APPARENT BOUNDARIES

OF

PROJECTIVE VARIETIES

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GUESS THE SEQUENCE!

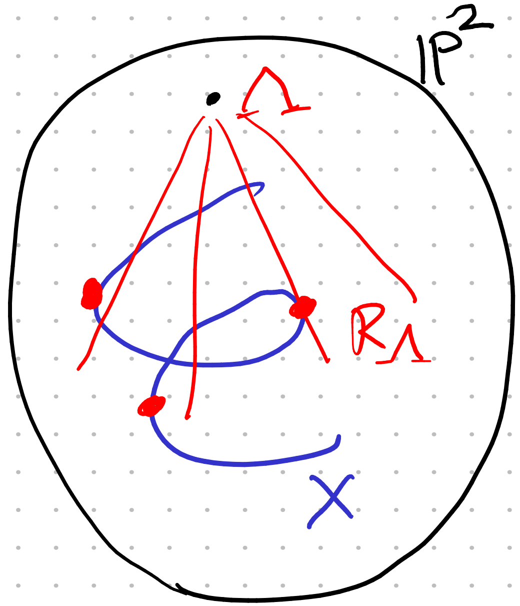
degree $\longrightarrow$

dim $\downarrow$

1	2	5	14	42	132	...	Catalans!	
	1	1	2	6	22	92	$\overline{422}$	...
				?			(OEIS...)	

$\square$ $\rightarrow$ Answer to an <u>enumerative problem</u> in A.G.

THE APPARENT BOUNDARY



$X \subset \mathbb{P}^n$ smooth proj $\dim X = r$

$\Lambda \subset \mathbb{P}^n$ linear subsp. of (generic)
$\dim = n-r-1$

$\pi_\Lambda : X \longrightarrow \mathbb{P}^r$ finite

R_Λ = Ram. div of (π_Λ)

$\hookrightarrow$ Apparent boundry of X
from Λ

QUESTIONS

Fix $X \subset \mathbb{P}^n$

- To what extent does R_Λ determine Λ?
- If Λ moves, must R_Λ also move?
- Are there multiple Λ that give the same R_Λ?
- Which divisors arise as R_Λ?

MOTIVATION:

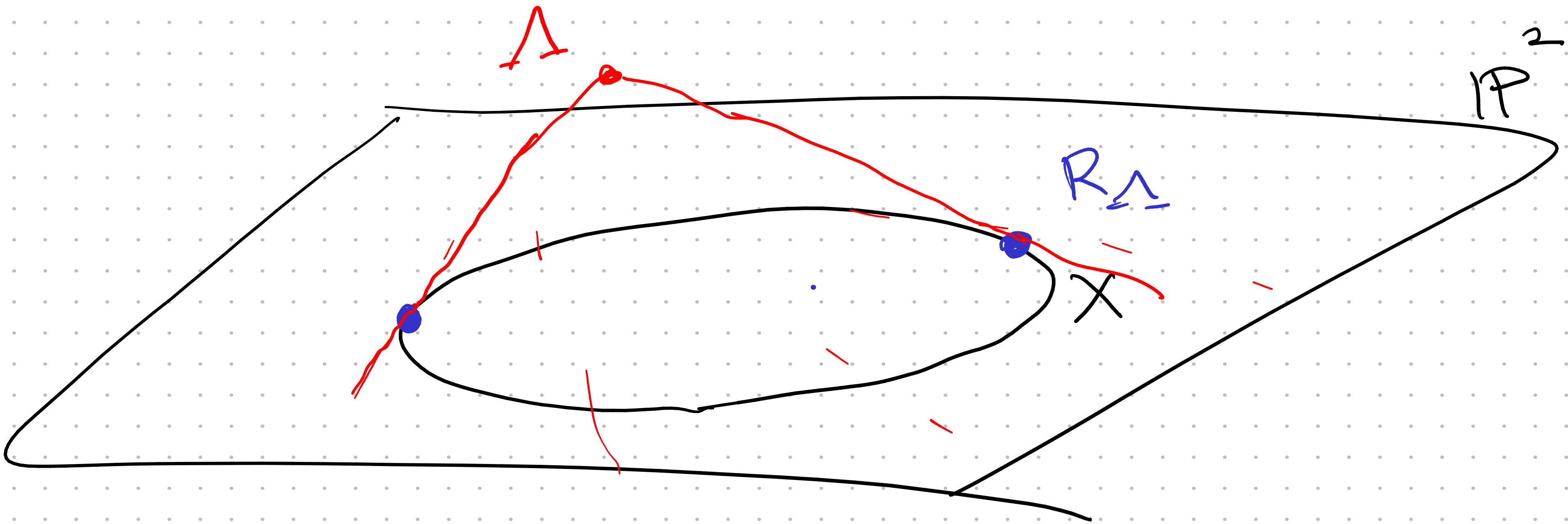
1. Hurwitz spaces of coverings
2. Kontsevich spaces of maps

} X a curve.

3. Stückrad-Vogel cycle (Flenner, Manaresi, Ciliberto, Zak)

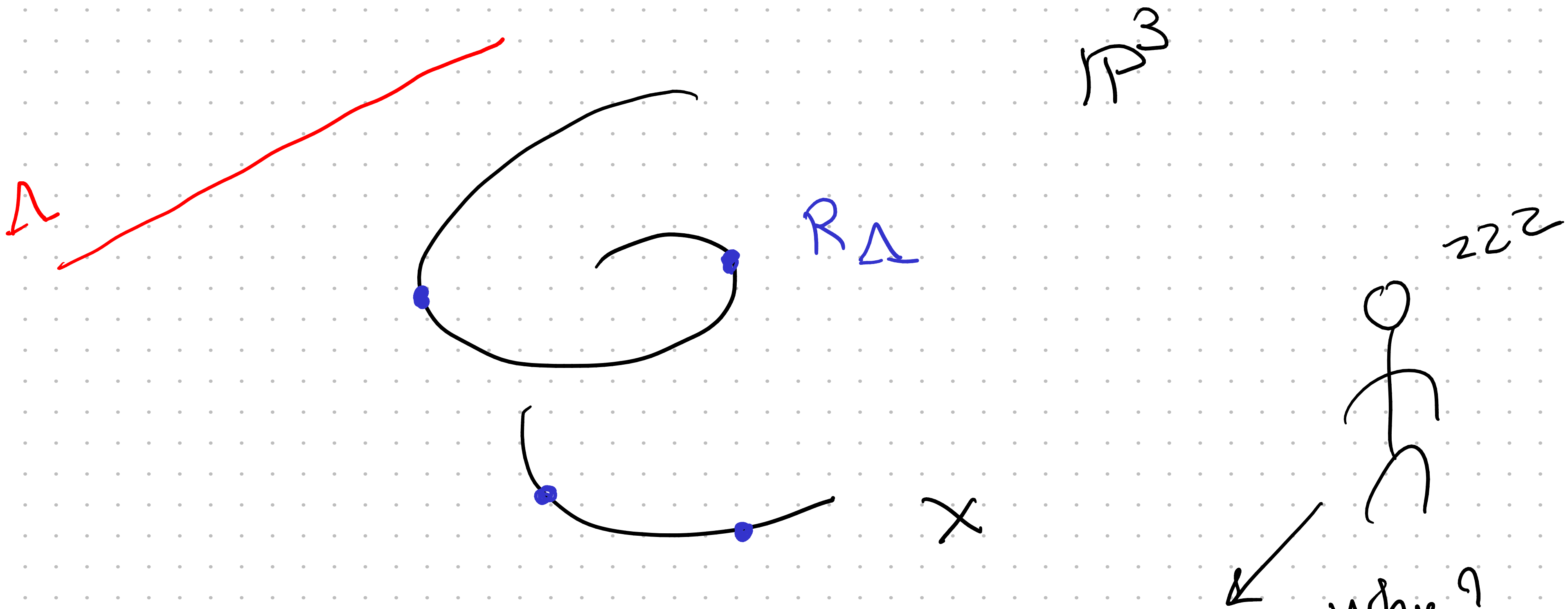
Intersection thy

EXAMPLE: PLANE CONIC



$\Lambda \longrightarrow R_\Lambda$ is of deg 1

EXAMPLE – TWISTED CUBIC



why?

Turns out, given 4 pts on X $\exists$ precisely 2 Λ that produce them.

QUESTIONS - MADE PRECISE $\quad X^r \subset \mathbb{P}^n$

$$\Lambda \rightsquigarrow R_\Lambda$$

$$\underline{\underline{Gr}}(\mathbb{P}^{n-r-1}, \mathbb{P}^n) \ni \Lambda \overset{\rho}{\dashrightarrow} R_\Lambda \in \underline{|K_X(r+1)|} \leftarrow \text{Proj. space}$$

Divisor class of R_Λ (Riemann-Hurwitz)

$$R_\Lambda \sim K_X + (r+1)H$$

Is ρ injective? surjective? finite? deg?

fiber dim?

DIMENSION COUNT

$\rho : Gr \dashrightarrow |K_X(r+1)|$

✓ ?

$$\dim Gr(\mathbb{P}^{n-r-1}, \mathbb{P}^n) = (n-r)(r+1)$$

Proposition:

$$\dim Gr \leq \dim |K_X(r+1)|$$

with equality iff

$$\deg X = n+1-r \leftarrow \dim X$$

$\hookrightarrow$ X is of minimal deg.

EXPECTED ANSWERS

$$\rho : Gr \dashrightarrow \mathbb{P}^{N-1}$$

① ρ is finite (generically) onto its image

② If X is of min deg, then ρ is finite & dominant.

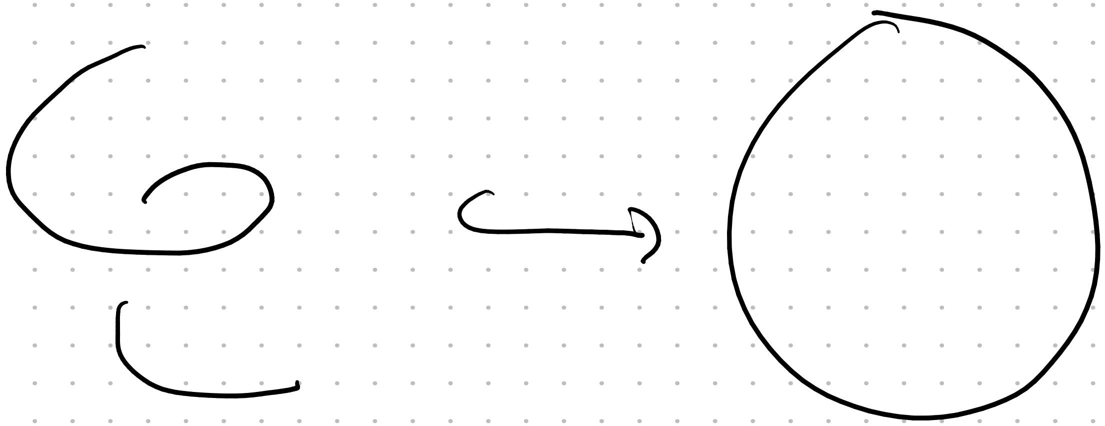
Almost true (not quite).

Focus on ②

VARIETIES OF MINIMAL DEGREE

1. Rat. normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$
2. Quadric hypersurfaces
3. Veronese $\mathbb{P}^2$ in $\mathbb{P}^5$
4. Scrolls $\|$

RATIONAL NORMAL CURVES

$X =$  $\mathbb{P}^n$

$$Gr(\mathbb{P}^{n-2}, \mathbb{P}^n) \xrightarrow{\ \rho\ } |\mathcal{O}(2n-2)| = \mathbb{P}^{2n-2}$$

1. ρ is regular!
2. $\rho^* \mathcal{O}(1) = \text{Plücker}$.

$$\Longrightarrow \deg \rho = [\text{Plücker}]^{\text{top}} = \deg Gr. = \frac{1}{n}\binom{2n-2}{n-1}$$

Catalan!

VARIETIES OF MINIMAL DEGREE

1. Rational normal curves
2. Quadric hypersurfaces
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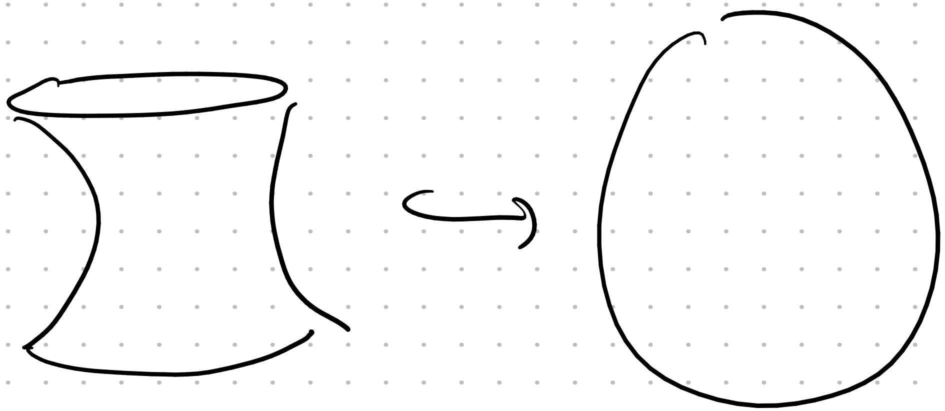
QUADRIC HYPERSURFACES

$$X = V(\underline{\underline{Q}})$$



$$\begin{tikzcd} Gr(\mathbb{P}^0, \mathbb{P}^n) = \mathbb{P}^n \arrow[r, dashed] & {|\mathcal{O}_X(1)|} \arrow[d, equal] \\ \mathbb{P}^n \arrow[r, dashed, "\rho"] & (\mathbb{P}^n)^* \end{tikzcd}$$

Here ρ is an iso. (deg 1)

$\hookrightarrow$ duality induced by Q.

VARIETIES OF MINIMAL DEGREE

1. Rational normal curves $\longrightarrow$ Catalans
2. Quadric hypersurfaces $\longrightarrow$ 1
3. Veronese $\mathbb{P}^2$ in $\mathbb{P}^5$
4. Scrolls.

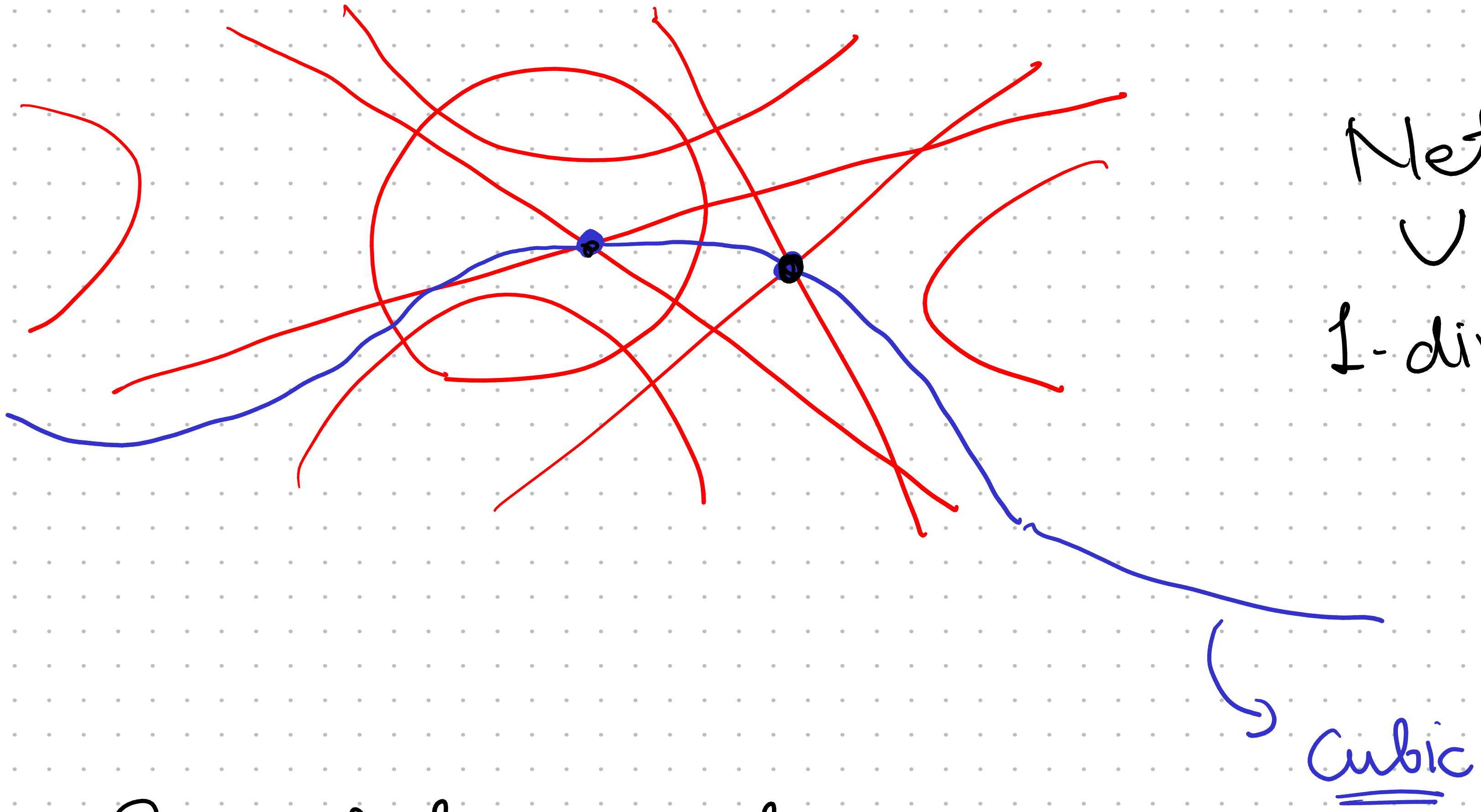
Veronese $\mathbb{P}^2$

$\mathbb{P}^2$

$$\mathbb{P}^2 \xrightarrow{\ |\mathcal{O}(2)|\ } \mathbb{P}^5$$

$$\begin{array}{ccc}
\rho : \quad Gr & \dashrightarrow & |\mathcal{O}_{\mathbb{P}^2}(3)| \\
\| & & \| \\
Gr(\underline{3}, H^0(\mathcal{O}(2))) & & \\
\| & & \\
\{\text{Net of conics}\} & \longrightarrow & \{\text{Cubics}\}
\end{array}$$

VERONESE $\mathbb{P}^2$: Net of conics $\longmapsto$ cubic.



Net = 2 dim

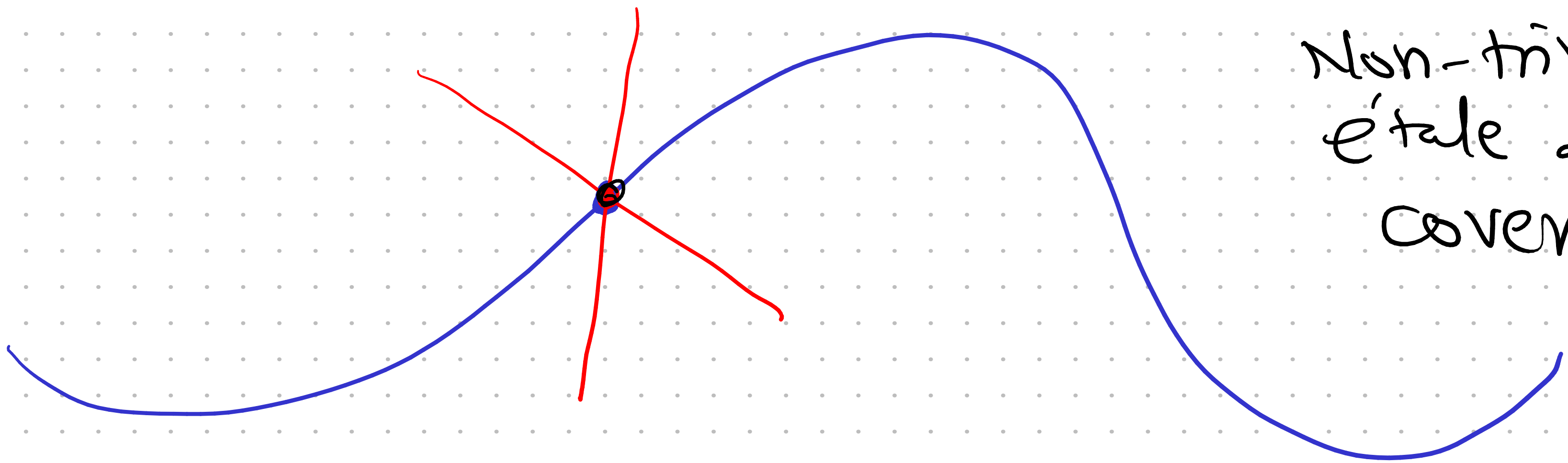
$\cup$

1-dim singulars

ρ : Net $\longrightarrow$ cubic

VERONESE $\mathbb{P}^2$: Net of conics $\longmapsto$ Discriminant cubic

$\longrightarrow$ cubic + Non-trivial étale 2 cover.



Cannot go back : Need $\boxed{\text{extra data}}$!

cubic has a double cover (2 lines)

Given cubic + non-triv double cover

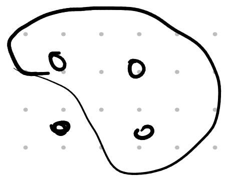
$\longrightarrow$ Can go back uniquely.

Veronese $\mathbb{P}^2$

deg 3

$$\begin{tikzcd}
\{\text{Net of conics}\} \arrow[rr, "\rho"] \arrow[dr] & & \{\text{cubic}\} \\
& \begin{array}{c}\text{cubic} + \\ \text{2-cover}\end{array} \arrow[ur] &
\end{tikzcd}$$

$$\deg \rho = \# \underbrace{\text{non-triv étale 2 covers}}_{\boxed{3.}}$$

$\mathbb{F}_2^{\oplus 2}$  $GL_2(\mathbb{F}_2) = S_3$

VARIETIES OF MINIMAL DEGREE

1. ~~Rational normal curves~~
2. ~~Quadric hypersurfaces~~
3. ~~Veronese $\mathbb{P}^2$ in $\mathbb{P}^5$~~
4. $\boxed{4}$ Scrolls.

SCROLLS — it's complicated!

There exist scrolls for which

ρ is not finite

$$\underline{\underline{Gr}} \longrightarrow \underline{\underline{\mathbb{P}^N}}$$

same dim

SCROLLS

Fix dim r

<u>Thm</u> :- For <u>generic</u> scrolls of $\deg \gg r$ $\sim 2r^2$

ρ is generically finite.

$\sum a_i$ is big

$|a_i - a_j| \leq 1$

Sharper in low dim

$r=2 \longrightarrow$ always finite

$r=3 \longrightarrow$ always finite

SCROLLS

$$E_0$$

$$\mathbb{P}\left(\underline{\underline{\mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(n)}}\right) \dashrightarrow \mathbb{P}^N$$

Write down the map in coordinates

It is birational (deg 1) ← <u>finite</u>

$$\mathcal{O}(1)\oplus\mathcal{O}(2)\oplus\mathcal{O}(n-1) \rightsquigarrow \boxed{E_0}$$

Anything

GRID OF ENUMERATIVE PROBLEMS

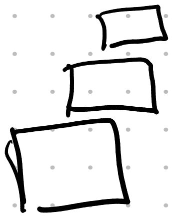
(1) monodromy gps?

degree $\rightarrow$

dim $\downarrow$

1 2 5 14 42 132 ... $\leftarrow r=1$

1 1 2 6 22 92 422 ... $\leftarrow r=2$

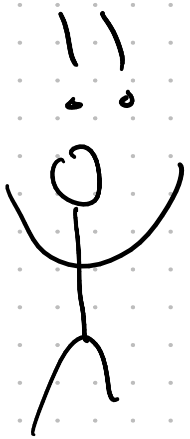


computer calc!

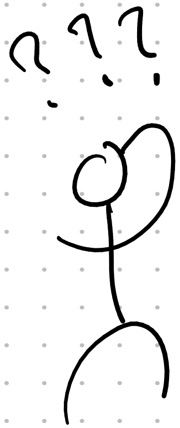
deg ρ for $\boxed{\text{generic}}$ scroll of given dim & degree

(2) Which $a_1, \dots, a_r$ give finite ρ?

GENERAL COMMENT



Enumerative geo. of curves $\rightsquigarrow$ GW / DT / PT theory



Enum. geo. in higher dim $\leftsquigarrow$ moduli in higher dim is hard!

SKETCH OF PROOF

THANK YOU!