Limits of plane quintics via covers of stacky curves

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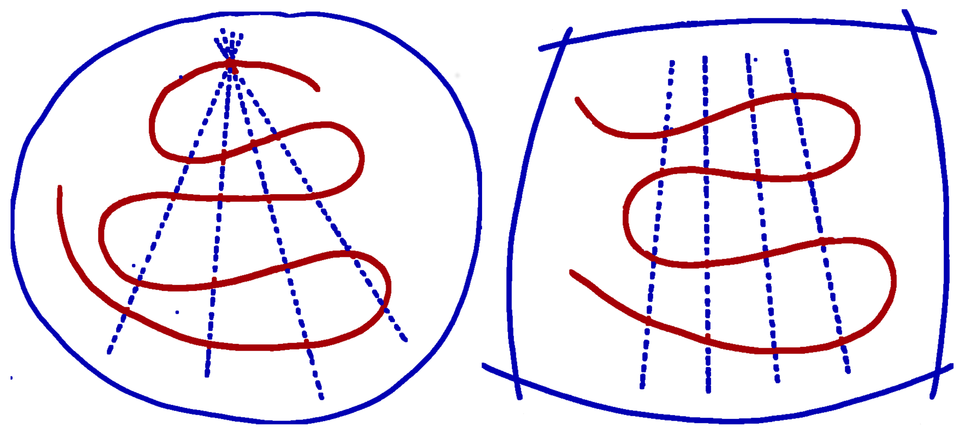
Sketch of the proof

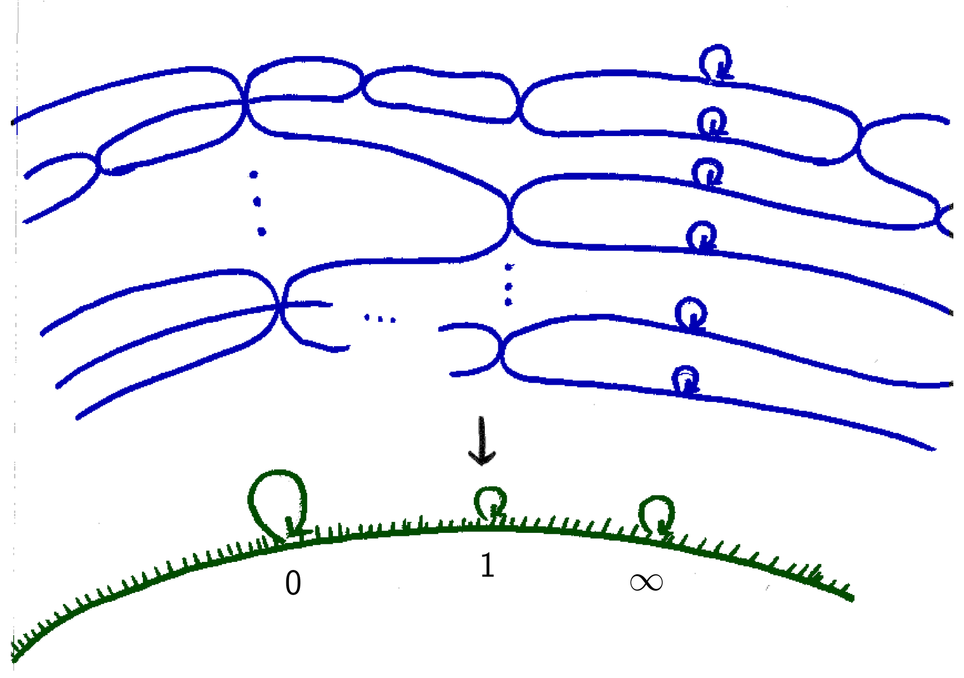
View a plane quintic C as a tetragonal curve on $\mathbf{F}_1$ by projecting from a point on it:



In turn, view this as a map from $\mathbf{P}^1$ to the stack $[\overline{M}_{0,4}/S_4]$. Denote the branch points of the projection $C \to \mathbf{P}^1$ by $p_1, \dots, p_{18}$. The map $\mathbf{P}^1 \dashrightarrow [\overline{M}_{0,4}/S_4]$ extends to a morphism

$$\mathbf{P}^1(\sqrt{p_1}, \dots, \sqrt{p_{18}}) \to [\overline{M}_{0,4}/S_4],$$

which is representable and finite of degree 18 on the coarse spaces. Here is a rough sketch of this finite cover.



The target has generic stabilizers isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ indicated by a fuzz and three points 0, 1, and ∞ with bigger stabilizers indicated by loops. The points on the source above ∞ have $\mathbf{Z}_2$ stabilizers, again indicated by loops.

Question

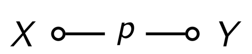
Which stable curves are limits of smooth plane curves?

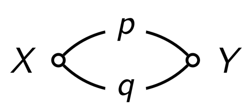
For curves of degree up to 4, every stable curve of that genus is such a limit. The first non-trivial case is of degree 5.

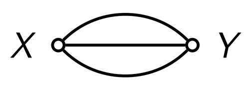
Theorem

Let $Q \subset \mathcal{M}_6$ be the locus of plane quintic curves. The generic points of the components of the boundary $\overline{Q} \cap \left(\overline{\mathcal{M}}_6 \setminus \mathcal{M}_6\right)$ of Q represent the following stable curves.

- With the dual graph 

  - A nodal plane quintic.
  - X hyperelliptic of genus 5.
- With the dual graph 

  - (X, p) the normalization of a cuspidal plane quintic, and Y of genus 1.
  - X of genus 2, Y Maroni special of genus 4, $p \in X$ a Weierstrass point, and $p \in Y$ a ramification point of $Y \xrightarrow{3} \mathbf{P}^1$.
  - X a plane quartic, Y hyperelliptic of genus 3, $p \in X$ a point on a bitangent, and $p \in Y$ a Weierstrass point.
  - X a plane quartic, Y hyperelliptic of genus 3, and $p \in X$ a hyperflex ($K_X = 4p$).
  - X hyperelliptic of genus 4, Y of genus 2, and $p \in X$ a Weierstrass point.
  - X of genus 1, and Y hyperelliptic of genus 5.
- With the dual graph 

  - X Maroni special of genus 4, Y of genus 1, and $p, q \in X$ on a fiber of the unique degree 3 map $X \to \mathbf{P}^1$.
  - X hyperelliptic of genus 3, Y of genus 2, and $p \in Y$ a Weierstrass point.
  - X of genus 2, Y a plane quartic, $p, q \in X$ hyperelliptic conjugate, and the line through p, q tangent to Y at a third point.
  - X hyperelliptic of genus 3, Y of genus 2, and $p, q \in X$ hyperelliptic conjugate.
- With the dual graph 

  - X hyperelliptic of genus 3, and Y of genus 1.

References

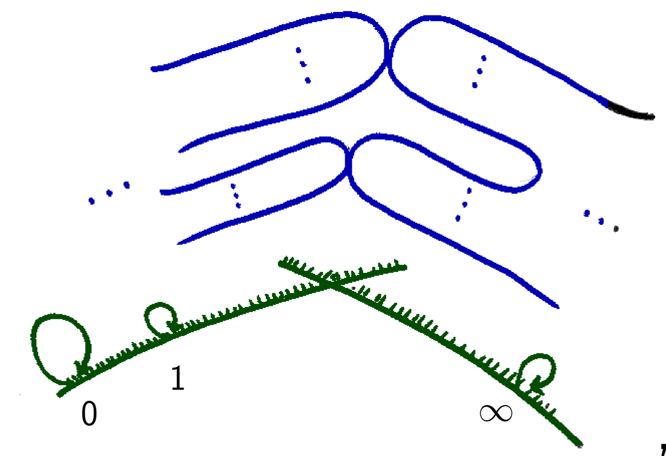
[1] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic).

[2] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88.

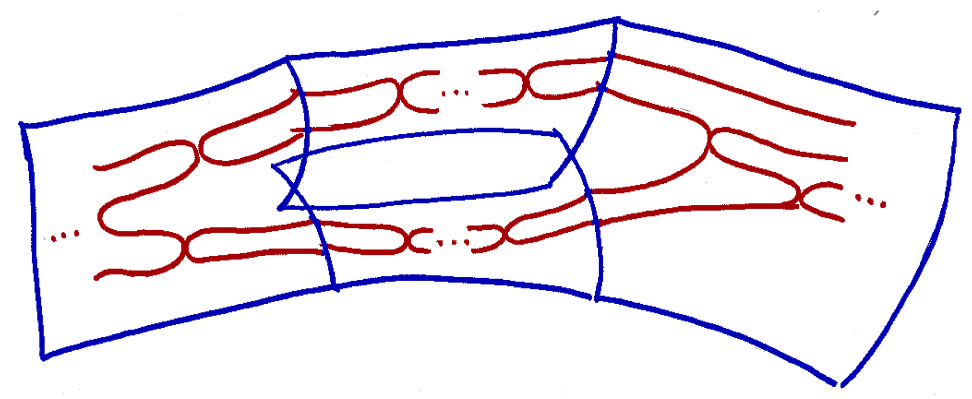
For more details, see the recent preprint *Covers of stacky curves and limits of plane quintics* (arXiv:1507.03252 [math.AG]).

Sketch (continued)

Compactify the space of such maps nicely and explicitly by combining the ideas of Harris–Mumford [2] and Abramovich–Vistoli [1]. The boundary points of this compactification are 'stacky admissible covers', such as the following



which correspond to degree four covers of nodal rational curves lying on degenerate scrolls, such as the following.



The stable images of such curves yield the list in the theorem.

Remarks

- The technique gives an explicit compactification of tetragonal curves on ruled surfaces. There is an issue of parity that we have suppressed in the poster.
- Other stacky target curves like $\overline{\mathcal{M}}_{1,1}$ yield nice and explicit compactifications of other interesting fibrations, like elliptic fibrations.