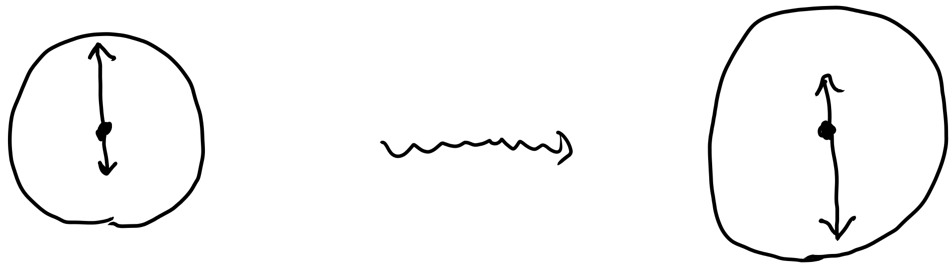
How to count using (co) homology ?

The Problem



Clock $\rightsquigarrow$ "Anti-clock"

6:00 $\rightsquigarrow$?? (invalid position).

Q1: How many clock positions remain valid after switching H & M hands? ≥ 11.

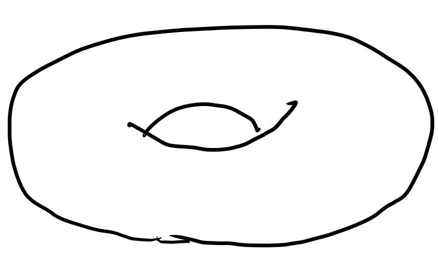
Q0: How many clock positions don't change after switching H & M? (11).

Reformulation – Topology / Geometry.

T = Set of configurations of H & M (valid or invalid).

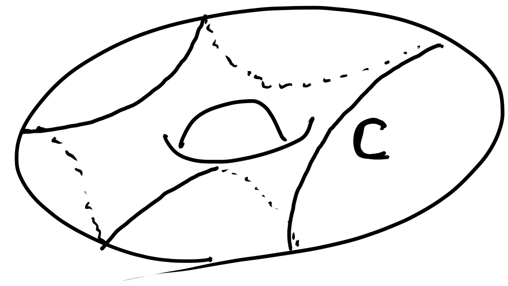
$= \{ (\text{Position of } H, \text{Position of } M) \}$

$\cong S^1 \times S^1 \leftarrow$ More than a set
A topological space
2-manifold
smooth orientable 2-manifold.



$T \supset \{\text{Valid clock positions}\} = C$

C is a smooth closed curve.



$T \supset \{\text{Switched valid clock positions}\} = C'$

C' is also a smooth closed curve.

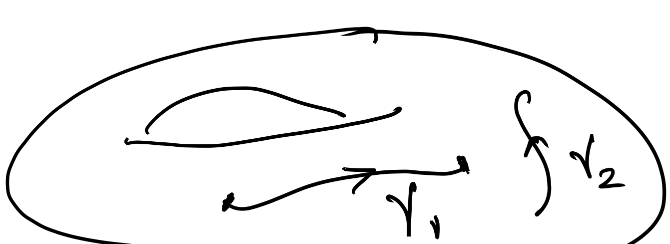
Problem : Find $\#(C \cap C')$.

General problem – Given a smooth oriented 2-manifold T want to calculate # intersection pts of curves.

Algebra?

Def: A curve on T is a continuous map $\gamma : I \to T$ $I = [0, 1]$

$C_1(T)$ = Free abelian group on the set of curves on T.

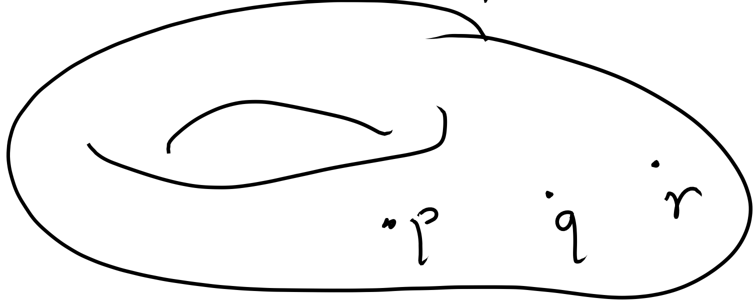


" $\gamma_1 + \gamma_2$ "

Elements of $C_1(T)$ called "1-chains"

Similarly

$C_0(T)$ = Free abelian gp on set of points of T



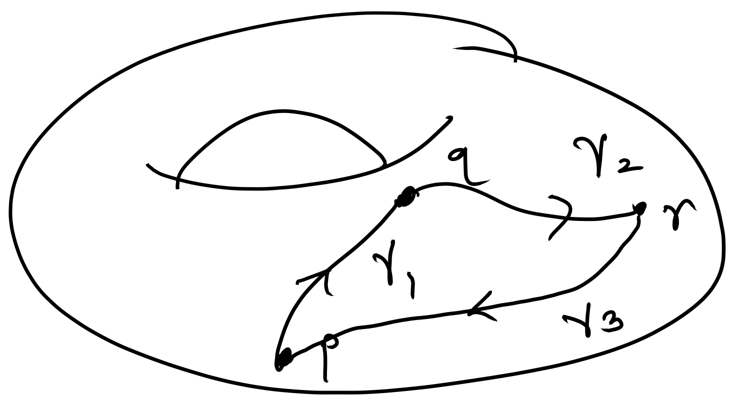
" $p + q - r$ "

Elements of $C_0(T)$ called 0-chains

Boundary map $\partial : C_1(T) \to C_0(T)$

$\partial(\gamma) = \gamma(1) - \gamma(0)$ (extend linearly)

ex.



$$\partial(\gamma_1 + \gamma_2) = q - p + r - q = r - p$$

$$\partial(\gamma_1 + \gamma_2 + \gamma_3) = 0$$

Also,

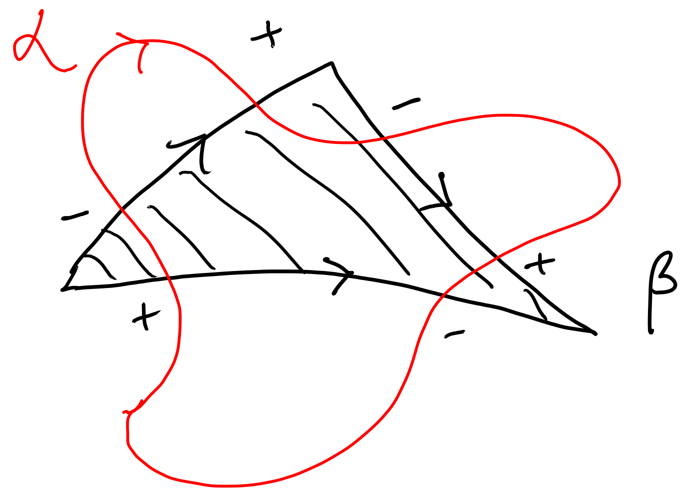
$C_2(T)$ = Free abelian gp on the set of triangles in T

Triangle = map from $\Delta \to T$.

Boundary map $\partial : C_2(T) \to C_1(T)$

$\partial(\Delta) =$ 

Crucial Observation:



α = a closed curve
β = the boundary of a triangle. Δ.

Then $\#\ \alpha$ enters Δ = $\#\ \alpha$ exits Δ.
(assuming no tangencies).

Def: Let C_1, C_2 be two 1-chains which intersect in a finite # points without tangency.

Set $C_1 \cdot C_2$ = # intersection points counted with sign.

$+1$ $\qquad$ -1.

① $C_1 \cdot C_2 = -C_2 \cdot C_1$

② $(C_1 + C_2) \cdot C_3 = C_1 \cdot C_3 + C_2 \cdot C_3$

(3) If C_1 is a cycle & C_2 is a

<u>Cor</u>: $C_1 \cdot C_2$ does not change if I change C_2 to $C_2 +$ boundary.

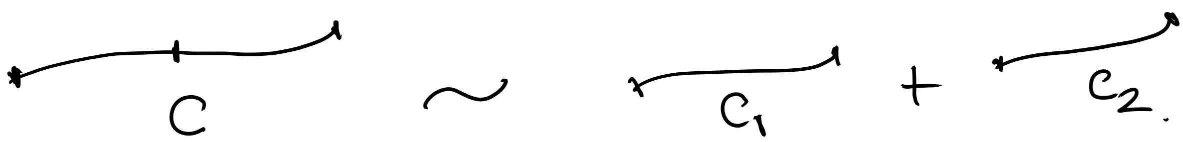
Call two 1-cycles equivalent if their difference is a boundary.

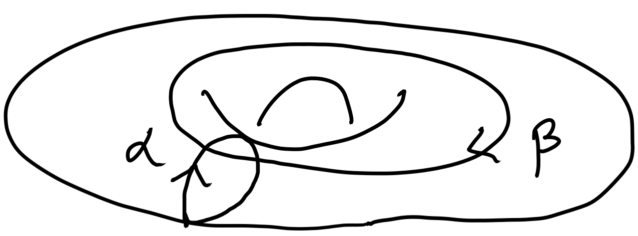
$H_1(T) :=$ Set of equiv. classes of 1-cycles.

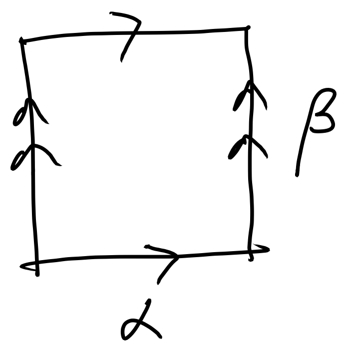
Then $C_1 \cdot C_2$ only depends on the equivalence class of C_1 and C_2!

<u>Obs</u>: 1) $C_1^{rev} \sim -C_1$

2)

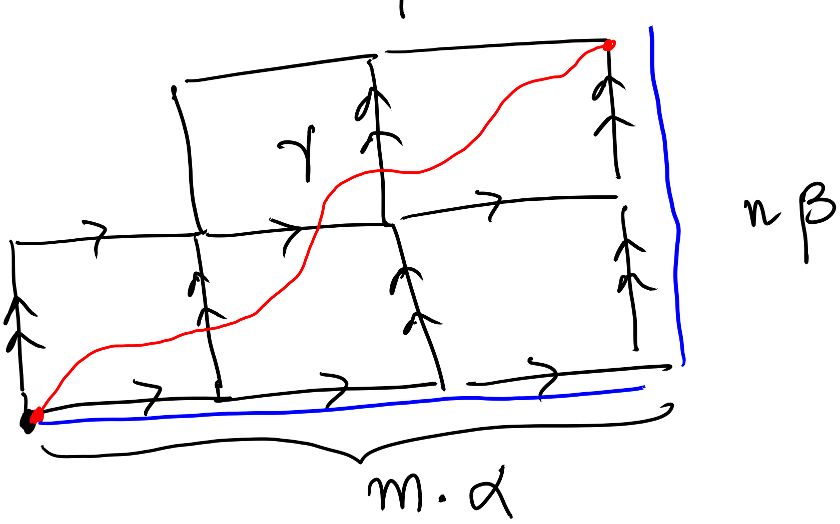






<u>Claim</u>: Any closed curve γ on T is equivalent to $m\alpha + n\beta$, $m, n \in \mathbb{Z}$.

<u>Pf</u>



$\square$

$\alpha \cdot \alpha = ?$

<u>Claim</u>: Given C_1 & C_2 $\exists$ C_1' & C_2' such that $C_1 \sim C_1'$ & $C_2 \sim C_2'$ so that C_1' & C_2' intersect finitely many times without tangencies.

<u>Set</u> $C_1 \cdot C_2 := C_1' \cdot C_2'$

$\alpha \cdot \alpha = 0$ $\qquad$ $\beta \cdot \beta = 0$

$\alpha \cdot \beta = 1$ $\qquad$ $\beta \cdot \alpha = -1$

$T \supset C$ = clock positions

$C \sim 12\alpha + \beta$

$T \supset C'$ = clock positions switched.

$C' \sim \alpha + 12\beta$

$$C \cdot C' = (12\alpha + \beta)(\alpha + 12\beta)$$
$$= 144\,\alpha\beta + \beta\alpha$$
$$= \underline{\underline{143}}$$

- Deduce that H & M overlap 11 times
- How many times is a mirror image of a clock position also a clock position?

Manifold M $\rightsquigarrow$ Ring formed out of submanifolds of M where product = signed intersection.

Geometry/Topology $\rightsquigarrow$ Algebra

Leads to extremely powerful invariants of geometric objects.