Australian National University

Student Number:

u							

Mathematical Sciences Institute

EXAMINATION: Semester 2 — Mid-Semester, 2020

Algebra 1

Math 2322,3104,6118

Exam Duration: (24 hours, which is) 1440 minutes.
Reading Time: n/a minutes.

Materials Permitted In The Exam Venue:

- This is a take home exam.
- You may consult the textbook, your own lecture notes, your own work from assignments and workshops, and anything available on the course Wattle site.
- **You may not consult with anyone else about the exam, and you may not use any other material, including anything from other textbooks or anything you find online.**

Materials To Be Supplied To Students:

- n/a

Instructions To Students:

- I will be available via zoom from 12-1 on Wednesday if you have any questions about the exam.
- You may handwrite or LaTex your solutions. **You do not need to write your solutions on a printout of the exam, though you may if you so choose.**
- While there is no official time limit other than 24 hours, you are not expected to spend more than 3 or 4 hours on the exam. You do not need to attempt every question in order to do well.
- *When you write your proofs, please write in complete sentences, and please be neat.*

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9
5	10	5	10	10	10	10	10	10

Total / 80

Question 1 ***5 pts***

Let x and y be the standard generators of the dihedral group D_n, so that x has order n, y has order 2, and $yx = x^{-1}y$. Write the element $xy^3x^{-2}y^{-2}x^5$ in the form $x^a y^b$.

Question 2 ***10 pts***

Give an example fitting the description, or explain why no such example exists.

(a) A non-abelian group of infinite order.

(b) An abelian group of order 8.

(c) A non-abelian group of order 8.

(d) A non-abelian group of order 49.

(e) Two groups of order 121 that are not isomorphic to each other. (You should explain briefly how to tell that they are not isomorphic).

Question 3 ***5 pts***

Write down all the elements of order 2 in the symmetric group S_4.

Question 4 ***10 pts***

Let $f, g : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ be the functions

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{x-1}{x}.$$

The functions f and g are invertible; denote the subgroup of functions they generate (under composition) by K. Prove that K is isomorphic to the symmetric group S_3.

Question 5 ***10 pts***

Let $G = GL_n(\mathbb{R})$ operate on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this operation.

(b) What is the stabilizer of the standard basis vector e_1?

Question 6 ***10 pts***

Let Z denote the centre of a group G, and let $\phi : G \longrightarrow G$ be an automorphism. Show that $\phi(Z) = Z$.

Question 7 ***10 pts***

Suppose that the class equation for a group G is 1+4+5+5+5.

(a) Does G have a subgroup of order 5? If so, is it a normal subgroup?

(b) Does G have a subgroup of order 4? If so, is it a normal subgroup?

Question 8 ***10 pts***

Let J and K be normal subgroups of a group G and suppose the intersection of J and K is the trivial subgroup. Let $k \in K$. Show that for every $j \in J$, we have $jk = kj$.

Question 9 ***10 pts***

Let G be a group of order 2020. Does G necessarily contain a proper normal subgroup?