Calculus III: Practice Midterm I

Name: _____

- Write your solutions in the space provided. Continue on the back if you need more space.

- You must show your work. Only writing the final answer will receive little credit.

- Partial credit will be given for incomplete work.

- The exam contains 6 problems.

- **Good luck!**

Question	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
6	12	
Total:	60	

1. Write true or false. No justification is needed.

    (a) (2 points) There is a vector $\mathbf{v}$ such that

$$\mathbf{v} \times \langle 1, 1, 1 \rangle = \langle 1, 2, 3 \rangle.$$

> **Solution:** False. (Because $\langle 1, 2, 3 \rangle$ is not perpendicular to $\langle 1, 1, 1 \rangle$.)

    (b) (2 points) The sixth power of $2e^{i\pi/6}$ is a real number.

> **Solution:** True. (Because the argument of the sixth power is π.)

    (c) (2 points) The surface described by $x^2 + y^2 - z^2 = 1$ is a hyperbolic paraboloid.

> **Solution:** False. (Because the traces are ellipses and hyperbolas, making it a hyperboloid.)

    (d) (2 points) The plane $2x + 4y + 6z = 9$ is perpendicular to the vector $\langle 1, 1, -1 \rangle$.

> **Solution:** False. (Because the normal direction to that plane is $\langle 2, 4, 6 \rangle$, which is not the same as $\langle 1, 1, -1 \rangle$.)

2. Determine whether the following vectors are parallel, perpendicular or neither. Explain why.

   (a) (3 points) $\langle 2, -3, 1 \rangle$ and $\langle 2, 1, -1 \rangle$.

> **Solution:** Since the vectors are not multiples of each other, they are not parallel. Since
> $$\langle 2, -3, 1 \rangle \cdot \langle 2, 1, -1 \rangle = 4 - 3 - 1 = 0,$$
> they are perpendicular.

   (b) (3 points) $2\,\mathbf{i} + \mathbf{j} - 4\,\mathbf{k}$ and $-14\,\mathbf{i} + 7\,\mathbf{j} + 14\,\mathbf{k}$.

> **Solution:** Since the vectors are not multiples of each other, they are not parallel. Since
> $$(2\,\mathbf{i} + \mathbf{j} - 4\,\mathbf{k}) \cdot (-14\,\mathbf{i} + 7\,\mathbf{j} + 14\,\mathbf{k}) = -28 + 7 - 56 \neq 0,$$
> they are not perpendicular either.

   (c) (4 points) $\langle 1, 1, 1 \rangle$ and $\langle 2, 1, 2 \rangle \times \langle 1, 0, 1 \rangle$.

> **Solution:** Let us compute the cross product:
> $$\langle 2, 1, 2 \rangle \times \langle 1, 0, 1 \rangle = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$
> $$= \mathbf{i} - \mathbf{k} = \langle 1, 0, -1 \rangle.$$
> Since $\langle 1, 1, 1 \rangle \cdot \langle 1, 0, -1 \rangle = 0$, the vectors are perpendicular.

3. (10 points) Do the four poins $(1, 1, 0)$, $(1, 1, -2)$, $(0, 2, -1)$ and $(5, -3, 0)$ lie on the same plane? Justify your answer.

Solution: There are many ways to do this problem. Let us denote the points by A, B, C, and D. Here are two strategies:

1. Find the equation of the plane passing through A, B, and C and check whether D satisfies it.

2. Find the volume of the parallelopiped formed by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$. If the points were on a plane, it would be zero (the paralellopiped will be flat), otherwise it would be non-zero.

First way: We first find the equation of the plane passing through A, B and C. This is the plane passing through $\langle 1, 1, 0 \rangle$ and with normal direction $\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC}$. We have

$$\overrightarrow{AB} = \langle 1, 1, -2 \rangle - \langle 1, 1, 0 \rangle = \langle 0, 0, -2 \rangle = -2\,\mathbf{k}$$
$$\overrightarrow{AC} = \langle 0, 2, -1 \rangle - \langle 1, 1, 0 \rangle = \langle -1, 1, -1 \rangle = -\,\mathbf{i} + \mathbf{j} - \mathbf{k}$$
$$\overrightarrow{n} = (-2\,\mathbf{k}) \times (-\,\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2\,\mathbf{k} \times \mathbf{i} - 2\,\mathbf{k} \times \mathbf{j} = 2\,\mathbf{j} + 2\,\mathbf{i}.$$

The plane through A and perpendicular to $\overrightarrow{n}$ is

$$(\langle x, y, z \rangle - \langle 1, 1, 0 \rangle) \cdot \langle 2, 2, 0 \rangle = 0$$
$$2(x - 1) + 2(y - 1) = 0$$
$$2x + 2y = 4 \text{ or equivalently } x + y = 2.$$

The fourth point $\langle 5, -3, 0 \rangle$ satisfies this equation, so it lies on this plane. In other words, the four points are coplanar.

Second way: We know that the volume of the paralellopiped formed by $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$. We need to check whether this is zero.

We have already computed

$$\overrightarrow{AB} \times \overrightarrow{AC} = 2\,\mathbf{i} + 2\,\mathbf{j} = \langle 2, 2, 0 \rangle.$$

Now, $\overrightarrow{AD} = \langle 5, -3, 0 \rangle - \langle 1, 1, 0 \rangle = \langle 4, -4, 0 \rangle$. Hence the triple product is

$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \langle 4, -4, 0 \rangle \cdot \langle 2, 2, 0 \rangle = 0.$$

In conclusion, the paralellopiped is flat and hence A, B, C, and D are coplanar.

4. (10 points) Find all the complex valued solutions of the equation

$$x^3 = i.$$

Express your answers both in polar and Cartesian forms.

Solution: Taking the magnitudes, we get

$$|x|^3 = 1.$$

Since $|x|$ is a positive real number, we conclude that $|x| = 1$. It remains to find the argument of x. Since $\arg i = \pi/2$, we must have

$$3 \arg x = \frac{\pi}{2} \text{ or } \left(\frac{\pi}{2} + 2\pi\right) \text{ or } \left(\frac{\pi}{2} + 4\pi\right) \text{ etc.}$$

Therefore,

$$\arg x = \frac{\pi}{6} \text{ or } \left(\frac{\pi}{6} + \frac{2\pi}{3}\right) \text{ or } \left(\frac{\pi}{6} + \frac{4\pi}{3}\right).$$

Hence, in polar form, the solutions to $x^3 = i$ are

$$x = e^{\frac{i\pi}{6}} \text{ or } e^{\frac{5i\pi}{6}} \text{ or } e^{\frac{9i\pi}{6}}.$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we get the Cartesian forms

$$x = \cos(\pi/6) + i\sin(\pi/6) \text{ or } \cos(5\pi/6) + i\sin(5\pi/6) \text{ or } \cos(9\pi/6) + i\sin(9\pi/6)$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{ or } \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{ or } \quad -i$$

Note: In class, we called $r(\cos\theta + i\sin\theta)$ the polar form. Since this is equal to $re^{i\theta}$ (which is more compact), it is more common to call $re^{i\theta}$ the polar form. In the exam, you may use either one.

5. Let P be the plane perpendicular to $\langle 1, 2, 3 \rangle$ and passing through the point $\langle 1, 0, 1 \rangle$.

    (a) (5 points) Find an equation for P.

> **Solution:** The equation is obtained from
>
> $$(\langle x, y, z \rangle - \langle 1, 0, 1 \rangle) \cdot \langle 1, 2, 3 \rangle = 0,$$
>
> which gives
> $$(x - 1) + 2y + 3(z - 1) = 0,$$
>
> or equivalenlty
> $$x + 2y + 3z = 4.$$

    (b) (5 points) Does the line given by $x(t) = 3t + 1$, $y(t) = 3$ and $z(t) = -t + 3$ intersect the plane P?

> **Solution:** The line intersects if and only if there exists a t such that $\langle x(t), y(t), z(t) \rangle$ satisfies the equation of the plane:
>
> $$x(t) + 2y(t) + 3z(t) = 4.$$
>
> Substituting from the parametric equations, we get
>
> $$(3t + 1) + 2 \cdot 3 + 3(-t + 3) = 4$$
> $$1 + 6 + 9 = 4,$$
>
> an equation that is never satisfied. Hence the line does not intersect the plane.

6. For which (real) values of a are the vectors $\langle 1, a, 2 \rangle$ and $\langle a, 4, 4 \rangle$

   (a) (3 points) parallel?

   > **Solution:** For the vectors to be parallel, they must be proportional. By looking at the third coordinate, we see that the second vector must be twice the first. So we must have
   > $$2\langle 1, a, 2 \rangle = \langle a, 4, 4 \rangle.$$
   > By comparing the first and the second coordinates, we get
   > $$a = 2 \text{ and } a = 2.$$
   > Hence $a = 2$.

   (b) (3 points) perpendicular?

   > **Solution:** For the vectors to be perpendicular, the dot product must be zero:
   > $$\langle 1, a2 \rangle \cdot \langle a, 4, 4 \rangle = a + 4a + 8 = 0,$$
   > which gives $a = -8/5$.

   (c) (6 points) For which a, does the first vector $\langle 1, a, 2 \rangle$ make an angle of $\pi/4$ with the vector $\mathbf{j}$?

   > **Solution:** One way of doing this is to use the dot product. If the angle is $\pi/4$, we must have
   > $$\mathbf{j} \cdot \langle 1, a, 2 \rangle = |\,\mathbf{j}||\langle 1, a, 2 \rangle| \cos(\pi/4).$$
   > Simplifying, we get
   > $$a = \sqrt{a^2 + 5} \cdot (1/\sqrt{2})$$
   > $$\sqrt{2}a = \sqrt{a^2 + 5}.$$
   > Squaring, we get
   > $$2a^2 = a^2 + 5$$
   > $$a^2 = 5.$$
   > So that $a = \pm\sqrt{5}$. We can discard $a = -\sqrt{5}$ since that would make the dot product negative (which corresponds to an obtuse angle). Hence $a = \sqrt{5}$.