Calculus I: Practice Midterm I

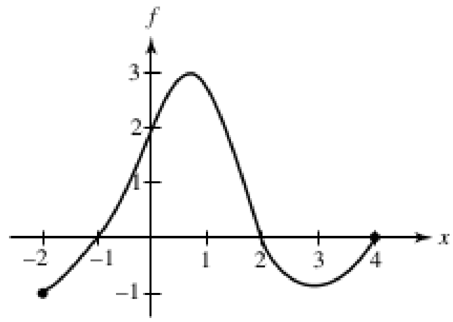
February 14, 2015

Name: ______________________________

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 6 problems.
- **Good luck!**

Question	Points	Score
1	7	
2	8	
3	9	
4	10	
5	8	
6	8	
Total:	50	

1. Below is the graph of a function f.



(a) (3 points) Use the graph to (approximately) compute the following:

(a) $f(-1)$ and $f(1)$.

Solution: $f(-1) = 0$, $f(0) = 2$, and $f(1) = 3$.

(b) All x such that $f(x) = 0$.

Solution: This is the set of x where the graph intersects the X-axis. These are -1, 2, and 4.

(c) The range of f.

Solution: The range of f is $[-1, 3]$.

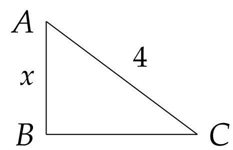
(d) (4 points) Let $g(x) = x^2 + 1$. What is $f(g(1))$? What is $g(f(1))$?

Solution:

$$f(g(1)) = f(2) = 0$$
$$g(f(1)) = g(3) = 3^2 + 1 = 10.$$

2. A 4 foot ladder is leaning against the wall. Denote by x the height of the top end of the ladder (as measured from the floor).

Solution: We first draw a picture. The wall is AB and the ladder is AC.



(a) (3 points) Express the distance of the bottom end of the ladder from the wall as a function of x.

Solution: Since $AB^2 + BC^2 = AC^2$, we get $BC^2 = 16 - x^2$. So $BC = \sqrt{16 - x^2}$, which is the distance of the bottom end of the ladder from the wall.

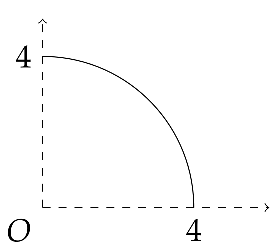
(b) (3 points) Find the domain and range of the function you found in the previous part.

Solution: The number x can range from 0 (when the ladder is horizontal) to 4 (when the ladder is vertical). So the domain is $[0, 4]$.

Some may argue that it is physically impossible for the ladder to be exactly vertical, so the domain should be $[0, 4)$. I will accept either answer (if justified).

(c) (2 points) Draw a rough sketch of the graph of the function.

Solution: The graph looks like the quarter of a circle. I'd give full credit even if you didn't get the exact shape right. But the graph should show a decreasing nonlinear graph passing through $(0, 4)$ and $(4, 0)$.



3. Let

$$f(x) = \frac{e^x}{1+e^x}.$$

It turns out that f has an inverse function.

(a) (3 points) Find $f^{-1}(1/2)$.

Solution: We want to solve the equation

$$\frac{1}{2} = \frac{e^x}{1+e^x}.$$

We get

$$1 + e^x = 2e^x \implies e^x = 1 \implies x = \ln 1 = 0.$$

Therefore, $f^{-1}(1/2) = 0$.

(b) (3 points) Find a formula for $f^{-1}(x)$.

Solution: To find a formula for the inverse function, let us write $y = f(x)$ and solve for x in terms of y. We have

$$\begin{aligned} y &= \frac{e^x}{1+e^x} \\ (1+e^x)y &= e^x \\ y + e^x y &= e^x \\ y &= e^x - e^x y \\ y &= (1-y)e^x \\ e^x &= \frac{y}{1-y} \\ x &= \ln\left(\frac{y}{1-y}\right) = \ln y - \ln(1-y). \end{aligned}$$

So $f^{-1}(y) = \ln y - \ln(1-y)$. We may change the name of the variable and also write the inverse function as $f^{-1}(x) = \ln x - \ln(1-x)$.

(c) (3 points) Write $f(x)$ as the composition of two functions.

Solution: Let $g(x) = e^x$ and $h(x) = \frac{x}{1+x}$. Then $f(x) = h(g(x))$.

4. Calculate each of the following limits, if it exists. Justify your answer.

(a) (3 points) $\lim_{x\to 0} |x|\sin(1/x)$.

Solution: We have
$$-1 \le \sin(1/x) \le 1.$$
Multiplying throughout by $|x|$, we get
$$-|x| \le |x|\sin(1/x) \le |x|.$$
Since $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$, we conclude using the squeeze theorem that $\lim_{x\to 0} |x|\sin(1/x) = 0$.

(b) (4 points) $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$

Solution: We have
$$|x-1| = \begin{cases} (x-1) & \text{if } x \ge 1 \\ -(x-1) & \text{if } x < 1. \end{cases}.$$
Therefore,
$$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^+} (x+1) = 2$$
$$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{(x-1)(x+1)}{-(x-1)} = \lim_{x\to 1^+} -(x+1) = -2.$$
Since the left hand limit and the right hand limit are unequal, the limit does not exist.

(c) (3 points) $\lim_{x\to +\infty} \arctan(e^x+2)$

Solution: Since $e^x + 2 \to +\infty$ as $x \to +\infty$, we get
$$\lim_{x\to +\infty} \arctan(e^x+2) = \lim_{y\to +\infty} \arctan(y) = \pi/2.$$

5. (8 points) Let

$$h(x) = \frac{2x^2 - 3x + 1}{x^2 - 1}$$

Find the horizontal and vertical asymptotes of $h(x)$.

Solution: We first compute the horizontal asymptotes. For this, we want to compute $\lim_{x\to+\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$. We have

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{2x^2 - 3x + 1}{x^2 - 1} &= \lim_{x\to\pm\infty} \frac{(2x^2 - 3x + 1)/x^2}{(x^2 - 1)/x^2} \\ &= \lim_{x\to\pm\infty} \frac{2 - 3/x + 1/x^2}{1 - 1/x^2} \\ &= \frac{2}{1} = 2.\end{aligned}$$

Note that we divided the numerator and the denominator by the dominant power of x, which is x^2. We get that $y = 2$ is a horizontal asymptote.

Next, we compute the vertical asymptotes. For this, we want to find a such that $\lim_{x\to a^\pm} h(x) = \pm\infty$. If $a^2 - 1 \neq 0$ then $\lim_{x\to a} h(x) = h(a) \neq \infty$. Therefore, the only possible vertical asymptotes are when $a = 1$ or $a = -1$.

For $a = -1$, the numerator $2x^2 - 3x + 1$ approaches 7 and the denominator $x^2 - 1$ approaches 0. So, the quotient $h(x)$ approaches $\pm\infty$. As a result $x = -1$ is a vertical asymptote.

For $a = 1$, the numerator $2x^2 - 3x + 1$ approaches 0 and the denominator $x^2 - 1$ also approaches 0. So we cannot conclude anything about $h(x)$ without doing something else. We factor the numerator and the denominator

$$\frac{2x^2 - 3x + 1}{x^2 - 1} = \frac{(x-1)(2x+1)}{(x-1)(x+1)} = \frac{2x+1}{x+1}.$$

Therefore,

$$\lim_{x\to 1} h(x) = \lim_{x\to 1} \frac{2x+1}{x+1} = \frac{3}{2} \neq \pm\infty.$$

As a result, $x = 1$ is *not* a vertical asymptote.

We conclude that the only horizontal asymptote is $y = 2$ and the only vertical asymptote is $x = -1$.

6. Let

$$f(x) = \frac{3x}{1+x}.$$

(a) (6 points) Find $f'(2)$ using the definition of the derivative.

Solution: We use the definition:

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{3x}{1+x} - \frac{6}{3}}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{3x-2(1+x)}{1+x}}{x-2} \\ &= \lim_{x\to 2} \frac{x-2}{(1+x)(x-2)} \\ &= \lim_{x\to 2} \frac{1}{1+x} \\ &= \frac{1}{3}. \end{aligned}$$

(b) (2 points) Is f increasing or decreasing near $x = 2$?

Solution: $f'(2) > 0$ means that f is increasing near $x = 2$ (the graph slopes upwards).