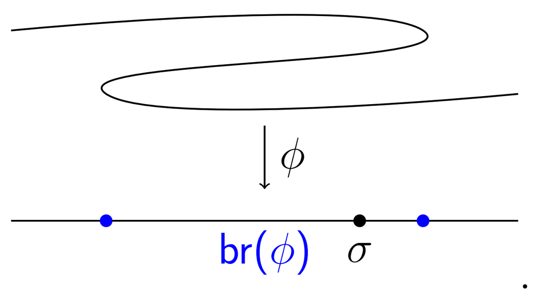
# Birational Geometry of the Space of Marked Trigonal Curves

We construct and study a sequence of modular compactifications of the Hurwitz space parametrizing $(\phi\colon C \to \mathbf{P}^1, \sigma)$, where $C$ is a smooth curve of genus $g$, $\phi$ a map of degree 3 and $\sigma \in \mathbf{P}^1$ a point away from the branch locus of $\phi$:




Our compactifications parametrize certain triple covers of $\mathbf{P}^1$ where all the branch points are allowed to coincide. Which covers are included is determined by two invariants: the Maroni invariant and the $\mu$ invariant.
We work over an algebraically closed field $k$ of characteristic 0.

## The Maroni Invariant

Consider a triple cover $\phi\colon C \to \mathbf{P}^1$, where $\rho_a(C) = g$. We have the sequence

$$0 \to O_{\mathbf{P}^1} \to \phi_* O_C \to V \to 0,$$

where

$$V \cong O_{\mathbf{P}^1}(-a) \oplus O_{\mathbf{P}^1}(-b),$$

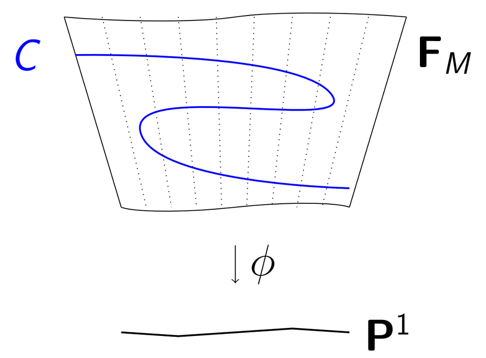
for some $a, b > 0$ with

$$a + b = g + 2.$$

Define the *Maroni invariant* $M$ by

$$M(\phi) = |b - a|.$$

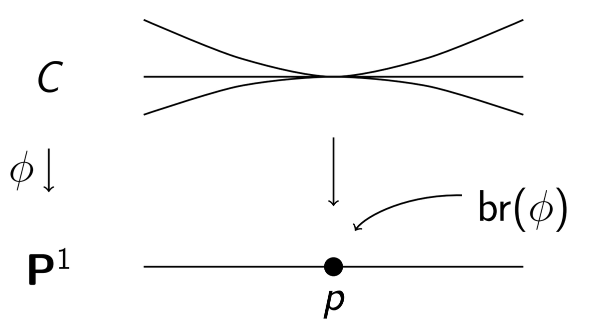
There is an embedding of $C$ into the Hirzebruch surface $\mathbf{F}_M$:




The Maroni invariant is congruent to $g$ modulo 2. It is upper semi-continuous in families of triple covers.

## The $\mu$ Invariant

Consider a triple cover $\phi\colon C \to \mathbf{P}^1$, with $\rho_a(C) = g$ and $\operatorname{supp} \operatorname{br}(\phi) = p$, for some $p \in \mathbf{P}^1$. We call such covers *punctually ramified.*




Clearly, the normalization $C^\nu$ is $\mathbf{P}^1 \cup \mathbf{P}^1 \cup \mathbf{P}^1$. We get the sequence

$$0 \to \phi_* O_C \to \phi^\nu_* O_{C^\nu} \to Q \to 0,$$

where $Q$ is supported at $p$. In fact,

$$Q \cong k[t]/t^a \oplus k[t]/t^b$$

for some $a, b > 0$ with

$$a + b = g + 2.$$

Define the $\mu$ invariant by

$$\mu(\phi) = |b - a|.$$

The $\mu$ invariant is congruent to $g$ modulo 2. It is lower semi-continuous in families of punctually ramified triple covers.

## The Main Theorem

Let $0 \leq l \leq g$ be an integer with $l \equiv g \pmod 2$.

### Definition

A triple cover $\phi\colon C \to \mathbf{P}^1$, where $C$ is a reduced, connected curve of arithmetic genus $g$, is *$l$-balanced* if
- $M(\phi) \leq l$, and
- if $\phi$ is punctually ramified, then $\mu(\phi) > l$.

Denote by $\mathscr{T}_{g;1}(l)$ the moduli stack of $(\phi\colon C \to \mathbf{P}^1, \sigma)$, where
- $\phi\colon C \to \mathbf{P}^1$ is an $l$-balanced triple cover, and
- $\sigma \in \mathbf{P}^1$ is a point away from $\operatorname{br}(\phi)$.

### Theorem

*$\mathscr{T}_{g;1}(l)$ is a Deligne–Mumford stack. It is connected, of finite type, smooth, and proper over $\operatorname{Spec} k$. It admits a projective coarse moduli space $\mathcal{T}_{g;1}(l)$.*

Thus, we obtain a sequence of birational models

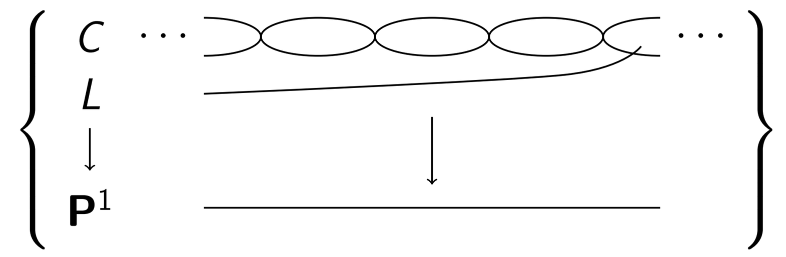
$$\mathcal{T}_{g;1}(g) \dashrightarrow \mathcal{T}_{g;1}(g-2) \dashrightarrow \cdots \dashrightarrow \mathcal{T}_{g;1}(0 \text{ or } 1).$$

We now describe the geometry of this sequence of maps.

## The Hyperelliptic Contraction: $\mathcal{T}_{g;1}(g) \to \mathcal{T}_{g;1}(g-2)$

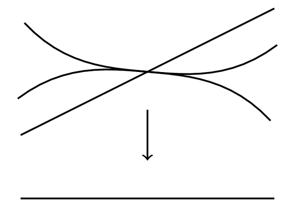
The exceptional locus of $\mathcal{T}_{g;1}(g) \dashrightarrow \mathcal{T}_{g;1}(g-2)$ is the *hyperelliptic divisor* $H$. Its generic point corresponds to a triple cover $\phi\colon C \cup L \to \mathbf{P}^1$.

$H =$




The rational map $\mathcal{T}_{g;1}(g) \dashrightarrow \mathcal{T}_{g;1}(g-2)$ extends to a morphism that contracts $H$ to a point. This point parametrizes a punctually ramified triple cover with a $D_{2g+2}$ singularity.

$$\begin{array}{c} \operatorname{Spec} k[x,y]/(y^2 - x^{2g})(y - x) \\ \downarrow \\ \operatorname{Spec} k[x] \end{array}$$

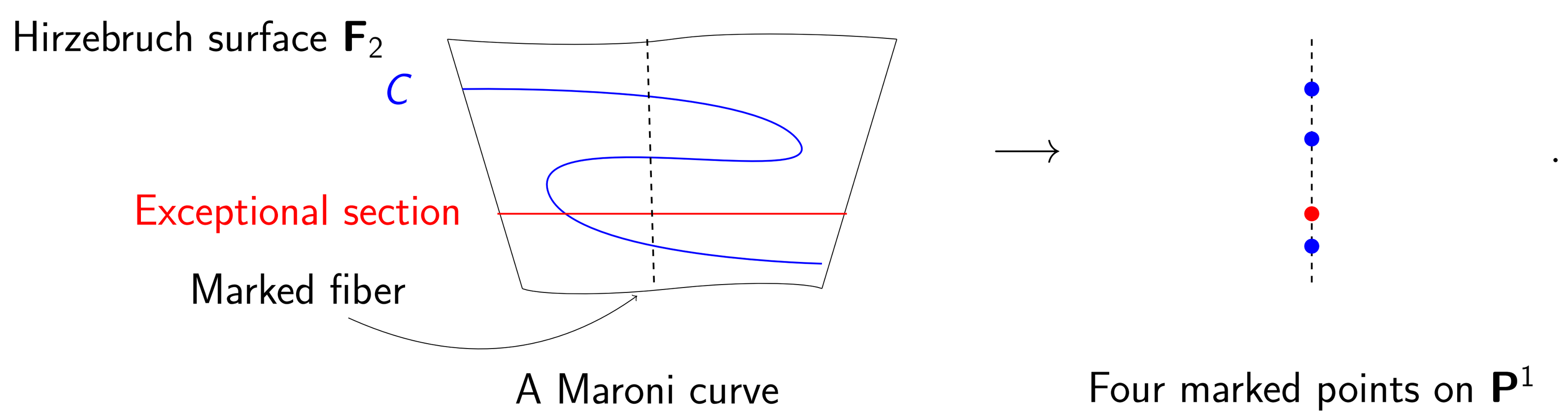


## The Maroni Contraction in Even Genus: $\mathcal{T}_{g;1}(2) \to \mathcal{T}_{g;1}(0)$

The exceptional locus of $\mathcal{T}_{g;1}(2) \dashrightarrow \mathcal{T}_{g;1}(0)$ consists of covers with Maroni invariant 2. This locus is a divisor, called the *Maroni divisor*.
The rational map $\mathcal{T}_{g;1}(2) \dashrightarrow \mathcal{T}_{g;1}(0)$ extends to a morphism, which contracts the Maroni divisor to a $\mathbf{P}^1$:

$$\text{Maroni divisor} \to \mathbf{P}^1.$$

The morphism from the Maroni divisor to $\mathbf{P}^1$ comes from a certain cross-ratio, explained in the following picture.




The final model $\mathcal{T}_{g;1}(0)$ is the quotient of a weighted projective space. Setting $g = 2h$, we have

$$\mathcal{T}_{g;1}(0) \cong \mathcal{P}\left(1, \ldots, h+1, 1, \ldots, h+1, 1, \ldots, h+1, 2, \ldots, h+1\right)/S_3.$$

## Chamber Decomposition of $\operatorname{Pic}_{\mathbf{Q}}$

For $0 < l < g$, the models $\mathcal{T}_{g;1}(l)$ are isomorphic to each other in codimension one. In particular, they have identical Picard groups. For $g \geq 4$, the group $\operatorname{Pic} \otimes \mathbf{Q}$ is generated (over $\mathbf{Q}$) by $\lambda$ and $\delta$.

$$\operatorname{Pic} \otimes \mathbf{Q} = \mathbf{Q}\langle \lambda, \delta \rangle.$$

### Theorem

*For $0 < l < g$ and $g \geq 4$, the nef cone of $\mathcal{T}_{g;1}(l)$ is bounded in the $\lambda$–$\delta$ plane by the rays spanned by $D_l$ and $D_{l+2}$, where*

$$D_l = \{(7g+6)\lambda - g\delta\} + \frac{l^2}{g+2} \cdot \{9\lambda - \delta\}.$$

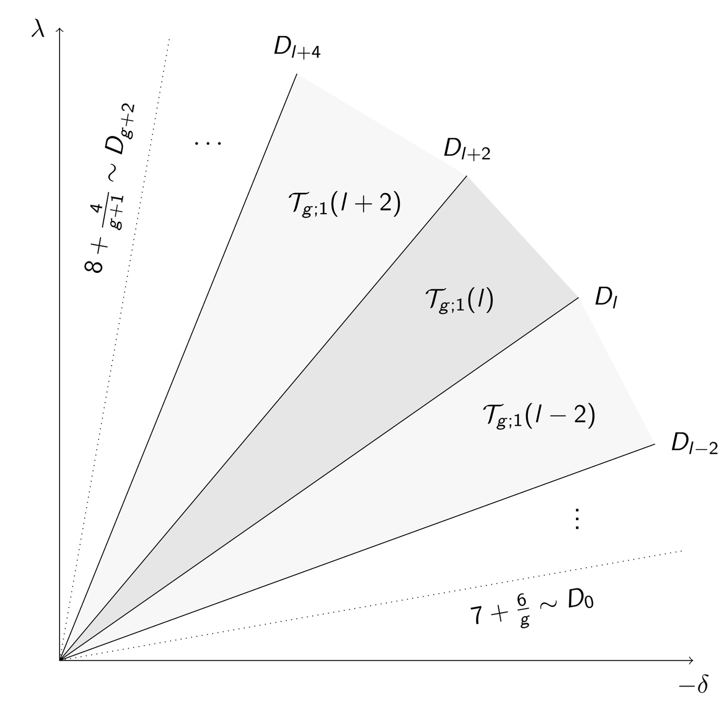



Figure: Decomposition of $\operatorname{Pic}_{\mathbf{Q}}$ into nef cones (not to scale!).

## How to Handle Ramified Marked Fibers?

It is natural to wonder if a similar picture holds for the other two ramification types of the marked fiber, namely (3) and (2, 1). We use the following technique to deal with ramification. For simplicity, we explain the case of total ramification. We have the correspondence

$$\left\{\begin{matrix}\text{Triple covers of } \mathbf{P}^1 \text{ totally ramified over } \sigma\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{(Certain) étale covers of the root stack } \mathbf{P}^1(\sqrt[3]{\sigma})\end{matrix}\right\}.$$

described by the diagram

$$\begin{array}{ccc} \operatorname{Spec} k[y] & \longleftarrow & \operatorname{Spec} k[y] \\ \downarrow {\scriptstyle x \mapsto y^3} & & \downarrow {\scriptstyle \text{étale}} \\ \operatorname{Spec} k[x] & \xleftarrow{x \mapsto t^3} & [\operatorname{Spec} k[t]/\mu_3] \end{array}$$

We handle the case of a marked fiber of ramification type (3) (resp. (2, 1)) by considering suitable étale covers of $\mathbf{P}^1(\sqrt[3]{\sigma})$ (resp. $\mathbf{P}^1(\sqrt{\sigma})$). After the right generalizations of the Maroni invariant and the $\mu$ invariant, a similar picture emerges.

## References


- D. Abramovich, A. Corti, and A. Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman.
- D. Abramovich and A. Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–75 (electronic), 2002.
- M. Bolognesi and A. Vistoli. Stacks of trigonal curves. *Arxiv E-Prints*, Mar. 2009.
- A. Deopurkar. Alternate compactifications of Hurwitz spaces. *Thesis*, In preparation.
- M. Fedorchuk. Moduli spaces of hyperelliptic curves with A and D singularities. *ArXiv e-prints*, July 2010.
- B. Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003.
- B. Poonen. The moduli space of commutative algebras of finite rank. *J. Eur. Math. Soc. (JEMS)*, 10(3):817–836, 2008.
- Z. E. Stankova-Frenkel. Moduli of trigonal curves. *J. Algebraic Geom.*, 9(4):607–662, 2000.