3. $z = f(x,y) = \sqrt{xy} \quad \Rightarrow \quad f_x(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{1}{2}\sqrt{y/x}$, $f_y(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{x/y}$, so $f_x(1,1) = \frac{1}{2}$ and $f_y(1,1) = \frac{1}{2}$. Thus an equation of the tangent plane is $z - 1 = f_x(1,1)(x-1) + f_y(1,1)(y-1) \quad \Rightarrow$ $z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$ or $x + y - 2z = 0$.

4. $z = f(x,y) = xe^{xy} \quad \Rightarrow \quad f_x(x,y) = xye^{xy} + e^{xy},\ f_y(x,y) = x^2e^{xy}$, so $f_x(2,0) = 1,\ f_y(2,0) = 4$, and an equation of the tangent plane is $z - 2 = f_x(2,0)(x-2) + f_y(2,0)(y-0) \quad \Rightarrow \quad z - 2 = 1(x-2) + 4(y-0)$ or $z = x + 4y$.

18. Let $f(x,y) = \sqrt{y + \cos^2 x}$. Then $f_x(x,y) = \frac{1}{2}(y + \cos^2 x)^{-1/2}(2\cos x)(-\sin x) = -\cos x \sin x / \sqrt{y + \cos^2 x}$ and $f_y(x,y) = \frac{1}{2}(y + \cos^2 x)^{-1/2}(1) = 1/\left(2\sqrt{y + \cos^2 x}\right)$. Both f_x and f_y are continuous functions for $y > -\cos^2 x$, so f is differentiable at $(0,0)$ by Theorem 8. We have $f_x(0,0) = 0$ and $f_y(0,0) = \frac{1}{2}$, so the linear approximation of f at $(0,0)$ is $f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + 0x + \frac{1}{2}y = 1 + \frac{1}{2}y$.

19. We can estimate $f(2.2, 4.9)$ using a linear approximation of f at $(2,5)$, given by $f(x,y) \approx f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5) = 6 + 1(x-2) + (-1)(y-5) = x - y + 9$. Thus $f(2.2, 4.9) \approx 2.2 - 4.9 + 9 = 6.3$.

21. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad f_x(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}},\ f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6) = \frac{3}{7},\ f_y(3,2,6) = \frac{2}{7},\ f_z(3,2,6) = \frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

22.

From the table, $f(40,20)=28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 14.3.4. Since $f_v(40,20)=\lim_{h\to 0}\dfrac{f(40+h,20)-f(40,20)}{h}$, we approximate this quantity with $h=\pm 10$ and use the values given in the table:

$$f_v(40,20)\approx\frac{f(50,20)-f(40,20)}{10}=\frac{40-28}{10}=1.2,\qquad f_v(40,20)\approx\frac{f(30,20)-f(40,20)}{-10}=\frac{17-28}{-10}=1.1$$

Averaging these values gives $f_v(40,20)\approx 1.15$. Similarly, $f_t(40,20)=\lim_{h\to 0}\dfrac{f(40,20+h)-f(40,20)}{h}$, so we use $h=10$ and $h=-5$:

$$f_t(40,20)\approx\frac{f(40,30)-f(40,20)}{10}=\frac{31-28}{10}=0.3,\qquad f_t(40,20)\approx\frac{f(40,15)-f(40,20)}{-5}=\frac{25-28}{-5}=0.6$$

Averaging these values gives $f_t(40,15)\approx 0.45$. The linear approximation, then, is

$$f(v,t)\approx f(40,20)+f_v(40,20)(v-40)+f_t(40,20)(t-20)\approx 28+1.15(v-40)+0.45(t-20)$$

When $v=43$ and $t=24$, we estimate $f(43,24)\approx 28+1.15(43-40)+0.45(24-20)=33.25$, so we would expect the wave heights to be approximately 33.25 ft.

33.

$dA=\dfrac{\partial A}{\partial x}\,dx+\dfrac{\partial A}{\partial y}\,dy=y\,dx+x\,dy$ and $|\Delta x|\le 0.1$, $|\Delta y|\le 0.1$. We use $dx=0.1$, $dy=0.1$ with $x=30$, $y=24$; then the maximum error in the area is about $dA=24(0.1)+30(0.1)=5.4\text{ cm}^2$.

38. Here $dV=\Delta V=0.3$, $dT=\Delta T=-5$, $P=8.31\,\dfrac{T}{V}$, so

$$dP=\left(\frac{8.31}{V}\right)dT-\frac{8.31\cdot T}{V^2}\,dV=8.31\left[-\frac{5}{12}-\frac{310}{144}\cdot\frac{3}{10}\right]\approx -8.83.$$ Thus the pressure will drop by about $8.83\,\text{kPa}$.

1. $z=x^2+y^2+xy,\ x=\sin t,\ y=e^t \quad\Rightarrow\quad \dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}=(2x+y)\cos t+(2y+x)e^t$

6. $w=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln(x^2+y^2+z^2)$, $x=\sin t$, $y=\cos t$, $z=\tan t\quad\Rightarrow$

$$\begin{aligned}\frac{dw}{dt}&=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=\frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t+\frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t)+\frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\cdot\sec^2 t\\&=\frac{x\cos t-y\sin t+z\sec^2 t}{x^2+y^2+z^2}\end{aligned}$$

12. $z = \tan(u/v),\ u = 2s + 3t,\ v = 3s - 2t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \sec^2(u/v)(1/v)\cdot 2 + \sec^2(u/v)(-uv^{-2})\cdot 3$$
$$= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2)$$
$$= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)$$

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v,\ y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u,\ \dfrac{\partial x}{\partial v} = \cos v,\ \dfrac{\partial y}{\partial u} = e^u,\ \dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)$$
$$= 2(1) + 5(0) = 2$$

23. $w = xy + yz + zx,\ x = r\cos\theta,\ y = r\sin\theta,\ z = r\theta \quad \Rightarrow$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r).$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$, so $\dfrac{\partial w}{\partial r} = (2+\pi)(0) + (0+\pi)(1) + (2+0)(\pi/2) = 2\pi$ and

$$\frac{\partial w}{\partial \theta} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2) = -2\pi.$$

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of $0.15°$C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.