A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS

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with Asilata Bapat
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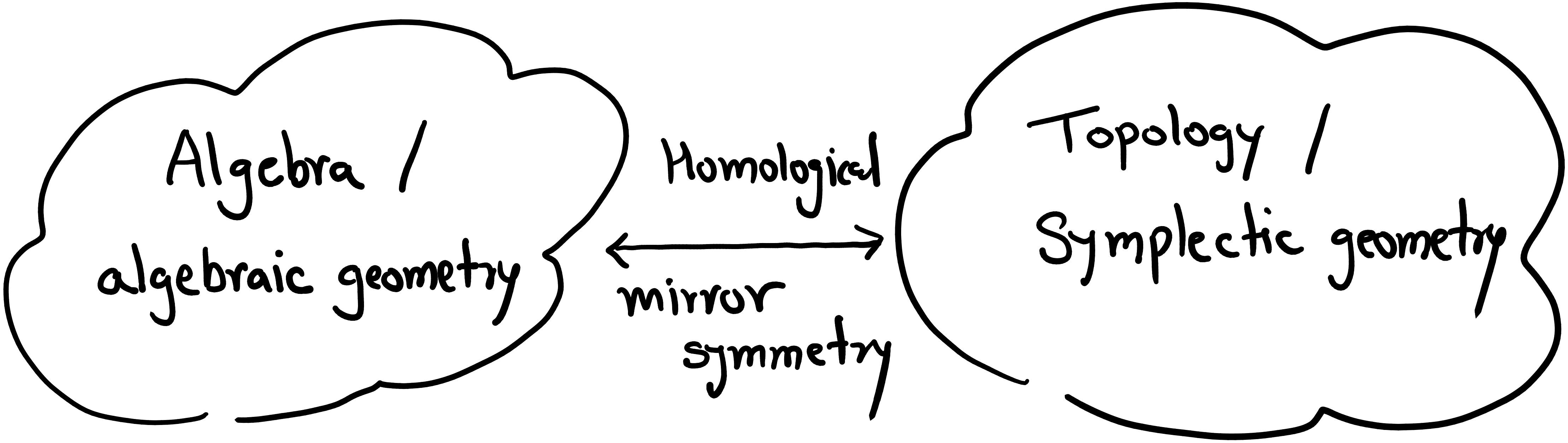
Guiding Principle -



Algebra / algebraic geometry

Homological mirror symmetry

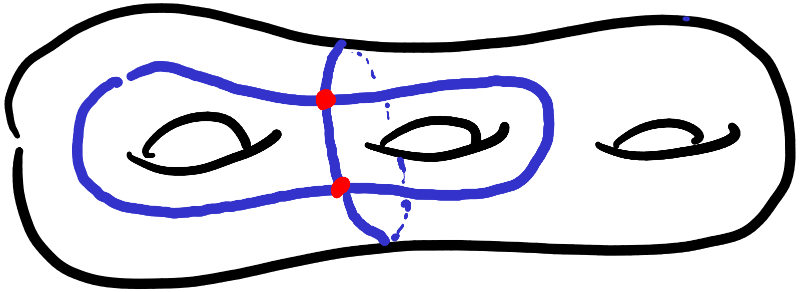
Topology / Symplectic geometry

$$D^b\mathrm{Coh}(W) \cong \mathrm{Fuk}(M)$$

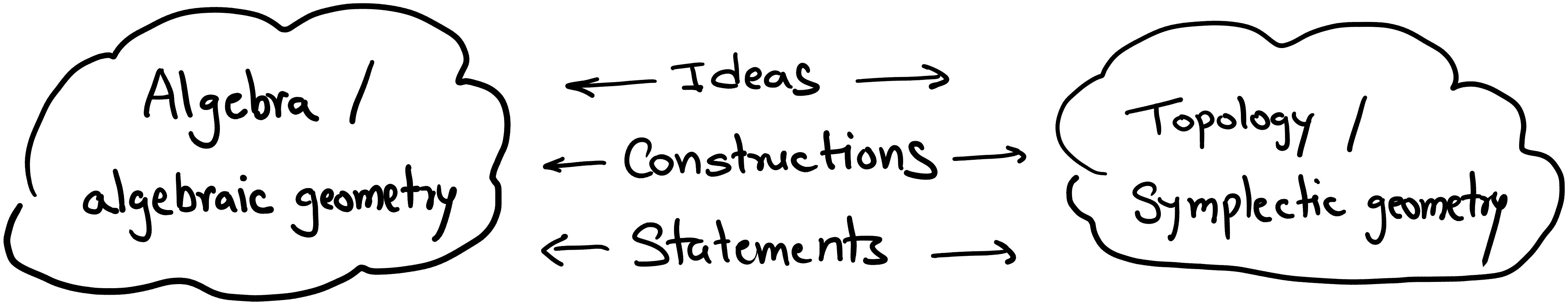
Guiding Principle -

$$D^b\mathrm{Coh}(W) \cong \mathrm{Fuk}(M)$$

Complexes of Modules / Sheaves ⟷



Guiding Principle -



Algebra / algebraic geometry

← Ideas →

← Constructions →

← Statements →

Topology / Symplectic geometry

Main Goal -

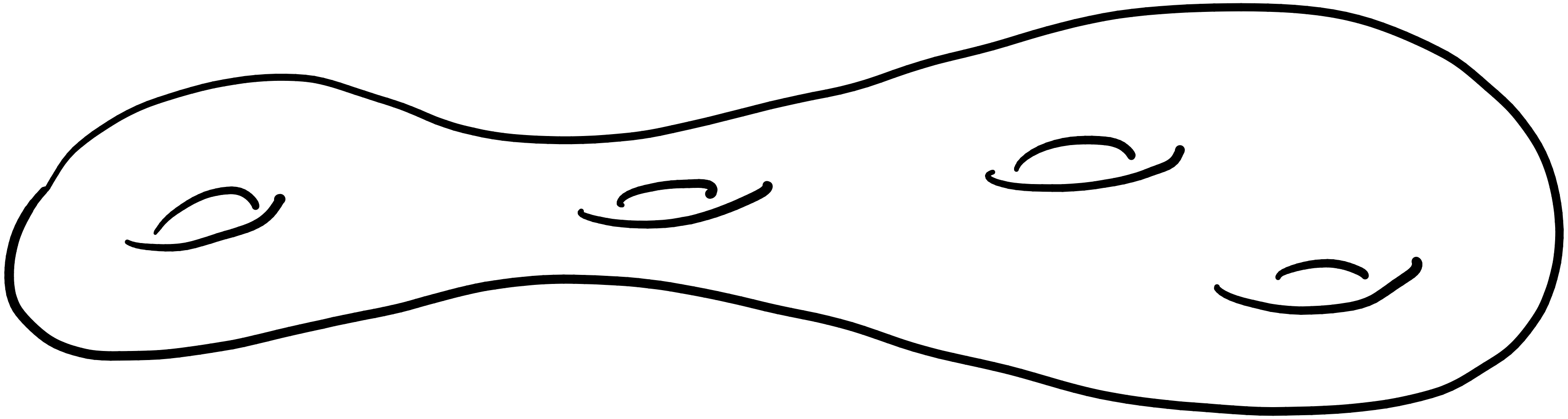
Construct $\overline{\mathrm{Stab}(\mathcal{C})} \longleftrightarrow \overline{\mathrm{Teich}(M)}$ (Thurston)

Plan : ① Thurston's construction

② Bridgeland stability conditions

③ Construction of $\overline{\mathrm{Stab}}$

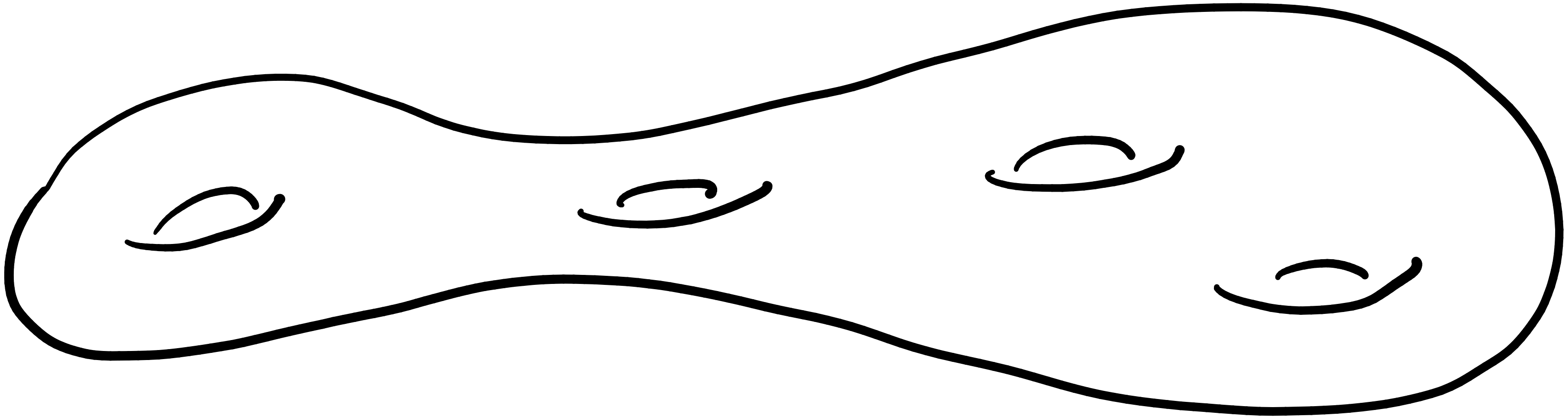
Teich(M) and its compactification

$M =$ 

$$\mathrm{Teich}(M) = \left\{ \begin{array}{l} \text{Normalised hyperbolic} \\ \text{metrics on } M \end{array} \right\}$$

$$\cong \mathbb{R}^{6g-6}$$

Teich(M) and its compactification

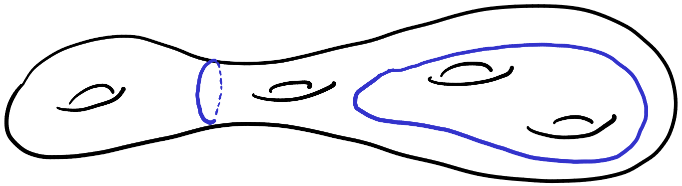


$$\mathrm{Teich}(M) \xrightarrow{\ell} \mathbb{RP}^{S}$$

$$S = \left\{ \begin{array}{l} \text{Simple closed} \\ \text{curves on } M \end{array} \right\} / \text{isotopy}$$

$$\mu \longmapsto [\, \cdots\cdots : \mathrm{Len}_{\mu}(\gamma) : \cdots\cdots \,]$$

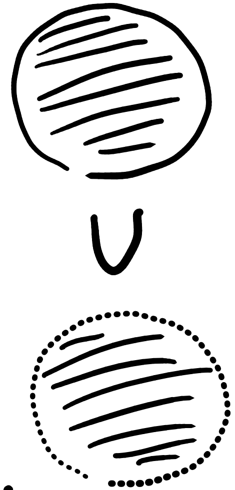
Teich(M) and its compactification



$$\text{Teich}(M) \xrightarrow{\ell} \mathbb{RP}^{S}$$

Theorem (Thurston):

1) ℓ is a homeo onto its image
2) The closure of the image is compact $\overline{\text{Teich}(M)}$
3) $$\begin{matrix} \overline{\text{Teich}(M)} \\ \cup \\ \text{Teich}(M) \end{matrix} \cong \begin{matrix} \overline{\mathbb{D}}^{6g-6} \\ \cup \\ \mathbb{D}^{6g-6} \end{matrix}$$



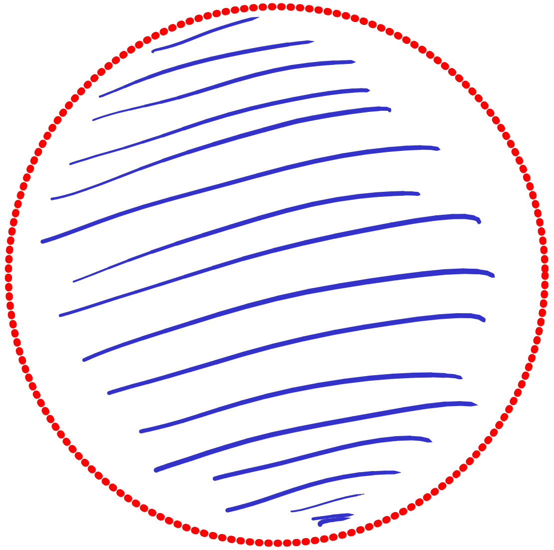
4) S itself appears as a dense subset of $\partial\overline{\text{Teich}(M)} \cong S^{6g-7}$

Teich(M) and its compactification

4) S itself appears as a dense subset of $\partial\overline{\mathrm{Teich}}(M) \cong S^{6g-7}$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "i"] & \mathbb{RP}^{\mathcal{S}} \\ \delta \arrow[r, mapsto] & {[\,\cdots : |\delta\cap\gamma| : \cdots\,]} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "i"] & \mathbb{RP}^{\mathcal{S}} \arrow[d, equal] \\ \mathrm{Teich}(M) \arrow[r, "\ell"] & \mathbb{RP}^{\mathcal{S}} \end{tikzcd}$$



Bridgeland Stability Conditions

$\mathcal{C}$ = $\mathbb{C}$- linear triangulated category

Def : A stability condition on $\mathcal{C}$ is

$$(Z, P)$$

"Slicing"

"Central charge"

satisfying some compatibility conditions.

Bridgeland Stability Conditions

$\sigma = ($ Central charge $\underset{\displaystyle\downarrow}{}$ Z , slicing $\mathcal{P}$ $)$

$$Z : K(\mathcal{C}) \longrightarrow \mathbb{C} \qquad \text{(Group hom)}$$

Bridgeland Stability Conditions

$\sigma = ($ Central charge Z, slicing $\mathcal{P}\,)$

Recall a t-structure on $\mathcal{C}$ is $\mathcal{A} \subset \mathcal{C}$ such that

$$\mathcal{C} = \cdots \underset{\mathcal{A}[2]}{\bullet} \text{---} \underset{\mathcal{A}[1]}{\bullet} \text{---} \underset{\mathcal{A}}{\bullet} \text{---} \underset{\mathcal{A}[-1]}{\bullet} \text{---} \underset{\mathcal{A}[-2]}{\bullet} \text{---} \cdots$$

($\mathbb{Z}$- indexed decomp)

Bridgeland Stability Conditions

$\sigma = ($ Central charge Z , slicing $\mathcal{P}$ $)$

$$\rotatebox{-90}{$\rightsquigarrow$}$$

For every $\phi \in \mathbb{R}$, a $P_\phi \subset \mathcal{C}$

such that

$\mathcal{C} =$

$P(\phi)$

(ℝ- indexed decomp)

$\rotatebox{90}{$\in$}$

$$\begin{tikzcd}
X \arrow[r, squiggly] & 0 = X_0 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & X_n = X \arrow[dl] \\
& & Z_1 \arrow[ul, dotted] & & \arrow[ul, dotted] & & Z_n \arrow[ul, dotted] & \\
& & P(\phi_1) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & & & & P(\phi_n) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}$$

Harder-Narasimhan
filtration

$(\phi_1 > \cdots > \phi_n)$

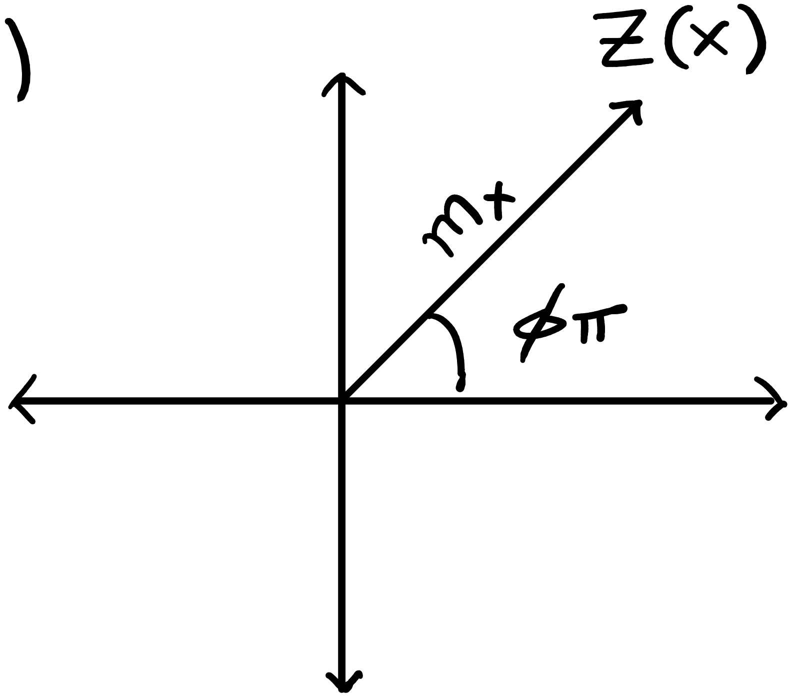
Bridgeland Stability Conditions

σ = (Central charge Z , slicing $\mathcal{P}$)

Compatibility : For $X \in P(\phi)$

$$Z(X) = \underset{\substack{\Big\downarrow \\ +\ \text{real}}}{m_X} \cdot e^{\pi i \phi}$$

"Mass of X wrt σ"



Bridgeland Stability Conditions

σ Stability condition

$X \in \mathcal{C}$ any object.

Def: $m_\sigma(X) = \sum_{HN} m_\sigma(Z_i)$

Analogy

metric μ curve γ

$\rightsquigarrow \mathrm{Len}_\mu(\gamma)$

(Aside - satisfies "triangle inequality"

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, dashed] & \end{tikzcd} \quad \Rightarrow \quad m(Y) \leq m(X) + m(Z)$$

)

Bridgeland Stability Conditions

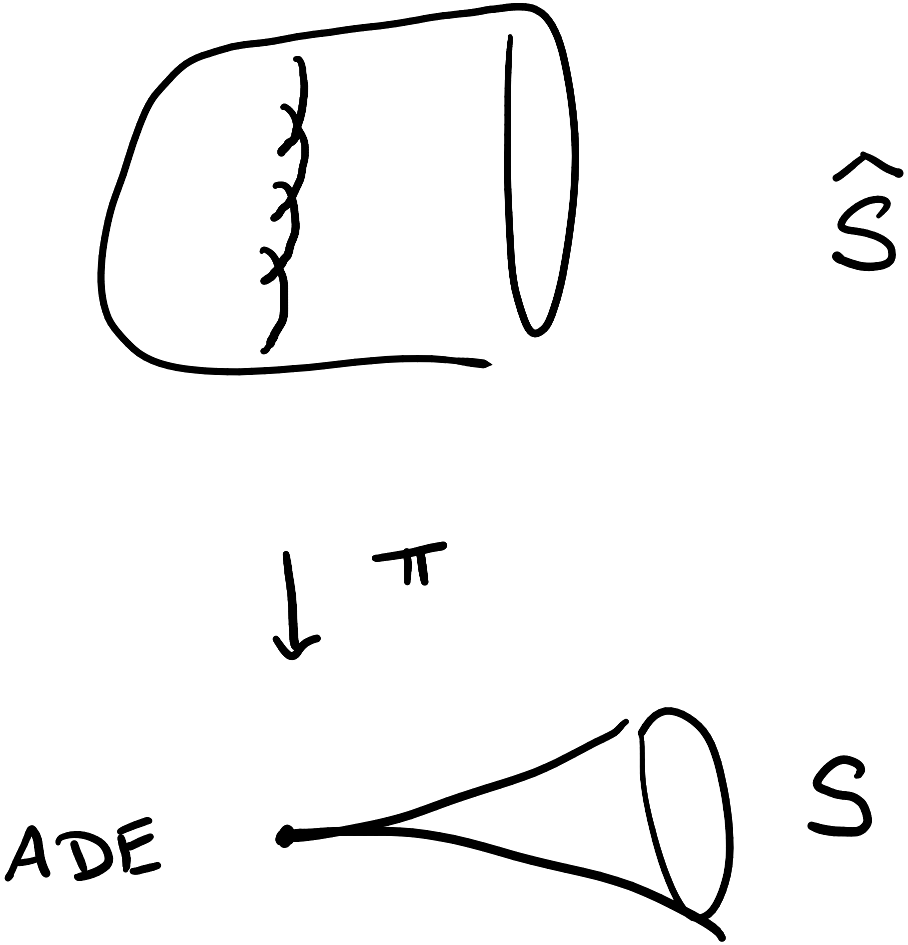
Theorem (Bridgeland) $\mathrm{Stab}(\mathcal{C})$ forms a manifold of real dimension $2 \cdot \mathrm{Rank}\, K(\mathcal{C})$.

(Conjecturally, contractible)

Analogy

$$\mathrm{Teich}(M) \simeq \mathbb{R}^{6g-6}$$

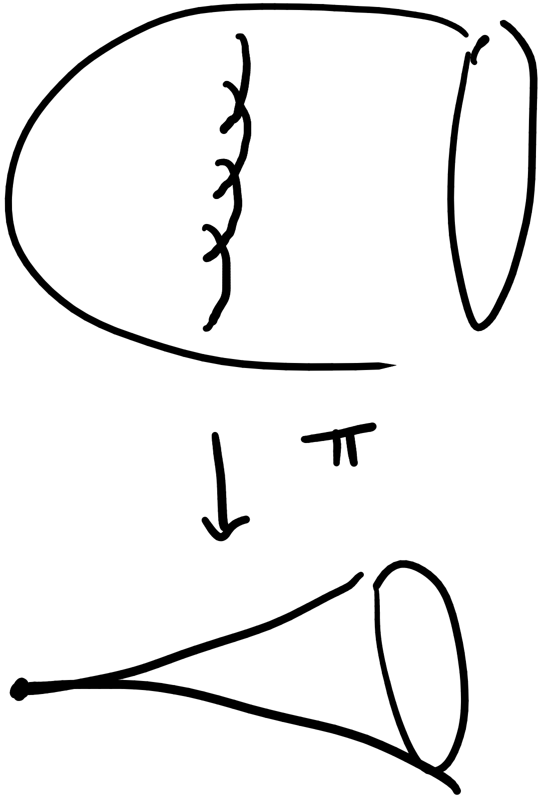
Categories



$$\mathcal{C} \subset D^b_{\mathrm{con}}(\hat{S})$$
$$\|$$
$$\{E \mid R\pi_* E = 0\}$$

$\mathcal{C}$ is a $\mathbb{C}$-linear, 2CY triangulated cat.
finite dim homs

Categories



Γ = Dual graph of $\mathrm{Exc}(\pi)$

$\mathcal{C} \ni P_i \qquad i \in \Gamma$

$\shortparallel$

$\mathcal{O}(-1)$ on i^{th} comp

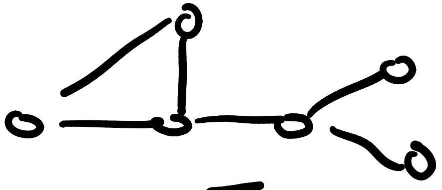
$\mathcal{C}$ is generated by P_i

$$\left.\begin{aligned} \mathrm{Hom}^*(P_i, P_j) &= \mathbb{C} \quad (\deg 1) \quad \text{if } i - j \\ &= 0 \quad \text{if } i \neq j \ \& \ i \not- j \\ &= \mathbb{C}[\varepsilon]/\varepsilon^2 \quad (\deg \varepsilon = 2) \quad \text{if } i = j \end{aligned}\right\} \circledast$$

P_i "spherical"

Categories

Γ finite simple graph



$\rightsquigarrow$ $\mathcal{C}_\Gamma$ $\mathbb{C}$-linear 2 CY triangulated cat. generated by P_i, $i \in \Gamma$ satisfying ⊛

$$\begin{tikzcd}
\mathcal{C}_\Gamma \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & \text{Artin-Tits Braid}(\Gamma) \arrow[d] \\
K(\mathcal{C}_\Gamma) \arrow[r, phantom, "\circlearrowleft"] & \text{Coxeter}(\Gamma)
\end{tikzcd}$$

Compactifying Stab - Motivation

$\mathcal{C}$ = $\mathbb{C}$-linear 2-CY finite dim homs
(e.g $\mathcal{C}_\Gamma$ or $D^b \mathrm{Coh}\, K3$)

$$\begin{tikzcd}
\mathcal{C} \arrow[r, rightsquigarrow] & \mathrm{Stab}\,\mathcal{C} \arrow[r, phantom, "\subset"] & \overline{\mathrm{Stab}\,\mathcal{C}} \\
\circlearrowright & \circlearrowright & \circlearrowright \\
G & G & G
\end{tikzcd}$$

Compactifying Stab - Construction

$\mathcal{C} \supset S$ = Sphericals

$\mathrm{Stab} \xrightarrow{m} \mathbb{RP}^S$

$\sigma \longmapsto [\cdots : m_\sigma(x) : \cdots]$

$S \xrightarrow{i} \mathbb{RP}^S$

$y \longmapsto [\cdots : \hom(x,y) : \cdots]$

Analogy

$\mathrm{Teich} \longrightarrow \mathbb{P}^S$

$\mu \longmapsto [\mathrm{Len}_\mu]$

$\mathcal{S} \longrightarrow \mathbb{P}^S$

$s \longmapsto [s \cap \gamma]$

Compactifying Stab - Construction

$$P\mathrm{Stab} \xrightarrow{m} \mathbb{RP}^S \quad , \quad S \xrightarrow{i} \mathbb{RP}^S$$

1) m is a homeomorphism onto its image.
2) Closure of the image is compact manifold with boundary.
3) S embeds (via i) as a dense subset of the boundary
4)

$$\begin{array}{c} \overline{\mathrm{Im}} \\ \cup \\ \overline{\mathrm{Im}}^{\circ} \end{array} \cong \begin{array}{c} \overline{\mathrm{Disk}} \\ \cup \\ \mathrm{Disk} \end{array}$$

Compactifying Stab - Construction

① – ④ :– Theorems in rank 2 [Bapat, –, Licata] (A_2 & $\hat{A}_1$)

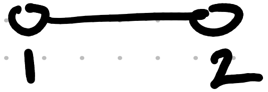
:– Work in progress in finite (ADE) & affine type

:– Conjectures for arbitrary $\mathcal{C}_\Gamma$

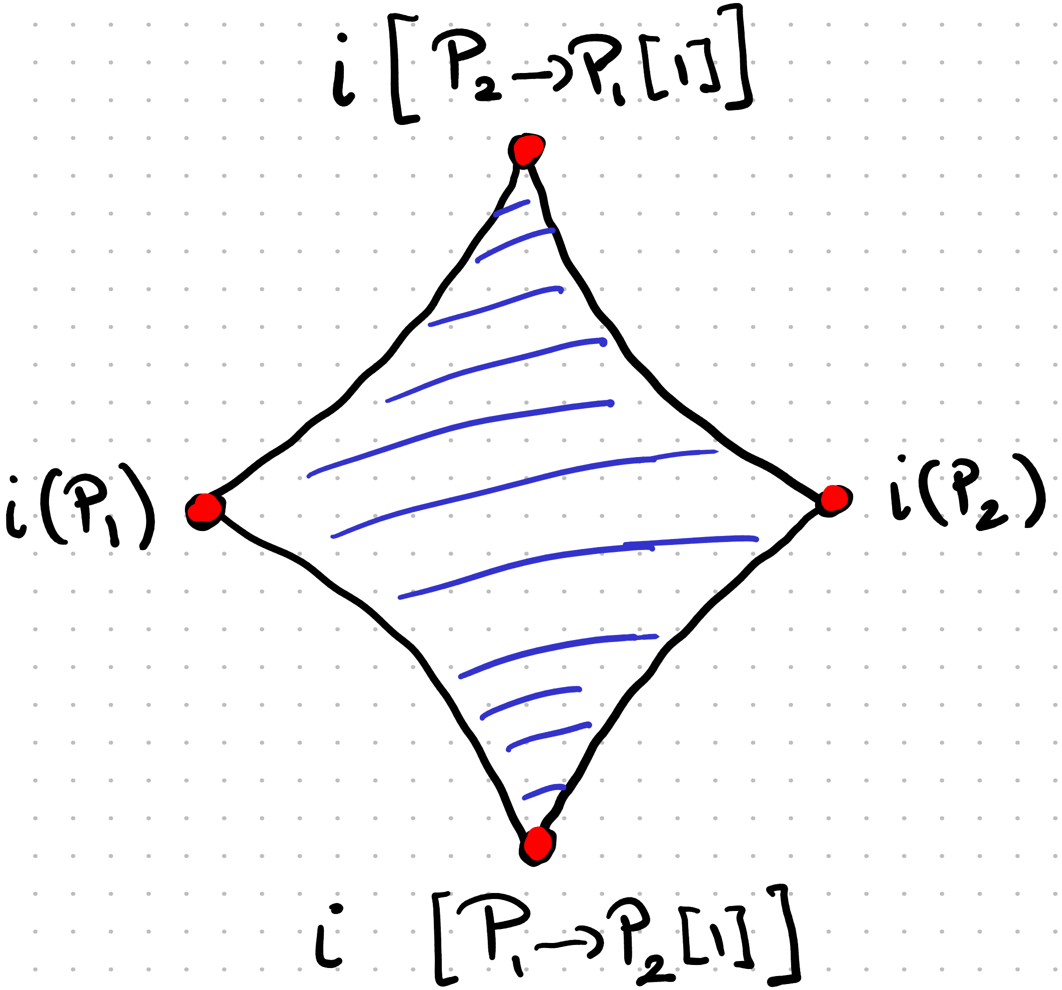
:– Dream / questions more generally.

Compactifying Stab - Picture

$\mathcal{C} = \mathcal{C}(A_2)$



$\heartsuit = \langle P_1, P_2 \rangle$

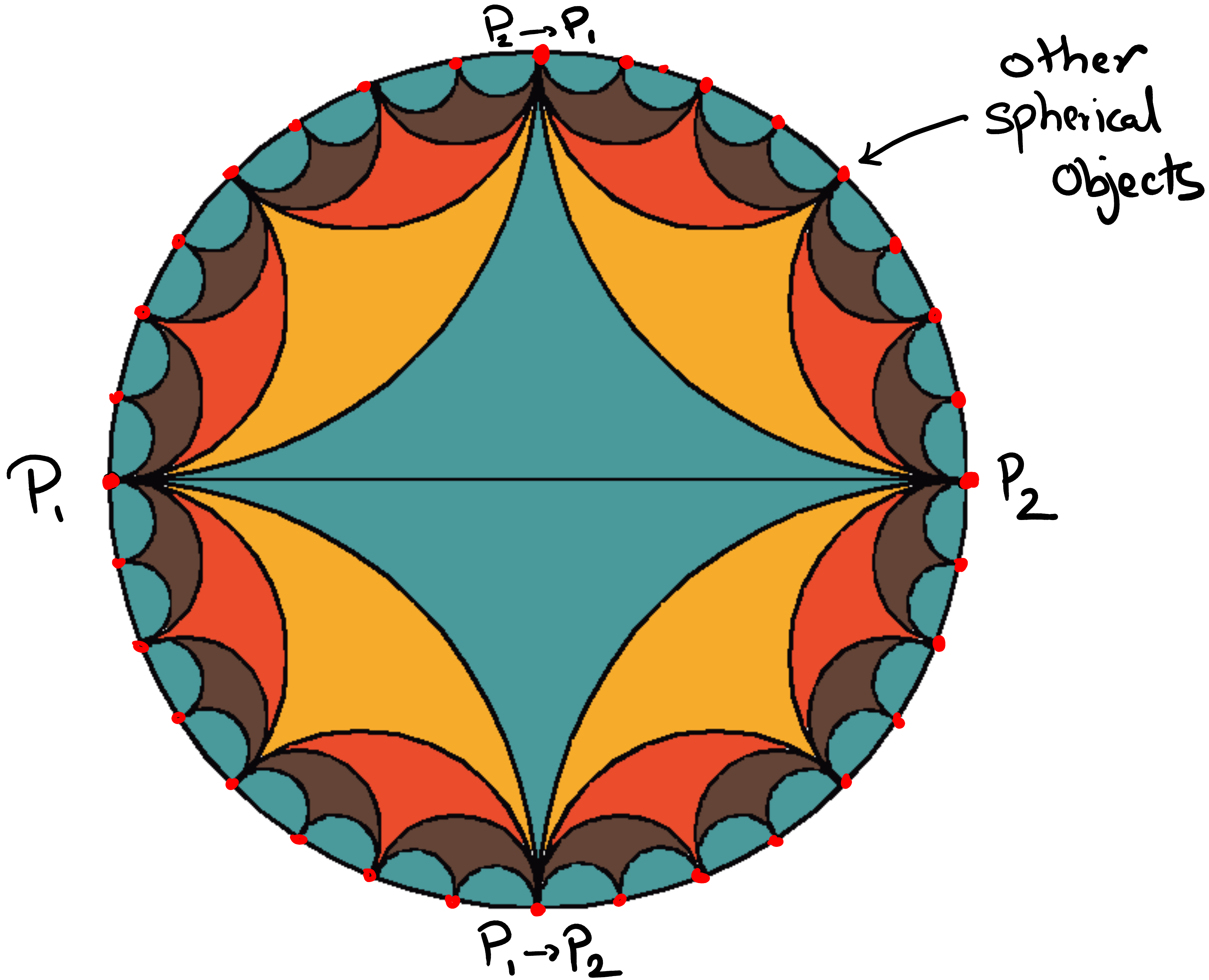


$\times\ B_2$

$\|$

$PSL_2(\mathbb{Z})$

Compactifying Stab - Picture



Compactifying Stab

Why is $X \in \overline{\mathrm{PStab}}$

Take $\sigma \in \mathrm{Stab}$

$$Y \to T_X Y \to \mathrm{Hom}(Y,X) \otimes X[1]$$

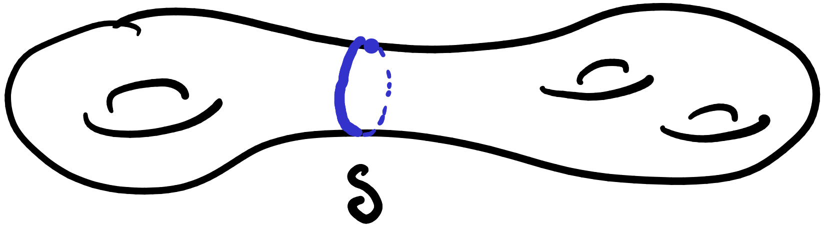
Let $\sigma_1 = T_X^{-1} \sigma$

$$m_{\sigma_1}(Y) = m_\sigma(T_X Y)$$

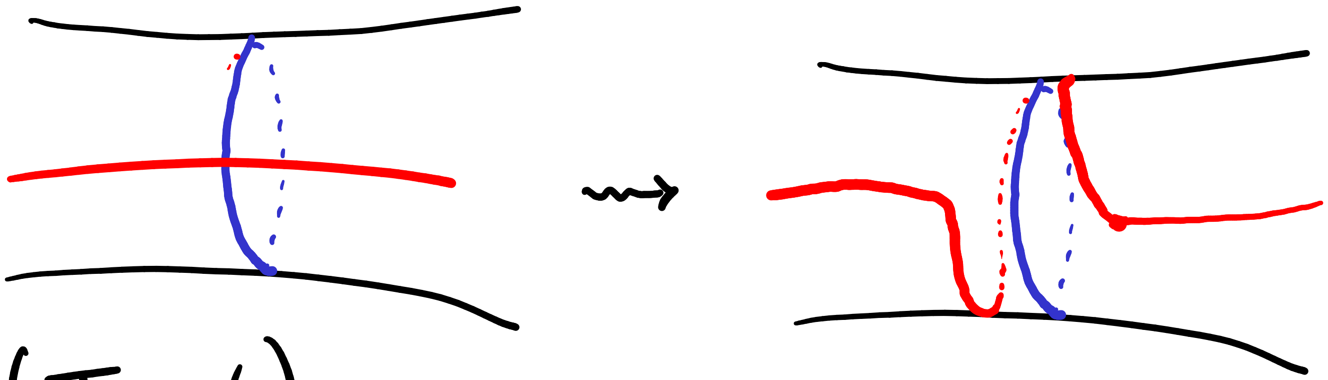
$$\approx m_\sigma(Y) + \hom(Y,X)\, m_\sigma(X)$$

Iterate: $m_{\sigma_n}(Y) \approx a + b\, n \hom(Y,X)$

Why is $\delta \in \overline{\text{Teich}}$



Let $\mu_1 = Tw_\delta^{-1}(\mu)$



$$\mathrm{Len}_{\mu_1}(\gamma) = \mathrm{Len}_{\mu}(Tw_\delta \gamma)$$

$$\sim \mathrm{Len}_{\mu}(\gamma) + \mathrm{Len}(\delta) \cdot |\delta \cap \gamma|$$

Iterate: $\mathrm{Len}_{\mu_n}(\gamma) \sim a + b\, n\, |\delta \cap \gamma|$

THANK YOU!