# Algebraic geometry

Over the complex numbers

$k = \mathbb{C}$.

Smooth proj var over $\mathbb{C}$ → projective Complex manifold → Compact Smooth manifold → Compact Topological Manifold

① equality

How close are these connections?

# ① GAGA I (Serre)

↳ Geometrié analytique &
geometrié algebrique.

**Thm:** X, Y smooth proj varieties

Alg maps $X \to Y$

$\Big\downarrow$ is bijective.

Analytic maps $X \to Y$

Every analytic map between projective analytic manifolds is algebraic!

$\underline{\mathbb{P}^1_{\mathbb{C}}} \longrightarrow \underline{\mathbb{P}^1_{\mathbb{C}}}$ analytic ?

<u>Ans</u>: $[x:y] \mapsto [F(x,y):G(x,y)]$

Homog poly.

$\|$

Algebraic maps!

<u>Contrast</u>:

$\mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ analytic?

$x \longmapsto p(x)$ poly ✓

$x \longmapsto \exp(x)$ ✓

$x \longmapsto$ Any convergent power series ✓

## GAGA 2 :

Every projective complex manifold $\hookrightarrow$ closed in $\mathbb{P}^n_{\mathbb{C}}$ is algebraic!

## GAGA :

$$\text{Smooth proj varieties}/\mathbb{C} \xrightarrow{\sim} \text{projective complex manifolds}$$

Smooth proj var / $\mathbb{C}$ $\xrightarrow{\sim}$ complex proj manifolds

$\cap$

compact complex manifold

<u>Theorems</u> (Kodaira embedding)

that tell you when a compact complex manifold can be embedded in projective space.

<u>dim 1</u> :— <u>All</u>!

Higher dimensions ← complicated.

in <u>dim 2</u> : <u>Not</u> <u>all</u>.

Smooth proj Variety / $\mathbb{C}$ $\longrightarrow$ Compact top. manifold

To what extent is this surj (A) & injective ? (B)

(A): Given a compact mfld $X$, when does it arise from algebraic geometry ?
(when is it the underlying space of a proj variety?)

Not always!

~~dim 0.~~

Easy reason :-

Every proj variety $\longrightarrow$ Orientable manifold ;

even dim.

smooth n dim variety $\rightsquigarrow$ 2n dim manifold.

X orientable, even dim.

Is X algebrizable ?

↳ there are more subtle restrictions :-

$X \rightsquigarrow$

dim $2n$

coh groups

(Topological)

$H^0(X,\mathbb{C})$

$\boxed{H^1(X,\mathbb{C})}$

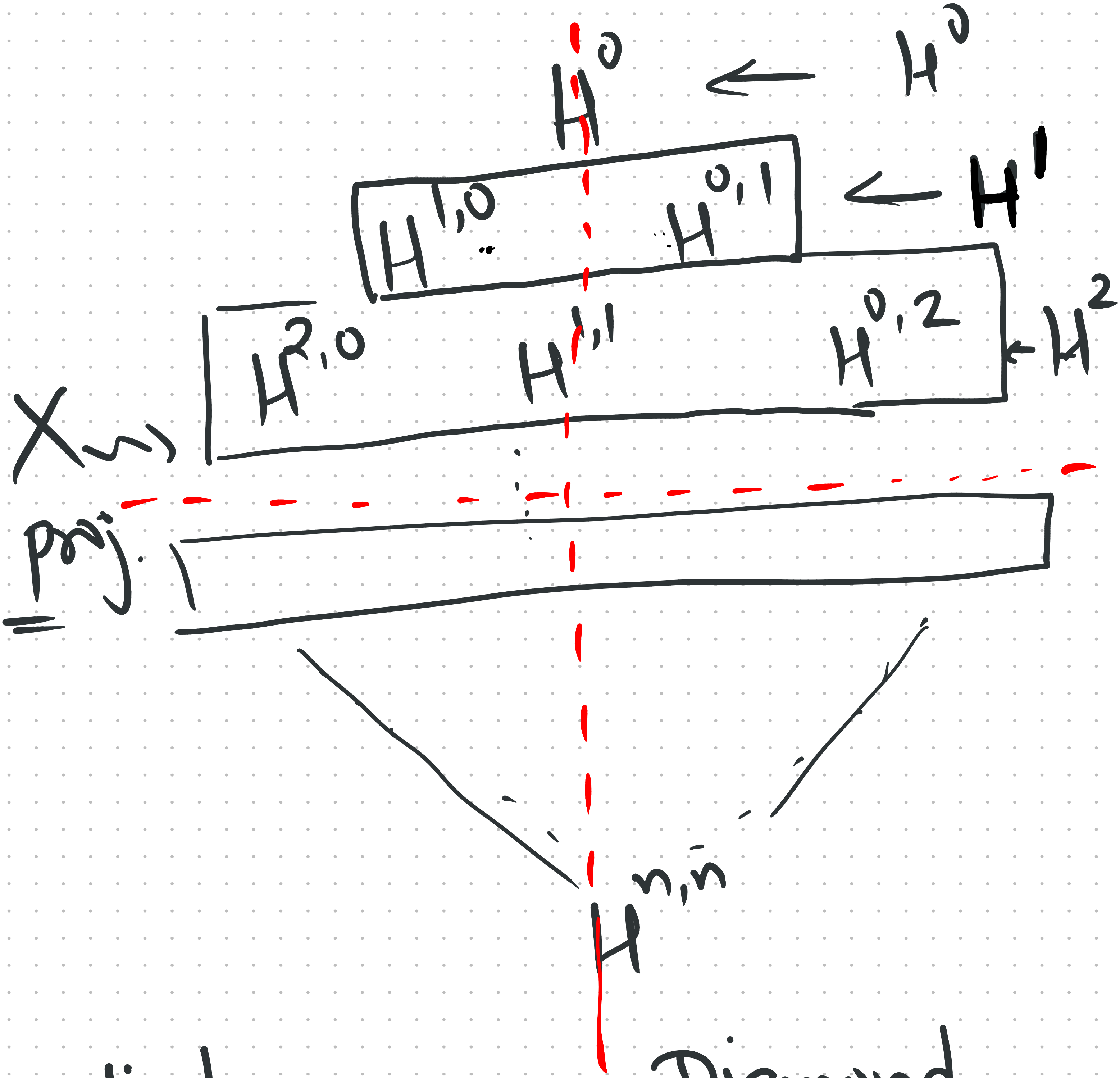
$\vdots$

$H^{2n-1}(X,\mathbb{C})$

$H^n(X,\mathbb{C})$

"palindrome"

If $X$ is projective, then the coh. groups are bi-graded.




palindrome $\longrightarrow$ Diamond

(top) (Algebraic)

"Hodge"

Ex.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\Rightarrow H^i(X,\mathbb{C})$ for $i$ odd must be even dim.

$\Downarrow$

This is a restriction on top. spaces that could be algebraic.

**Q:** Given a even dim orientable top mfold X

(A) Is X algebraic

(B) In how many ways?

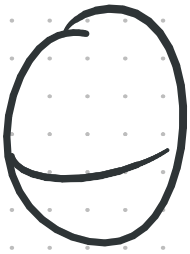
**X connected.**

$\dim_{\mathbb{C}} X = 0 \Rightarrow X = \cdot$ easy.

$\dim_{\mathbb{C}} X = 1 \Rightarrow X$ is a surface.

X is a compact orientable topological surface.

↳ Topologists have classified up to homeomorphism, these are:



genus 0

sphere.

genus 1

torus

genus 2

genus 3

, . . . . .

Given genus $g$.

$X$

(A) Is there a smooth proj var whose underlying top space is homeo to $X$? YES!

(B) How many?

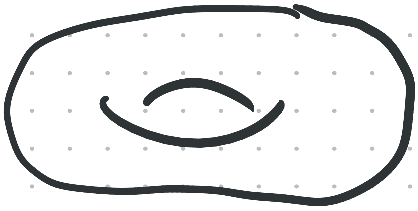
↳ Complicated

↳ depends on $g$.

$\underline{g=0}$ (B) Only one!

The only smooth proj variety whose underlying top space is a sphere is $\mathbb{P}^1$

$\underline{g=1}$ (B) Infinitely many.

$X \rightsquigarrow$ 

Thm: In this case "elliptic"

$X$ is a cubic curve in $\mathbb{P}^2$.

$$X_{a,b} = V(ZY^2 = X^3 + aXZ^2 + bZ^3)$$

() a & b are not unique

$X_{a,b}$ & $X_{a',b'}$ are isomorphic iff.

$$j(a,b) = j(a',b')$$

$$\Leftrightarrow \frac{a^3}{4a^3 + 27b^2} = \frac{a'^3}{4a'^3 + 27b'^2}$$

$\Rightarrow$ Elliptic curves are classified by the j-invariant

can be any complex number.

So: the X whose underlying top space is 

are classified up to iso by a complex number $j(X) \in \mathbb{C}$

$\vdots$

$\mathbb{C} = \mathbb{A}^1$

For $g \geq 2$.

What do alg. curves of genus 2 look like?

genus 3

$\vdots$

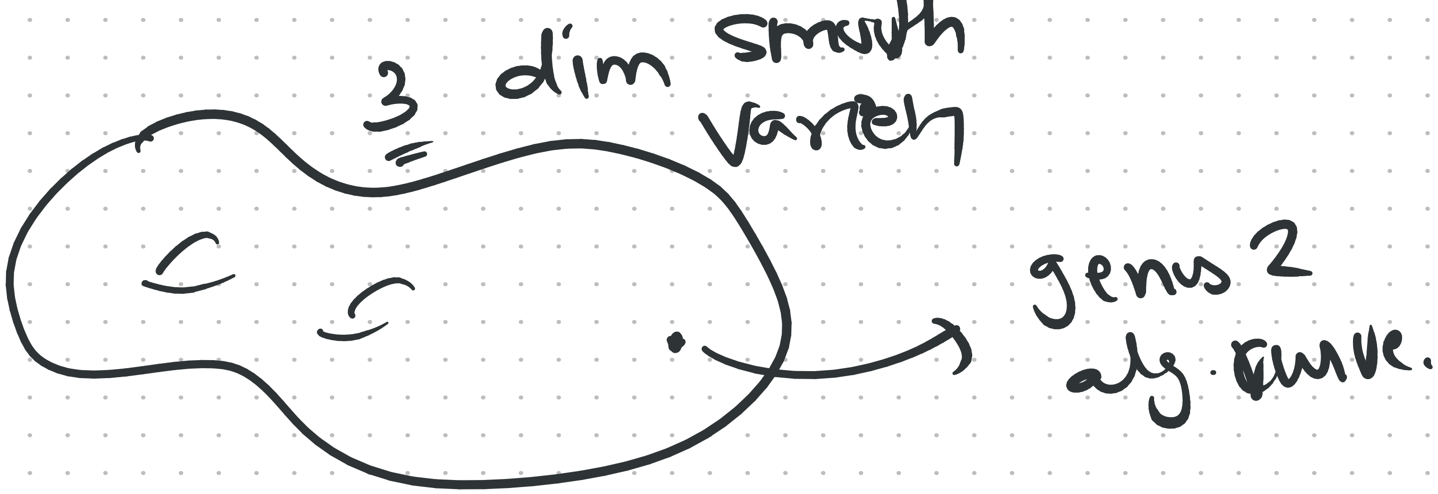
For $g \geq 2$

$$\left\{ \begin{array}{c} \text{Algebraic curves} \\ \text{"of genus } g\text{"} \end{array} \right\}.$$

$\uparrow$ param by the pts of a smooth variety* (q. proj) & dim $3g-3$.

$M_g$

$g=2$ : $\longrightarrow$ $3g-3 = 3$.



* Not quite a variety
↳ (Deligne-Mumford Stack)

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Also in higher dim.

Fix a top. type $X$.

$$\left\{ \begin{array}{c} \text{Iso classes of} \\ \text{alg. varieties} \\ \text{whose top} \\ \text{type is } X \end{array} \right\}$$

Non-trivial

themselves form an alg. variety

"Moduli space"

q-proj.