x_k), h_k(x_k) \rangle$$

with $\deg h_i = i + 1$.

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$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{2k+1} & x_1^{2k+1} & \dots & x_k^{2k+1} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2k+1} \end{pmatrix},$$

- Pick a_i, eliminate w_i, and all but one $x_i \rightsquigarrow$ Legendre polynomial.
- For $m = 5$, pick a_i, eliminate $w_i \rightsquigarrow$

$$\left\langle x_0 + x_1 + x_2, 5x_1^2 + 5x_1x_2 + 5x_2^2 - 3, 5x_2^3 - 3x_2 \right\rangle$$

▸ Summary

Better answers for numerical methods?

Eliminating the w_i

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{2k+1} & x_1^{2k+1} & \dots & x_k^{2k+1} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2k+1} \end{pmatrix}$$

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implies

$$\operatorname{rk} \begin{pmatrix} 1 & 1 & \dots & 1 & a_0 \\ x_0 & x_1 & \dots & x_k & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{2k+1} & x_1^{2k+1} & \dots & x_k^{2k+1} & a_{2k+1} \end{pmatrix} < k+1$$

Better answers for numerical methods?

Eliminating the w_i

$$\operatorname{rk}\left(\begin{array}{ccccc} 1 & 1 & \dots & 1 & a_0 \\ x_0 & x_1 & \dots & x_k & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^k & \dots & \dots & x_k^k & a_k \\ \hline x_0^{k+1} & \dots & \dots & x_k^{k+1} & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^{2k+1} & x_1^{2k+1} & \dots & x_k^{2k+1} & a_{2k+1} \end{array}\right) < k+1$$

Better answers for numerical methods?

Eliminating the w_i

$$\operatorname{rk}\begin{pmatrix} V & A \\ W & B \end{pmatrix} < k+1$$

Better answers for numerical methods?

Eliminating the w_i

$$\operatorname{rk}\begin{pmatrix} V & A \\ W & B \end{pmatrix} < k+1$$

equivalent to

$$\operatorname{rk}\begin{pmatrix} V & A \\ 0 & B - WV^{-1}A \end{pmatrix} < k+1,$$

that is

$$B - WV^{-1}A = 0.$$

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Proposition

WV^{-1} is a matrix of polynomials.

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Eliminating the w_i

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equivalent to

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Proposition

WV^{-1} is a matrix of polynomials.

The Gröbner basis of $B - WV^{-1}A = 0$ has the form

$$\langle h_0(x_0, \ldots, x_k), h_1(x_1, \ldots, x_k), \ldots, h_k(x_k)\rangle.$$

Summary

- (Enrico) Studying ranks of tensors is related to studying secant varieties of suitable X.
- (Today) Studying quadrature equations of polynomials is *also* related to studying secant varieties of suitable X.
- Better understanding the polynomial equations arising in the context of secant varieties, especially from the point of view of numerical analysis, will be valuable for multiple goals.