Moduli of curves : Nov 18.

Let k be alg. closed., C/k a proper, connected curve, $P_i \in C(k)$ points for $i = 1, \dots, n$. We say that $(C, P_1, \dots, P_n)$ is stable if.

(1) C is at worst-nodal (prestable).

(2) $P_i \in C$ lies in the smooth locus (i.e. away from nodes).

(3) $P_i \neq P_j$ for $i \neq j$.

(4) $\mathrm{Aut}(C, P_1, \dots, P_n)$ is finite

(eqv. every component (the normalization of) has at least 3 special points.

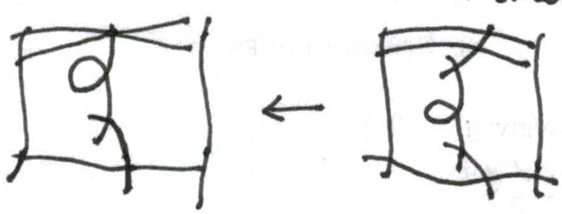
$$\overline{M}_{g,n} = \left\{ \begin{array}{c} C \\ \downarrow \pi \\ S \end{array} \Big) P_1, \dots, P_n \;\middle|\; \text{family of stable } n\text{-pointed curves} \right\}.$$

Thm: $\overline{M}_{g,n} \to \mathrm{spec}\,\mathbb{Z}$ ~~stable~~ smooth and proper DM stack.

Last time: stable reduction for $\overline{M}_g$ with smooth generic fiber.

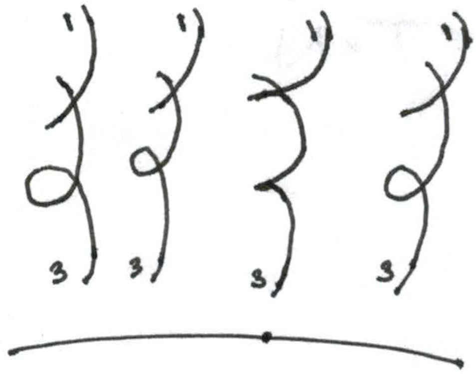
Easy consequence, stable reduction for $\overline{M}_{g,n}$ with smooth generic fiber.

How? : • First forget the sections and do semi-stable reduction.



- Then make further blow ups to separate the sections
- Contract unstable components on the central fiber.
  (first -1 curves, then image under $K(P_1 + \dots + P_n)$)

Stable reduction with singular gen. fiber.

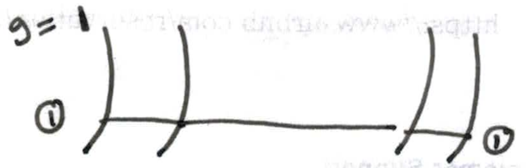


$\Delta^* = \mathrm{spec}\,K$. $C \to K$ nodal curve

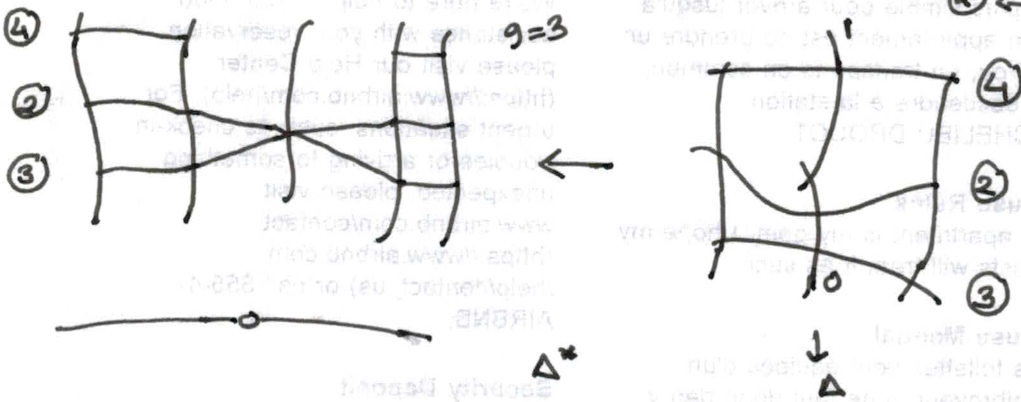
$C^\nu \xrightarrow{\pi} C$ normalisation.

Extend K so that each pt of $\pi^{-1}(\text{nodes } C)$ is defined over K.

Label these points ①, ②, ...

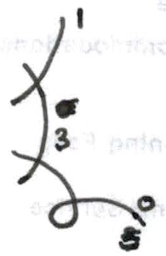


C/Δ^* obtained from C by gluing ① $\leftrightarrow$ ④, ② $\leftrightarrow$ ③.



Now do stable reduction for these pointed curves.

Again glue ① $\leftrightarrow$ ④, ② $\leftrightarrow$ ③. over Δ. $\Rightarrow$ Stable reduction.



$\Leftarrow$ New central fiber

Proof of separatedness: Skip (Not hard using geometry of surfaces and birational maps among them.)

Local Structure of $\overline{M}_g$ / k alg. closed.

$$
\begin{tikzcd}
C_A \arrow[d] & C_\varepsilon \arrow[l, hook'] \arrow[d] & C_0 \arrow[l, hook'] \arrow[d] & \\
\operatorname{Spec} A & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[l, hook'] & \operatorname{Spec} k \arrow[l, hook'] \arrow[r] & \overline{M}_g
\end{tikzcd}
$$

$\underline{\mathrm{Art}}_k$ = Category of Artin-local k-algebras.

We will consider functors $F : \mathrm{Art}_k \to$ sets.

Example ①. X_0/k a scheme.

$\mathrm{Def}_{X_0} : \underline{\mathrm{Art}}_k \to$ Sets.

$A \mapsto$ iso. classes of deformations of X_0 over A.

i.e. ($X_A \to A$ flat and an iso. $X_A \otimes_A k \xrightarrow{\sim} X_0$).

② Let R be a complete local k-algebra.

$h_R : \mathrm{Art}_k \to$ sets.

$A \mapsto$ Ring Homs $(R \to A)$.

~~Def~~ All our functors will satisfy $F(k)$ = Singleton set.

Def :(1) $F : \underline{\mathrm{Art}}_k \to$ Sets is pro-representable if $F \cong h_R$ for some R.

~~Eg.~~

(2) F has a versal family if there is a formally smooth map $h_R \to F$.

(3) A versal family is mini-versal if it induces an iso on $k[\varepsilon]/\varepsilon^2$.

Ex: What does formally smooth $G \to F$ mean?

Lifting criterion: $\tilde{A} \to A \to 0$ in Art_k.

$$\left.\begin{array}{ccc} g_A \in G(A) & & \text{extending } f_A \\ \downarrow & & \rotatebox{90}{$\supseteq$} \\ f_A \in F(A) & , & f_{\tilde{A}} \in F(\tilde{A}) \end{array}\right\} \text{given.}$$

Then $\exists$ $g_{\tilde{A}} \in G(\tilde{A})$ extending g_A and mapping to $f_{\tilde{A}}$.

$$\left[\quad G(\tilde{A}) \to G(A) \times_{F(A)} F(\tilde{A}) \text{ is surjective} \right].$$

Ex: $\mathcal{X}$ an algebraic stack, $x : \operatorname{spec} k \to \mathcal{X}$.

$F_x : Art_k \to Sets$

$A \mapsto$ maps $\operatorname{spec} A \to \mathcal{X}$ along with iso $\operatorname{spec} k \to \mathcal{X}$ with x.

$U \xrightarrow{\pi} \mathcal{X}$ an atlas. $u \in U$ over x.

Then $\hat{O}_{U,u} \to F_x$ is a versal family.

If π is an étale atlas, then a mini-versal family.

Schlessinger: Criteria for a functor to have veral / mini-versal families.

Prop:
$$\begin{tikzcd} h_R \arrow[r, "①"] \arrow[d, dashed] & F \\ h_S \arrow[ur, "②"'] & \end{tikzcd}$$
① versal, ② any. Then $\exists$ ③.

If ② smooth, then ③ also smooth.

So all versal families are alike "up to smooth parameters".

Versal family $\longleftrightarrow$ smooth local chart for F.

Scratch Work

<u>Example</u>: $X_0 = \operatorname{Spec} k[x,y]/xy$.

$R = k[[t]]$.

Consider $h_R \to \underline{\mathrm{Def}}\, X_0$ given by $(xy-t) \subset R[x,y]$.

<u>Prop</u>: This is a versal family (in fact miniversal).

<u>Pf</u>: Given $0 \to k \xrightarrow{\varepsilon} \tilde A \to A \to 0$ in Art_k.

$$\begin{array}{ccc} C_{\tilde A} & \text{and} & C_A \xrightarrow{\sim} A[x,y]/(xy-a) \\ \downarrow & & \downarrow \\ \tilde A & & A \end{array}$$

(map $k[[t]] \to A$ eqv. to $t \mapsto a$).

Want to lift to $C_{\tilde A} \xrightarrow{\sim} \tilde A[x,y]/(xy-\tilde a)$.

We know that $C_{\tilde A} \subset \tilde A[x,y]$ is defined by some equation $\tilde g(x,y)$.

Also, from $C_A \xrightarrow{\sim} A[x,y]/(xy-a)$ we get

$$A[x,y]/g(x,y) \xleftrightarrow{\sim} A[x,y]/(xy-a)$$

i.e. $X \in A[x,y]$, $Y \in A[x,y]$ reducing to x, y s.t.

$$g(X,Y) = U\,(xy-a) \qquad U \in A[x,y] \text{ unit.} \qquad U_0 = 1$$

Lift X, Y, a to $\tilde A[x,y]$ and $\tilde A$ arbitrarily. (also U).

Then $\tilde g(\tilde X, \tilde Y) = \tilde U\,(xy-\tilde a) + \varepsilon\, e(x,y)$ ← error.

$= \tilde U\,(xy-\tilde a) + \underline{\varepsilon a} + \underline{\varepsilon b x} + \underline{\varepsilon c y} + \varepsilon(xy) f(x,y)$.

Change $\tilde X \to \tilde X + \varepsilon b$, $\tilde Y \to \tilde Y + \varepsilon$, $\tilde a \to \tilde a + \varepsilon a$, $\tilde U \to \tilde U + \varepsilon f(x,y)$.

i.e. can eat all the errors by wiggling the params. $\square$.

<u>Generalization</u>: $f(x,y)=0 \subset \mathbb{A}^2$ isolated sing at $(0,0)$.

$g_1, \dots, g_r \in k[x,y]$ basis of $k[x,y]/(f_x, f_y)$

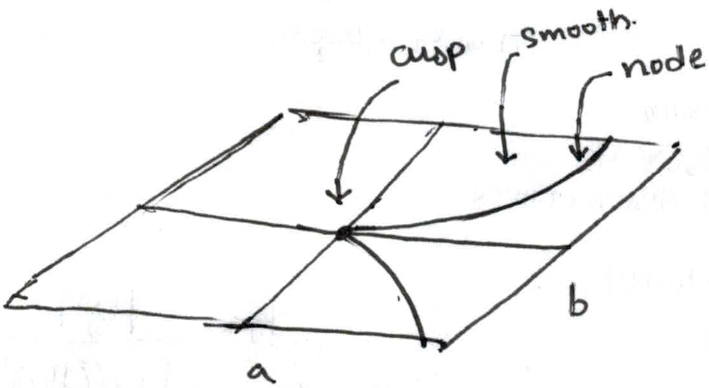
$k[[t_1, \dots, t_r]] = \Lambda$.

$\Lambda[x,y] / f(x,y) + \sum t_i g_i$ is a (mini) versal family.

<u>Same</u> proof.

<u>Ex.</u> $y^2 - x^3 = f(x,y)$, $\quad f_x = 3x^2$, $f_y = 2y$ $\quad \underline{\underline{1, x,}}$

$\underline{y^2 - x^3 + ax + b.}$ ~~univer~~ Versal deformation.



$4a^2 + 27b^3$

~~Moduli of curves: Nov 18~~

~~Last time: Stable reduction for generically smooth families~~

~~$\overline{M}_{g,n} = \{(C, P_1, \dots, P_n) \xrightarrow{\pi} S \mid \pi$ flat, P_i sections s.t. over each geometric point,~~