Calculus 1: Review

- Limits
- Derivatives
- Integrals

Limits

$$\lim_{x \to a} f(x)$$

where $f(x)$ is a formula involving the basic functions.

- Plug in $x = a$.
- Simplify and then plug in $x = a$.
- $0/0$ or ∞/∞: Use l'Hôpital.

$$\lim_{x \to a} \frac{N(x)}{D(x)} = \lim_{x \to a} \frac{N'(x)}{D'(x)}.$$

- $0 \cdot \infty, \infty - \infty, 0^0, 1^\infty$, or other indeterminate form: manipulate (take log, bring something in the denominator, etc) and use l'Hôpital.
- $f(x)$ is defined piecewise: work on each side and see if you get the same answer on each side.
- Squeeze theorem.

Derivatives

- Derivatives of basic functions.
- Basic rules: sum, product, quotient, chain.
- Chain rule:

$$f(g(x))' = f'(g(x))g'(x).$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- Implicit differentiation: don't forget $\frac{dy}{dx}$.

$$x^3 + y^3 \rightsquigarrow 3x^2 + 3y^2 \frac{dy}{dx}.$$

- Logarithmic differentiation:

$$\begin{aligned} y &= \text{Complicated product or exponents} \\ \ln y &= \text{Sum or product (easier)} \\ \frac{y'}{y} &= \text{Derivative of easier expression.} \end{aligned}$$

Applications of derivatives

- Basic interpretation

$$f'(a) = \text{Slope of the tangent to } y = f(x) \text{ at } (a, f(a)).$$

- Sign of f' and direction of f
  - $f' > 0 \Leftrightarrow f$ increasing.
  - $f' < 0 \Leftrightarrow f$ decreasing.
  - $f' = 0 \Leftrightarrow f$ has a critical point.
- Sign of f'' and concavity of f:
  - $f'' > 0 \Leftrightarrow f'$ increasing $\Leftrightarrow f$ concave up.
  - $f'' < 0 \Leftrightarrow f'$ decreasing $\Leftrightarrow f$ concave down.

Applications of derivatives

- Linear approximation: $f(x) \approx f(a) + (x - a)f'(a)$, where a is an "easy number" close to x.
- Related rates: find an expression relating different quantities and differentiate the expression.
- Optimization:
  - Construct $f(x)$
  - Set $f'(x) = 0$ to get critical points.
  - Check critical and end points for closed interval.
  - Check sign of f' on parts of the domain for non-closed interval.
- Newton's method to solve $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Integrals

$$\int_a^b f(x)dx.$$

- Find antiderivative $F(x)$. Then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

- Basic (anti)derivatives.
- Basic rule: sum rule (and taking out constant multipliers).

Integrals

- Substitution.
  - Find a good $u =$ Function of x.
  - Replace dx by $du/$Derivative of the function.
  - Rewrite the rest in terms of u.
  - For definite integrals, also rewrite a and b.

Integrals

$$\int_a^b f(x)dx.$$

- Interpretation: (signed) area under $y = f(x)$ from $x = a$ to $x = b$.
- $$A(t) = \int_a^t f(x)dx \implies A'(t) = f(t).$$

Integrals

- For unsigned area between $f(x)$ and $g(x)$:

$$\int_a^b |f(x) - g(x)| dx.$$

  To evaluate:
  - Divide $[a, b]$ in regions where $f(x) - g(x)$ has specific sign.
    - First find x's where $f(x) - g(x) = 0$.
    - Then find the signs in the intermediat regions.
  - On each region, integrate $f(x) - g(x)$ or $g(x) - f(x)$ depending on the sign (positive or negative).
- Average value of $f(x)$ on $[a, b]$

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

Integrals

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$