Australian National University

Student Number:

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Mathematical Sciences Institute

EXAMINATION: Semester 2 — Final 2022

MATH2322/3104/6118 — Algebra 1

Exam Duration: 120 minutes.

Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- No books, notes, reference materials are permitted.
- No electronic aids are permitted e.g. laptops, phones, calculators.

Materials To Be Supplied To Students:

- Scribble paper (last 3 pages; ask if you need more).

Instructions To Students:

- The exam is worth a total of 60 points, with the value of each question as shown.
- Unless asked otherwise, you must justify your answers. Please be neat and concise.
- You may use any result from class, homework, or workshops as long as it does not trivialise the question.
- You must cite what you use either by name ("Using the Nullstellensatz…") or by recalling the statement ("Since every maximal ideal is prime…").
- The last 3 pages are blank. You may detach them and use them for scratch work.

Q1	Q2	Q3	Q4	Q5
16	8	12	10	14

Total / 60

Question 1 *16 pts*

Give an example or state that no such example exists.
No justification is necessary.

(a) Two elements of S_4 that are in different conjugacy classes.

(b) An ideal of $\mathbf{Q}[\![t]\!]$ that is not principal.

(c) An element of $\mathbf{F}_5[x]$ that generates the ideal $\langle x^2 - 1, x^2 + 3x + 1\rangle$.
Recall that $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$.

(d) A surjective homomorphism $\langle a, b | a^2, b^2, ab\rangle \to S_3$.

(e) A nilpotent element in $\mathbf{Z}/14\mathbf{Z}$.
Recall that an element x is nilpotent if $x^n = 0$ for some positive integer n.

(f) A ring R and an R-module M that is not free.

(g) A homomorphism $\mathbf{Q}[x]/(x^2 + x + 1) \longrightarrow \mathbf{F}_3$.

(h) Let $M \subset \mathbf{Z}^3$ be the set of (a, b, c) with $a + b + c = 0$. A basis of M as a $\mathbf{Z}$-module.

Question 2 ***8 pts***

(a) Let R be a ring. Give the definition of a *prime ideal* of R.
Give an example of a ring R and prime ideal $I \subset R$ that is not a maximal ideal.
No justification is necessary. *4 pts*

(b) Let $n = 2^3 \cdot 5^2$ and let G be a group of order n. Deduce all that you can about subgroups of G of order 25. *4 pts*

Question 3 ***12 pts***

Prove or give a counterexample with justification.

(a) Every finite field is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime number p. *6 pts*

(b) Up to isomorphism, $\mathbf{Z}/210\mathbf{Z}$ is the only abelian group of order 210. *6 pts*

Question 4 ***10 pts***

Let $G = \mathrm{SL}_2(\mathbf{Z}/4\mathbf{Z})$. This is the subgroup of $\mathrm{GL}_2(\mathbf{Z}/4\mathbf{Z})$ consisting of matrices of determinant 1. Let $M = (\mathbf{Z}/4\mathbf{Z})^2$ be the set of column vectors of length 2 with entries in $\mathbf{Z}/4\mathbf{Z}$. Consider the action of G on M by multiplication on the left.

(a) Choose any non-zero element of M and explicitly describe its orbit and stabiliser.
Beyond a clear description of the orbit and the stabiliser, no justification is necessary.
5 pts

(b) Determine the order of G.
Justify your answer. *5 pts*

Question 5 — *14 pts*

Let $S \subset \mathbf{R}$ be the subset defined by

$$S = \{a + b\sqrt{2} \mid a, b \in \mathbf{Z}\}.$$

(a) Describe a homomorphism $\mathbf{Z}[x] \to \mathbf{R}$ whose image is S. What is the kernel of this homomorphism?
No justification is necessary. *4 pts*

(b) Consider the map $\phi\colon S \oplus S \to S \oplus S$ given by left multiplication by the matrix

$$\begin{pmatrix} 3+\sqrt{2} & 1 \\ 4 & \sqrt{2} \end{pmatrix}$$

Describe the cokernel of ϕ as a product of cyclic S-modules. Recall that a cyclic module is a module isomorphic to S/I for an ideal $I \subset S$.
No justification is necessary. *4 pts*

(c) Consider the ideal $I = \langle 1 + 3\sqrt{2} \rangle \subset S$. Is S/I isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime number p? If it is, provide a p and an isomorphism $S/I \to \mathbf{Z}/p\mathbf{Z}$. If not, explain why not. *6 pts*

Scratch work

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